An improved Hardy-Sobolev inequality in $W^{1,p}$ and its application to Schrodinger operator

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1. Introduction. Let $0 \in \Omega \subset I\!R^n$ be a domain and $n \geq 3$. Then Hardy-Sobolev inequality states that for all $u \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u|^p - \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} \geq 0\,, \tag{1.1}$$

and $\left(\frac{n-p}{p}\right)^p$ is the best constant in (1.1) and it is never achieved. In view of this, is there a scope of improving this inequality by replacing the zero term by some nontrivial functional of u in (1)? Recently there is a considerable interest in this question and one of the important improvements is obtained by Brezis and Vázquez [7]. They showed that if Ω is a bounded domain, then there exists a constant $C > 0$ such that for all $u \in H_0^1(\Omega)$,

$$\int_\Omega |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} \geq C \int_\Omega |u|^2. \tag{1.2}$$

Furthermore if $\lambda(\Omega)$ denotes the best choice for the constant C in inequality (1.2), then $\lambda(\Omega)$ is never achieved. Again this shows that there is a scope of further improving this inequality and in this direction Brezis and Vázquez [7] raised the following question: "what is the best possible reminder term that one can expect in (1.2)?"

Recently this question was answered in [2] and [4,5], where the following inequality was proved.

Let Ω be a bounded domain and $1 < p \leq n$. Let R be large enough (to be precised later). Then there exist a constant $C > 0$ depending on n, p and $R > 0$

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such that

$$\int_\Omega |\nabla u|^p - \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} \geq C \int_\Omega \frac{|u|^p}{|x|^p \left(\log R/|x|\right)^\gamma}\,, \tag{1.3}$$

for every $u \in W_0^{1,p}(\Omega)$ if and only if
(i) $\gamma \geq 2$ when $1 < p < n$,
(ii) $\gamma \geq n$ when $p = n$.

Moreover it was shown in [3] that in case (i), $C = \frac{1}{4}$ if $p = 2$ and $\gamma = 2$ and $C = \left(\frac{n-1}{n}\right)^n$ in case (ii), these being the best possible constants. Furthermore, the perturbed eigenvalue problem corresponding to the Euler-Lagrange equations associated to (1.3) was studied in [3], where one can find the conditions on the perturbed coefficients to guarantee the existence of an eigenvalue of the corresponding operator in $H_0^1(\Omega)$. This was extended to $W_0^{1,p}(\Omega)$ by [10].

Our interest in this direction is twofold:
(A) What is the corresponding result of (1.3) in $W^{1,p}(\Omega)$ if Ω is bounded?
(B) What is the corresponding result if $\Omega = I\!R^n$?

Since the constant functions are in $W^{1,p}(\Omega)$, it is clear that some negative term should be added to the right hand side of (1.3). The first result in this direction was obtained by Lieb and Yau [9] for the case $p = 2$ when Ω is a ball. More recently this question was answered in [1] for the case $p = 2$ for every bounded domain Ω. The proof in the above paper heavily depends on (1.3) and the Hilbert space nature of $W_0^{1,2}(\Omega)$. This method will not allow an extension to $W^{1,p}$ spaces for $p \neq 2$.

In this paper we answer the two above questions by using the nature of the Green's function for the Laplace operator. In Section 3, we extend these results to arbitrary compact Riemannian manifolds. Furthermore we study the eigenvalue problem for the perturbed Schrödinger operator.

In order to state our main results, let us introduce the following notation: let $e^{(0)} = 1,\ e^{(1)} = e,\ e^{(k)} = e^{e^{(k-1)}}$ for $k \geq 1$. Let $a > 0$ and define

$$\begin{aligned} \log_{(1)} a &= \log(a), \\ \log_{(k)}(a) &= \log(\log_{(k-1)}(a))\,, \ \text{ if } a > e^{(k-1)} \ \text{ for } k \geq 2\,, \\ \log^{(k)}(a) &= \prod_{j=1}^{k} \log_{(j)}(a)\,, \ \text{ if } \ a > e^{(k-1)}. \end{aligned}$$

Finally, we define the space $\mathcal{D}^{1,2}(I\!R^n)$ by:

$$\mathcal{D}^{1,2}(I\!R^n) := \{u \in L^{2N/(N-2)}(I\!R^N)\ ;\ \nabla u \in L^2(I\!R^N)\}\,,$$

then, our main results are the following.

Theorem 1.1. Let $k \geq 1$ be an integer and $1 < p \leq n$. Then,

(1) Let Ω be a bounded domain with smooth boundary and $0 \in \Omega$. If $R > e^{(k-1)} \sup_{x\in\Omega} |x|$, there exist constants $C = C(p,n) > 0$ and $\lambda = \lambda(\Omega, p, R) < 0$ such that for all $u \in W^{1,p}(\Omega)$,

$$\begin{aligned} \int_\Omega |\nabla u|^p \ - \ \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} \geq \\ C \int_\Omega \left(\sum_{j=1}^{k} \frac{1}{\left(\log^{(j)} R/|x|\right)^2} \right) \frac{|u|^p}{|x|^p} + \lambda \int_{\partial\Omega} |u|^p\,,\ (p < n) \end{aligned} \tag{1.4}$$

$$\begin{aligned} \int_\Omega |\nabla u|^n \ - \ \left(\frac{n-1}{n}\right) \int_\Omega \frac{|u|^n}{\left(|x| \log \frac{R}{|x|}\right)^n} \geq \\ C \int_\Omega \left(\sum_{j=2}^{k} \frac{1}{\left(\log^{(j)} R/|x|\right)^n} \right) \frac{|u|^n}{|x|^n} + \lambda \int_{\partial\Omega} |u|^n;\ (p = n) \end{aligned} \tag{1.5}$$

(2) Let $R > e^{(k-1)}$ and $\rho(x) = \max(R/|x|, R|x|)$. Then there exist constants $C = C(p,n) > 0$ and $\lambda = \lambda(R) < 0$ such that for all $u \in W^{1,p}(I\!R^n)$, if $p < n$,

$$\int_\Omega |\nabla u|^p \;-\; \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} \geq$$

$$C \int_{I\!R^n} \left(\sum_{j=1}^{k} \frac{1}{\left(\log^{(j)} \rho(x)\right)^2} \right) \frac{|u|^p}{|x|^p} + \lambda \int_{|x|=1} |u|^p \,, \qquad (1.6)$$

and if $p = n$,

$$\int_{I\!R^n} |\nabla u|^n \;-\; \left(\frac{n-1}{n}\right)^n \int_{I\!R^n} \frac{|u|^n}{\left(|x| \log \rho(x)\right)^n} \geq$$

$$C \int_{I\!R^n} \left(\sum_{j=2}^{k} \frac{1}{\left(\log^{(j)} \rho(x)\right)^n} \right) \frac{|u|^n}{|x|^n} + \lambda \int_{|x|=1} |u|^n. \qquad (1.7)$$

Moreover, if $p = 2$, then $C = 1/4$ is the best constant in inequalities (1.4) to (1.7). For further improvements of these inequalities see Section 3.

Motivated by the results of [6], [3], [1], we also study the eigenvalue problem for the corresponding perturbed Schrödinger operator. Our results also extend to the p-Laplacian as it was done in [S].

Let $B(r)$ denote the ball of radius r centered at the origin and let k be some integer, $k \geq 1$ ($k \geq 2$, if $n = 2$). Let q, η, f be measurable functions on $I\!R^n$ satisfying the following hypothesis:

(H_1) $\qquad 0 \;\leq\; q(x) \leq 1,\ \eta \in L^\infty(I\!R^n \setminus B(r)), \quad \forall\, r > 0$

(H_2) $\qquad 0 \leq \eta(x) \;\leq\; \frac{1}{4} \sum_{l=1}^{k} \frac{1}{(|x| \log^{(l)} \rho(x))^2}\,,$ $\qquad$ if $n \geq 3$.

$\qquad\qquad\qquad\quad \leq\; \frac{1}{4} \sum_{l=2}^{k} \frac{1}{(|x| \log^{(l)} \rho(x))^2}\,,$ $\qquad$ if $n = 2$.

(H_3) $f \in L^{N/2}(I\!R^N) \cap L^q_{loc}(I\!R^N \setminus \{0\})$, for some $q > N/2$ and

(i) $\qquad \limsup_{\substack{|x|\to 0 \\ \text{or } |x|\to\infty}} f(x)\,|x|^2\,(\log^{(k+1)} \rho(x))^2 = 0\,,$

and there exist a constant $A > 0$, and a neighborhood U of $|x| = 1$ such that

$$(ii) \qquad f(x) \geq A \ \text{ for } x \in U\,.$$

For q, η, f and ρ as above and $u \in \mathcal{D}^{1,2}(I\!R^n)$, define

$$I_{q,\eta}(u) = \begin{cases} \int\limits_{I\!R^n} \left(|\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \frac{q\,u^2}{|x|^2} - \eta u^2 \right) & \text{if} \quad n \geq 3\,, \\ \\ \int\limits_{I\!R^n} \left(|\nabla u|^2 - \frac{1}{4} \frac{q\,u^2}{(|x| \log \rho(x))^2} - \eta u^2 \right) & \text{if} \quad n = 2\,, \end{cases} \tag{1.8}$$

$$L_{q,\eta}(u) = \begin{cases} -\Delta u - \left(\frac{n-2}{2}\right)^2 \frac{qu}{|x|^2} - \eta u & \text{if} \quad n \geq 3\,, \\ \\ -\Delta u - \frac{1}{4} \frac{qu}{(|x| \log \rho(x))^2} - \eta u & \text{if} \quad n = 2\,, \end{cases} \tag{1.9}$$

$$\lambda_1(q, \eta) = \inf \left\{ I_{q,\eta}(u); \int\limits_{I\!R^n} f u^2 = 1 \right\}\,, \tag{1.10}$$

and consider the following eigenvalue problem

$$\begin{cases} L_{q,\eta}\phi &= \lambda f \phi \quad \text{in } I\!R^n\,, \\ \\ \phi &\in \mathcal{D}'(I\!R^n)\,, \quad \phi \neq 0 \end{cases} \tag{1.11}$$

Then we have the following

Theorem 1.2. Let $0 < s < 1$ and q, η, f satisfy (H_1) to (H_3). Then,
(i) The spectrum of $L_{sq,s\eta}$ in $L^2(f)$ is discrete.

(ii) Let $\lambda_1(sq, s\eta) \leq \lambda_2(sq, s\eta) \leq \ldots$ be the eigenvalues of $L_{sq,s\eta}$ and $\phi_1(sq, s\eta)$, $\phi_2(sq, s\eta), \ldots$ be the corresponding eigenfunctions. Then after a normalization, we have:

$$\varphi_1(sq, s\eta) > 0\,, \quad \varphi_i(sq, s\eta) \underset{s \to 1}{\longrightarrow} \varphi_i(q, \eta) \quad \text{in } W^{1,r}_{loc}(I\!R^n) \quad \text{for all } 1 < r < 2\,, \quad i \geq 1\,,$$

$$\lambda_i(sq, s\eta) \underset{s \to 1}{\longrightarrow} \lambda_i(q, \eta) \quad \text{for all} \quad i \geq 1\,,$$

where $(\phi_i(q, \eta), \lambda_i(q, \eta))$ satisfies (1.11).

(iii) Let q satisfy

$$\liminf_{\substack{|x|\to 0\\ \text{or } |x|\to\infty}} \begin{cases} (\log\rho(x))^2\left\{(1-q(x))-\frac{4|x|^2}{(n-2)^2}\eta(x)\right\} > \frac{3}{(n-2)^2}\,, & \text{if } n\geq 3\,,\\ (\log\log\rho(x))^2\{(1-q(x))-4|x|^2(\log\rho(x))^2\eta(x)\} > 3\,, & \text{if } n=2, \end{cases} \tag{1.12}$$

then after a normalization $\phi_i(sq,s\eta)\xrightarrow[s\to 1]{}\phi_i(q,\eta)$ in $\mathcal{D}^{1,2}(I\!R^n)$ for all $i\geq 1$.

(iv) Let $R_1>0$ and $\eta\geq 0$ in $B(R_1)$ or $I\!R^n\setminus B(R_1)$. Let q satisfy either

$$\sup_{x\in B(R_1)} \begin{cases} (\log\rho(x))^2((1-q(x))-\frac{4}{(n-2)^2}|x|^2\eta(x))\leq 3/(n-2)^2 & \text{if} \quad n\geq 3\,,\\ (\log\log\rho(x))^2((1-q(x))-4(|x|\log\rho(x))^2\eta(x))\leq 3 & \text{if} \quad n=2\,, \end{cases} \tag{1.13}$$

or

$$\sup_{x\in I\!R^n\setminus B(R_1)} \begin{cases} (\log\rho(x))^2((1-q(x))-\frac{4}{(n-2)^2}|x|^2\eta(x)) \leq \\ \qquad\qquad 3/(n-2)^2\,, & \text{if } n\geq 3\,,\\ (\log\log\rho(x))^2((1-q(x))-4(|x|\log\rho(x))^2\eta(x))\leq \\ \qquad\qquad 3\,, & \text{if } n=2, \end{cases} \tag{1.14}$$

then (1.11) does not admit any solution $\phi\in\mathcal{D}^{1,2}(I\!R^n)$, $\phi\geq 0$.

(v) Assume that η and f satisfy the stronger assumption,

$$\lim_{\substack{|x|\to 0\\ \text{or } |x|\to\infty}} \begin{cases} (\log\rho(x))^{-1}(\eta(x)+|f(x)|) &=& 0\,, \text{ if } n\geq 3\,,\\ (\log\log\rho(x))^{-1}(\eta(x)+|f(x)|) &=& 0\,, \text{ if } n=2\,, \end{cases} \tag{1.15}$$

and let $\phi_{1,s}>0$ be the normalized eigenfunction corresponding to $\lambda_1(s,\eta)$. If $R_1>0$, there exist two constants $C_1,C_2>0$ such that for $x\in B(R_1)$,

$$\begin{aligned} (C_1,C_1) &\leq |x|^{\frac{n-2}{2}[1-\sqrt{1-s}]}(\phi_{1,s}(x),|x||\nabla\phi_{1,s}(x)|)\leq (C_2,C_2)\,, \quad \text{if } n\geq 3\,,\\ (C_1,C_1) &\leq (\log R/|x|)^{\frac{1}{2}(-1+\sqrt{1-s})}(\phi_{1,s}(x),|x||\nabla\phi_{1,s}(x)|)\leq (C_2,C_2)\,, \quad \text{if } n=2, \end{aligned} \tag{1.16}$$

and for $x \in I\!R^n \setminus B(R_1)$,

$$\begin{aligned}(C_1, C_1) &\leq |x|^{\frac{n-2}{2}[1+\sqrt{1-s}]}\ (\phi_{1,s}(x), |x||\nabla\phi_{1,s}(x)|) \leq (C_2, C_2)\,, \ \text{ if } n \geq 3\,,\\ (C_1, C_1) &\leq (\log R|x|)^{-\frac{1}{2}(1+\sqrt{1-s})}\ (\phi_{1,s}(x),\ |x||\nabla\phi_{1,s}(x)|) \leq (C_2, C_2)\,, \ \text{ if } n = 2.\end{aligned} \tag{1.17}$$

2. Proof of the theorems.

First recall some of the basic results which are needed in the proof of the theorems. Let $1 < p < \infty$ and $M > 1$. There exist constants $\alpha_1, \alpha_2 > 0$ such that for all $a, b \in I\!R^n, |a| = 1$ we have

$$|a+b|^p - 1 - p\langle a, b\rangle \geq \alpha_1|b|^2 + \alpha_2|b|^p\,, \quad \text{if } \ p \geq 2\,, \tag{2.1}$$

$$|a+b|^p - 1 - p\langle a, b\rangle \geq \begin{cases} \alpha_1|b|^2 & \text{if} \quad |b| \leq M, \quad 1 < p \leq 2\,, \\ \alpha_2|b|^p & \text{if} \quad |b| \geq M, \quad 1 < p < 2\,. \end{cases} \tag{2.2}$$

Furthermore if $p = 2$, then $\alpha_1 = \alpha_2 = \frac{1}{2}$ is the best choice in (2.1)-(2.2). Note also that a simple computation shows that if $|S^{n-1}|$ denotes the volume of the unit sphere S^{n-1} in $I\!R^n$, then for all $k \geq 1$,

$$\text{div}\ \left(\frac{x}{|x|^n}\right) = -\ \text{div}\ (\frac{\log^{(k)} R/|x|}{|x|^{n-2}}\nabla \log_{(k+1)} R/|x|) = |S^{n-1}|\,\delta_0, \tag{2.3}$$

where δ_0 is the Dirac distribution at the origin.

Lemma 2.1. Let R and ρ be as in Theorem 1.1. Let $w_1 \in C^1(\overline{\Omega}), W_1 \in C_0^1(I\!R^n)$ and define the sequence $\{w_i\}_i$ and $\{W_i\}_i$ by

$$w_2 = (\log R/|x|)^{-1/2} w_1\,,\ \ldots, w_k = (\log_{(k-1)} R/|x|)^{-1/2} w_{k-1}\,, \ldots$$

$$W_2 = (\log \rho(x))^{-1/2} W_1\,, \ldots, W_k = (\log_{(k-1)} \rho(x))^{-1/2} W_{k-1}\,, \ldots$$

The following identities hold:

$$\begin{aligned}\int_\Omega \frac{|\nabla w_1|^2}{|x|^{n-2}} = \frac{1}{4}\int_\Omega \left(\sum_{j=1}^k \frac{1}{(\log^{(j)} R/|x|)^2}\right)\frac{w_1^2}{|x|^n} - \frac{1}{2}\int_{\partial\Omega}\left(\sum_{j=1}^k \frac{\langle x\cdot\nu\rangle}{(\log^{(j)} R/|x|)}\right)\frac{w_1^2}{|x|^n} \\ + \int_\Omega \frac{(\log^{(k)} R/|x|)|\nabla w_{k+1}|^2}{|x|^{n-2}}\,,\end{aligned} \tag{2.4}$$

$$\int\limits_{I\!R^n} \frac{|\nabla W_1|^2}{|x|^{n-2}} = \frac{1}{4}\int\limits_{I\!R^n} \left(\sum_{j=1}^{k} \frac{1}{(\log^{(j)}\rho(x))^2}\right) \frac{W_1^2}{|x|^n} - \frac{1}{2}(\sum_{j=1}^{k} \frac{1}{(\log^{(j)} R)}) \int\limits_{|x|=1} W_1^2$$
$$+ \int\limits_{I\!R^n} (\log^{(k)}\rho(x)) \frac{|\nabla W_{k+1}|^2}{|x|^{n-2}}. \qquad (2.5)$$

Proof. We will only prove one of the equalities since the other one follows in the same way. Let us prove (2.5). Let us start with the case $k = 1$ and then (2.5) will follow by induction on k. Since

$$\nabla \log \rho(x) = \begin{cases} -\frac{x}{|x|^2} & \text{if} \quad |x| < 1 \\ \frac{x}{|x|^2} & \text{if} \quad |x| > 1\,, \end{cases}$$

by (2.3) and $W_2(0) = 0$ we have

$$\begin{aligned}
\int\limits_{I\!R^n} \tfrac{|\nabla W_1|^2}{|x|^{n-2}} &= \int\limits_{I\!R^n} \tfrac{|\nabla(\log\rho(x))^{1/2} W_2 + (\log\rho(x)^{1/2}\nabla W_2|^2}{|x|^{n-2}} \\
&= \tfrac{1}{4}\int\limits_{I\!R^n} \tfrac{(\log\rho(x))^{-1} W_2^2}{|x|^n} - \tfrac{1}{2}\int\limits_{|x|<1} \langle \tfrac{x}{|x|^n}, \nabla W_2^2\rangle \\
&\quad + \tfrac{1}{2}\int\limits_{|x|>1} \langle \tfrac{x}{|x|^n}, \nabla W_2^2\rangle + \int\limits_{I\!R^n} \tfrac{\log\rho(x)|\nabla W_2|^2}{|x|^{n-2}} \\
&= \tfrac{1}{4}\int\limits_{I\!R^n} \tfrac{W_1^2}{|x|^n(\log\rho(x))^2} - \int\limits_{|x|=1} \tfrac{\langle x\cdot\nu\rangle}{|x|^n} W_2^2 + \int\limits_{I\!R^n} \tfrac{(\log\rho(x))|\nabla W_2|^2}{|x|^{n-2}} \\
&= \tfrac{1}{4}\int\limits_{I\!R^n} \tfrac{W_1^2}{|x|^n(\log\rho(x))^2} - \tfrac{1}{\log R}\int\limits_{|x|=1} W_1^2 + \int\limits_{I\!R^n} \tfrac{(\log\rho(x))|\nabla W_2|^2}{|x|^{n-2}}.
\end{aligned}$$

Now observe that $\nabla(\log_{(k)}\rho(x)) = \left(\frac{1}{\log^{(k-1)}\rho(x)}\right)\nabla\log\rho(x)$. Hence,

$$\begin{aligned}
\int\limits_{I\!R^n} \tfrac{|\nabla W_{k+1}|^2 \log^{(k)}\rho(x)}{|x|^{n-2}} &= \int\limits_{I\!R^n} \tfrac{\log^{(k)}\rho(x)}{|x|^{n-2}} |\nabla(\log_{(k+1)}\rho(x))^{1/2} W_{k+2} + \\
&\quad (\log_{(k+1)}\rho(x))^{1/2}\nabla W_{k+2})^2 \\
&= \tfrac{1}{4}\int\limits_{I\!R^n} \tfrac{W_{k+2}^2}{|x|^n \log^{(k+1)}\rho(x)} - \tfrac{1}{2}\int\limits_{|x|<1} \langle \tfrac{x}{|x|^n}, \nabla W_{k+2}^2\rangle \\
&\quad + \tfrac{1}{2}\int\limits_{|x|>1} \langle \tfrac{x}{|x|^n}, \nabla W_{k+2}^2\rangle + \int\limits_{I\!R^n} \tfrac{(\log^{(k+1)}\rho(x))|\nabla W_{k+2}|^2}{|x|^{n-2}} \\
&= \tfrac{1}{4}\int\limits_{I\!R^n} \tfrac{W_1^2}{|x|^n(\log^{(k+1)}\rho(x))^2} - \tfrac{1}{(\log^{(k+1)} R)}\int\limits_{|x|=1} W_1^2 + \\
&\quad \int\limits_{I\!R^n} \tfrac{(\log^{(k+1)}\rho(x))|\nabla W_{k+2}|^2}{|x|^{n-2}}
\end{aligned}$$

Hence by induction (2.5) follows. This proves the lemma. □

Proof of Theorem 1.1. We will prove this in several steps.

Step 1. Let $2 \le p < n$ and $0 \le u \in C^1(\overline{\Omega})$. Let $v = |x|^{\frac{n-p}{p}} u$, then $v(0) = 0$ and from (2.1),

$$\begin{aligned}
|\nabla u|^p &= \left| -\left(\tfrac{n-p}{p}\right)|x|^{-\left(\frac{n-p}{p}\right)-1}\tfrac{x}{|x|}\,v + |x|^{-\left(\frac{n-p}{p}\right)}\nabla v\right|^p \\
&= \left(\tfrac{n-p}{p}\right)^p |x|^{-n} v^p \left|\tfrac{x}{|x|} - \left(\tfrac{p}{n-p}\right)\tfrac{|x|\nabla v}{v}\right|^p \\
&\ge \left(\tfrac{n-p}{p}\right)^p \tfrac{v^p}{|x|^n} \\
&\quad \left\{1 - p\left(\tfrac{p}{n-p}\right)\tfrac{\langle x,\nabla v\rangle}{v} + \alpha_1\left(\tfrac{p}{n-p}\right)^2 \tfrac{|x|^2|\nabla v|^2}{v^2} + \alpha_2\left(\tfrac{p}{n-p}\right)^p \tfrac{|x|^p|\nabla v|^p}{v^p}\right\} \\
&= \left(\tfrac{n-p}{p}\right)^p \tfrac{v^p}{|x|^n} - \left(\tfrac{n-p}{p}\right)^{p-1}\langle \tfrac{x}{|x|^n}, \nabla v^p\rangle + \\
&\quad \tfrac{4\alpha_1}{p^2}\left(\tfrac{n-p}{p}\right)^{p-2}\tfrac{|\nabla v^{(p/2)}|^2}{|x|^{n-2}} + \alpha_2\tfrac{|\nabla v|^p}{|x|^{n-p}} \\
&\ge \left(\tfrac{n-p}{p}\right)^p \tfrac{u^p}{|x|^p} - \left(\tfrac{n-p}{p}\right)^{p-1}\langle \tfrac{x}{|x|^n}, \nabla v^p\rangle + \tfrac{4\alpha_1}{p^2}\left(\tfrac{n-p}{p}\right)^{p-2}\tfrac{|\nabla v^{(p/2)}|^2}{|x|^{n-2}}\,.
\end{aligned} \tag{2.6}$$

We now integrate the above inequality over Ω, taking $w_1 = v^{p/2}$ in (2.4) and remembering that $v(0) = 0$. We thus obtain

$$\begin{aligned}
&\int_\Omega |\nabla u|^p - \left(\tfrac{n-p}{p}\right)^p \tfrac{u^p}{|x|^p} \ge -\left(\tfrac{n-p}{p}\right)^{p-1}\int_\Omega \langle \tfrac{x}{|x|^n}, \nabla v^p\rangle + \tfrac{4\alpha_1}{p^2}\left(\tfrac{n-p}{p}\right)^{p-2}\int_\Omega \tfrac{|\nabla w_1|^2}{|x|^{n-2}} \\
&\ge -\left(\tfrac{n-p}{p}\right)^{p-1} |S^{n-1}|\, v^p(0) - \left(\tfrac{n-p}{p}\right)^{p-1}\int_{\partial\Omega}\tfrac{\langle x\cdot\nu\rangle}{|x|^n}v^p \\
&\quad + \tfrac{4\alpha_1}{p^2}\left(\tfrac{n-p}{p}\right)^{p-2}\left\{\tfrac{1}{4}\int_\Omega\left(\sum_{j=1}^k \tfrac{1}{(\log^{(j)} R/|x|)^2}\right)\tfrac{w_1^2}{|x|^n} - \tfrac{1}{2}\int_{\partial\Omega}\left(\sum_{j=1}^k \tfrac{\langle x\cdot\nu\rangle}{\log^{(j)} R/|x|}\right)\tfrac{w_1^2}{|x|^n}\right\} \\
&\ge C_1\int_\Omega\left(\sum_{j=1}^k \tfrac{1}{(\log^{(j)} R/|x|)^2}\right)\tfrac{u^p}{|x|^p} + \lambda\int_{\partial\Omega} u^p,
\end{aligned}$$

where $C_1 = \frac{\alpha_1}{p^2}\left(\frac{n-p}{p}\right)^{p-2}$,

and $-\lambda = \sup\limits_{x\in\partial\Omega}\frac{|\langle x\cdot\nu\rangle|}{|x|^n}\left\{\left(\frac{n-p}{p}\right)^{p-1} + \frac{2\alpha_1}{p^2}\left(\frac{n-p}{p}\right)^{p-2}\sum\limits_{j=1}^k \frac{1}{\log^{(j)} R/|x|}\right\}$.

Step 2. Let $p = n$ and $0 \le u \in C^1(\overline{\Omega})$ and define $v = (\log R/|x|)^{\frac{1-n}{n}} u$. Then

$v(0) = 0$ and from (2.1) we have

$$\begin{aligned}
|\nabla u|^p &= \left| -\left(\tfrac{n-1}{n}\right)(\log R/|x|)^{-1/n}\tfrac{x}{|x|^2}v + (\log R/|x|)^{\frac{n-1}{n}}\nabla v\right|^n \\
&= \left(\tfrac{n-1}{n}\right)^n \tfrac{(\log R/|x|)^{-1}}{|x|^n}\, v^n \left|\tfrac{x}{|x|} - \tfrac{|x|(\log R/|x|)\nabla v}{v}\right|^n \\
&\geq \left(\tfrac{n-1}{n}\right)\tfrac{(\log R/|x|)^{-1}}{|x|^n}\, v^n \left\{1 - n(\log R/|x|)\,\langle x, \tfrac{\nabla v}{v}\rangle\right. \\
&\quad \left. +\alpha_1 |x|^2 (\log R/|x|)^2 \tfrac{|\nabla v|^2}{v^2} + \alpha_2 |x|^n (\log R/|x|)^n \tfrac{|\nabla v|^n}{v^n}\right\} \\
&\geq \left(\tfrac{n-1}{n}\right)^n \left(\log \tfrac{R}{|x|}\right)^{-1} \tfrac{v^n}{|x|^n} - \left(\tfrac{n-1}{n}\right)^n \langle \tfrac{x}{|x|}, \nabla v^n\rangle \\
&\quad +\tfrac{4\alpha_1}{n^2}(\tfrac{n-1}{n})^n (\log R/|x|)\,|\nabla(v^{n/2})|^2 + \alpha_2 \left(\tfrac{n-1}{n}\right)^n (\log R/|x|)^{n-1}\,|\nabla v|^n.
\end{aligned} \tag{2.7}$$

Hence using (2.4) as in Step 1 we finally find

$$\int_\Omega \left(|\nabla u|^n - \left(\frac{n-1}{n}\right)^n \frac{u^n}{|x|^n\,(\log R/|x|)^n}\right) \geq C_1 \int_\Omega \left(\sum_{j=2}^k \frac{1}{(\log^{(j)} R/|x|)^n}\right)\frac{u^n}{|x|^n} + \lambda \int_{\partial\Omega} u^n.$$

Step 3. Let $1 < p < 2$. In this case the analysis is more delicate and we use Sard's Lemma. To begin with we assume that $u \in C^\infty(\overline{\Omega})$. Let $v = |x|^{\frac{n-p}{p}} u$ and choose $M > 1$ such that

$$M^{p-1} > \frac{2\,\alpha_1}{p\,\alpha_2}\left(\frac{n-p}{p}\right)^{p-2} \sup_{x\in\overline{\Omega}} \left(\sum_{j=1}^k \frac{1}{\log^{(j)} R/|x|}\right).$$

By perturbing v, and if necessary $\partial\Omega$ also, we can assume that M^2 is a regular value for $\frac{|\nabla v|^2|x|^2}{v^2}$. Let

$$\begin{aligned}
\Omega^+ &= \left\{x \in \overline{\Omega} : \tfrac{|x|\,|\nabla v|}{v} < M\right\} \\
\Omega^- &= \left\{x \in \overline{\Omega} : \tfrac{|x|\,|\nabla v|}{v} \geq M\right\} \\
\Gamma &= \left\{x \in \overline{\Omega} : \tfrac{|x|\,|\nabla v|}{v} = M\right\}
\end{aligned}$$

Furthermore we can assume that $0 \in \Omega^+$ and that Γ intersects $\partial\Omega$ transversely. Hence, from (2.2), we have

$$|\nabla u|^p \geq \left(\frac{n-p}{p}\right)^p |x|^{-n} v^p \left\{1 - p\left(\frac{p}{n-p}\right)\frac{\langle x, \nabla v\rangle}{v} + \alpha_2 \left(\frac{p}{n-p}\right)^p \frac{|x|^p |\nabla v|^p}{v^p}\chi_{\Omega^-} \right.$$
$$\left. + \alpha_1 \left(\frac{p}{n-p}\right)^2 \frac{|x|^2 |\nabla v|^2}{v^2}\chi_{\Omega^+}\right\}$$

Integrating this we obtain

$$\int_\Omega |\nabla u|^p - \left(\frac{n-p}{p}\right)^p \frac{u^p}{|x|^p} \geq -\left(\frac{n-p}{p}\right)^{(p-1)} \int_{\partial\Omega} \frac{\langle x\cdot\nu\rangle\, v^p}{|x|^n} \;+\; \alpha_2 \int_{\Omega^-} \frac{|\nabla v|^p}{|x|^{n-p}}$$
$$+ \frac{4\alpha_1}{p^2}\left(\frac{n-p}{p}\right)^{p-2} \int_{\Omega^+} \frac{|\nabla v^{p/2}|^2}{|x|^{n-2}}. \tag{2.8}$$

Let $A = \frac{4\alpha_1}{p^2}\left(\frac{n-p}{p}\right)^{p-2}$. Then, from (2.4) we have

$$\begin{aligned}\int_\Omega |\nabla u|^p \;-\; & \left(\tfrac{n-p}{p}\right)^p \tfrac{u^p}{|x|^p} + \left(\tfrac{n-p}{p}\right)^{p-1} \int_{\partial\Omega} \tfrac{\langle x\cdot\nu\rangle u^p}{|x|^p} \geq A \int_{\Omega^+} \tfrac{|\nabla v^{p/2}|^2}{|x|^{n-2}} + \alpha_2 \int_{\Omega^-} \tfrac{|\nabla v|^p}{|x|^{n-p}} \\ & \geq \tfrac{A}{4} \int_{\Omega^+} \left(\sum_{j=1}^k \tfrac{1}{(\log^{(j)} R/|x|)^2}\right) \tfrac{v^p}{|x|^n} - \tfrac{A}{2} \int_{\Omega^+\cap\partial\Omega} \left(\sum_{j=1}^k \tfrac{\langle x,\nu\rangle}{\log^{(j)} R/|x|}\right) \tfrac{v^p}{|x|^n} \\ & -\tfrac{A}{2} \int_\Gamma \left(\sum_{j=1}^k \tfrac{\langle x\cdot\nu\rangle}{(\log^{(j)} R/|x|)}\right) \tfrac{v^p}{|x|^n} + \alpha_2 \int_{\Omega^-} \tfrac{|\nabla v|^p}{|x|^{n-p}}.\end{aligned}$$

Since ν is the outer normal on Γ with respect to Ω^+, it is the inner normal on Γ with respect to Ω^- and we have

$$\begin{aligned}\int_{\Omega^-} \langle \frac{x}{|x|^n}, \;\; \nabla \;\; & \left(\sum_{j=1}^k \frac{v^p}{(\log^{(j)} R/|x|)}\right)\rangle = \\ & -\int_\Gamma \left(\sum_{j=1}^k \frac{\langle x\cdot\nu\rangle}{\log^{(j)} R/|x|}\right)\frac{v^p}{|x|^n} + \int_{\Omega^-\cap\partial\Omega} \left(\sum_{j=1}^k \frac{\langle x,\nu\rangle}{\log^{(j)} R/|x|}\right)\frac{v^p}{|x|^n}.\end{aligned}$$

Let J denote the LHS of the previous inequality. Then substituting the above identity in the previous inequality one gets,

$$\begin{aligned} J \;\; \geq \;\; & \tfrac{A}{4} \int_{\Omega^+} \left(\sum_{j=1}^k \tfrac{1}{(\log^{(j)} R/|x|)^2}\right) \tfrac{v^p}{|x|^n} + \tfrac{A}{2} \int_{\Omega^-} \left(\sum_{j=1}^k \tfrac{1}{(\log^{(j)} R/|x|)^2}\right) \tfrac{v^p}{|x|^n} - \\ & \tfrac{A}{2} \int_{\partial\Omega} \left(\sum_{j=1}^k \tfrac{\langle x\cdot\nu\rangle}{\log^{(j)} R/|x|}\right) \tfrac{v^p}{|x|^n} + \int_{\Omega^-} \left\{ \tfrac{pA}{2} \left(\sum_{j=1}^k \tfrac{1}{\log^{(j)} R/|x|}\right) \tfrac{\langle x\cdot\nabla v\rangle\, v^{p-1}}{|x|^n} + \alpha_2 \tfrac{|\nabla v|^p}{|x|^{n-p}} \right\}.\end{aligned}$$

Now on Ω^-, $\frac{|x|\,|\nabla v|}{v} \geq M$. Therefore on Ω^-,

$$\begin{aligned}
\alpha_2 \tfrac{|\nabla v|^p}{|x|^{n-p}} &+ \tfrac{pA}{2}\left(\sum_{j=1}^{k} \tfrac{1}{(\log^{(j)} R/|x|)}\right) \tfrac{\langle x\cdot\nabla v\rangle\, v^{p-1}}{|x|^n} \\
&\geq \tfrac{|\nabla v|\, v^{p-1}}{|x|^{n-1}} \left\{ \alpha_2 \left(\tfrac{|x|\,|\nabla v|}{v}\right)^{p-1} - \left(\tfrac{pA}{2}\right) \sum_{j=1}^{k} \tfrac{1}{\log^{(j)} R/|x|} \right\} \\
&\geq \tfrac{|\nabla v|\, v^{p-1}}{|x|^{n-1}} \left\{ \alpha_2 M^{p-1} - \left(\tfrac{pA}{2}\right) \sum_{j=1}^{k} \tfrac{1}{\log^{(j)} R/|x|} \right\} \\
&\geq 0.
\end{aligned}$$

So,

$$J \geq \frac{A}{4} \int\limits_{\Omega} \left(\sum_{j=1}^{k} \frac{1}{(\log^{(j)} R/|x|)^2}\right) \frac{v^p}{|x|^n} - \frac{A}{2} \int\limits_{\partial\Omega} \left(\sum_{j=1}^{k} \frac{\langle x\cdot\nu\rangle}{\log^{(j)} R/|x|}\right) \frac{v^p}{|x|^n}.$$

This proves the theorem. □

In order to prove Theorem 1.2, we need a few more lemmas.

Lemma 2.2. Let f satisfy (H_3). Then there exists a constant $\widetilde{\lambda} \in \mathbb{R}$ such that for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$: if $n \geq 3$,

$$\int\limits_{\mathbb{R}^n} \left(|\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \frac{u^2}{|x|^2}\right) \geq \frac{1}{4} \int\limits_{\mathbb{R}^n} \left(\sum_{j=1}^{k} \frac{1}{\left(\log^{(j)} \rho(x)\right)^2}\right) \frac{u^2}{|x|^2} + \widetilde{\lambda} \int\limits_{\mathbb{R}^n} f u^2\,, \tag{2.9}$$

and if $n = 2$,

$$\int\limits_{\mathbb{R}^2} \left(|\nabla u|^2 - \frac{1}{4} \frac{u^2}{|x|^2 \log \rho(x)^2}\right) \geq \frac{1}{4} \int\limits_{\mathbb{R}^2} \left(\sum_{j=2}^{k} \frac{1}{(\log^{(j)} \rho(x))^2}\right) \frac{u^2}{|x|^2} + \widetilde{\lambda} \int\limits_{\mathbb{R}^n} f u^2. \tag{2.10}$$

Proof. It is enough to prove (2.9) and (2.10) will follow in a similar way. Observe that by assumption $(H_3)(i)$, the embedding $\mathcal{D}^{1,2}(\mathbb{R}^n) \hookrightarrow L^2(f)$ is compact. Moreover, from $(H_3)(ii)$, for every $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^n)$,

$$\int\limits_{|x|=1} u^2 \leq \varepsilon \int\limits_{\mathbb{R}^n} |\nabla u|^2 + C(\varepsilon) \int\limits_{\mathbb{R}^n} f u^2\,.$$

Let $s = \frac{1}{(1-\lambda\varepsilon)}$. Since $\lambda < 0$, $s \in (0,1)$. Then from (1.6) and from the above inequality we obtain

$$\int_{I\!R^n} |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 s \int_{I\!R^n} \frac{u^2}{|x|^2} - \frac{s}{4} \int_{I\!R^n} \left(\sum_{j=1}^{k} \frac{1}{(\log^{(j)} \rho(x))^2} \right) \frac{u^2}{|x|^2} \geq \lambda\, C(\varepsilon) \int_{I\!R^n} f u^2 \,. \tag{2.11}$$

Let $A_k(x) = \frac{1}{4} \sum_{j=1}^{k} \frac{1}{(\log^{(j)} \rho(x))^2}$ and

$$\lambda(s) = \inf \left\{ \int_{I\!R^n} |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 s \frac{u^2}{|x|^2} - s A_k(x) \frac{u^2}{|x|^2}); \quad \int_{I\!R^n} f u^2 = 1 \right\}.$$

From (2.11) and by the definition of $\lambda(s)$, there exists a constant C independent of s such that $\lambda\, C(\varepsilon) \leq \lambda(s) < C$. Moreover, from (1.6), for any $\delta > 0$, there exists $C(\delta) > 0$ such that for all $u \in \mathcal{D}^{1,2}(I\!R^n)$,

$$\begin{aligned} \int_{I\!R^n} |\nabla u|^2 - \left(\tfrac{n-2}{2}\right)^2 s \tfrac{u^2}{|x|^2} - s A_k(x) \tfrac{u^2}{|x|^2} &\geq (1-s) \int_{I\!R^n} |\nabla u|^2 + s\lambda \int_{|x|=1} u^2 \\ &\geq (1-s) \int_{I\!R^n} |\nabla u|^2 + s\lambda\delta \int_{I\!R^n} |\nabla u|^2 + \\ &\quad C(\varepsilon) C(\delta) s\lambda \int_{I\!R^n} f u^2 \\ &= (1 - s + s\lambda\delta) \int_{I\!R^n} |\nabla u|^2 + \\ &\quad C(\varepsilon)\, C(\delta)\, s\, \lambda \int_{I\!R^n} f u^2 \end{aligned}$$

Hence if $\{u_k\}$ is a minimizing sequence for $\lambda(s)$, then from the above inequality $\{u_k\}$ is bounded and hence $u_k \rightharpoonup u_s$ weakly in $\mathcal{D}^{1,2}(I\!R^n)$ for some u_s. Since the embedding $\mathcal{D}^{1,2} \hookrightarrow L^2(f)$ is compact, u_s is a minimizer for $\lambda(s)$ and satisfies

$$\begin{cases} -\Delta u_s - \left(\frac{n-2}{2}\right)^2 \frac{s u_s}{|x|^2} - \frac{s A_k u_s}{|x|^2} = \lambda(s) f u_s \\ \int_{I\!R^n} f u_s^2 = 1\,. \end{cases}$$

From (H_3), choose $0 < \delta < 1$ such that $f(x) \geq A$ for $x \in D(\delta) = \{x; 1-\delta < |x| < 1+\delta\}$. Let $0 \leq \varphi \in C_0^\infty(D(\delta))$ such that $\varphi = 1$ on $D(\delta/2)$. Since $\int_{I\!R^n} f u_s^2 = 1$, $\int_{D(\delta)} u_s^2 \leq 1/A$. Hence multiply the above equation by $\varphi^2 u_s$ and integrate by parts to obtain

$$\int_{D(\delta/2)} |\nabla u_s|^2 \leq \int_{I\!R^n} \varphi^2 |\nabla u_s|^2 = O\left(\int_{D(\delta)} u_s^2 \right) = O(1). \tag{2.12}$$

Hence u_s is uniformly bounded in $\mathcal{D}^{1,2}(D(\delta/2))$. Therefore $\left\{\int\limits_{|x|=1} u_s^2\right\}_s$ is bounded as $s \to 1$, whence the existence of $C_0 \in I\!R$ such that

$$\varliminf_{s\to 1} \lambda \int\limits_{|x|=1} u_s^2 \geq C_0.$$

Therefore from (1.6) we have

$$\begin{aligned}
\widetilde{\lambda} = \varliminf_{s\to 1} \lambda(s) &= \varliminf_{s\to 1} \left\{ \int\limits_{I\!R^n} |\nabla u_s|^2 - \left(\tfrac{n-2}{2}\right)^2 s\, \tfrac{|u_s|^2}{|x|^2} - \tfrac{s\, A_k\, |u_s|^2}{|x|^2} \right\} \\
&\geq \varliminf_{s\to 1} \lambda \int\limits_{|x|=1} |u_s|^2 \\
&\geq C_0.
\end{aligned}$$

This proves the lemma. □

Proof of Theorem 1.2. By Lemma 2.2 and assumption (H2), for $u \in \mathcal{D}^{1,2}(I\!R^n)$, $s \in (0,1)$,

$$\begin{aligned}
\int\limits_{I\!R^n} |\nabla u|^2 - \left(\tfrac{n-2}{2}\right)^2 \tfrac{squ^2}{|x|^2} - s\eta u^2 &\geq (1-s) \int\limits_{I\!R^n} |\nabla u|^2 \\
&\quad + s \left(\int\limits_{I\!R^n} |\nabla u|^2 - \left(\tfrac{n-2}{2}\right)^2 \tfrac{u^2}{|x|^2} - \tfrac{A_k u^2}{|x|^2} \right) \\
&\geq (1-s) \int\limits_{I\!R^n} |\nabla u|^2 + \widetilde{\lambda} s \int\limits_{I\!R^n} f u^2.
\end{aligned}$$

So, since $\tilde{\lambda} < 0$ and $s < 1$, $L_{sq,s\eta} - \widetilde{\lambda} f$ is a coercive operator and hence by the spectral theorem, it has an infinite discrete spectrum $\{\lambda_i(sq, s\eta)\}_{i\in I\!N}$ with corresponding eigenfunctions $\{\phi_i(sq, s\eta)\}_{i\in I\!N}$. The rest of the proof follows as in [3] except for the convergence of the eigenfunctions $\phi_i(sq, s\eta)$ for $i \geq 2$. We will give the proof of this below.

Step 1. For $r > 0$, let $\Omega(r) = I\!R^n \setminus B(r)$ and define

$$a(r) = \inf_{u\in H_0^1(B(r))} \left\{ \int\limits_{B(r)} |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \frac{u^2}{|x|^2} - \frac{A_k(x)}{|x|^2}\, u^2;\ \int\limits_{B(r)} f(x)\, |u|^2 = 1 \right\}$$

$$b(r) = \inf_{u\in H_0^1(\Omega(r))} \left\{ \int\limits_{\Omega(r)} |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \frac{u^2}{|x|^2} - \frac{A_k(x)}{|x|^2}\, u^2;\ \int\limits_{\Omega(r)} f(x)\, |u|^2 = 1 \right\}$$

then by part (i) of hypothesis (H_3), we have

$$\lim_{r\to 0}\ a(r)=\infty,\ \lim_{r\to\infty}\ b(r)=\infty.$$

Indeed, by the definition of A_k and Theorem 1.1, for all $u\in H_0^1(B(r)$, we have

$$\int\limits_{B(r)}\Big(|\nabla u|^2-\Big(\frac{n-2}{2}\Big)^2\,\frac{u^2}{|x|^2}-\frac{A_k(x)\,u^2}{|x|^2}\Big)\,dx\geq C\int\limits_{B(r)}\frac{u^2}{|x|^2(\log^{(k+1)}\rho(x))^2}$$

$$\geq\ \frac{C}{\varepsilon(r)}\int\limits_{B(r)}f(x)\,|u(x)|^2\,dx\,,$$

where $\varepsilon(r):=\sup_{B(r)}f(x)\,|x|^2\,(\log^{(k+1)}\rho(x))^2$. Hence, $\lim\limits_{r\to 0}\ a(r)\geq\lim\limits_{r\to 0}\ \frac{C}{\varepsilon(r)}=+\infty$, by hypothesis $(H_3)(i)$.

Step 2. Let $u_0(x)=|x|^{-\frac{(n-2)}{2}}(\log\rho(x))^{-\delta_1}$, where thanks to (1.12), $\delta_1>1/2$ is chosen so that

$$\liminf_{\substack{|x|\to 0\\ \text{or }|x|\to\infty}}\ (\log\rho(x))^2\left\{(1-q(x))-\frac{4\,|x|^2\,\eta(x)}{(n-2)^2}\right\}>\frac{4\delta_1(1+\delta_1)}{(n-2)^2}\,.$$

There exist two constants $R_1,R_2,\ R_1<1<R_2$ such that for all $\lambda<0$, on $D(R_1,R_2)=\{x;R_1<|x|<R_2\}$

$$\begin{aligned}(L_{q,\eta}-\lambda f)\,u_0\ &=\ \big(\tfrac{n-2}{2}\big)^2\,\tfrac{u_0}{(|x|\log\rho(x))^2}\Big[(\log\rho(x))^2\Big((1-q(x))-\tfrac{4}{(n-2)^2}|x|^2\eta\Big)\\ &\quad-\tfrac{4}{(n-2)^2}\delta_1(1+\delta_1)-\tfrac{4\,\lambda}{(n-2)^2}(|x|\log\rho(x))^2 f\Big]\\ &\geq 0\end{aligned}\ .$$

Now, let $\phi_i(sq,s\eta)$ be the i^{th} eigenfunction in (1.11) with $\|\phi_i(sq,s\eta)\|_{\mathcal{D}^{1,2}}=1$ and such that $\lim\limits_{s\to 1}\lambda_i(sq,s\eta)=\lambda_i(q,\eta)$. By elliptic regularity, $\phi_i(sq,s\eta)$ is uniformly bounded in a neighbourhood of $\partial D(R_1,R_2)$. Let

$$M_1=\sup_{\substack{|x|=R_1\\ s\in(0,1)}}\phi_i(sq,s\eta),\quad M_2=\sup_{\substack{|x|=R_2\\ s\in(0,1)}}\phi_i(sq,s\eta)$$

and let

$$W_1=\phi_i(sq,s\eta)-\frac{M_1}{u_0(R_1)}u_0,\quad W_2=\phi_i(sq,s\eta)-\frac{M_2}{u_0(R_2)}u_0.$$

Then $W_1 \le 0$ on $\partial B(R_1)$ and $W_2 \le 0$ on $\partial B(R_2)$. Furthermore,

$$(L_{sq,s\eta} - \lambda_i(sq,s\eta) f)\, W_1 \le -(1-s)\frac{M_1\, u_0}{u_0(R_1)} \left(\eta + \left(\frac{n-2}{2}\right)^2 \frac{q}{|x|^2} \right) \le 0 \quad \text{in } B(R_1)\,,$$

$$(L_{sq,s\eta} - \lambda_i(sq,s\eta) f)\, W_2 \le -(1-s)\frac{M_2\, u_0}{u_0(R_2)} \left(\eta + \left(\frac{n-2}{2}\right)^2 \frac{q}{|x|^2} \right) \le 0 \quad \text{in } I\!R^n \backslash B(R_2)\,.$$

Now, since $W_1^+ \in H_0^1(B(R_1))$ and $W_2^+ \in H_0^1(I\!R^n \setminus B(R_2))$, if we multiply the first inequality by W_1^+ and the second one by W_2^+ and we integrate by parts we obtain:

$$\begin{aligned} \int\limits_{B(R_1)} (a(R_1) - \lambda_i(sq,s\eta))\, f\, |W_1^+|^2 \;&\le\; \int\limits_{B(R_1)} |\nabla W_1^+|^2 - \\ &\quad \left(\left(\tfrac{n-2}{2}\right)^2 \tfrac{s\,q}{|x|^2} - s\eta - \lambda_i(sq,s\eta)\, f \right) |W_1^+|^2 \\ &= \int\limits_{B(R_1)} W_1^+ (L_{sq,s\eta} - \lambda_i(sq,sn)\, f\,)\, W_i^+ \\ &\le\; 0. \end{aligned}$$

But from Step 1, $a(R_1) \to \infty$ as $R_1 \to 0$. So, for R_1 small enough, $a(R_1) - \lambda_i(sq,s\eta) > 0$ in $B(R_1)$ and hence $W_1^+ = 0$ a.e. This implies that $\phi_i(sq,s\eta) \le M_1\, u_0$ in $B(R_1)$. Now apply the same method to $-\phi_i$ to obtain $|\phi_i(sq,s\eta)| \le M_1 u_0$ in $B(R_1)$. Similarly, $|\phi_i(sq,s\eta)| \le M_2 u_0$ on $I\!R^n \setminus B(R_2)$ if R_2 is sufficiently large. Hence by elliptic regularity, there exists a constant $M > 0$ such that

$$|\phi_i(sq,s\eta)| \le M u_0 \quad \text{in} \quad I\!R^n.$$

Assume that up to subsequences, $\phi_i(sq,s\eta)$ converges weakly to $\phi_i(q,\eta)$ in $\mathcal{D}^{1,2}(I\!R^n)$ and a.e. in $I\!R^n$ as s goes to 1. Since $u_0 \in \mathcal{D}^{1,2}(I\!R^n)$, by the dominated convergence theorem we have

$$\lim_{s\to 1} \int\limits_{I\!R^n} \frac{\phi_i^2(sq,s\eta)}{|x|^2} = \int\limits_{I\!R^n} \frac{\phi_i^2(q,\eta)}{|x|^2}\,,$$

and by our hypothesis on η and f, we also have

$$\lim_{s\to 1} \int\limits_{I\!R^n} s\,\eta\, \phi_i^2(sq,s\eta) = \int\limits_{I\!R^n} \eta\, \phi_i^2(q,\eta)\,,$$

$$\lim_{s\to 1}\int\limits_{I\!R^n} \lambda_i(sq,s\eta)\, f\, \phi_i^2(sq,s\eta) = \int\limits_{I\!R^n} \lambda_i(q,\eta)\, f\, \phi_i^2(q,\eta)\,.$$

Hence, by using the equations satisfied by $\phi_i(sq,s\eta)$ and by $\phi_i(q,\eta)$ we obtain:

$$\lim_{s\to 1}\int_{I\!R^n} |\nabla\phi_i(sq,s\eta)|^2 = \int_{I\!R^n} |\nabla\phi_i(q,\eta)|^2\,,$$

and therefore, $\phi_i(sq,s\eta) \underset{s\to 1}{\longrightarrow} \phi_i(q,\eta)$ strongly in $\mathcal{D}^{1,2}(I\!R^n)$. This proves the theorem. □

3. Extensions. The boundary integral on the right hand side of (1.4) and (1.5) can be replaced by an integral over the whole domain. The proof follows exactly as in Lemma 2.2 and hence we will just briefly sketch it.

1. Replacement of boundary integrals in (1.4) and (1.5) by integrals over Ω.

Let Ω be a bounded domain of $I\!R^n$ and

$$H(u) = \begin{cases} \left(\frac{n-p}{p}\right)^p \int\limits_\Omega \frac{|u|^p}{|x|^p} & \text{if} \quad n > p\,, \\ \left(\frac{n-1}{n}\right)^n \int\limits_\Omega \frac{|u|^n}{|x|^n (\log R/|x|)^n} & \text{if} \quad n = p\,, \end{cases}$$

$$A(u) = C \begin{cases} \int\limits_\Omega \sum\limits_{j=1}^{k} \frac{1}{(\log^{(j)} R/|x|)^2} \frac{|u|^p}{|x|^p} & \text{if} \quad n > p\,, \\ \int\limits_\Omega \left(\sum\limits_{j=2}^{k} \frac{1}{(\log^{(j)} R/|x|)^n} \frac{|u|^n}{|x|^n} \right) & \text{if} \quad n = p\,, \end{cases}$$

where C is the constant of inequality (1.3).

Step 1. Let $\varepsilon > 0$, then there exist $C(\varepsilon) > 0$ such that

$$\int\limits_{\partial\Omega} |u|^p \le \varepsilon \int\limits_\Omega |\nabla u|^p + C(\varepsilon) \int\limits_\Omega |u|^p\,.$$

holds for all $u \in W^{1,p}(\Omega)$. Substituting this in (1.4) and (1.5) yields for $s = \frac{1}{(1-\lambda\varepsilon)} < 1$

$$\int_\Omega |\nabla u|^p - sH(u) - sA(u) \geq \lambda C(\varepsilon) s \int_\Omega |u|^p .$$

Step 2. Let

$$\lambda_s = \inf \left\{ \int_\Omega |\nabla u|^p - sH(u) - sA(u); \int_\Omega |u|^p = 1 \right\},$$

By Step 1, $\lambda_s \geq \lambda C(\varepsilon) > -\infty$. Without loss of generality, assume that $p < n$. Then from Step 1 and as in Lemma 2.2, the corresponding operator is coercive and hence from Ekeland's principle λ_s is achieved by some some function $u_s \geq 0$ which satisfies

$$-\operatorname{div}\left(|\nabla u_s|^{p-2}\nabla u_s\right) - \left(\frac{n-p}{p}\right)^p \frac{s\,|u_s|^{p-2}\,u_s}{|x|^p} - \left(\sum_{j=1}^{k} \frac{C\,s}{\log^{(j)} R/|x|)^2}\right) \frac{|u_s|^{p-2}\,u_s}{|x|^p} = \lambda_s |u_s|^{p-2}\,u_s.$$

Hence by regularity, $\{u_s\}$ is bounded in $W^{1,p}(\Omega \setminus B(r))$ for some $r > 0$. Hence $\{\int_{\partial\Omega} u_s^p\}$ is bounded. From (1.4) and the above equation,

$$\lambda \int_{\partial\Omega} u_s^p \leq \lambda_s \int_\Omega u_s^p = \lambda_s.$$

and hence λ_s is bounded uniformly. Clearly, by monotonicity, $\lambda_s \to \lambda_1$ as $s \to 1$. So, we have the following inequalities for all $u \in W^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^p - H(u) - A(u) \geq \lambda_1 \int_\Omega |u|^p. \tag{3.1}$$

2. Vázquez and Zuazua [11] (see also [8]) proved the following inequality. Let $1 < q < 2$, then there exists a constant $C > 0$ such that for all $u \in H_0^1(\Omega)$,

$$\int_\Omega |\nabla u|^2 - \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} \geq C \|u\|^2_{W^{1,q}(\Omega)} . \tag{3.2}$$

First they prove the above inequality when Ω a ball and for radial functions u. Then by using spherical harmonics they decompose the functions u into their radial and non radial part. Next they prove the inequality for non radial functions and then extend it to artitrary Ω.

Here we can also obtain an extension of (3.2) by using the term $\frac{|\nabla v|^p}{|x|^{n-p}}$ in (2.6). For the sake of simplicity assume that $1 < p < n$ and $k = 1$. Then for all $u \in W_0^{1,p}(\Omega)$, (2.4) and (2.6) imply

$$\int\limits_\Omega \left(|\nabla u|^p - \left(\frac{n-p}{p}\right)^p \frac{|u|^p}{|x|^p} - \frac{C_1}{(\log R/|x|)^2} \frac{|u|^p}{|x|^p} \right) \geq C_2 \int\limits_\Omega \frac{(\log R/|x|)|\nabla w|^2}{|x|^{n-2}}\,, \tag{3.3}$$

where $w = (\log R/|x|)^{-1/2} v^{p/2} = (\log R/|x|)^{-1/2} |x|^{\frac{n-p}{2}} u^{p/2}$. But

$$|\nabla u^{p/2}| = O\left(\frac{(\log R/|x|)^{1/2}}{|x|^{\frac{n-p}{2}}} \nabla w + \frac{(\log R/|x|)^{1/2}}{|x|^{\frac{n-p}{2}+1}} w \right),$$

therefore,

$$\int\limits_\Omega |\nabla(u^{p/2})|^q = O\left(\int\limits_\Omega \left(\frac{(\log R/|x|)^{q/2} |\nabla w|^q}{|x|^{q\left(\frac{n-p}{2}\right)}} + \frac{(\log R/|x|)^{q/2} |w|^q}{|x|^{q\left(\frac{n-p}{2}+1\right)}} \right) \right). \tag{3.4}$$

In order to estimate the right hand side of (3.4), we need some estimates regarding the weighted Hardy's inequality.

Let $w \in C_0^\infty(\Omega)$ and $k \geq 0, \alpha \geq 0$. Then,

$$\begin{aligned} \int\limits_\Omega \frac{(\log R/|x|)^\alpha |w|^q}{|x|^k} &= \frac{1}{n} \int\limits_\Omega \frac{(\operatorname{div} x)(\log R/|x|)^\alpha |w|^q}{|x|^k} \\ &= -\frac{q}{n} \int\limits_\Omega \frac{|w|^{q-2} w \langle \nabla w, x \rangle (\log R/|x|)^\alpha}{|x|^k} \\ &\quad + \frac{k}{n} \int\limits_\Omega \frac{|w|^q (\log R/|x|)^\alpha}{|x|^k} + \frac{\alpha}{n} \int\limits_\Omega \frac{|w|^q (\log R/|x|)^\alpha}{|x|^k (\log R/|x|)}. \end{aligned}$$

Let $k < n$ and choose $R_0 > 0$ such that $\frac{\alpha}{n} \sup\limits_{x \in \Omega} \frac{1}{(\log R_0/|x|)} < \frac{1}{2}\left(1 - \frac{k}{n}\right)$. Then for

$R \geq R_0$, the above identity gives

$$\begin{aligned}\frac{1}{2}\left(1-\frac{k}{n}\right)\int\limits_{\Omega}\frac{|w|^q(\log R/|x|)^\alpha}{|x|^k} &\leq \frac{q}{n}\int\limits_{\Omega}\frac{|w|^{q-1}|\nabla w|(\log R/|x|)^\alpha}{|x|^{k-1}}\\ &\leq \frac{q}{n}\left(\int\limits_{\Omega}\frac{|w|^q(\log R/|x|)^\alpha)}{|x|^k}\right)^{\frac{q-1}{q}}\\ &\quad\left(\int\limits_{\Omega}\frac{|\nabla w|^q(\log R/|x|)^\alpha}{|x|^{k-q}}\right)^{\frac{1}{q}}\end{aligned}$$

and hence there exists a constant $C = C(K, N, \alpha) > 0$ such that

$$\int\limits_{\Omega}\frac{|w|^q(\log R/|x|)^\alpha}{|x|^k} \leq C\int\limits_{\Omega}\frac{|\nabla w|^q\,(\log R/|x|)^\alpha}{|x|^{k-q}}. \tag{3.5}$$

Let $k = q\left(\frac{n-p}{2}+1\right)$ and q be chosen so that $k < n$, that is, $1 < q < \frac{2n}{n-p+2}$. Then, (3.4) and (3.5) with $\alpha = q/2$, $R \geq R_0$ imply

$$\int\limits_{\Omega}|\nabla(u^{p/2})|^q = O\left(\int\limits_{\Omega}\frac{(\log R/|x|)^{q/2}|\nabla w|^q}{|x|^{q\frac{(n-p)}{2}}}\right).$$

Let $1 < q < 2$, $\frac{n}{q} > \frac{2-p}{2-q}$. The above inequality yields

$$\begin{aligned}\int\limits_{\Omega}|\nabla(u^{p/2})|^q &= O\left(\left(\frac{(\log R/|x|)|\nabla w|^2}{|x|^{n-2}}\right)^{q/2}\left(\int\limits_{\Omega}|x|^{\frac{(p-2)q}{2-q}}dx\right)^{\frac{2-q}{2}}\right)\\ &= O\left(\left(\int\limits_{\Omega}\frac{(\log R/|x|)\,|\nabla w|^2}{|x|^{n-2}}\right)^{q/2}\right).\end{aligned} \tag{3.6}$$

Combining this with (3.3) we obtain the following:

Theorem 3.1. Let $1 < p < n$, $1 < q < 2$, $\frac{n}{q} > \frac{2-p}{2-q}$, then there exists R_0 such that for all $R \geq R_0$,

$$\int\limits_{\Omega}|\nabla u|^p - \left(\frac{n-p}{p}\right)^p\frac{|u|^p}{|x|^p} - \frac{C_1}{(\log R/|x|)^2}\frac{|u|^p}{|x|^p} \geq C\,\|u^{p/2}\|^2_{W_0^{1,q}(\Omega)}, \tag{3.7}$$

for all $u \in W_0^{1,p}(\Omega)$.

Notice that when $p = 2$, (3.6) is Zuazua-Vázquez's inequality. A similar inequality can be obtained when $p = n$. For that purpose one must use the function $\log\log R/|x|$ instead of $\log R/|x|$.

3. Let (M, g) be a compact Riemannian manifold of dimension ≥ 2 and let Δ_g denote the Laplace-Beltrami operator associated to g and $G(x, y)$, the Green's function corresponding to the operator $-\Delta_g + 1$. Assume moreover that $0 \in M$ and define $G(x) := G(0, x)$. Then we have the following Hardy-Sobolev inequality.

If $1 < p < dim\ M = n$ and k is a nonnegative integer, there exist constants $C > 0$ and $\lambda < 0$ such that for all $u \in W^{1,p}(M)$,

$$\int_M |\nabla u|^p - \left(\frac{n-p}{p(n-2)}\right)^p \left|\frac{\nabla G}{G}\right|^p |u|^p \quad - \quad C\left(\sum_{j=1}^{k} \frac{1}{(\log^{(j)} G)^2}\right) \left|\frac{\nabla G}{G}\right|^p |u|^p \geq \lambda \int_M |u|^p, \quad \text{if } p < n$$

$$\int_M |\nabla u|^n - \left(\frac{n-1}{n}\right)^n \left|\frac{\nabla G}{G}\right|^n |u|^n \quad - \quad C\left(\sum_{j=1}^{k} \frac{1}{(\log^{(j)} G)^n}\right) \left|\frac{\nabla G}{G}\right|^n |u|^n \geq \lambda \int_M |u|^n, \quad \text{if } p = n.$$

If M has a boundary, then a boundary integral appears on the right hand side of the above inequality. Furthermore the corresponding perturbed Euler-Lagrange equation can be studied. The condition to be put on q in order to prove existence and non existence results remains same as that used in the case of a bounded domain in $\mathbb{R}^n$.

Acknowledgement. The first author would like to thank Olivier Druet for pointing out the extension to compact manifolds.

Both authors acknowledge funding from the Indo-French Centre for promotion of Advanced Research, under project 1901-02.

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