Multibump Orbits Continued from the Anti-integrable Limit for Lagrangian Systems

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Abstract

A continuous-time anti-integrable limit is considered in the context of time-dependent Lagrangian systems on the torus. The anti-integrable limit is the singular (or adiabatic) limit in the singularly (adiabatically, resp.) perturbed problems. This paper presents an implicit function theorem version of the results of Bolotin and MacKay (1997 *Nonlinearity* **10** 1015-1029) for multibump orbits.

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1 Introduction and main result

In 1997, Bolotin and MacKay [8] studied a Lagrangian system with kinetic energy minus potential energy having the following form

$$\frac{\epsilon^2}{2}|v|^2 - V(x,\tau), \qquad ((x,v),\tau) \in T\mathbb{T} \times \mathbb{R}. \tag{1.1}$$

(Actually, in their paper, the configuration space can be any non-simply connected compact manifold.) They called the singular limit $\epsilon \to 0$ the *anti-integrable limit*. In the special case when V is periodic in τ, by assuming that the integral of corresponding adiabatic Poincaré-Melnikov function [16] has global minimum points at $\tau = n_0 + \mathbb{Z}$ for some $n_0 \in \mathbb{R}$, when ϵ is small they proved existence of multibump orbits, then obtained the Bernoulli shift property [27]. Their work remarkably extended the concept of anti-integrability [4, 19] of symplectic twist maps to Lagrangian systems. However, their method does not work if n_0 is a maximum point. Due to this, MacKay proposed that problem (1.1) may be approached by the implicit function theorem. This led to the work of the present paper.

The small mass system (1.1) can be converted to a system of slowly variation in potentials, namely with the following Lagrangian

$$L(x,v,s,t,\epsilon) = \frac{1}{2}|v|^2 - V(x,\epsilon s + t) \tag{1.2}$$

by introducing a fast time $s = \tau/\epsilon$ and letting the potential has a phase t when $s = 0$. The corresponding Newton's equation of (1.2) reads:

$$\ddot{q}(s) + D_x V(q(s), \epsilon s + t) = 0.$$

Below, we state our assumptions on the potential function, which guarantee existence of homoclinic trajectories when $\epsilon = 0$. Then we state our main theorem.

We assume for any fixed s, t and ϵ that the origin is the unique non-degenerate global maximum point of the C^3-potential function V and that there is a neighbourhood U of the origin, $U = \{x : |x| < r \text{ for some } r > 0\}$, and constants $K > k > 0$ such that

$$\begin{aligned} D_x^2 V(0,\epsilon s + t) \quad &:= -\lambda_{t,\epsilon}(s), && (1.3)\\ -K \leq \quad D_x^2 V(x,\epsilon s + t) \quad &\leq -k \ \forall x \in U. && (1.4) \end{aligned}$$

Thus the potential is concave in U. We also assume

$$V(x,\epsilon s + t) > V(y,\epsilon s + t) \ \forall\ x \in \partial U \text{ and } y \in \mathbb{T} \setminus U. \tag{1.5}$$

When $\epsilon = 0$, we have an autonomous potential $V(x,t)$. By conservation of energy, the phase portrait consists of closed curves. For fixed t, there are two classes of homoclinic trajectories to the origin. They come in one-parameter families due to time-translation invariance. We need a notion to specify which one we will be talking about.

Definition 1.1. *A homoclinic trajectory* $\Gamma : \mathbb{R} \to \mathbb{T}$, $s \mapsto \Gamma(s)$*, is said to be of type* $(x, 0)$ *if* $\Gamma(0) = x$.

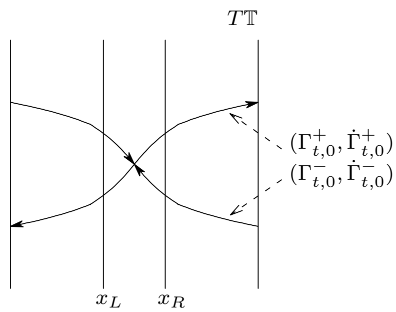

Figure 1: Phase portraits of two homoclinics for $L(x, v, s, t, 0)$: $\Gamma^-_{t,0}$ is such that $\Gamma^-_{t,0}(0) = x_L$, $\Gamma^-_{t,0}(T^*_{t,0}) = x_R$, thus is of type $(x_L, 0)$; $\Gamma^+_{t,0}$ is such that $\Gamma^+_{t,0}(0) = x_R$, $\Gamma^+_{t,0}(T^*_{t,0}) = x_L$, thus is of type $(x_R, 0)$.

Hence, when $\epsilon = 0$, the system possesses two homoclinic trajectories: one is of type $(x_L, 0)$, the other one is of type $(x_R, 0)$, where x_L and x_R are the boundary points of U. (See figure 1.) Let $f(t)$ be the area enclosed by any of these two orbits with the zero section of $T\mathbb{T}$, then $f(t)$ is C^2 and is crucial for proving existence of homoclinic orbits when $\epsilon \neq 0$ [12, 25, 23, 33, 14, 8]. In most cases it is assumed to have only non-degenerate critical points. In fact, Wiggins, Kaper and Kovačič showed that the derivative of this function is the *adiabatic Poincaré-Melnikov functions* [31, 16].

We do not require the potential to be periodic in t but assume it is bounded below. Then, the function f is bounded above, and condition (1.4) guarantees it is bounded away from zero as well. These facts together with (1.5) imply the time $T^*_{t,0}$ needed for a homoclinic trajectory to travel from a boundary point to another one of U is finite and bounded away from zero. Let

$$\Delta^*_T := \overline{\bigcup_{t\in\mathbb{R}} T^*_{t,0}}.$$

Generically, we can assume f is non-constant. Let $\mathfrak{N} = \{n_i\}_{i\in\mathcal{I}}$ be an arbitrary increasing sequence consisting of uniformly non-degenerate critical points of f, where $\mathcal{I}$ is an index set defined by

$$\mathcal{I} := [l, l+1, \cdots, m-1, m], \quad -\infty \le l \le m \le \infty, \tag{1.6}$$

with $m-l+1$ equal to the cardinality of such sequence. To be precise, we assume there are positive constants c_1, c_2 and Δ_n such that if $n \in \mathfrak{N}$ then $f'(n) = 0$ and $c_2 \ge |f''(n)| \ge c_1$ and that for all $i \in \mathcal{I}$, we have $|n_{i+1} - n_i| \ge \Delta_n$.

Before stating our theorem, we first divide the real line into time intervals $\Omega_{i,j,\epsilon}$: We fix a small open interval Δ_t containing 0 such that

$$c_3 \le \Delta_n - (\sup \Delta_t - \inf \Delta_t) \tag{1.7}$$

for some constant $c_3 > 0$. Suppose i, $j \in \mathcal{I}$ and t_i, $t^*_{i,\epsilon}$, t_j, $t^*_{j,\epsilon} \in \Delta_t$, and suppose $T^*_{i,\epsilon} > 0$ belonging to a bounded open interval Δ_T containing Δ^*_T, then for sufficiently small ϵ, define

$$\begin{aligned} n_{i,j} &:= t_i + n_i - t_j - n_j, \\ n^*_{i,j,\epsilon} &:= t^*_{i,\epsilon} + n_i - t^*_{j,\epsilon} - n_j, \end{aligned}$$

and define the following intervals

$$\begin{aligned}
\Omega^-_{i,j,\epsilon} &:= \left[\frac{n^*_{i-1,j,\epsilon}}{\epsilon}+\frac{n^*_{i,i-1,\epsilon}}{2\epsilon}, \frac{n^*_{i,j,\epsilon}}{\epsilon}\right],\\
\Omega^*_{i,j,\epsilon} &:= \left[\frac{n^*_{i,j,\epsilon}}{\epsilon}, \frac{n^*_{i,j,\epsilon}}{\epsilon}+T^*_{i,\epsilon}\right],\\
\Omega^+_{i,j,\epsilon} &:= \left[\frac{n^*_{i,j,\epsilon}}{\epsilon}+T^*_{i,\epsilon}, \frac{n^*_{i,j,\epsilon}}{\epsilon}+\frac{n^*_{i+1,i,\epsilon}}{2\epsilon}\right],\\
\Omega_{i,j,\epsilon} &:= \Omega^-_{i,j,\epsilon}\cup\Omega^*_{i,j,\epsilon}\cup\Omega^+_{i,j,\epsilon}.
\end{aligned}$$

For j fixed and ϵ non-zero, $\Omega^-_{i,j,\epsilon}$, $\Omega^*_{i,j,\epsilon}$, $\Omega^+_{i,j,\epsilon}$ are ordered intervals in the sense that $\max\Omega^-_{i,j,\epsilon}=\min\Omega^*_{i,j,\epsilon}$, $\max\Omega^*_{i,j,\epsilon}=\min\Omega^+_{i,j,\epsilon}$, $\max\Omega^+_{i,j,\epsilon}=\min\Omega^-_{i+1,j,\epsilon}$ for all $i\in\mathcal{I}$. As a rule, in the above if $n_i\in\min\mathfrak{N}$ then $n_{i-1}=-\infty$; if $n_i\in\max\mathfrak{N}$ then $n_{i+1}=\infty$.

Theorem 1.2. *Suppose $\Gamma^-_{n_i,0}$ and $\Gamma^+_{n_i,0}$ are homoclinic trajectories of respectively type $(x_L,0)$ and $(x_R,0)$ for $L(x,v,s,n_i,0)$. For a small neighbourhood U satisfying (1.4) and (1.5) and for any increasing sequence $\{n_i\}_{i\in\mathcal{I}}$ and any sequence $\left\{\Gamma_{n_i,0}\in\{\Gamma^-_{n_i,0},\Gamma^+_{n_i,0}\}\right\}_{i\in\mathcal{I}}$, we fix a small open interval Δ_t containing zero satisfying (1.7) and fix an open interval Δ_T containing Δ^*_T. Then there are $\epsilon_0>0$ and sequences $\{t^*_{i,\epsilon}\in\Delta_t\}_{i\in\mathcal{I}}$, $\{T^*_{i,\epsilon}\in\Delta_T\}_{i\in\mathcal{I}}$ such that for any $0<\epsilon<\epsilon_0$ and any given $n_j\in\{n_i\}_{i\in\mathcal{I}}$ there is a $t^*_{j,\epsilon}\in\{t^*_{i,\epsilon}\}_{i\in\mathcal{I}}$ so that $L(x,v,s,t^*_{j,\epsilon}+n_j,\epsilon)$ admits a unique multi-bump trajectory $\Upsilon_{j,\epsilon}(s)$ which is close to $\Gamma_{n_i,0}(s-n^*_{i,j,\epsilon}/\epsilon)$ when $s\in\Omega_{i,j,\epsilon}$ and satisfies $\Upsilon_{j,\epsilon}(n^*_{i,j,\epsilon}/\epsilon)=\Gamma_{n_i,0}(0)$, $\Upsilon_{j,\epsilon}(n^*_{i,j,\epsilon}/\epsilon+T^*_{i,\epsilon})=\Gamma_{n_i,0}(T^*_{n_i,0})$, in particular $\Upsilon_{j,\epsilon}(s)\in\overline{U}$ when $s\in\Omega^-_{i,j,\epsilon}\cup\Omega^+_{i,j,\epsilon}$, but $\Upsilon_{j,\epsilon}(s)\in\overline{\mathbb{T}\setminus U}$ when $s\in\Omega^*_{i,j,\epsilon}$. Moreover, $t^*_{i,\epsilon}$ and $T^*_{i,\epsilon}$ are C^1 in ϵ, uniformly in i. As $\epsilon\to 0$, $t^*_{i,\epsilon}$ converges to 0, $T^*_{i,\epsilon}$ converges to $T^*_{n_i,0}$, and $\Upsilon_{j,\epsilon}(s)$ C^1-converges to $\Gamma_{n_j,0}(s)$ uniformly on any compact subset of $\mathbb{R}$.*

The set $\mathfrak{N}$ can be $\{n_1,n_2,\cdots\}$, $\{\cdots,n_{-2},n_{-1}\}$ or $\{n_1,\cdots,n_N\}$, then $\Upsilon_{j,\epsilon}$ is respectively backwards asymptotic, forwards asymptotic or homoclinic to the origin, in particular, if $\mathfrak{N}\equiv\{n_j\}$ then $\Upsilon_{j,\epsilon}$ is a one-bump homoclinic trajectory.

Recently, Rabinowitz and Coti Zelati [24], Alessio, Bertotti and Montecchiari [1] also obtained multibump type trajectories for some classes of time-dependent Lagrangian systems, and same as in [8] no non-degeneracy conditions nor time-periodicity were required. However, our non-degeneracy condition, which is the same as the one required by the Poincaré-Melnikov method, is necessary and sufficient [31] for a transversal homoclinic orbit to persist under perturbations.

Applying the Birkhoff-Smale theory to time-periodic systems, transversal homoclinics imply existence of multibump orbits and the Bernoulli property. But, in general time-dependence cases, this implication is not obvious, in particular, if the Poincaré-Melnikov function has only finite zeros. As pointed out in [32, 14, 16] that the Poincaré-Melnikov function only detects *primary intersection points* [32], but our gluing technique allows us to detect higher order (transversal) intersection points. Note also that in [26, 29] higher-order Poincaré-Melnikov functions were constructed for similar purpose.

The idea we have to prove theorem 1.2 is quite simple: We divide the configuration space $\mathbb{T}$ into two parts, a neighbourhood U of the origin and $\mathbb{T}\setminus U$, then show that for any two points x_{i-1}, $y_i\in\partial U$ there is a unique trajectory confined in U spending a given time Δs_i connecting them, also for an $x_i \in \partial U$ ($x_i \neq y_i$) there is a unique trajectory outside U spending a given time T_i connecting y_i and x_i. (See figure 2.) We adjust T_i so that these two trajectories have no velocity discontinuity when they join at y_i. This smoothing process will be made simultaneously for all integer i, and we also adjust Δs_i to get a smooth (C^2)-trajectory. The result for the limit $\epsilon \to 0$ in theorem 1.2 will follow automatically.

Theorem 1.2 has a higher dimensional analogue, as can be seen by the result of [8], and the proof will be in spirit not too different from the one here. The main difference is that we also have to adjust all positions of x_i and y_i on the boundary ∂U. See [10, 9, 11] for related topics.

In the next section, some persistence results for trajectories inside and outside U are given; with them we then preform our smoothing process in section 3. Sections 4 and 5 are devoted to the proof of our main theorem.

2 Persistence of non-degenerate trajectories

2.1 Local results

We state the following theorem first.

Theorem 2.1. *For the Lagrangian (1.2) and a fixed small neighbourhood U satisfying (1.4) and (1.5), there exists $\epsilon_0 > 0$ such that if $0 \leq \epsilon < \epsilon_0$ then:*

- *except the origin, there is no other trajectories confined in U for all $s \in \mathbb{R}$;*
- *for any $s_0 \in \mathbb{R}$, $x \in U$, there is a unique trajectory $\omega^+_{t,\epsilon} : s \mapsto \omega^+_{t,\epsilon}(s)$ with $\omega^+_{t,\epsilon}(s_0) = x$ and is confined in U for all $s \geq s_0$. In particular, $\omega^+_{t,\epsilon}$ and its derivative tend to zero exponentially fast at a rate at least $e^{-\sqrt{k}(s-s_0)}$ as s tends to ∞;*
- *for any $s_1 \in \mathbb{R}$, $y \in U$, there is a unique trajectory $\omega^-_{t,\epsilon} : s \mapsto \omega^-_{t,\epsilon}(s)$ with $\omega^-_{t,\epsilon}(s_1) = y$ and is confined in U for all $s \leq s_1$. In particular, $\omega^-_{t,\epsilon}$ and its derivative tend to zero exponentially fast at a rate at least $e^{\sqrt{k}(s-s_1)}$ as s tends to $-\infty$;.*

If write $\omega^+_{t,\epsilon}$ and $\omega^-_{t,\epsilon}$ as functions of (x, s_0, t, ϵ) and (y, s_1, t, ϵ) respectively, then they depend C^2 on their variables in the C^2_{loc}-sense.

A proof of the theorem based on the contraction mapping theorem can be found in [11], while the theorem is implied by Fenichel's result [15, 33] (see also [23]) and is an alternative way to describe the persistence of a hyperbolic invariant set and it stable and unstable manifolds. Because of the theorem above, one is able to define two C^2 *action functions* (cf. [7])

$$S^+_\epsilon(x, s_0, t) := \int_{s_0}^{\infty} L(\omega^+_{t,\epsilon}(s), \dot{\omega}^+_{t,\epsilon}(s), s, t, \epsilon)\, ds, \tag{2.1}$$

$$S^-_\epsilon(y, s_1, t) := \int_{-\infty}^{s_1} L(\omega^-_{t,\epsilon}(s), \dot{\omega}^-_{t,\epsilon}(s), s, t, \epsilon)\, ds. \tag{2.2}$$

When ϵ is small, the union of $(\omega^+_{t,\epsilon}(s), \dot{\omega}^+_{t,\epsilon}(s))$ for all $s \geq s_0$ is a curve in the local weak stable manifold of the origin, while the union of $(\omega^-_{t,\epsilon}(s), \dot{\omega}^-_{t,\epsilon}(s))$ for all $s \leq s_1$ is a curve in the local weak unstable manifold. The functions

$S^+_\epsilon(x, s_0, t)$ and $S^-_\epsilon(y, s_1, t)$ act respectively as generating functions of the local stable manifold $W^+_{loc,t}$ and the unstable manifold $W^-_{loc,t}$ of the origin. Indeed, if keep s_0, s_1 and t fixed, then

$$\begin{aligned} W^+_{loc,t} &= \left\{(x, v) \in T_U\mathbb{T}, \quad v = -D_x S^+_\epsilon(x, s_0, t)\right\}, \\ W^-_{loc,t} &= \left\{(y, v) \in T_U\mathbb{T}, \quad v = D_y S^-_\epsilon(y, s_1, t)\right\}. \end{aligned}$$

The two formulae below can be derived from the first variation formula for Hamilton's action:

$$\begin{aligned} \lim_{\epsilon\to 0} D_t S^+_\epsilon(x, s_0, t) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} H(x, \dot\omega^+_{t,\epsilon}(s_0), s_0, t, \epsilon), \\ \lim_{\epsilon\to 0} D_t S^-_\epsilon(y, s_1, t) &= -\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} H(y, \dot\omega^-_{t,\epsilon}(s_1), s_1, t, \epsilon), \end{aligned}$$

where H is the C^3-Hamiltonian function associated with the Lagrangian L. In the above we have identified the velocity with the momentum and have used the fact that $0 = H(x, \dot\omega^+_{t,0}(s_0), s_0, t, 0) = H(y, \dot\omega^-_{t,0}(s_1), s_1, t, 0)$.

The following theorem gives an improved description of [8] (see also [30]) for trajectories which stay in U for an extended time. Its proof is located in Appendix section. In case a system is autonomous, it can also be derived from Shilnikov's lemma [28, 13] or from the λ-lemma [22].

Theorem 2.2. *For any $\Delta s > 0$ and a sufficiently small neighbourhood U of the origin, there exists ϵ_0 such that if $0 \le \epsilon < \epsilon_0$ then for any $s_0 < s_1 \in \mathbb{R}$, $s_1 - s_0 \ge \Delta s$, any x, $y \in U$, the Lagrangian system (1.2) has a unique trajectory $q_{t,\epsilon} : s \mapsto q_{t,\epsilon}(s)$ such that $q_{t,\epsilon}(s) \in U$ for all $s \in [s_0, s_1]$ and satisfies $q_{t,\epsilon}(s_0) = x$, $q_{t,\epsilon}(s_1) = y$. Write $q_{t,\epsilon}$ as a function of $(x, y, s_0, s_1, t, \epsilon)$, then it is a C^2-function. If, in addition, Δs is large enough and ϵ is small enough, it has the representation*

$$q_{t,\epsilon}(s) = \omega^+_{t,\epsilon}(s) + \omega^-_{t,\epsilon}(s) + \Psi(s_0, s_1, s, t, \epsilon)e^{-\sqrt{k}(s_1 - s_0)/2} \tag{2.3}$$

for some uniformly bounded C^2-function Ψ. Moreover, $q_{t,\epsilon}(s) \to \omega^+_{t,\epsilon}(s)$ as $s_1 \to \infty$, $q_{t,\epsilon}(s) \to \omega^-_{t,\epsilon}(s)$ as $s_0 \to -\infty$, and $q_{t,\epsilon}(s) \to 0$ as $s_0 \to -\infty$ and $s_1 \to \infty$, all uniformly on any compact time interval under consideration.

From the above theorem, one can define another *action function* S_ϵ by

$$S_\epsilon(x, y, s_0, s_1, t) := \int_{s_0}^{s_1} L(q_{t,\epsilon}(s), \dot q_{t,\epsilon}(s), s, t, \epsilon)\, ds. \tag{2.4}$$

Then S_ϵ also C^2 (cf. [2]), in particular,

$$\begin{aligned} &\lim_{\epsilon\to 0} D_t S_\epsilon(x, y, s_0, s_1, t) \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (H(x, \dot q_{t,\epsilon}(s_0), s_0, t, \epsilon) - H(y, \dot q_{t,\epsilon}(s_1), s_1, t, \epsilon)) \end{aligned} \tag{2.5}$$

because $H(x, \dot q_{t,0}(s_0), s_0, t, 0) = H(y, \dot q_{t,0}(s_1), s_1, t, 0)$.

When Ω is bounded, the fact that $q_{t,\epsilon}$ depends C^2 on ϵ implies S_ϵ depends C^2 on ϵ. When Ω is unbounded, it can be shown that the second derivatives of $\omega^{\pm}_{t,\epsilon}(s)$ with respect to ϵ are at most of order $O(s^2)$ [11, 18, 6], but $\omega^{+}_{t,\epsilon}(s)$ ($\omega^{-}_{t,\epsilon}(s), resp.$) forwards (backwards, resp.) converges to zero exponentially in s. This means the integrals (2.1), (2.2) and their partial derivatives with respect to ϵ to second order are finite, and depend C^2 on ϵ. So we have

Proposition 2.3. *The action functions S_ϵ, S^-_ϵ and S^+_ϵ are C^2-dependent of ϵ when $\epsilon < \epsilon_0$.*

2.2 Global results

For trajectories outside the neighbourhood U of the origin, in the Appendix section we shall prove

Theorem 2.4. *If $q^*_{n,0}$ is a homoclinic trajectory for $L(x,v,s,n,0)$ such that $q^*_{n,0}(0) = y_n \in \partial U$, $q^*_{n,0}(T^*_{n,0}) = x_n \in \partial U$ for some $T^*_{n,0} > 0$, then there are neighbourhoods $U_y \ni y_n$, $U_x \ni x_n$ so that for any $y \in U_y$, $x \in U_x$, any T, t, ϵ with $T - T^*_{n,0}$, $t - n$, ϵ sufficiently small, there is a unique trajectory $q^*_{t,\epsilon}$ for $L(x,v,s,t,\epsilon)$ satisfying $q^*_{t,\epsilon}(0) = y$, $q^*_{t,\epsilon}(T) = x$. Moreover, regard the trajectory $q^*_{t,\epsilon}$ as a function of (y,x,T,t,ϵ) then it depends C^2 on its variables.*

Similar to the foregoing results, the following C^2 action function defined by

$$S^*_\epsilon(y,x,T,t) := \int_0^T L(q^*_{t,\epsilon}(s), \dot{q}^*_{t,\epsilon}(s), s, t, \epsilon)\ ds \tag{2.6}$$

with $q^*_{t,\epsilon}$ as in theorem 2.4 is single valued. Because $H(y, \dot{q}^*_{t,0}(0), 0, t, 0) - H(y, \dot{q}^*_{t,0}(T), T, t, 0) = 0$, we get

$$\begin{aligned} \lim_{\epsilon\to 0} D_t S^*_\epsilon(y,x,T,t) &= D_t S^*_0(y,x,T,t) \\ &= \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \Big(H(y, \dot{q}^*_{t,\epsilon}(0), 0, t, \epsilon) - H(x, \dot{q}^*_{t,\epsilon}(T), T, t, \epsilon) \Big). \end{aligned} \tag{2.7}$$

For a trajectory $q^*_{t,\epsilon} : \mathbb{R} \to \mathbb{T}$ with $q^*_{t,\epsilon}(0) = y$ and $q^*_{t,\epsilon}(T) = x$, one can define its value of Hamiltonian as $\mathcal{H}^x_\epsilon(T,t)$ when it arrives at x, namely

$$\mathcal{H}^x_\epsilon(\cdot, t) : T \mapsto H(x, \dot{q}^*_{t,\epsilon}(T), T, t, \epsilon). \tag{2.8}$$

The following lemma, which is proved in the Appendix section, gives an energy-travelling time relation.

Lemma 2.5. *Suppose ϵ is small and $q^*_{t,\epsilon}$ is a trajectory satisfying $q^*_{t,\epsilon}(0) = y$, $q^*_{t,\epsilon}(T^*) = x$. If $\dot{q}^*_{t,\epsilon}(s) \neq 0\ \forall s \in [0, T^*]$, then there is a neighbourhood $U_T \subset \mathbb{R}$ of T^* such that $D_T \mathcal{H}^x_\epsilon(T,t) < 0$ for all $T \in U_T$.*

Let $\Gamma_{n,0}$ be a homoclinic trajectory for $L(x,v,s,n,0)$ and $\Gamma_{t,0}$ be another for $L(x,v,s,t,0)$, we would like to have a notion about continuation of homoclinic trajectories with respect to t.

Definition 2.6. *A family of type $(y,0)$ homoclinic trajectories $\{\Gamma_{t,0}\}$ parameterised by t is the C^2-continuation of $\Gamma_{n,0}$ (with respect to t) if there is $\delta t > 0$ such that for all $t \in (n - \delta t, n + \delta t)$ the mapping $t \mapsto \Gamma_{t,0} \in C^2(\mathbb{R}, \mathbb{T})$ is C^2.*

With this definition, suppose $\Gamma_{t,0}$ is C^2-continued from $\Gamma_{n,0}$ and is of type $(y,0)$ then, for each t close to n, $\Gamma_{t,0}$ can be written as

$$\Gamma_{t,0}(s) = \begin{cases} \omega^-_{t,0}(s) & s \in (-\infty, 0] \\ q^*_{t,0}(s) & s \in [0, \mathfrak{T}(t)] \\ \omega^+_{t,0}(s) & s \in [\mathfrak{T}(t), \infty) \end{cases} \tag{2.9}$$

with $\omega^{\pm}_{t,0}$, $q^*_{t,0}$ indicating trajectories for $L(x,v,s,t,0)$ such that $\omega^-_{t,0}(0) = y = q^*_{t,0}(0)$ and $q^*_{t,0}(\mathfrak{T}(t)) = x = \omega^+_{t,0}(\mathfrak{T}(t))$ for some C^2-function $\mathfrak{T}$. The product path

$$\begin{cases} \omega^-_{n,0}(s) & s \in (-\infty, 0] \\ q^*_{n,0}(s) & s \in [0, \mathfrak{T}(n)] \\ \omega^+_{n,0}(s) & s \in [\mathfrak{T}(n), \infty) \end{cases}$$

is identical to $\Gamma_{n,0}$. According to theorems 2.1 and 2.4, $\omega^-_{t,0}$ is C^2-continued from $\omega^-_{n,0}$ with respect to t and with y, $s_1(= 0$ here), ϵ fixed; $q^*_{t,0}$ is C^2-continued from $q^*_{n,0}$ with respect to $(T,t)(= (\mathfrak{T}(t),t)$ here) and with y, x, ϵ fixed; $\omega^+_{t,0}$ is C^2-continued from $\omega^+_{n,0}$ with respect to $(s_0,t)(= (\mathfrak{T}(t),t)$ here) and with x, ϵ fixed. The homoclinic trajectory $\Gamma_{t,0}$ has Hamilton's action

$$I(\Gamma_{t,0}, t) := \int_{-\infty}^{\infty} L(\Gamma_{t,0}(s), \dot{\Gamma}_{t,0}(s), s, t, 0)\, ds. \tag{2.10}$$

The following results will be useful in the future.

Proposition 2.7. *Let $\{\Gamma_{t,0}\}$ be the C^2-family of homoclinic trajectories of type $(y,0)$ having representation (2.9).*
(a) If $D_T\mathcal{H}^x_0(\mathfrak{T}(t),t) \neq 0$ then

$$D\mathfrak{T}(t) = -\frac{D_t\mathcal{H}^x_0(\mathfrak{T}(t),t)}{D_T\mathcal{H}^x_0(\mathfrak{T}(t),t)}.$$

(b)

$$\begin{aligned} \left.\frac{d^2}{dt^2}\right|_{t=n} I(\Gamma_{t,0},t) &= \left.D^2_t\right|_{t=n} \left(S^-_0(y,0,t) + S^*_0(y,x,\mathfrak{T}(n),t) + S^+_0(x,\mathfrak{T}(n),t)\right) \\ &+ \left.D^2_{Tt}\right|_{t=n} S^*_0(y,x,\mathfrak{T}(n),t)\ D\mathfrak{T}(n). \end{aligned} \tag{2.11}$$

Proof: (a): $\mathcal{H}^x_0(\mathfrak{T}(t),t)$ equals zero for all t because $q^*_{t,0}$ is a segment of $\Gamma_{t,0}$, which tends to zero as time goes on, thus

$$0 = \frac{d}{dt}\mathcal{H}^x_0(\mathfrak{T}(t),t) = D_T\mathcal{H}^x_0(\mathfrak{T}(t),t)\ D\mathfrak{T}(t) + D_t\mathcal{H}^x_0(\mathfrak{T}(t),t).$$

(b): Now for all t, $I(\Gamma_{t,0},t) = S^-_0(y,0,t) + S^*_0(y,x,\mathfrak{T}(t),t) + S^+_0(x,\mathfrak{T}(t),t)$ by (2.1), (2.2), (2.6). Thus for all t,

$$\begin{aligned} \frac{d}{dt}I(\Gamma_{t,0},t) &= D_tS^-_0(y,0,t) \\ &+ D_TS^*_0(y,x,\mathfrak{T}(t),t)\ D\mathfrak{T}(t) + D_tS^*_0(y,x,\mathfrak{T}(t),t) \\ &+ D_{s_0}S^+_0(x,\mathfrak{T}(t),t)\ D\mathfrak{T}(t) + D_tS^+_0(x,\mathfrak{T}(t),t). \end{aligned} \tag{2.12}$$

Because $D_TS^*_0(y,x,\mathfrak{T}(t),t) = 0 = D_{s_0}S^+_0(x,\mathfrak{T}(n),t)$ for all n and t, the right hand side of (2.12) equals $D_t\left(S^-_0(y,0,t) + S^*_0(y,x,\mathfrak{T}(t),t) + S^+_0(x,\mathfrak{T}(t),t)\right)$. Taking derivative again, we have (2.11). $\square$

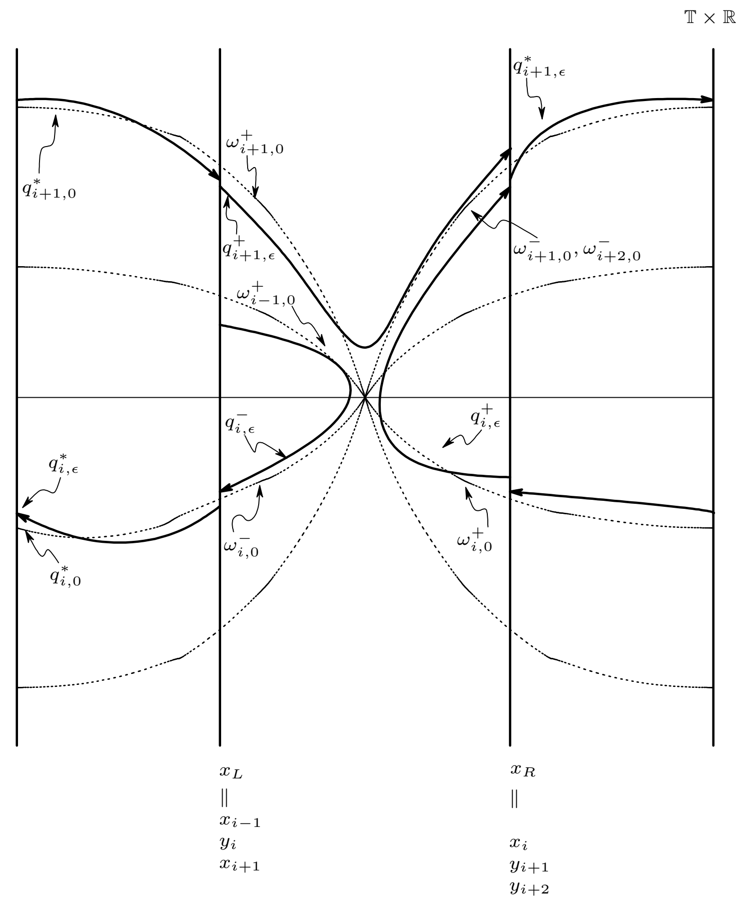

Figure 2: In the figure, we assume the system is periodic and its adiabatic Poincaré-Melnikov function has exactly two simple zeros. There are two separatrices corresponding to these two zeros and they are the dotted lines in the phase space. The solid line shows a broken orbits.

3 Smoothing broken trajectories

In this section we construct a set of variational functionals whose critical points give multi-bump trajectories.

Suppose $\mathcal{I}$ is the index set as defined in (1.6). Let $Z_i \subset \mathbb{R}^2$ be a copy of $(\Delta_t \times \Delta_T)$ with Δ_t and Δ_T as in theorem 1.2, and $Z := \prod_{i\in\mathcal{I}} Z_i$ be the space of sequences $(z_l, z_{l+1}, \cdots, z_{m-1}, z_m)$,

$$z_i = (t_i, T_i) \in \Delta_t \times \Delta_T \ \forall\ i \in \mathcal{I},$$

equipped with the sup norm $\|z\| = \sup_i\{|z_i|\}$ for $z \in Z$. For an increasing sequence $\{n_i\}_{i\in\mathcal{I}} = \mathfrak{N}$ and a prescribed sequence $\{y_i \in \{x_L, x_R\}\}_{i\in\mathcal{I}}$, we define a functional W_ϵ on Z, which is similar to the one used in [8] as well as to the ones in the Aubry-Mather theory [3, 20],

$$W_\epsilon(z) = \sum_{i\in\mathcal{I}} h^-(z_{i-1}, z_i, \epsilon) + h^*(z_i, \epsilon) + h^+(z_i, z_{i+1}, \epsilon), \tag{3.1}$$

with

$$\begin{aligned} h^-(z_{i-1}, z_i, \epsilon) &:= S_\epsilon\big(q^-_{i,\epsilon}(\frac{n_{i-1,i}}{2\epsilon}), y_i, \frac{n_{i-1,i}}{2\epsilon}, 0, t_i + n_i\big), \\ h^*(z_i, \epsilon) &:= S^*_\epsilon\big(y_i, x_i, T_i, t_i + n_i\big), \\ h^+(z_i, z_{i+1}, \epsilon) &:= S_\epsilon\big(x_i, q^+_{i,\epsilon}(\frac{n_{i+1,i}}{2\epsilon}), T_i, \frac{n_{i+1,i}}{2\epsilon}, t_i + n_i\big). \end{aligned}$$

Here, (see figure 2)

- S_ϵ and S^*_ϵ are respectively the action functions (2.4) and (2.6);
- $x_i \in \{x_L, x_R\}$ with $x_i \neq y_i \ \forall\ i$;
- $q^-_{i,\epsilon}$, $q^*_{i,\epsilon}$ and $q^+_{i,\epsilon}$ are trajectories for $L(x, v, s, t_i + n_i, \epsilon)$ such that

$$\begin{aligned} q^-_{i,\epsilon}(T_{i-1} + n_{i-1,i}/\epsilon) = x_{i-1}, &\qquad q^-_{i,\epsilon}(0) = y_i, \\ q^*_{i,\epsilon}(0) = y_i, &\qquad q^*_{i,\epsilon}(T_i) = x_i, \\ q^+_{i,\epsilon}(T_i) = x_i, &\qquad q^+_{i,\epsilon}(n_{i+1,i}/\epsilon) = y_{i+1}; \end{aligned}$$

- as a rule, if $\min\{\mathcal{I}\} = l > -\infty$, set $n_{l-1} = -\infty$; if $\max\{\mathcal{I}\} = m < \infty$, set $n_{m+1} = \infty$;
- in case $n_{i-1} = -\infty$ then $q^-_{i,\epsilon} = \omega^-_{i,\epsilon} \equiv \omega^-_{t_i+n_i,\epsilon}$ with $\omega^-_{i,\epsilon}(0) = y_i$, and $h^-(z_{i-1}, z_i, \epsilon) := S^-_\epsilon(y_i, 0, t_i + n_i)$ (S^-_ϵ was defined in (2.2)) ; in case $n_{i+1} = \infty$ then $q^+_{i,\epsilon} = \omega^+_{i,\epsilon} \equiv \omega^+_{t_i+n_i,\epsilon}$ with $\omega^+_{i,\epsilon}(T_i) = x_i$, and $h^+(z_i, z_{i+1}, \epsilon) := S^+_\epsilon(x_i, T_i, t_i + n_i)$ (S^+_ϵ was defined in (2.1)).

It would be much clear what W_ϵ is if write h^+, h^* and h^- as the integrals:

$$h^+(z_i, z_{i+1}, \epsilon) = \int_{T_i}^{\frac{t_{i+1}+n_{i+1}-t_i-n_i}{2\epsilon}} L(q^+_{i,\epsilon}(s), \dot{q}^+_{i,\epsilon}(s), s, t_i + n_i, \epsilon)\, ds, \tag{3.2}$$

$$h^*(z_i, \epsilon) = \int_0^{T_i} L(q^*_{i,\epsilon}(s), \dot{q}^*_{i,\epsilon}(s), s, t_i + n_i, \epsilon), \tag{3.3}$$

$$h^-(z_{i-1}, z_i, \epsilon) = \int_{\frac{t_{i-1}+n_{i-1}-t_i-n_i}{2\epsilon}}^{0} L(q^-_{i,\epsilon}(s), \dot{q}^-_{i,\epsilon}(s), s, t_i + n_i, \epsilon)\, ds, \tag{3.4}$$

Definition 3.1. *We call a point z^*_ϵ a* **critical point** *of W_ϵ if at which the partial derivative $D_{z_i} W_\epsilon$ is identically zero for every $i \in \mathcal{I}$.*

The following notation is similar to the one used in [27]: For $n \in \mathbb{R}$ and $\epsilon \neq 0$, we call the following map

$$n \star_\epsilon : C^2(\mathbb{R}, \mathbb{T}) \to C^2(\mathbb{R}, \mathbb{T}),\ q \mapsto n \star_\epsilon q$$

the *time-n/ϵ-translation*, where $n \star_\epsilon q(s) = q(s - n/\epsilon)$.

Theorem 3.2. *Let $i \in \mathcal{I}$. When $\epsilon \neq 0$, for a given increasing sequence $\{n_i\}_i$, a critical point $z^*_\epsilon = (\cdots, z^*_{i,\epsilon} = (t^*_{i,\epsilon}, T^*_{i,\epsilon}), \cdots)$ of W_ϵ corresponds to a sequence of C^2-trajectories $\{\Upsilon_{i,\epsilon} : \mathbb{R} \to \mathbb{T}\}_i$ for a sequence of Lagrangians $\{L(x, v, s, t^*_{i,\epsilon} + n_i, \epsilon)\}_i$ such that*

$$
\begin{aligned}
\Upsilon_{i,\epsilon}(s) = &\vdots \\
= &\, n^*_{i-1,i,\epsilon} \star_\epsilon q^+_{i-1,\epsilon}(s) \qquad s \in \Omega^+_{i-1,i,\epsilon}, \\
= &\, q^-_{i,\epsilon}(s) \qquad s \in \Omega^-_{i,i,\epsilon}, \\
= &\, q^*_{i,\epsilon}(s) \qquad s \in \Omega^*_{i,i,\epsilon}, \\
= &\, q^+_{i,\epsilon}(s) \qquad s \in \Omega^+_{i,i,\epsilon}, \\
= &\, n^*_{i+1,i,\epsilon} \star_\epsilon q^-_{i+1,\epsilon}(s) \qquad s \in \Omega^-_{i+1,i,\epsilon}, \\
= &\, n^*_{i+1,i,\epsilon} \star_\epsilon q^*_{i+1,\epsilon}(s) \qquad s \in \Omega^*_{i+1,i,\epsilon}, \\
= &\, n^*_{i+1,i,\epsilon} \star_\epsilon q^+_{i+1,\epsilon}(s) \qquad s \in \Omega^+_{i+1,i,\epsilon}, \\
= &\, n^*_{i+2,i,\epsilon} \star_\epsilon q^-_{i+2,\epsilon}(s) \qquad s \in \Omega^-_{i+2,i,\epsilon}, \\
= &\, \ddots
\end{aligned}
\tag{3.5}
$$

This theorem is to be understood in the following way: On every time interval $\Omega_{j,i,\epsilon}$, there are pieces of trajectories connecting x_{j-1} and y_j via $n^*_{j,i,\epsilon} \star_\epsilon q^-_{j,\epsilon}$ and then leaving y_j for x_j via $n^*_{j,i,\epsilon} \star_\epsilon q^*_{j,\epsilon}$, then leaving x_j for y_{j+1} via $n^*_{j,i,\epsilon} \star_\epsilon q^+_{j,\epsilon}$ etc etc. In Morse's sense $\Upsilon_{i,\epsilon}$ would be a "broken trajectory", which in general is not a true trajectory. (For example, the solid line in figure 2.) But we smoothed it by adjusting its arrival time at y_j by t_j and simultaneously adjusting its arrival time at x_j by T_j. This theorem points out that if (t_i, T_i) is close enough to $(t^*_{i,\epsilon}, T^*_{i,\epsilon})$ for every i then there is a true trajectory shadowing the broken one. We shall see later that if ϵ is small, there is only one such a true trajectory. The above gives an idea of proof; for a complete proof see [11] or [8].

4 Anti-integrable orbits

The following theorem says a critical point of the functional W_0 gives a sequence of homoclinic orbits for the unperturbed Lagrangian.

Theorem 4.1. *For a given increasing sequence $\{n_i\}$ and a prescribed sequence $\{y_i \in \{x_L, x_R\}\}$, $i \in \mathcal{I}$, the functional W_0 has exactly one critical point $z^*_0 = (\cdots, (0, T^*_{i,0} \equiv T^*_{n_i,0}), \cdots)$, which corresponds to a sequence of homoclinic trajectories $\{\Gamma_{i,0}\}$ in such a way that for each i the C^2-curve*

$$
\Gamma_{i,0}(s) := \begin{cases} \omega^-_{i,0}(s) & s \in (-\infty, 0] \\ q^*_{i,0}(s) & s \in [0, T^*_{i,0}] \\ \omega^+_{i,0}(s) & s \in [T^*_{i,0}, \infty) \end{cases}
$$

is a type-$(y_i, 0)$ homoclinic trajectory for $L(x, v, s, n_i, 0)$, and Hamilton's action of which is equal to the Maupertuis action $f(n_i)$.

Theorems 3.2 and 4.1 show that we have defined a variational problem via a one parameter family of variational functionals W_ϵ to find a certain class of trajectories $\Upsilon_{i,\epsilon}$ of a Lagrangian system. When $\epsilon \neq 0$, pieces of these trajectories obtained in theorem 3.2 are mutually coupled together via formula (3.5) to give whole trajectories. While $\epsilon = 0$, these trajectories are decoupled since the variational functional becomes, with respect to i, decoupled and the dynamical system becomes sequential copies of integrable systems. Hence we say the Lagrangian system constrained by the variational functional (3.1) is *anti-integrable* in Aubry's sense [5]. And, we call the limit $\epsilon = 0$ the *anti-integrable limit*.

Theorem 4.1 will be achieved by making use of the following proposition.

Proposition 4.2. *$h^-(z_{i-1}, z_i, \epsilon)$, $h^*(z_i, \epsilon)$ and $h^+(z_i, z_{i+1}, \epsilon)$ uniformly converge to actions $S_0^-(y_i, 0, t_i + n_i)$, $S_0^*(y_i, x_i, T_i, t_i + n_i)$ and $S_0^+(x_i, T_i, t_i + n_i)$ respectively as $\epsilon \to 0$. In particular, $h^-(z_{i-1}, z_i, 0)$ and $h^+(z_i, z_{i+1}, 0)$ are dependent of z_i only.*

Proof: For a given $i \in \mathcal{I}$, we only need to prove the cases $n_{i-1} \neq -\infty$ or $n_{i+1} \neq \infty$, otherwise the proposition is implied by proposition 2.3. Since $q_{i,\epsilon}^+$ has the representation (2.3), if on the time interval $[T_i, n_{i+1,i}/2\epsilon]$, $L(q_{i,\epsilon}^+(s), \dot{q}_{i,\epsilon}^+(s), s, t_i + n_i, \epsilon)$ is expanded around $\big(\omega_{i,\epsilon}^+(s), \dot{\omega}_{i,\epsilon}^+(s)\big)$ for all s to the first order then the values of the first order terms are of order $O(\exp(-\sqrt{k}\ n_{i+1,i}/2\epsilon))$. Thus by (3.2),

$$
\begin{aligned}
&h^+(z_i, z_{i+1}, \epsilon) = \\
&\qquad \int_{T_i}^{n_{i+1,i}/2\epsilon} L(\omega_{i,\epsilon}^+(s), \dot{\omega}_{i,\epsilon}^+(s), s, t_i + n_i, \epsilon)\ ds + O\left(\left(\frac{n_{i+1,i}}{2\epsilon}\right) e^{-\sqrt{k}\ n_{i+1,i}/2\epsilon}\right) \\
&\qquad \xrightarrow{\epsilon \to 0} \int_{T_i}^{\infty} L(\omega_{i,0}^+(s), \dot{\omega}_{i,0}^+(s), s, t_i + n_i, 0)\ ds = S_0^+(x_i, T_i, t_i + n_i)
\end{aligned}
$$

uniformly in z_i for all i. The case h^- can be proved likewise, and the case h^* is trivial. $\square$

In view of the proposition above and theorem 3.2, we have theorem 4.1, with the uniqueness of such a z_0^* by virtue of our construction of the functional W_0.

5 Persistence of anti-integrable orbits

Our next step is to show that those orbits corresponding to z_ϵ^* in theorem 3.2 exist and, in the C_{loc}^2-sense, are close to chains of homoclinics corresponding to z_0^* in theorem 4.1.

Let $W(\cdot, \epsilon) := W_\epsilon$. We need to formalise the criticality of $W(\cdot, \epsilon)$. To this end, we regard $D_z W(z, \epsilon)$ as a sequence in $\mathbb{R}_i^2$, $(\cdots, D_{z_i} W(z, \epsilon), \cdots)$, $D_z W$ is then a map

$$D_z W : Z \times (\mathbb{R}^+ \cup \{0\}) \to \mathbb{R}^{2\mathcal{I}},\ (z, \epsilon) \mapsto D_z W(z, \epsilon),$$

where the Cartesian product $\mathbb{R}^{2\mathcal{I}} := \prod_{i \in \mathcal{I}} \mathbb{R}_i^2$ has topology induced by the sup norm. By our formalism above, that a point is a critical point of the functional $W(\cdot, \epsilon)$ for our definition 3.1 precisely means it is a zero of the map $D_z W(\cdot, \epsilon)$. $D_z^2 W(z, \epsilon)$ then is a linear map

$$D_z^2 W(z, \epsilon) : \mathbb{R}^{2\mathcal{I}} \to \mathbb{R}^{2\mathcal{I}},\ \delta z \mapsto D_z^2 W(z, \epsilon) \delta z.$$

We represent $D_z^2 W(z,\epsilon)$ in matrix form $D^2_{z_j z_i} W(z,\epsilon)$, which behaves in a manner that the i-th component of $D_z^2 W(z,\epsilon)\delta z$ is equal to

$$\begin{aligned}\left(D_z^2 W(z,\epsilon)\delta z\right)_i &= \sum_{j\in\mathcal{I}} D^2_{z_j z_i} W(z,\epsilon)\delta z_j \\ &= \sum_j \begin{pmatrix} D^2_{t_j t_i} W(z,\epsilon) & D^2_{t_j T_i} W(z,\epsilon) \\ D^2_{T_j t_i} W(z,\epsilon) & D^2_{T_j T_i} W(z,\epsilon)\end{pmatrix}\begin{pmatrix}\delta t_j \\ \delta T_j\end{pmatrix} \qquad (5.1)\end{aligned}$$

in which $D^2_{t_j t_i} W(z,\epsilon)$ is an abbreviation for $D^2_{t_j t_i} W((\cdots,(t_k,T_k),\cdots),\epsilon)$; similarly for $D^2_{t_j T_i} W(z,\epsilon)$, $D^2_{T_j t_i} W(z,\epsilon)$ and $D^2_{T_j T_i} W(z,\epsilon)$. By definitions of W, h^-, h^* and h^+, (5.1) equals

$$\begin{aligned} &\begin{pmatrix} D^2_{t_{i-1},t_i}\left(h^-(z_{i-1},z_i,\epsilon)+h^+(z_{i-1},z_i,\epsilon)\right) & 0 \\ D^2_{T_{i-1},t_i} h^+(z_{i-1},z_i,\epsilon) & 0\end{pmatrix}\begin{pmatrix}\delta t_{i-1} \\ \delta T_{i-1}\end{pmatrix} \\ +&\begin{pmatrix} \begin{array}{l} D^2_{t_i}\left(h^*(z_i,\epsilon)\right. \\ +h^-(z_{i-1},z_i,\epsilon)+h^-(z_i,z_{i+1},\epsilon) \\ \left.+h^+(z_{i-1},z_i,\epsilon)+h^+(z_i,z_{i+1},\epsilon)\right)\end{array} & D^2_{t_i T_i}\left(h^*(z_i,\epsilon)+h^+(z_i,z_{i+1},\epsilon)\right) \\ D^2_{T_i t_i}\left(h^*(z_i,\epsilon)+h^+(z_i,z_{i+1},\epsilon)\right) & D^2_{T_i}\left(h^*(z_i,\epsilon)+h^+(z_i,z_{i+1},\epsilon)\right)\end{pmatrix}\begin{pmatrix}\delta t_i \\ \delta T_i\end{pmatrix} \\ +&\begin{pmatrix} D^2_{t_{i+1},t_i}\left(h^-(z_i,z_{i+1},\epsilon)+h^+(z_i,z_{i+1},\epsilon)\right) & D^2_{t_{i+1},T_i} h^+(z_i,z_{i+1},\epsilon) \\ 0 & 0\end{pmatrix}\begin{pmatrix}\delta t_{i+1} \\ \delta T_{i+1}\end{pmatrix} \qquad (5.2)\end{aligned}$$

Proposition 5.1. *The map $D_z W : (z,\epsilon)\mapsto D_z W(z,\epsilon)$ is C^1 when ϵ is small.*

Proof: Firstly we show it is continuous at $(z,0)$ for all z. From the first variation formula for Hamilton's action, we know

$$\begin{aligned} D_{t_i} W(z,\epsilon) &= \frac{1}{\epsilon} H(y_i,\dot q^*_{i,\epsilon}(0),0,t_i+n_i,\epsilon) - \frac{1}{\epsilon} H(x_i,\dot q^*_{i,\epsilon}(T_i),T_i,t_i+n_i,\epsilon) \\ &- \frac{1}{\epsilon} H(y_i,\dot q^-_{i,\epsilon}(0),0,t_i+n_i,\epsilon) + \frac{1}{\epsilon} H(x_i,\dot q^+_{i,\epsilon}(T_i),T_i,t_i+n_i,\epsilon),\end{aligned}$$

because the terms on the right hand side of the equality come only from the contribution of variation of t_i in the integrands in (3.2) - (3.4), while the contribution from variation in the integration domain cancel out. Also,

$$D_{T_i} W(z,\epsilon) = -H(x_i,\dot q^*_{i,\epsilon}(T_i),T_i,t_i+n_i,\epsilon) + H(x_i,\dot q^+_{i,\epsilon}(T_i),T_i,t_i+n_i,\epsilon),$$

and it is easy to see that $D_{T_i} W(z,\epsilon)$ depends C^1 on (z,ϵ). Because of the facts that

$$\lim_{\epsilon\to 0} q^{\pm}_{i,\epsilon}(s) = \omega^{\pm}_{i,0}(s) \quad \text{and} \quad D_\epsilon|_{\epsilon=0}\dot q^{\pm}_{i,\epsilon}(s) = D_\epsilon|_{\epsilon=0}\dot\omega^{\pm}_{i,\epsilon}(s) \qquad (5.3)$$

by theorem 2.2 and that $\dot q^*_{i,\epsilon}$ depends C^2 on ϵ and that $H(y_i,\dot q^*_{i,0}(0),0,t_i+n_i,0) = H(x_i,\dot q^*_{i,0}(T_i),T_i,t_i+n_i,0)$ and $H(y_i,\dot q^-_{i,0}(0),0,t_i+n_i,0) = H(x_i,\dot q^+_{i,0}(T_i),T_i,t_i+n_i,0)$, the limit $\lim_{\epsilon\to 0} D_{t_i} W(z,\epsilon)$ exists and equals

$$\begin{aligned} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} &\Big(H(y_i,\dot q^*_{i,\epsilon}(0),0,t_i+n_i,\epsilon) - H(y_i,\dot q^-_{i,\epsilon}(0),0,t_i+n_i,\epsilon) \\ &-H(x_i,\dot q^*_{i,\epsilon}(T_i),T_i,t_i+n_i,\epsilon) + H(x_i,\dot q^+_{i,\epsilon}(T_i),T_i,t_i+n_i,\epsilon)\Big). \qquad (5.4)\end{aligned}$$

But (5.4) equals $D_{t_i}W(z,0)$ by (2.5) and (2.7) and by the fact that the convergences in (5.3) are uniform in i, so D_zW is continuous.

Again, because the convergence in (5.3) is exponentially fast, it can be shown that the convergences of $\lim_{\epsilon\to 0} D^2_{t_i}W(z,\epsilon)$ and $\lim_{\epsilon\to 0} D^2_{T_it_i}W(z,\epsilon)$ are uniform in z and i, thus $\lim_{\epsilon\to 0} D^2_{z_i}W(z,\epsilon) = D^2_{z_i}W(z,0)$. Also $D^2_{\epsilon z_i}W(z,0)$ exists. Now, for fixed ϵ, $D_{z_i}W(z,\epsilon)$ actually depends only on z_{i-1}, z_i and z_{i+1}, therefore, by a result of mathematical analysis, $D_{z_i}W$ is differentiable at $(z,0)$, so is D_zW. $\square$

Lemma 5.2. *Let $z_0^* = (\cdots,(0,T^*_{i,0}),\cdots)$ be such that $D_zW(z_0^*,0)=0$. Then*
(a) Let δ be the Kronecker δ matrix. The map $D_z^2W(z_0^*,0)$ in matrix form reads

$$D^2_{z_jz_i}W(z_0^*,0) = \delta_{ji}\begin{pmatrix} f''(n_i) - \frac{\left(D_t\mathcal{H}_0^x(T^*_{i,0},n_i)\right)^2}{D_T\mathcal{H}_0^x(T^*_{i,0},n_i)} & -D_t\mathcal{H}_0^x(T^*_{i,0},n_i) \\ -D_t\mathcal{H}_0^x(T^*_{i,0},n_i) & -D_T\mathcal{H}_0^x(T^*_{i,0},n_i)\end{pmatrix}. \quad (5.5)$$

The eigenvalues of the matrix are real, distinct, and non-zero.
(b) The linear map $D_z^2W(z_0^*,0):\mathbb{R}^{2\mathcal{I}}\to\mathbb{R}^{2\mathcal{I}}$ is an isomorphism.

Proof: (a): Now the system is autonomous, by proposition 4.2, we know that for all z

$$\begin{aligned} & D^2_{t_{i-1},t_i}h^-(z_{i-1},z_i,0) = D^2_{t_{i-1},t_i}h^+(z_{i-1},z_i,0) = D^2_{t_i}h^-(z_i,z_{i+1},0) \\ = \; & D^2_{t_i}h^+(z_{i-1},z_i,0) = D^2_{T_{i-1},t_i}h^+(z_{i-1},z_i,0) = D^2_{t_{i+1},t_i}h^-(z_i,z_{i+1},0) \\ = \; & D^2_{t_{i+1},t_i}h^+(z_i,z_{i+1},0) = D^2_{t_{i+1},T_i}h^+(z_i,z_{i+1},0) = 0. \end{aligned}$$

Also, $D_{T_i}h^+(z_i,z_{i+1},0) = H(x_i,\dot\omega^+_{t_i+n_i,0}(T_i),T_i,t_i+n_i,0) = 0$ for all t_i and T_i by the first variation formula for actions, therefore, $D^2_{t_i,T_i}h^+(z_i,z_{i+1},0) = D^2_{T_i,t_i}h^+(z_i,z_{i+1},0) = D^2_{T_i}h^+(z_i,z_{i+1},0) = 0$ for all z. Hence, from (5.2)

$$\begin{aligned} & D^2_{z_jz_i}W(z_0^*,0) \\ = \; & \delta_{ji}\begin{pmatrix} D^2_{t_i}\left(h^-(z^*_{i-1,0},z^*_{i,0},0)+h^*(z^*_{i,0},0)+h^+(z^*_{i,0},z^*_{i+1,0},0)\right) & D^2_{t_iT_i}h^*(z^*_{i,0},0) \\ D^2_{T_it_i}h^*(z^*_{i,0},0) & D^2_{T_i}h^*(z^*_{i,0},0)\end{pmatrix}. \end{aligned}$$

By proposition 4.2, $h^-(z^*_{i-1,0},z^*_{i,0},0)$, $h^*(z^*_{i,0},0)$ and $h^+(z^*_{i,0},z^*_{i+1,0},0)$ are respectively $S_0^-(y_i,0,n_i)$, $S_0^*(y_i,x_i,T^*_{i,0},n_i)$ and $S_0^+(x_i,T^*_{i,0},n_i)$. Thus by proposition 2.7,

$$\begin{aligned} & D^2_{t_i}\left(h^-(z^*_{i-1,0},z^*_{i,0},0)+h^*(z^*_{i,0},0)+h^+(z^*_{i,0},z^*_{i+1,0},0)\right) \\ = \; & \frac{d^2}{dt^2}I(\Gamma_{n_i,0},n_i) - D^2_{Tt}S_0^*(y_i,x_i,T^*_{i,0},n_i)\; D\mathfrak{T}(n_i). \end{aligned}$$

But by definition (2.8) and proposition 2.7,

$$\begin{aligned} D^2_{Tt}S_0^*(y_i,x_i,T^*_{i,0},n_i) &= -D_t\mathcal{H}_0^x(T^*_{i,0},n_i), \\ D\mathfrak{T}(n_i) &= -\frac{D_t\mathcal{H}_0^x(T^*_{i,0},n_i)}{D_T\mathcal{H}_0^x(T^*_{i,0},n_i)}, \\ D^2_TS_0^*(y_i,x_i,T^*_{i,0},n_i) &= -D_T\mathcal{H}_0^x(T^*_{i,0},n_i). \end{aligned}$$

Besides, $I(\Gamma_{n_{i,0}}, n_i) \equiv f(n_i)$, so we have (5.5).

The last assertion follows by virtue of the symmetry of the matrix, nonzeroness of $D_T\mathcal{H}_0^x(T_{i,0}^*, n_i)$ (by lemma 2.5) and $f''(n_i)$.

(b): This can be proved by the Banach isomorphism theorem: $D_z^2 W(z,\epsilon)$ is a continuous map by proposition 5.1 (or can be seen from the boundedness of the norm $\|D_z^2 W(z,\epsilon)\delta z\|$) and assertion (a) implies it is also a bijection. $\square$

5.1 Proof of theorem 1.2

The sequence of homoclinic trajectories $\{\Gamma_{n_i,0}\}_{i\in\mathcal{I}}$ in theorem 1.2 determines a sequence of points $\{y_i\}_{i\in\mathcal{I}}$ on the boundary ∂U. Using this sequence and the given sequence $\{n_i\}$, we can construct our variational functional W. Assume ϵ sufficiently small and z_0^* is the critical point of W_0. From lemma 5.2, proposition 5.1 and the implicit function theorem, we know there exists a unique z^* as a C^1-function of ϵ such that $z^*(0) = z_0^*$, $D_z W(z^*(\epsilon),\epsilon) = 0$. In view of our construction of the functional W, theorem 1.2 then essentially follows. Since there is a unique sequence $\{t_{i,\epsilon}^*, T_{i,\epsilon}^*\}_i = z^*(\epsilon)$ with which the product path $\Upsilon_{j,\epsilon}$ defined by formula (3.5) is a trajectory for $L(x,v,s,t_{j,\epsilon}^* + n_j,\epsilon)$ for each $j \in \mathcal{I}$ provided ϵ is sufficiently small.

On $\Omega_{i,j,\epsilon}$, $\Upsilon_{j,\epsilon}$ is close to $n_{i,j,\epsilon}^* \star_\epsilon \Gamma_{n_i,0}$ in the C^2-topology because $n_{i,j,\epsilon}^* \star_\epsilon q_{i,\epsilon}^-$, $n_{i,j,\epsilon}^* \star_\epsilon q_{i,\epsilon}^*$ and $n_{i,j,\epsilon}^* \star_\epsilon q_{i,\epsilon}^+$ on $\Omega_{i,j,\epsilon}$ are close respectively to $n_{i,j,\epsilon}^* \star_\epsilon q_{n_i,0}^-$, $n_{i,j,\epsilon}^* \star_\epsilon q_{n_i,0}^*$ and $n_{i,j,\epsilon}^* \star_\epsilon q_{n_i,0}^+$ in the C^2-topology. $\Upsilon_{j,\epsilon}(n_{i,j,\epsilon}^*/\epsilon) = \Gamma_{n_i,0}(0)$ because $n_{i,j,\epsilon}^* \star_\epsilon q_{i,\epsilon}^*(n_{i,j,\epsilon}^*/\epsilon) = q_{i,\epsilon}^*(0) = q_{n_i,0}^*(0)$. Also, $\Upsilon_{j,\epsilon}(n_{i,j,\epsilon}^*/\epsilon + T_{i,\epsilon}^*) = n_{i,j,\epsilon}^* \star_\epsilon q_{i,\epsilon}^*(n_{i,j,\epsilon}^*/\epsilon + T_{i,\epsilon}^*) = q_{i,\epsilon}^*(T_{i,\epsilon}^*) = q_{n_i,0}^*(T_{i,0}^*) = \Gamma_{n_i,0}(T_{i,0}^*)$.

As $\epsilon \to 0$, $\Omega_{i,j,\epsilon}$ diverges to $-\infty$ if $i < j$ but to ∞ if $i > j$. Therefore, by (3.5), $\Upsilon_{j,\epsilon}$ C^1-converges to $\Gamma_{n_j,0}$ uniformly on any bounded interval as $\epsilon \to 0$ for each $j \in \mathcal{I}$. $\square$

6 Appendix

Proof of theorem 2.2:

We construct a curve $\mu_\epsilon(s) := \omega_{t,\epsilon}^+(s) + \omega_{t,\epsilon}^-(s) + \nu_\epsilon^+(s) + \nu_\epsilon^-(s)$ with ν_ϵ^+, ν_ϵ^- being the unique solutions of the equation

$$\ddot{\xi}(s) - \lambda_{t,\epsilon}(s)\xi(s) = 0$$

subject to boundary conditions $\nu_\epsilon^+(s_0) = -\omega_{t,\epsilon}^-(s_0)$, $\nu_\epsilon^+(s_1) = 0$, $\nu_\epsilon^-(s_0) = 0$, $\nu_\epsilon^-(s_1) = -\omega_{t,\epsilon}^+(s_1)$. Since $|\omega_{t,\epsilon}^+(s)|$ (or $|\omega_{t,\epsilon}^-(s)|$) decays exponentially in forward (resp. backward) time and $\max\{|\nu_\epsilon^+(s)|,\ |\nu_\epsilon^-(s)|\} \lesssim O\left(\exp\left(-\sqrt{k}(s_1 - s_0)\right)\right)$, it can be verified that $\mu_\epsilon \in C^2([s_0,s_1],U)$. We want to show within a closed ball $\overline{B}(\mu_\epsilon,\delta)$ of radius δ centred at μ_ϵ in the C^2-topology there exists a unique trajectory $q_{t,\epsilon}$ such that $q_{t,\epsilon}(s_0) = \mu_\epsilon(s_0)$, $q_{t,\epsilon}(s_1) = \mu_\epsilon(s_1)$ and show δ is of order $O(\exp(-\sqrt{k}(s_1 - s_0)/2))$.

Such a $q_{t,\epsilon}$ can be obtained if the map $G_{t,\epsilon}$ maps $\overline{B}(\mu_\epsilon,\delta)$ into itself and is contracting, with

$$\begin{aligned} G_{t,\epsilon} : Q^2([s_0,s_1],U) &\to Q^2([s_0,s_1],U), \\ q &\mapsto q - D_qF(\mu_\epsilon,t,\epsilon)^{-1}F(q,t,\epsilon), \end{aligned}$$

and

$$Q^2([s_0,s_1],U) := \{q \in C^2([s_0,s_1],U),\ q(s_0)=x,\ q(s_1)=y\},$$

$$\begin{aligned} F : Q^2([s_0,s_1],U)\times\mathbb{R}\times(\mathbb{R}^+\cup\{0\}) &\to C^0([s_0,s_1],\mathbb{R}), \\ F(q,t,\epsilon)(s) &= \ddot{q}(s) + D_xV(q(s),\epsilon s+t). \end{aligned}$$

$D_qF(\mu_\epsilon,t,\epsilon)$ is invertible [11] because it is a continuous bijection. The bijectivity comes from the negative definiteness of the Hessian of $V(\cdot,\tau)$ by (1.4). (The same reason as the non-degeneracy of geodesics on a Riemannian manifold with negative sectional curvature [17].)

We work out $\|F(\mu_\epsilon,t,\epsilon)\|_0$ first:

$$\begin{aligned} F(\mu_\epsilon,t,\epsilon)(s) = & -D_xV(\omega^+_{t,\epsilon}(s),\epsilon s+t) - D_xV(\omega^-_{t,\epsilon}(s),\epsilon s+t) \\ & +\lambda_{t,\epsilon}(s)\nu^+_\epsilon(s) + \lambda_{t,\epsilon}(s)\nu^-_\epsilon(s) \\ & +D_xV(\omega^+_{t,\epsilon}(s)+\omega^-_{t,\epsilon}(s)+\nu^+_\epsilon(s)+\nu^-_\epsilon(s),\epsilon s+t). \end{aligned}$$

The last term can be expanded as

$$\begin{aligned} & D_xV(\omega^+_{t,\epsilon}(s),\epsilon s+t) + D^2_xV(\omega^+_{t,\epsilon}(s)+\phi_1(s)\nu^+_\epsilon(s),\epsilon s+t)\nu^+_\epsilon(s) \\ + \quad & D^2_xV(\omega^+_{t,\epsilon}(s)+\nu^+_\epsilon(s)+\phi_0(s)(\omega^-_{t,\epsilon}(s)+\nu^-_\epsilon(s)),\epsilon s+t)(\omega^-_{t,\epsilon}(s)+\nu^-_\epsilon(s)) \end{aligned}$$

and $D_xV(\omega^-_{t,\epsilon}(s),\epsilon s+t)$ can be expanded as $D^2_xV(\phi_2(s)\omega^-_{t,\epsilon}(s),\epsilon s+t)\omega^-_{t,\epsilon}(s)$ for some $\phi_0(s)$, $\phi_1(s)$, and $\phi_2(s)\in(0,1)$, therefore on the time interval $[s_0,(s_0+s_1)/2]$,

$$\begin{aligned} & |F(\mu_\epsilon,t,\epsilon)(s)| \\ \le\ & 2K\left(|\omega^-_{t,\epsilon}(s)|+|\nu^+_\epsilon(s)|+|\nu^-_\epsilon(s)|\right) \\ \le\ & 2CK(|y|e^{-\sqrt{k}(s_1-s_0)/2}+|x|e^{-3\sqrt{k}(s_1-s_0)/2}+|y|e^{-\sqrt{k}(s_1-s_0)}) \\ \le\ & C_1e^{-\sqrt{k}(s_1-s_0)/2} \end{aligned}$$

for some constants C and C_1. Similarly, $|F(\mu_\epsilon,t,\epsilon)(s)|$ satisfies the same inequality on the time interval $[(s_0+s_1)/2,s_1]$.

It is shown in [11] that $G_{t,\epsilon}$ has a contraction constant $1/2$ on $\overline{B}(\mu_\epsilon,\delta)$ if

$$\delta \le \left(2\sup_{x\in U,s\in[s_1-s_0]}\left|D^3_xV(x,\epsilon s+t)\right|\left\|D_qF(\mu_\epsilon,t,\epsilon)^{-1}\right\|\right)^{-1}. \tag{6.1}$$

Assume the above inequality is satisfied, then

$$\|G_{t,\epsilon}(q)-\mu_\epsilon\|_2 \le \frac{\delta}{2} + \|D_qF(\mu_\epsilon,t,\epsilon)^{-1}\|C_1e^{-\sqrt{k}(s_1-s_0)/2}.$$

We need the right hand side of the inequality above is at most δ so the contraction mapping theorem is applicable. This implies δ should be

$$\delta \ge 2C_1\|D_qF(\mu_\epsilon,t,\epsilon)^{-1}\|e^{-\sqrt{k}(s_1-s_0)/2}. \tag{6.2}$$

Since $\|D_qF(\mu_\epsilon,t,\epsilon)^{-1}\|$ is bounded for $\epsilon \leq \epsilon_0$, if $s_1 - s_0$ is large enough, both (6.1) and (6.2) can be fulfilled. Thence there exists a unique trajectory $q_{t,\epsilon}$ shadowing $\omega^+_{t,\epsilon} + \omega^-_{t,\epsilon}$ such that

$$
\begin{aligned}
&\|q_{t,\epsilon} - \omega^+_{t,\epsilon} - \omega^-_{t,\epsilon}\|_2 \\
&\leq \quad \|\nu^+_\epsilon\|_2 + \|\nu^-_\epsilon\|_2 + 2C_1\|D_qF(\mu_\epsilon,t,\epsilon)^{-1}\| \exp\left(-\sqrt{k}(s_1 - s_0)/2\right) \\
&\leq \quad C_3 \exp(-\sqrt{k}(s_1 - s_0)/2)
\end{aligned}
$$

for some constant C_3. So, $q_{t,\epsilon}$ can be decomposed into the desired form with Ψ a uniformly bounded C^2-function as asserted. □

Proof of theorem 2.4:

Because we allow the travelling time T to vary in theorem 2.4, it will be convenient to normalise the time scale: $s \mapsto s/T := \tilde{s}$, $q(s) \mapsto \gamma(\tilde{s}) := q(T\tilde{s})$, and instead consider a Lagrangian L_γ of the form

$$L_\gamma(\gamma(\tilde{s}), \dot{\gamma}(\tilde{s}), \tilde{s}, T, t, \epsilon) = \frac{|\dot{\gamma}(\tilde{s})|^2}{2T^2} - V(\gamma(\tilde{s}), T\epsilon\tilde{s} + t)$$

and its associated C^3-Hamilton's functional

$$A : \Pi \to \mathbb{R},\ \gamma \mapsto A(\gamma) := T \int_0^1 L_\gamma(\gamma(\tilde{s}), \dot{\gamma}(\tilde{s}), \tilde{s}, T, t, \epsilon)\ d\tilde{s}$$

on the Sobolev space Π consisting of all absolutely continuous curves $\gamma : [0,1] \to \mathbb{T}$, $\gamma(0)$, $\gamma(1) \in \partial U$, with bounded $W^{1,2}$-norm.

Now $\gamma^*(\tilde{s}; T^*_{n,0}, n, 0) = q^*_{n,0}(T^*_{n,0}\tilde{s})$ is a trajectory for $L_\gamma(x, v, \tilde{s}, T^*_{n,0}, n, 0)$ with non-zero velocity, thus $D^2A(\gamma^*(\cdot; T^*_{n,0}, n, 0))$ is invertible [21, 9]. Since A is C^3 dependent of its parameters T, t and ϵ, then by the implicit function theorem, this implies that there is a unique C^2-continuation $(T, t, \epsilon) \mapsto \gamma^*(\cdot; T, t, \epsilon) \in \Pi$ as a trajectory for $L_\gamma(x, v, \tilde{s}, T, t, \epsilon)$ on a small neighbourhood of $(T^*_{n,0}, n, 0)$ such that $\gamma^*(0; T, t, \epsilon) = \gamma^*(0; T^*_{n,0}, n, 0)$ and $\gamma^*(1; T, t, \epsilon) = \gamma^*(1; T^*_{n,0}, n, 0)$. Theorem 2.4 follows.

Proof of theorem 2.5:

Let $\mathcal{H}$ be the value of the Hamiltonian associated with $L(x, v, s, t, \epsilon)$ and let $q^* : [0, T] \to \mathbb{T}$ be a trajectory, then $\dot{q}^*(s) = \sqrt{2(\mathcal{H}(s) - V(q^*(s), \epsilon s + t))}$. As long as $\dot{q}^*(s) \neq 0$ for $s \in [0, T]$, $q^*(s)$ is C^2-diffeomorphic to s. Thence, one can calculate the travelling time T by

$$T = \frac{1}{\sqrt{2}} \int_{q^*} \left(\sqrt{\mathcal{H}(s(x)) - V(x, \epsilon s(x) + t)}\right)^{-1}\ dx. \tag{6.3}$$

where the integral path is along q^* from $y = q^*(0)$ to $x = q^*(T)$.

When $\epsilon = 0$, the Hamiltonian is preserved along the trajectory, and is denoted by a constant E_t. In our case $E_t > V(x, t)$ along the trajectory, then we obtain a mapping by making use of formula (6.3)

$$E_t \mapsto T_t(E_t) = \frac{1}{\sqrt{2}} \int_{q^*} \left(\sqrt{E_t - V(x, t)}\right)^{-1}\ dx.$$

The function T_t is a locally C^∞-function of E_t, and its first derivative at E_t is negative definite:

$$DT_t(E_t) < 0.$$

Recall from (2.8) that $\mathcal{H}_\epsilon^x(T,t)$ is the value of the Hamiltonian of the trajectory with $q^*_{t,\epsilon}(0) = y$ and $q^*_{t,\epsilon}(T) = x$ when it arrives at x. Due to the inequality above we see $\mathcal{H}_0^x(T,t)$ is locally diffeomorphic to T satisfying $D_T\mathcal{H}_0^x(T,t) < 0$. This proves the theorem for the case $\epsilon = 0$. When ϵ is not zero but small, $q^*_{t,\epsilon}$ is the C^2-continuation of $q^*_{t,0}$ with $q^*_{t,0}(0) = q^*_{t,\epsilon}(0)$ and $q^*_{t,0}(T^*) = q^*_{t,\epsilon}(T^*)$, thence $D_T\mathcal{H}_\epsilon^x(T^*,t) < 0$, by C^2-dependence on ϵ. The lemma then follows by the inverse function theorem. □

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