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Rigorous nonperturbative Ornstein-Zernike theory for Ising ferromagnets

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Abstract. – We rigorously derive the Ornstein-Zernike asymptotics of the pair-correlation functions for finite-range Ising ferromagnets in any dimensions and at any temperature above critical.

The celebrated heuristic argument by Ornstein and Zernike [1] implies that the asymptotic form of the truncated two-point density correlation function of simple fluids away from the critical region is given by

$$G(\vec{r}\,) \simeq \frac{A_\beta}{\sqrt{|\vec{r}\,|^{d-1}}} e^{-|\vec{r}\,|\xi_\beta}, \tag{1}$$

where the value of the inverse correlation length ξ_β depends only on the density ρ, the inverse temperature β and the spatial dimension d. The original OZ approach hinges on the assumption that the so-called direct correlation function $C(\cdot)$, which is *de facto* introduced through the renewal-type relation

$$G(\vec{r}\,) = C(\vec{r}\,) + \rho \int_{\mathbb{R}^d} C(\vec{r} - \vec{r}_1) G(\vec{r}_1)\, \mathrm{d}\vec{r}_1, \tag{2}$$

is of an appropriately short range.

Because of the physical significance of both the conclusions and of the underlying heuristic assumptions, a number of works (see, *e.g.*, [2–6]) were devoted to attempts to put the theory on a rigorous basis, that is to derive (1) directly from the microscopic picture of intermolecular interactions. Most of these works, however, were based on expansion/perturbation techniques

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and required technical low-density or high-/low-temperature assumptions and, thereby, addressed the situation when the parameters are far away from the critical region. Since the OZ theory was, above all, intended to describe the phenomenon of critical scattering it would be of interest to devise such rigorous approach to (1) which would rely only on qualitative features of noncriticality such as, for example, finite compressibility (or finite susceptibility in the context of ferromagnetic lattice models).

In this letter, we present a fully nonperturbative derivation of the direction-dependent analog of (1) for finite-range ferromagnetic Ising model above the critical temperature in any dimension. Let $\boldsymbol{J} = \{J_v\}_{v\in\mathbb{Z}^d}$, be a collection of nonnegative real numbers such that $J_v = J_{-v}$ and $J_v = 0$ if $|v| > R$, where R is some finite number. The (formal) Hamiltonian is then of the form

$$H(\sigma) = -\frac{1}{2}\sum_{x,y\in\mathbb{Z}^d} J_{y-x}\sigma_x\sigma_y \,. \tag{3}$$

Our approach pertains to the high-temperature region $\beta < \beta_{\mathrm{c}} = \beta_{\mathrm{c}}(\boldsymbol{J}, d)$, which is the set of all β such that the susceptibility $\chi_\beta = \sum_{x\in\mathbb{Z}^d}\langle\sigma_0\sigma_x\rangle_\beta$ is finite. By the Simon-Lieb argument $\chi_\beta < \infty$ implies strict exponential decay of the two-point function. Equivalently, the series

$$\chi_\beta(t) = \sum_{x\in\mathbb{Z}^d} e^{(t,x)}\langle\sigma_0\sigma_x\rangle_\beta \,, \tag{4}$$

where $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{R}^d$, has a nonempty domain of convergence for each subcritical value of the inverse temperature $\beta < \beta_{\mathrm{c}}$. Note that, by an important result of Aizenman, Barsky and Fernández [7], β_{c} is actually the usual critical temperature: The spontaneous magnetization is positive whenever $\beta > \beta_{\mathrm{c}}$.

In the sequel we shall use $\boldsymbol{K}_\beta$ to denote the domain of convergence of (4). From a purely geometric point of view, the direction-dependent inverse correlation length ξ_β is the support function of $\boldsymbol{K}_\beta$. In particular, the dependence of $\xi_\beta(n)$ on the direction $n \in \mathbb{S}^{d-1}$ is encoded in the geometry of $\partial\boldsymbol{K}_\beta$.

Theorem 1 *In any dimension $d \geq 1$ and for any ferromagnetic model* (3)*, the asymptotic decay of the two-point correlation function in the high-temperature region $\beta < \beta_{\mathrm{c}}(\boldsymbol{J}, d)$ is given by*

$$\langle\sigma_0\sigma_x\rangle_\beta \simeq \frac{\Psi_\beta(n_x)}{\sqrt{|x|^{(d-1)}}} e^{-\xi_\beta(n_x)|x|}(1 + o(1)) \,, \tag{5}$$

where $n_x = x/|x| \in \mathbb{S}^{d-1}$ is the unit vector in the direction of x, and the function Ψ_β is strictly positive and analytic. Moreover, the inverse correlation length $\xi_\beta(n)$ is an analytic function of the direction $n \in \mathbb{S}^{d-1}$ in the sense that the boundary $\partial\boldsymbol{K}_\beta$ of $\boldsymbol{K}_\beta$ is locally analytic and strictly convex. Furthermore, the Gaussian curvature κ_β of $\partial\boldsymbol{K}_\beta$ is uniformly positive,

$$\bar{\kappa}_\beta = \min_{t\in\partial\boldsymbol{K}_\beta} \kappa_\beta(t) > 0.$$

Alternatively, strict convexity and analyticity of the direction-dependent inverse correlation length $\xi_\beta(\cdot)$ could be formulated in terms of the geometry of the unit sphere $\partial\boldsymbol{U}_\beta$ in the ξ_β-norm.

A full proof of Theorem 1 can be found in [8].

In principle, our approach pertains to any model in which the pair-correlation function $g(x)$ admits a suitable graphical representation of the type

$$g(x) = \sum_{\lambda:0\to x} q(\lambda) \,, \tag{6}$$

where the sum runs over a family of paths or, more generally, pathlike objects connecting 0 and x, possibly with compatibility constraints, *e.g.*, some form of self-avoidance; let us call such paths admissible. The weights $q(\,\cdot\,)$ are supposed to be strictly positive and to possess a variation of the following four properties:

- *Strict exponential decay of the two-point function.* There exists $C_1 < \infty$ such that, for all $x \in \mathbb{Z}^d \setminus \{0\}$,
$$g(x) = \sum_{\lambda:0\to x} q(\lambda) \le C_1 e^{-\xi(x)}\,, \tag{7}$$
where $\xi(x) = -\lim_{k\to\infty}(k|x|)^{-1}\log g([kx])$ is the inverse correlation length.
- *Finite energy condition.* For any pair of compatible paths λ and η define the conditional weight
$$q(\lambda \mid \eta) = q(\lambda \amalg \eta)/q(\eta),$$
where $\lambda \amalg \eta$ denotes the concatenation of λ and η. Then there exists a universal finite constant $C_2 < \infty$ such that the conditional weights are controlled in terms of path sizes $|\lambda|$ as
$$q(\lambda \mid \eta) \ge e^{-C_2|\lambda|}. \tag{8}$$
- *Splitting property.* There exists $C_3 < \infty$, such that, for all $x, y \in \mathbb{Z}^d \setminus \{0\}$ with $x \ne y$,
$$\sum_{\lambda:0\to x\to y} q(\lambda) \le C_2 \sum_{\lambda:0\to x} q(\lambda) \sum_{\lambda:x\to y} q(\lambda)\,. \tag{9}$$
- *Exponential mixing.* There exists $C_4 < \infty$ and $\theta \in (0,1)$ such that, for any four paths λ, η, γ_1 and γ_2, with $\lambda \amalg \eta \amalg \gamma_1$ and $\lambda \amalg \eta \amalg \gamma_2$ both admissible,
$$\frac{q(\lambda \mid \eta \amalg \gamma_1)}{q(\lambda \mid \eta \amalg \gamma_2)} \le \exp\left[C_4 \sum_{\substack{x\in\lambda\\ y\in\gamma_1\cup\gamma_2}} \theta^{|x-y|}\right]. \tag{10}$$

Many models enjoy a graphical representation of correlation functions of the form (6). In perturbative regimes, cluster expansions provide a generic example. Nonperturbative examples include the random-cluster representation for Potts (and other) models [9], random line representation for Ising [10–12], or more generally, random walk representation of N-vector models [13], etc. However, it might not always be easy, or even possible, to establish properties (7), (8), (9) and (10) for the corresponding weights.

Before proceeding to explain how such an expansion is used in order to prove Theorem 1, let us first discuss the similar but much simpler case of self-avoiding walks (SAW), which has been treated, though with a somewhat different approach, in [14, 15].

Self-avoiding walks. Here we are interested in exact asymptotics of the following quantity:
$$g_\beta^{\mathrm{SAW}}(x) = \sum_{\lambda:0\to x} e^{-\beta|\lambda|}\,,$$
where the sum runs over all finite-range SAW connecting 0 and x, $|\lambda|$ is the length (*i.e.* the number of steps) of λ, and $\beta > 0$. It is known [16] that g_β^{SAW} is well defined and, accordingly, that (7) holds for all β as soon as $\beta > \beta_{\mathrm{c}}^{\mathrm{SAW}}(d)$. The remaining three properties trivially follow with $C_2 = \beta$, $C_3 = 1$ and $C_4 = 0$.

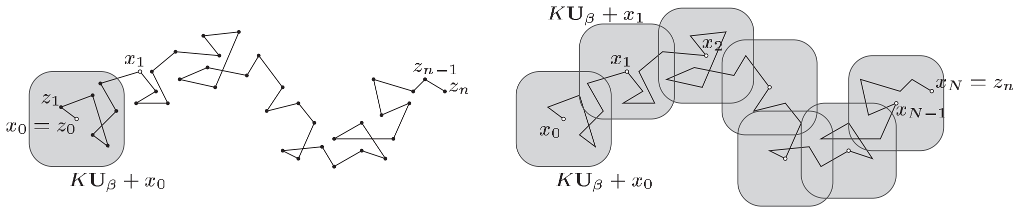

Fig. 1 – The construction of the K-skeleton $\lambda_K = (x_0, \dots, x_N)$ of a path $\lambda = (z_0, \dots, z_n)$. Set $x_0 \equiv z_0$; then define iteratively x_{k+1} to be the first point of λ outside the set $K\boldsymbol{U}_\beta + x_k$. When this procedure stops, set $x_N \equiv z_n$.

We are now going to decompose a path $\lambda : 0 \to x$ into elementary irreducible pieces. The notion of irreducibility is adjusted to the direction of the target point x through the geometry of the inverse correlation length ξ_β^{SAW}: Let $\boldsymbol{K}_\beta^{\mathrm{SAW}}$ be the convex set supported by ξ_β^{SAW}. Choose a dual point $t \in \partial\boldsymbol{K}_\beta^{\mathrm{SAW}}$ such that $(t, x) = \xi_\beta^{\mathrm{SAW}}(x)$. We say that a vertex $i \in \lambda$ is a *regeneration point* of λ if the hyperplane through i, orthogonal to t, cuts λ into two pieces. A path is then said to be *irreducible* if it does not contain any regeneration points. Since SAW-weights possess the factorization property $e^{-\beta|\gamma \amalg \lambda|} = e^{-\beta|\gamma|} e^{-\beta|\lambda|}$, we arrive to the following Ornstein-Zernike–type equation, or in the probabilistic jargon a *renewal equation*:

$$g_\beta^{\mathrm{SAW}}(x) = \sum_{y \in \mathbb{Z}^d} c_\beta^{\mathrm{SAW}}(x-y) g_\beta^{\mathrm{SAW}}(y) \,, \tag{11}$$

where the direct two-point function is defined via the summation of path weights over *irreducible* paths,

$$c_\beta^{\mathrm{SAW}}(y) = \sum_{\substack{\lambda:0 \to y \\ \text{irreducible}}} e^{-\beta|\lambda|} \,.$$

The short-range nature of c_β^{SAW} finds then its precise mathematical expression in the claim that the direct correlation length is strictly smaller than $1/\xi_\beta^{\mathrm{SAW}}(x)$:

$$\liminf_{k \to \infty} -(k|x|)^{-1} \log c_\beta^{\mathrm{SAW}}([kx]) > \xi_\beta^{\mathrm{SAW}}(x) \,. \tag{12}$$

It is very easy to establish (12) at large values of β. The main difficulty is to give a nonperturbative proof. As one gets close to $\beta_{\mathrm{c}}^{\mathrm{SAW}}$, paths typically have a very complicated messy structure at the lattice scale, and one expects them to behave properly only at distances large compared to the correlation length. It is thus natural to introduce a coarse-grained description of the microscopic paths. To this end, we choose some big number K, and construct the K-*skeleton* λ_K of a path λ as described in fig. 1. Notice that $K\boldsymbol{U}_\beta$ are balls of radius K in the metric ξ_β^{SAW}, so both the scale and the geometry of the inverse correlation length ξ_β literally set up the stage for our path coarse-graining procedures. Given a target point x, we use a dual direction $t \in \partial\boldsymbol{K}_\beta^{\mathrm{SAW}}$; $(t, x) = \xi_\beta(x)$, to measure the amount of backtracking by different skeleton steps: For any $v \in \mathbb{Z}^d$, define the surcharge cost $\mathfrak{s}_t(v) = \xi_\beta^{\mathrm{SAW}}(v) - (t, v) \geq 0$. Accordingly, the surcharge cost of a skeleton $\lambda_K : 0 \mapsto x$ is defined as the sum of all the surcharge costs of all its steps, $\mathfrak{s}_t(\lambda_K) = \sum_k \mathfrak{s}_t(x_k - x_{k-1})$.

The following energy-entropy argument explains what is gained by the above change of scale: By the splitting property the weight of a skeleton $\lambda_K : 0 \mapsto x$ is bounded above as

$$q_\beta^{\mathrm{SAW}}(\lambda_K) = \sum_{\lambda \sim \lambda_K} e^{-\beta|\lambda|} \le e^{-\xi_\beta^{\mathrm{SAW}}(x) - \mathfrak{s}_t(\lambda_K)}.$$

On the other hand, the total number of all K-skeletons of length N is of the order $(K^{d-1})^N = \exp[cN \log K]$. Therefore, energy dominates entropy as soon as K is large. In particular, up to exponentially small probabilities, only a negligible fraction of λ_K-steps are backtracking (have surcharges of order K) with respect to x. Because of the finite-energy condition (8) such predominant forward structure of paths on the fixed finite scale K implies the mass gap property (12).

Repeated iterations of (11) give

$$g_\beta^{\mathrm{SAW}(x)} e^{\xi_\beta(x)} = \sum_{n=1}^{\infty} \sum_{y_1+y_2+\cdots+y_n=x} \prod_{i=1}^{n} c_\beta^{\mathrm{SAW}}(y_i) e^{(t,y_i)}$$

which, in probabilistic terms, is tantamount to independence of different irreducible paths. Since $c_\beta^{\mathrm{SAW}}(y)e^{(t,y)}$ is a probability measure on $\mathbb{Z}^d$ when $t \in \partial \boldsymbol{K}_\beta^{\mathrm{SAW}}$, the proof of the OZ formula for self-avoiding walks reduces to a local limit computation for sums of independent random variables with exponentially decaying tails. A very similar construction applies to all subcritical short-range Bernoulli bond percolation models [17, 18].

Ising model. A convenient graphical representation for the pair-correlation functions of the Ising model is the random-line representation [10–12],

$$g_\beta(x) = \langle \sigma_x \sigma_y \rangle_\beta = \sum_{\lambda : x \to y} q_\beta(\lambda),$$

where the sum runs over admissible paths connecting x and y, making jumps only between sites u and v such that $J_{u-v} > 0$, and q_β is some strictly positive weight. It can then be proved that properties (7), (8), (9) and (10) hold for this representation. We would like to proceed similarly to what was done for SAW. The major new difficulty arising now is that for the Ising model the weights do not factorize: $q_\beta(\lambda \amalg \gamma) \neq q_\beta(\lambda) q_\beta(\gamma)$. The closest expression to (11) one can write now is

$$g_\beta(x) = \sum_{y \in \mathbb{Z}^d} \sum_{\substack{\lambda : y \to x \\ \text{irreducible}}} q_\beta(\lambda) \sum_{\substack{\gamma : 0 \to y \\ \gamma \amalg \lambda \text{ admissible}}} q_\beta(\gamma \mid \lambda). \qquad (13)$$

The admissibility constraint does not create serious problems, for example it is automatically satisfied across the regeneration points. However, the presence of the conditional weight $q_\beta(\gamma \mid \lambda)$ instead of $q_\beta(\gamma)$ destroys the independence and, consequently, dramatically changes the probabilistic structure of the equation.

At this point one is compelled to adjust the traditional point of view on the nature of the OZ formula and to try to understand it not through the spectral theory of the renewal-type equations (2) or (11), but rather in a broader context of local limit properties of one-dimensional systems of objects (irreducible paths) under appropriate mixing conditions. As we shall explain below, such an approach leads to a reformulation of the problem in terms of the statistical mechanics of Ruelle's operator for shifts over countable alphabets. The

OZ formula (5) is recovered in this way from a local limit analysis based on the analytic perturbation theory of the corresponding Perron-Frobenius eigenvalue.

For the moment, let us postpone the exact definition of path irreducibility and focus on the induced irreducible representation of $g_\beta(x)$:

$$g_\beta(x) = \sum_{n\geq 1} \sum_{\substack{\lambda:0\to x \\ \lambda=\lambda_1 \amalg \ldots \amalg \lambda_n \\ \lambda_k \text{ irreducible}, k=1,\ldots,n}} q_\beta(\lambda_1 \amalg \ldots \amalg \lambda_n). \tag{14}$$

Repeated iterations of (13) suggest to rewrite the weights as

$$q_\beta(\lambda_1 \amalg \ldots \amalg \lambda_n) = q_\beta(\lambda_n) \prod_{k=1}^{n-1} q_\beta(\lambda_k \mid \lambda_{k+1} \amalg \ldots \amalg \lambda_n).$$

Introducing a dummy empty path $\emptyset$ and using it to extend finite strings of irreducible paths $(\lambda_1, \ldots, \lambda_n)$ to infinite strings $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n, \emptyset, \emptyset, \ldots)$, we rewrite (14) in a more uniform and compact form:

$$g_\beta(x) = \sum_{n\geq 1} \sum_{\underline{\lambda}:0\mapsto x} \exp\left[\psi\left(\tau^k \underline{\lambda}\right)\right], \tag{15}$$

where τ is the shift $\tau\underline{\lambda} = \tau(\lambda_1, \lambda_2, \ldots) = (\lambda_2, \ldots)$ and the potential ψ is defined via $e^{\psi(\underline{\lambda})} = q_\beta(\lambda_1 \mid \lambda_2 \amalg \ldots)$ under the convention $q_\beta(\lambda \mid \emptyset \amalg \emptyset \amalg \ldots) = q_\beta(\lambda)$.

The expansion (15) suggests to introduce the normalized operator L as follows:

$$Lf(\underline{\lambda}) = \sum_{\nu \text{ irreducible}} e^{\psi(\nu,\underline{\lambda})+(t,V(\nu))} f(\nu, \underline{\lambda}), \tag{16}$$

where $V(\nu) \in \mathbb{Z}^d$ is the displacement along the path ν, and, as before, $t \in \partial \boldsymbol{K}_\beta$ satisfies $(t, x) = \xi_\beta(x)$.

Expression (15) falls into the framework of the classical Gaussian local limit theory once the potential ψ happens to be Hölder continuous: Given two different strings $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots)$ and $\underline{\lambda}' = (\lambda_1', \lambda_2', \ldots)$ of irreducible paths define $\boldsymbol{i}(\underline{\lambda}, \underline{\lambda}') = \min\{k \geq 1 : \lambda_k \neq \lambda_k'\}$. We would like to find $\theta \in (0, 1)$ such that

$$|\psi|_\theta = \inf_k \inf_{\boldsymbol{i}(\underline{\lambda},\underline{\lambda}')>k} \frac{\left|\psi(\underline{\lambda}) - \psi(\underline{\lambda}')\right|}{\theta^k} < \infty. \tag{17}$$

Clearly, the exponential mixing property (10) alone is not sufficient to ensure (17): Even if $k \gg 1$ the sum $\sum_{u\in\lambda_1} \sum_{v\in\lambda_k} \theta^{|v-u|}$ is beyond control unless one imposes further restrictions on the geometry of irreducible paths. This is precisely the motivation behind the following refined definition of irreducible paths: Given $t \in \partial \boldsymbol{K}_\beta$ and a renormalization scale K, let us say that i_l is a break point of a path $\lambda = (i_1, \ldots, i_n)$ if i_l is a regeneration point of λ and

$$\{i_{l+1}, \ldots, i_n\} \subset i_l + KU_\beta + \mathcal{C}_\delta(t),$$

where $\mathcal{C}_\delta(t) = \{v \in \mathbb{Z}^d : (t, v) > (1 - \delta)\xi_\beta(v)\}$ is a positive cone along the direction of t. A path λ is *irreducible* if it does not contain break points. If $\lambda_1 \amalg \ldots \amalg \lambda_n$ is the concatenation of such irreducible paths, then for every $1 \leq l < k \leq n$,

$$\sum_{u\in\lambda_l} \sum_{v\in\lambda_k} \theta^{|v-u|} \leq C_5(K)\theta^{k-l},$$

which, in view of (10), already implies that the potential ψ in (16) satisfies the Hölder condition (17). It remains to check that up to exponentially negligible weights typical paths $\gamma : 0 \mapsto x$ contain a density of break points. However, coarse-graining procedures similar to those developed above for SAW, and based on the properties (7), (8) and (9), show that there exist $M < \infty$ and $\nu > 0$ such that

$$\sum_{\substack{\lambda:0\to y\\ \text{irreducible}}} q_\beta(\lambda) \le M e^{-\nu|x|} e^{-\xi_\beta(x)}.$$

This is the analogue of the mass separation result (12).

Both the local limit asymptotics of Theorem 1 and the geometry of $\boldsymbol{K}_\beta$ can be now read from the analytic dependence of the leading eigenvalue $\rho(z)$ of L_z on the perturbation $z \in \mathbb{C}^d$ in

$$L_z f(\underline{\lambda}) = L\big(e^{(z,V(\lambda_1))} f\big).$$

Indeed, $\log \rho(z)$ is nothing but the limiting log-moment generation function of $1/n \sum_1^n V(\lambda_i)$. On the other hand, since $\boldsymbol{K}_\beta$ is the domain of convergence of (4), the equation of $\partial \boldsymbol{K}_\beta$ in a neighborhood of t is given by $\{t + z \in \mathbb{R}^d : \rho(z) = 1\}$. In other words, locally the surface $\partial \boldsymbol{K}_\beta$ is just a level set of ρ through t. Furthermore, choosing perturbations z of the form $z = p_1 n_x + i\boldsymbol{p}$, where $\boldsymbol{p}$ lies in the tangent hyperplane to $\partial \boldsymbol{K}_\beta$ at t, we recover the results of Paes-Leme (see [3], Proposition 4.2) on the analyticity of the 1-particle mass shells.

* * *

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