Analytic Representation of Relativistic Wave Equations II: The Square-Root Operator Case

Tepper L. Gill[1,2] and W. W. Zachary[1]

[1]Department of Electrical & Computer Engineering
[2]Department of Mathematics
Howard University
Washington, D. C. 20059

Abstract

In this paper, using the theory of fractional powers for operators, we construct the most general (analytic) representation for the square-root operator of relativistic quantum theory. We allow for arbitrary, but time-independent, vector potential and mass terms. Our representation is uniquely determined by the Green's function for the corresponding Schrödinger equation. *We find that the square-root operator is represented by a nonlocal composite of (at least) three singularities.* To our knowledge, this is the first example of a physically relevant operator with these properties. In the standard interpretation, the particle component has two negative parts and one (hard core) positive part, while the antiparticle component has two positive parts and one (hard core) negative part. This effect is confined within a Compton wavelength such that, at the point of singularity, they cancel each other providing a finite result. Furthermore, the operator looks (almost) like the identity outside a Compton wavelength, but has a residual instantaneous connection with all other particles in the universe at each point in time.

In addition to the possibility that the square-root operator may be used to represent the inside of hadrons, it is also possible that the residual attractive (particle) part may be the long sought connection between the internal particle composition and the cause for gravitation interaction in matter. If this view were correct, then we would expect matter and antimatter to be gravitationally attractive among themselves and gravitationally repulsive with each other. This would make physical sense if we take seriously the interpretation of antimatter as matter with its time reversed (as opposed to hole theory).

I. Background

Introduction

In the transition from nonrelativistic to relativistic quantum theory, the Hamiltonian

$$H = \frac{\left[\mathbf{p} - (e/c)\mathbf{A}\right]^2}{2m} + V$$

is replaced by the square-root equation:

$$H = \sqrt{c^2\left[\mathbf{p} - (e/c)\mathbf{A}\right]^2 + m^2c^4} + V.$$

It is quite natural to expect that the first choice for a relativistic wave equation would be:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\sqrt{c^2\left[\mathbf{p} - (e/c)\mathbf{A}\right]^2 + m^2c^4} + V\right]\psi,$$

where $\mathbf{p} = -i\hbar\nabla$. However, no one knew how to directly relate this equation to physically important problems. Furthermore, this equation is nonlocal, meaning, in the terminology of the times (1920-30), that it represents a power series in the momentum operator. Historically, Schrödinger[1], Gordon[2], Klein[3] and others[4,5,6] attempted to circumvent this problem by starting with the relationship:

$$(H - V)^2 = c^2\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right)^2 + m^2c^4,$$

which led to the Klein-Gordon equation. At that time, the hope was to construct a relativistic quantum theory that would provide a natural extension of the nonrelativistic case. However, the problems with the Klein-Gordon equation were so great, that all involved became frustrated and it was dropped from serious consideration for a few years.

Dirac[7] argued that the proper equation should be first order in both the space and time variables, in order to be a true relativistic wave equation. This approach led to the well-known Dirac equation, which was the subject of the first paper in this series. It was shown in that paper that, although the Dirac equation is first order in the space and time variables, it has a hidden nonlocal time behavior, which means that these variables are not on the same "*physical*" footing.

Purpose

In a survey article on relativistic wave equations, Foldy[8] pointed out that, in the absence of interaction, the above equation "gives a perfectly good wave equation for the description of a (spin zero) free particle". Foldy[9] had shown in an earlier paper that the square-root form:

$$H = \boldsymbol{\beta}\sqrt{c^2\mathbf{p}^2 + m^2c^4}\,,$$

provides a canonical representation for particles of all finite spin. However, when $\mathbf{A}$ is not zero, the noncommutativity of $\mathbf{p}$ and $\mathbf{A}$ "appeared" to make it impossible to give an unambiguous meaning to the radical operator.

The purpose of this paper is to show that the square-root operator has a unique analytic representation, which is defined for variable (but otherwise arbitrary) time independent $\mathbf{A}$ and m. We focus on the general square-root operator equation:

$$S[\psi] = \mathbf{H}_s\psi = \left\{\boldsymbol{\beta}\sqrt{c^2\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right)^2 - e\hbar c\boldsymbol{\Sigma}\cdot\mathbf{B} + m^2c^4}\right\}\psi\,. \tag{1a}$$

Where β and Σ are the Dirac matrices: $\beta = \begin{bmatrix}\mathbf{I} & 0\\ 0 & -\mathbf{I}\end{bmatrix}$, $\Sigma = \begin{bmatrix}\sigma & 0\\ 0 & \sigma\end{bmatrix}$; $\mathbf{I}$ and σ are the identity and Pauli matrices, respectively. In Section II, we construct the general analytic representation for (1a). Since this representation is complicated, we explore a number of simplifications that allow for physical interpretation in Sections III-IV. In Section IIV, we construct the general solution to the equation:

$$i\hbar\frac{\partial}{\partial t}\psi = \left\{\boldsymbol{\beta}\sqrt{c^2\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right)^2 - e\hbar c\boldsymbol{\Sigma}\cdot\mathbf{B} + m^2c^4}\right\}\psi\,. \tag{1b}$$

In section IIIV, we show that if we treat the potential energy as a part of the mass, there is an alternate connection between the Dirac and square-root equations. This approach has a number of advantages compared to the conventional approach. In the conclusion, we summarize our results and discuss open problems. In the appendix, we summarize the basic theory of semigroups of operators and fractional powers of closed operators so that the paper is self-contained.

II General Representation

In this section, we construct a general analytic representation for equation (1a). To make our approach clear, set $\mathbf{G} = -\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right)^2$ and $\omega^2 = m^2c^2 - \tfrac{e\hbar}{c}\boldsymbol{\Sigma}\cdot\mathbf{B}$. Except for time independence, we leave the form of the vector potential and the mass unspecified. We assume that $-\mathbf{G} + \omega^2$ satisfies the conditions required to be a generator of a unitary group. Using the above notation, we can write (1a) as

$$S[\psi] = \left\{ c\boldsymbol{\beta}\sqrt{-\mathbf{G}+\omega^2} \right\}\psi. \tag{2}$$

Using the theory of fractional powers of closed operators, (see the Appendix equation (A8)) it can be shown that for generators of unitary groups we can write (2) as: (using $\sqrt{\mathbf{F}} = (1/\sqrt{\mathbf{F}})\mathbf{F}$)

$$S[\psi] = \tfrac{c\boldsymbol{\beta}}{\pi}\int_0^\infty \left[(\lambda+\omega^2)-\mathbf{G}\right]^{-1}(-\mathbf{G}+\omega^2)\psi \tfrac{d\lambda}{\sqrt{\lambda}}, \tag{3}$$

where $\left[(\lambda+\omega^2)-\mathbf{G}\right]^{-1}$ is the resolvent associated with the operator $(\mathbf{G}-\omega^2)$. The resolvent can be computed directly if we can find the fundamental solution to the equation

$$\frac{\partial Q(\mathbf{x},\mathbf{y};t)}{\partial t} + (\mathbf{G}-\omega^2)Q(\mathbf{x},\mathbf{y};t) = \delta(\mathbf{x}-\mathbf{y}). \tag{4}$$

It is shown in Schulman[11] that the equation:

$$i\hbar\frac{\partial \overline{Q}(\mathbf{x},\mathbf{y};t)}{\partial t} + (\tfrac{1}{2M}\mathbf{G}-V)\overline{Q}(\mathbf{x},\mathbf{y};t) = \delta(\mathbf{x}-\mathbf{y}) \tag{5}$$

has the general solution

$$\overline{Q}(\mathbf{x},\mathbf{y};t) = \left(\frac{M}{2\pi i\hbar t}\right)^{3/2}\exp\left\{\frac{it}{\hbar}\left[\frac{M}{2}\left(\frac{\mathbf{x}-\mathbf{y}}{t}\right)^2 - V(\mathbf{y})\right] + \frac{ie}{\hbar c}(\mathbf{x}-\mathbf{y})\cdot\left[\left(\tfrac{\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y})}{2}\right)\right]\right\}, \tag{6}$$

provided that $\mathbf{A}$ and m are time-independent. We can use either the average, $\frac{1}{2}[\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y})]$, or the midpoint $\mathbf{A}[\frac{1}{2}(\mathbf{x}+\mathbf{y})]$ in the last term of equation (6). We have chosen the average for later convenience. Either choice gives a solution. However, this ambiguity has nothing to do with the square-root operator. Furthermore, it does not appear in the path-integral formulation (see section IIV). If we set $\omega^2/i\hbar = V$ and $M = i\hbar/2$, we see that

$$Q(\mathbf{x},\mathbf{y};t) = \left(\frac{1}{4\pi t}\right)^{3/2}\exp\left\{-\frac{(\mathbf{x}-\mathbf{y})^2}{4t} - \frac{\omega^2 t}{\hbar^2} + \frac{ie}{\hbar c}(\mathbf{x}-\mathbf{y})\cdot\left[\left(\tfrac{\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y})}{2}\right)\right]\right\} \tag{7}$$

solves (4). Using (7), we can compute $\left[(\lambda+\omega^2)-\mathbf{G}\right]^{-1}$ from

$$\left[(\lambda+\omega^2)-\mathbf{G}\right]^{-1} f(\mathbf{x}) = \int_0^\infty e^{-\lambda t}\left[\int_{\mathbf{R}^3} Q(\mathbf{x},\mathbf{y};t) f(\mathbf{y})d\mathbf{y}\right]dt. \tag{8}$$

If we interchange the order of integration in (8) and use (7), we get

$$\left[(\lambda+\omega^2)-\mathbf{G}\right]^{-1} f(\mathbf{x})$$
$$= \int_{\mathbf{R}^3} \exp\left\{\tfrac{ie(\mathbf{x}-\mathbf{y})}{\hbar c}\cdot\left[\left(\tfrac{\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y})}{2}\right)\right]\right\}\left\{\int_0^\infty \exp\left[-\frac{(\mathbf{x}-\mathbf{y})^2}{4t}-\frac{\omega^2 t}{\hbar^2}-\lambda t\right]\frac{dt}{(4\pi t)^{3/2}}\right\} f(\mathbf{y})d\mathbf{y}. \tag{9}$$

Using a table of Laplace transforms, the inner integral can be computed to get

$$\int_0^\infty \exp\left[-\frac{(\mathbf{x}-\mathbf{y})^2}{4t}-\frac{\omega^2 t}{\hbar^2}-\lambda t\right]\frac{dt}{(4\pi t)^{3/2}} = \frac{1}{4\pi}\frac{\exp\left[-\sqrt{(\lambda+\mu^2)}\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|},$$

where $\mu^2=\mu^2(\mathbf{y})=\left(\omega^2(\mathbf{y})/\hbar^2\right)$. Equation (3) now becomes

$$S[\psi](\mathbf{x})$$
$$= \tfrac{c\boldsymbol{\beta}}{4\pi^2}\int_0^\infty\left\{\int_{\mathbf{R}^3}\exp\left\{\tfrac{ie(\mathbf{x}-\mathbf{y})}{\hbar c}\cdot\left[\left(\tfrac{\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y})}{2}\right)\right]\right\}\frac{\exp\left[-\sqrt{(\lambda+\mu^2)}\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}(-\mathbf{G}+\omega^2)\psi(\mathbf{y})d\mathbf{y}\right\}\tfrac{d\lambda}{\sqrt{\lambda}}. \tag{10}$$

Once again, we interchange the order of integration in (10) and perform the computations to get ($\mathbf{K}_1[\mathrm{z}]$ is the modified Bessel function of the third kind and first order)

$$\int_0^\infty\left\{\frac{\exp\left[-\sqrt{(\lambda+\mu^2)}\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}\right\}\frac{d\lambda}{\sqrt{\lambda}} = \frac{4\mu\Gamma(\frac{3}{2})}{\pi^{1/2}}\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}. \tag{11}$$

Thus, if we set $\mathbf{a}=\frac{e}{\hbar c}\mathbf{A}(\mathbf{y}),\ \mathbf{a}_1=\frac{e}{2\hbar c}\left[\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y})\right]$, we get

$$S[\psi](\mathbf{x})=\frac{c\boldsymbol{\beta}}{2\pi^2}\int_{\mathbf{R}^3}\exp\left[i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})\right]\frac{\mu\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}(-\mathbf{G}+\omega^2)\psi(\mathbf{y})d\mathbf{y}. \tag{12}$$

Now $-\mathbf{G}+\omega^2=\hbar^2\left(-\Delta+2i\mathbf{a}\cdot\nabla+i\nabla\cdot\mathbf{a}+\mathbf{a}^2+\mu^2\right)$ so that (12) becomes

$$S[\psi](\mathbf{x})=\tfrac{\hbar^2 c\boldsymbol{\beta}}{2\pi^2}\int_{\mathbf{R}^3}e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\frac{\mu\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}\left(-\Delta+2i\mathbf{a}\cdot\nabla+i\nabla\cdot\mathbf{a}+\mathbf{a}^2+\mu^2\right)\psi(\mathbf{y})d\mathbf{y}. \tag{13}$$

Before continuing, we recall a few standard results that will be used to compute (13). We assume that $f,g\in\mathbf{H}_0^2(\mathbf{D}),\ \mathbf{D}\subseteq\mathbf{R}^3,\partial\mathbf{D}$ smooth. From the divergence theorem we have that ($\boldsymbol{\nu}$ is the outward normal on $\partial\mathbf{D}$)

$$\int_D fg_{y_i}d\mathbf{y}=\int_{\partial D}(fg)\nu_i d\mathbf{S}-\int_D gf_{y_i}d\mathbf{y}, \tag{14}$$

$$\int_D f(\mathbf{a}\cdot\nabla)g d\mathbf{y} = \int_{\partial D}(fg)(\mathbf{a}\cdot\mathbf{v})d\mathbf{S} - \int_D g(\mathbf{a}\cdot\nabla)f d\mathbf{y}. \tag{15}$$

We will also need the Green's identity in the form

$$\int_D f\Delta g d\mathbf{y} = \int_{\partial D} fg_v d\mathbf{S} - \int_{\partial D} gf_v d\mathbf{S} + \int_D g\Delta f d\mathbf{y}.$$

Let us now consider a ball $\mathbf{B}_\rho(\mathbf{x})$ of radius ρ about $\mathbf{x}$ so that $\mathbf{R}^3 = \mathbf{R}^3_\rho \cup \mathbf{B}_\rho(\mathbf{x})$, where $\mathbf{R}^3_\rho = \left(\mathbf{R}^3 \setminus \mathbf{B}_\rho(\mathbf{x})\right)$, so that $\partial\mathbf{R}^3_\rho = \left(\partial\mathbf{R}^3 \setminus \partial\mathbf{B}_\rho(\mathbf{x})\right)$. Let $\mathbf{v}$ be the outward normal on $\partial\mathbf{R}^3_\rho$. It follows that $-\mathbf{v}$ is the outward normal on $\mathbf{B}_\rho(\mathbf{x})$ and $\mathbf{y} = \mathbf{x} - \mathbf{v}\rho$ on $\partial\mathbf{B}_\rho(\mathbf{x})$. Using (14) and (15), we can write (13) as ($u = \mu\|\mathbf{x}-\mathbf{y}\|$ and $\mathbf{T}[\psi](\mathbf{x}) = \frac{2\pi^2\boldsymbol{\beta}}{\hbar^2 c}\mathcal{S}[\psi](\mathbf{x})$)

$$\begin{aligned}
\mathbf{T}[\psi]_\rho &= \int_{\mathbf{R}^3_\rho} e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\left[\mathbf{K}_1(u)/u\right]\left(-\Delta + 2i\mathbf{a}\cdot\nabla + i\nabla\cdot\mathbf{a} + \mathbf{a}^2 + \mu^2\right)\psi d\mathbf{y} \\
&= \int_{\mathbf{R}^3_\rho} e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\left[\mathbf{K}_1(u)/u\right]\left(\mu^2 + i\nabla\cdot\mathbf{a} + \mathbf{a}^2\right)\psi d\mathbf{y} \\
&\quad -2i\int_{\mathbf{R}^3_\rho}\mathbf{a}\cdot\nabla\left\{e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\left[\mathbf{K}_1(u)/u\right]\right\}\psi d\mathbf{y} - \int_{\mathbf{R}^3_\rho}\Delta\left\{e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\left[\mathbf{K}_1(u)/u\right]\right\}\psi d\mathbf{y} \\
&\quad -\int_{\partial\mathbf{R}^3_\rho}\left\{e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\left[\mathbf{K}_1(u)/u\right]\right\}_{-v}\psi d\mathbf{S} + \int_{\partial\mathbf{R}^3_\rho}\left\{e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\left[\mathbf{K}_1(u)/u\right]\right\}\psi_{-v} d\mathbf{S} \\
&\quad +2i\int_{\partial\mathbf{R}^3_\rho} e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\left[\mathbf{K}_1(u)/u\right]\psi(\mathbf{a}\cdot\mathbf{v})d\mathbf{S}.
\end{aligned} \tag{16}$$

It is clear that the surface integrals vanish on $\partial\mathbf{R}^3$, so we need only consider them on $\partial\mathbf{B}_\rho(\mathbf{x})$. It is easy to check that the integrands in the last two terms are continuous so they vanish as $\rho \to 0$. Easy analysis shows that the only possible nonvanishing part of the remaining surface term is ($d\mathbf{S} = \rho^2\sin\theta\, d\theta d\phi = \rho^2 d\Omega$)

$$\int_{\partial\mathbf{B}_\rho}\exp\left[i\mathbf{a}_1\cdot\mathbf{v}\rho\right]u^2\left[\mathbf{K}_1(u)/u\right]_{-v}\psi d\Omega, \tag{17}$$

where on $\partial\mathbf{B}_\rho(\mathbf{x})$, $\|\mathbf{x}-\mathbf{y}\| = \rho,\ (x_i - y_i) = v_i$ & $u = \mu\rho$. We also have

$$\left[\frac{\mathbf{K}_1(u)}{u}\right]_{-v} = -\mathbf{v}\cdot\nabla\left[\frac{\mathbf{K}_1(u)}{u}\right] = -\sum_{i=1}^{3} v_i\frac{d}{du}\left[\frac{\mathbf{K}_1(u)}{u}\right]\frac{\partial u}{\partial y_i},$$

and

$$\frac{d}{du}\left[\frac{\mathbf{K}_1(u)}{u}\right] = -\frac{\mathbf{K}_2(u)}{u},\ \ \&\ \ \frac{\partial u}{\partial y_i} = \frac{\partial\mu}{\partial y_i}\|\mathbf{x}-\mathbf{y}\| - \mu\frac{(x_i - y_i)}{\|\mathbf{x}-\mathbf{y}\|},$$

so that

$$\left[\frac{\mathbf{K}_1(u)}{u}\right]_{-v} = \frac{\mathbf{K}_2(u)}{u}\left[\rho\mathbf{v}\cdot\nabla\mu - \mu\right]. \tag{18}$$

Assuming that $\mu \in \mathbf{H}_0^2(\mathbf{R}^3)$, it is easy to see that $u^2\left[\mathbf{K}_2(u)/u\right]\mathbf{v}\cdot\nabla\mu\rho$ is continuous as $\rho \to 0$, so that the surface integral of this term vanishes. Thus, we only need consider

$$\lim_{\rho\to 0}\int_{\|\mathbf{x}-\mathbf{y}\|=\rho} \exp\left\{i\mathbf{a}_1\left[\mathbf{x}-\mathbf{v}\rho/2\right]\cdot\mathbf{v}\rho\right\}\mu(\mathbf{x}-\mathbf{v}\rho)\psi(\mathbf{x}-\mathbf{v}\rho)u\mathbf{K}_2(u)d\Omega. \tag{19}$$

The first three factors in the integrand of (19) are continuous as $\rho \to 0$. However, as $u = \mu\rho$ and $\mathbf{K}_2(u) \approx 1/u^2$ near zero, we see that, because of the last term, the integral diverges like $1/u$. This is the first of several divergent integrals that arise in the analytic representation of the square-root operator. For later use, we represent it as

$$4\pi\int_{\mathbf{R}^3} e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2 \frac{\mathbf{K}_2\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}\delta(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})d\mathbf{y}. \tag{20}$$

In the limit as $\rho \to 0$, equation (16) becomes:

$$\begin{aligned}
\mathbf{T}[\psi] &= \int_{\mathbf{R}^3}\left(\mu^2 + i\nabla\cdot\mathbf{a} + \mathbf{a}^2\right)e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}\psi d\mathbf{y} \\
&-2i\int_{\mathbf{R}^3}\mathbf{a}\cdot\nabla\left\{e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}\right\}\psi d\mathbf{y} - \int_{\mathbf{R}^3}\Delta\left\{e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}\right\}\psi d\mathbf{y} \\
&+\int_{\mathbf{R}^3} e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\frac{\mathbf{K}_2\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}4\pi\delta(\mathbf{x}-\mathbf{y})\psi d\mathbf{y}.
\end{aligned} \tag{21}$$

The above expression can be further refined after computation of the middle two terms. The calculations are long but straightforward, so we only provide intermediate steps omitting details. Using $\Delta(fg) = f\Delta g + f\Delta g + 2\nabla f\cdot\nabla g$, the third term becomes

$$\begin{aligned}
&\int_{\mathbf{R}^3}\Delta\left\{e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}\right\}\psi d\mathbf{y} \\
&= \int_{\mathbf{R}^3}\Delta\left\{e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\right\}\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\mu\|\mathbf{x}-\mathbf{y}\|}\psi d\mathbf{y} + \int_{\mathbf{R}^3} e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\Delta\left\{\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\mu\|\mathbf{x}-\mathbf{y}\|}\right\}\psi d\mathbf{y} \\
&+2\int_{\mathbf{R}^3}\nabla\left\{e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2\right\}\cdot\nabla\left\{\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\mu\|\mathbf{x}-\mathbf{y}\|}\right\}\psi d\mathbf{y}.
\end{aligned} \tag{22}$$

For further refinement, we need the following: ($u = \mu\|\mathbf{x}-\mathbf{y}\|$, $\mathrm{w} = e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\mu^2$)

$$\Delta\left[\mathbf{K}_1(u)/u\right]=\left[\mathbf{K}_3(u)/u\right](\nabla u)^2-\left[\mathbf{K}_2(u)/u\right](\Delta u),$$

$$\nabla\left[\mathbf{K}_1(u)/u\right]=-\nabla u\left[\mathbf{K}_2(u)/u\right],$$

$$\nabla u=\left(\|\mathbf{x}-\mathbf{y}\|\nabla\mu-\mu\left[(\mathbf{x}-\mathbf{y})/\|\mathbf{x}-\mathbf{y}\|\right]\right)=u\left(\tfrac{\nabla\mu}{\mu}\right)-\mu^2\,\tfrac{(\mathbf{x}-\mathbf{y})}{u}, \tag{23}$$

$$(\nabla u)^2=\mu^2+\|\mathbf{x}-\mathbf{y}\|^2(\nabla\mu)^2-2\mu\left[\nabla\mu\cdot(\mathbf{x}-\mathbf{y})\right],$$

$$\Delta u=\|\mathbf{x}-\mathbf{y}\|\Delta\mu-2\left[\nabla\mu\cdot(\mathbf{x}-\mathbf{y})-\mu\right]/\|\mathbf{x}-\mathbf{y}\|,$$

$$\begin{aligned}K_3[u]/u&=K_1[u]/u+4K_2[u]/u^2,\\K_2[u]/u&=K_0[u]/u+2K_1[u]/u^2.\end{aligned} \tag{24}$$

Using (23), we can compute (22) and the second term of (21) to get ($\mathbf{z}=\mathbf{x}-\mathbf{y}$)

$$\begin{aligned}\int_{\mathbf{R}^3}\Delta\left\{w\tfrac{\mathbf{K}_1[u]}{u}\right\}\psi d\mathbf{y}&=\int_{\mathbf{R}^3}\Delta w\tfrac{\mathbf{K}_1[u]}{u}\psi d\mathbf{y}+\int_{\mathbf{R}^3}w\left[\mu^2+u^2\tfrac{(\nabla\mu)^2}{\mu^2}-2\left[\nabla\mu\cdot\mathbf{u}\right]\right]\tfrac{\mathbf{K}_3[u]}{u}\psi d\mathbf{y}\\&-\int_{\mathbf{R}^3}w\left[u\tfrac{\Delta\mu}{\mu}-2\tfrac{[\nabla\mu\cdot\mathbf{u}-\mu^2]}{u}\right]\tfrac{\mathbf{K}_2[u]}{u}\psi-2\int_{\mathbf{R}^3}\left(u\left(\tfrac{\nabla w\cdot\nabla\mu}{\mu}\right)-\mu\tfrac{\nabla w\cdot\mathbf{u}}{u}\right)\tfrac{\mathbf{K}_2[u]}{u}\psi d\mathbf{y},\end{aligned} \tag{25}$$

$$\int_{\mathbf{R}^3}\mathbf{a}\cdot\nabla\left\{w\tfrac{\mathbf{K}_1[u]}{u}\right\}\psi d\mathbf{y}=+\int_{\mathbf{R}^3}\left(\mathbf{a}\cdot\nabla w\right)\tfrac{\mathbf{K}_1[u]}{u}\psi d\mathbf{y}-\int_{\mathbf{R}^3}w\left\{u\tfrac{(\mathbf{a}\cdot\nabla\mu)}{\mu}-\mu\tfrac{\mathbf{a}\cdot\mathbf{u})}{u}\right\}\tfrac{\mathbf{K}_2[u]}{u}\psi d\mathbf{y}. \tag{26}$$

In order to complete our representation for the square-root operator, we need to compute ∇w & Δw:

$$\begin{aligned}\nabla w&=w\left\{2\tfrac{\nabla\mu}{\mu}+i\left[\nabla\left(\mathbf{a}_1\cdot\mathbf{z}\right)\right]\right\}=w\left\{2\tfrac{\nabla\mu}{\mu}+i\left[(\mathbf{z}\cdot\nabla)\cdot\mathbf{a}_1\right]-i\mathbf{a}_1\right\},\\\Delta w&=w\left\{2\left[\tfrac{\Delta\mu}{\mu}+\left(\tfrac{\nabla\mu}{\mu}\right)^2\right]+4i\left[\tfrac{1}{\mu}\nabla\left(\mathbf{a}_1\cdot\mathbf{z}\right)\cdot\nabla\mu\right]+i\left[\Delta\left(\mathbf{a}_1\cdot\mathbf{z}\right)\right]-\left[\nabla\left(\mathbf{a}_1\cdot\mathbf{z}\right)\right]^2\right\}.\end{aligned} \tag{27}$$

It follows that

$$\begin{aligned}\mathbf{z}\cdot\nabla w&=w\left\{2\tfrac{\mathbf{z}\cdot\nabla\mu}{\mu}+i\left[(\mathbf{z}\cdot\nabla)\mathbf{a}_1\right]\cdot\mathbf{z}-i\mathbf{z}\cdot\mathbf{a}_1\right\}\\\left(\tfrac{\nabla w\cdot\nabla\mu}{\mu}\right)&=w\left\{2\tfrac{(\nabla\mu)^2}{\mu^2}+i\left[(\mathbf{z}\cdot\nabla)\mathbf{a}_1\right]\cdot\left(\tfrac{\nabla\mu}{\mu}\right)-i\mathbf{a}_1\cdot\left(\tfrac{\nabla\mu}{\mu}\right)\right\},\end{aligned} \tag{28}$$

and

$$
\begin{aligned}
\mathbf{S}[\psi] = \frac{\hbar^2 c\boldsymbol{\beta}}{2\pi^2}\Bigg\{ & \int_{\mathbf{R}^3}\left(\mu^2 + i\nabla\cdot\mathbf{a} + \mathbf{a}^2\right)e^{i\mathbf{a}_1\cdot\mathbf{z}}\mu^2\,\frac{\mathbf{K}_1\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\psi\,d\mathbf{y} + \int_{\mathbf{R}^3} e^{i\mathbf{a}_1\cdot\mathbf{z}}\mu^3\,\frac{\mathbf{K}_2\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}4\pi\delta(\mathbf{z})\psi\,d\mathbf{y} \\
& - 2i\int_{\mathbf{R}^3}\left(\mathbf{a}\cdot\nabla w\right)\frac{\mathbf{K}_1\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\psi\,d\mathbf{y} + 2i\int_{\mathbf{R}^3} w\left\{\|\mathbf{z}\|\left(\mathbf{a}\cdot\nabla\mu\right) - \mu\,\tfrac{\mathbf{a}\cdot\mathbf{z})}{\|\mathbf{z}\|}\right\}\frac{\mathbf{K}_2\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\psi\,d\mathbf{y} \\
& - \int_{\mathbf{R}^3}\Delta w\frac{\mathbf{K}_1\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\psi\,d\mathbf{y} - \int_{\mathbf{R}^3} w\left[\mu^2 + \|\mathbf{z}\|^2(\nabla\mu)^2 - 2\mu\left[\nabla\mu\cdot\mathbf{z}\right]\right]\frac{\mathbf{K}_3\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\psi\,d\mathbf{y} \\
& + \int_{\mathbf{R}^3} w\left[\|\mathbf{z}\|\Delta\mu - 2\,\tfrac{[\nabla\mu\cdot\mathbf{z}-\mu]}{\|\mathbf{z}\|}\right]\frac{\mathbf{K}_2\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\psi\,d\mathbf{y} + 2\int_{\mathbf{R}^3}\left(\|\mathbf{z}\|(\nabla w\cdot\nabla\mu) - \mu\,\tfrac{\nabla w\cdot\mathbf{z}}{\|\mathbf{z}\|}\right)\frac{\mathbf{K}_2\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\psi\,d\mathbf{y}\Bigg\}.
\end{aligned}
\tag{29}
$$

Equation (29) is quite general, and allows us to explore the consequences of all possible combinations of vector and scalar potentials. However, at this level, the equation is so complicated that physical interpretation is almost impossible. Thus, it is necessary to look at some special cases, which will be explored in the next few sections.

III. The Free Particle Case (μ constant and $\mathbf{A} = 0$)

In order to see what new physical insights our analytic representation provides, we consider the simplest case, when μ is constant and $\mathbf{A} = 0$ (free particle). In this case, (29) reduces to

$$
S[\psi](\mathbf{x}) = -\frac{\mu^2\hbar^2 c\boldsymbol{\beta}}{\pi^2}\int_{\mathbf{R}^3}\left[\frac{1}{\|\mathbf{x}-\mathbf{y}\|} - 2\pi\delta(\mathbf{x}-\mathbf{y})\right]\frac{\mathbf{K}_2\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}\psi(\mathbf{y})d\mathbf{y}. \tag{30}
$$

Using $K_2[u]/u = K_0[u]/u + 2K_1[u]/u^2$, we see that (30) has the representation

$$
S[\psi](\mathbf{x}) = -\frac{\mu^2\hbar^2 c\boldsymbol{\beta}}{\pi^2}\int_{\mathbf{R}^3}\left[\frac{1}{\|\mathbf{x}-\mathbf{y}\|} - 2\pi\delta(\mathbf{x}-\mathbf{y})\right]\left\{\frac{\mathbf{K}_0\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|} + \frac{2\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\mu\|\mathbf{x}-\mathbf{y}\|^2}\right\}\psi(\mathbf{y})d\mathbf{y}. \tag{31}
$$

Gill[12] first derived equation (31) using the method of fractional powers of closed operators. In order to understand the physical implications of (31), it will be helpful to review some properties of the Bessel functions $K_0[u]$, $K_{1/2}[u]/u^{1/2}$ and $K_1[u]/u$. If $\mathbf{x} \neq \mathbf{y}$, the effective kernel in (31) is

$$
\frac{\mathbf{K}_0\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|^2} + \frac{2\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\mu\|\mathbf{x}-\mathbf{y}\|^3}. \tag{32}
$$

(Note that the integral of $\left\|\mathbf{x}-\mathbf{y}\right\|^{-2}$ is finite over $\mathbf{R}^3$.) We follow Gradshteyn and Ryzhik[12]. For $0 < u << 1$, we have that:

$$\left.\begin{array}{l} \mathbf{K}_1[u]/u = c_1[1+\theta_1(u)]u^{-2} \\ \mathbf{K}_{1/2}[u]/u^{1/2} = [\sqrt{\pi/2}]u^{-1} \\ \mathbf{K}_0[u] = c_0[1+\theta_0(u)]\ln u^{-1} \end{array}\right\}, \tag{33a}$$

where $\theta_0(u),\ \theta_1(u) \downarrow 0,\ u \downarrow 0$. On the other hand, for $u >> 1$, we have:

$$\left.\begin{array}{l} \dfrac{\mathbf{K}_1[u]}{u} = c_1[1+\theta_1'(u)]\dfrac{\exp\{-u\}}{u^{3/2}} \\ \mathbf{K}_{1/2}[u]/u^{1/2} = [\sqrt{\pi/2}]\dfrac{\exp\{-u\}}{u} \\ \mathbf{K}_0[u] = c_0[1+\theta_0'(u)]\dfrac{\exp\{-u\}}{u^{1/2}} \end{array}\right\}. \tag{33b}$$

In this case, the functions $\theta_0'(u),\ \theta_1'(u)$ converge $\downarrow 0$, as $u \uparrow \infty$.

Recall that $g^2 \exp\{-u\}/u$ is the well-known Yukawa potential[13], conjectured in 1935 in order to account for the short range of the nuclear interaction that was expected to have massive exchange particles (where g represents the "charge" of the exchange field). Yukawa assumed that the range of the exchange field was $1/\mu \cong 1.4$ fermi, which led to a mass value of about 170 times that of the electron. Anderson and Neddermeyer[14] discovered what was believed to be Yukawa's meson with a mass of 207 times that of an electron in 1935. However, this particle interacted so weakly with nuclei and had such a long lifetime, it was rejected as a participant in the nuclear interaction. Finally, in 1947 Lattes et al[15] identified the π-meson (pion) with all the expected properties.

Looking at equation (33a) in the strength of singularity sense, we see that when $0 < u << 1$,

$$\frac{\mathbf{K}_1[u]}{u} > \frac{\mathbf{K}_{1/2}[u]}{u^{1/2}} > \mathbf{K}_0[u]. \tag{34}$$

The $\mathbf{K}_0[u]$ term is the weakest possible singularity in that: $\lim_{u\to 0}\{u^{\varepsilon}\mathbf{K}_0[u]\},\ \varepsilon > 0$. (In fact, it is an integrable singularity.) On the other hand, the $\mathbf{K}_1[u]/u$ term has the strongest singularity (as it diverges like $1/u^2$), while the Yukawa term is halfway between them. From equation (33b) we see that for large u, inequality (34) is reversed so that:

$$\mathbf{K}_0\left[u\right] > \frac{\mathbf{K}_{1/2}\left[u\right]}{u^{1/2}} > \frac{\mathbf{K}_1\left[u\right]}{u}. \tag{35}$$

Although all three terms in (35) have an exponential cutoff, the $\mathbf{K}_0[u]$ term has the longest range (see (33b)). If we include the $-\mu^2$ term in equation (32), it becomes:

$$-\mu^2\frac{\mathbf{K}_0\left[\mu\left\|\mathbf{x}-\mathbf{y}\right\|\right]}{\left\|\mathbf{x}-\mathbf{y}\right\|^2} - 2\mu\frac{\mathbf{K}_1\left[\mu\left\|\mathbf{x}-\mathbf{y}\right\|\right]}{\left\|\mathbf{x}-\mathbf{y}\right\|^3}. \tag{36}$$

Thus, $-\mu^2\mathbf{K}_0[u]$ has an extra factor of μ, compared to the $\mathbf{K}_1[u]/u$ term, giving a value of $5\times 10^{13}\,cm^{-1}$, assuming the mass is that of an electron. Hence, although this term is the weakest of all possible singularities near $\mathbf{x}=\mathbf{y}$, it is asymptotically stronger by a factor of at least 10^9.

Equation (31) is the first example of a physically relevant operator, which has a "natural" representation as a composite of three singularities. In the standard interpretation, the particle component has two negative and one (hard core) positive part, while the antiparticle component has two positive and one (hard core) negative part. This effect is confined within a Compton wavelength such that, at the point of singularity, they cancel each other providing a finite result. Furthermore, the operator looks (almost) like the identity outside a Compton wavelength, but has a residual instantaneous connection with all the particles in the universe at each point in time.

These results suggest that, the square-root operator represents the inside of an extended object, while the Dirac operator represents the outside.

IV. The Constant Case (A and μ constant)

When $\mathbf{A}$ and μ are constant, we get a more interesting but still simple case. Since $\nabla\cdot\mathbf{A}=0$ and $\nabla\times\mathbf{A}=0$, equation (29) becomes: ($\mathbf{a}_1=\mathbf{a}$, $\mu^2 = m^2c^2/\hbar^2$)

$$\begin{aligned}\mathcal{S}[\psi](\mathbf{x}) = &-\tfrac{\mu^2\hbar^2 c\boldsymbol{\beta}}{\pi^2}\int_{\mathbf{R}^3} e^{i\mathbf{a}\cdot(\mathbf{x}-\mathbf{y})}\left[\frac{1}{\left\|\mathbf{x}-\mathbf{y}\right\|} - 2\pi\delta\left(\mathbf{x}-\mathbf{y}\right)\right]\left\{\frac{\mathbf{K}_0\left[\mu\left\|\mathbf{x}-\mathbf{y}\right\|\right]}{\left\|\mathbf{x}-\mathbf{y}\right\|} + \frac{2\mathbf{K}_1\left[\mu\left\|\mathbf{x}-\mathbf{y}\right\|\right]}{\mu\left\|\mathbf{x}-\mathbf{y}\right\|^2}\right\}\psi(\mathbf{y})d\mathbf{y} \\ &-\frac{\mu\hbar^2 c\boldsymbol{\beta}\mathbf{a}^2}{\pi^2}\int_{\mathbf{R}^3} e^{i\mathbf{a}\cdot(\mathbf{x}-\mathbf{y})}\frac{\mathbf{K}_1\left[\mu\left\|\mathbf{x}-\mathbf{y}\right\|\right]}{\left\|\mathbf{x}-\mathbf{y}\right\|}\psi(\mathbf{y})d\mathbf{y}\end{aligned} \tag{37}$$

Thus, when $\mathbf{A}$ and μ are both constant, we get one extra attractive but nonsingular term, along with an additional multiplicative exponential factor.

V. The Constant Field Case (μ, B constant)

We get the (physically more interesting) case of a constant magnetic field $\mathbf{B}$, when μ is constant and $\mathbf{A} = \frac{1}{2}\mathbf{y}\times\mathbf{B}$. Since $\nabla\cdot\mathbf{A} = 0$, equation (29) becomes:

$$\begin{aligned} S[\psi](\mathbf{x}) &= -\frac{\mu^2\hbar^2 c\boldsymbol{\beta}}{\pi^2}\int_{\mathbf{R}^3} e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\left[\frac{1}{\|\mathbf{x}-\mathbf{y}\|} - \delta(\|\mathbf{x}-\mathbf{y}\|)\right]\frac{\mathbf{K}_2[\mu\|\mathbf{x}-\mathbf{y}\|]}{\|\mathbf{x}-\mathbf{y}\|}\psi(\mathbf{y})d\mathbf{y}. \\ &+2i\frac{\mu^2\hbar^2 c\boldsymbol{\beta}}{\pi^2}\int_{\mathbf{R}^3} e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\left\{(\mathbf{x}-\mathbf{y})\cdot\left[(\mathbf{a}_1-\mathbf{a}) - (\mathbf{x}-\mathbf{y})\cdot\nabla\mathbf{a}_1\right]\right\}\frac{\mathbf{K}_2[\mu\|\mathbf{x}-\mathbf{y}\|]}{\|\mathbf{x}-\mathbf{y}\|^2}\psi(\mathbf{y})d\mathbf{y} \qquad (38) \\ &+\frac{\mu\hbar^2 c\boldsymbol{\beta}}{\pi^2}\int_{\mathbf{R}^3} e^{i\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y})}\left\{\mathbf{a}^2 - 2\mathbf{a}_1\cdot\mathbf{a} + 2\left[(\mathbf{x}-\mathbf{y})\cdot\nabla\mathbf{a}_1\right]\cdot\mathbf{a}\right\}\frac{\mathbf{K}_1[\mu\|\mathbf{x}-\mathbf{y}\|]}{\|\mathbf{x}-\mathbf{y}\|}\psi(\mathbf{y})d\mathbf{y}. \end{aligned}$$

Equation (38) can be further simplified if we note that:

$$(\mathbf{a}_1 - \mathbf{a}) = \tfrac{e\hbar}{4c}(\mathbf{x}-\mathbf{y})\times\mathbf{B}, \qquad (39a)$$

and

$$(\mathbf{x}-\mathbf{y})\cdot\nabla\mathbf{a}_1 = \tfrac{1}{2}(\mathbf{x}-\mathbf{y})\cdot\nabla\mathbf{a} = \tfrac{e\hbar}{4c}(\mathbf{x}-\mathbf{y})\times\mathbf{B}. \qquad (39b)$$

A simple calculation shows that the middle term vanishes and, from the fact that

$$2\left[(\mathbf{x}-\mathbf{y})\cdot\nabla\mathbf{a}_1\right]\cdot\mathbf{a} = \tfrac{e^2\hbar^2}{4c^2}(\mathbf{x}\times\mathbf{B})\cdot(\mathbf{y}\times\mathbf{B}) - \tfrac{e^2\hbar^2}{4c^2}\|\mathbf{y}\times\mathbf{B}\|^2, \qquad (40a)$$

we have

$$-2\mathbf{a}_1\cdot\mathbf{a} = -\left\{\tfrac{e^2\hbar^2}{4c^2}(\mathbf{x}\times\mathbf{B})\cdot(\mathbf{y}\times\mathbf{B}) + \tfrac{e^2\hbar^2}{4c^2}\|\mathbf{y}\times\mathbf{B}\|^2\right\}. \qquad (40b)$$

It is easy to show that the part in brackets of the last term reduces to $-\mathbf{a}^2$. Thus, equation (38) becomes: ($\mathbf{a}_1\cdot(\mathbf{x}-\mathbf{y}) = -\mathbf{x}\cdot\mathbf{a}$)

$$\begin{aligned} S[\psi](\mathbf{x}) &= -\frac{\mu^2\hbar^2 c\boldsymbol{\beta}}{\pi^2}\int_{\mathbf{R}^3} e^{-i\mathbf{x}\cdot\mathbf{a}}\left[\frac{1}{\|\mathbf{x}-\mathbf{y}\|} - 2\pi\delta(\mathbf{x}-\mathbf{y})\right]\left\{\frac{\mathbf{K}_0[\mu\|\mathbf{x}-\mathbf{y}\|]}{\|\mathbf{x}-\mathbf{y}\|} + \frac{2\mathbf{K}_1[\mu\|\mathbf{x}-\mathbf{y}\|]}{\mu\|\mathbf{x}-\mathbf{y}\|^2}\right\}\psi(\mathbf{y})d\mathbf{y} \qquad (41) \\ &- \frac{\mu\hbar^2 c\boldsymbol{\beta}}{\pi^2}\int_{\mathbf{R}^3} e^{-i\mathbf{x}\cdot\mathbf{a}}\mathbf{a}^2\frac{\mathbf{K}_1[\mu\|\mathbf{x}-\mathbf{y}\|]}{\|\mathbf{x}-\mathbf{y}\|}\psi(\mathbf{y})d\mathbf{y}. \end{aligned}$$

It follows that there is only the one new term as in Section IV; the term is attractive and non-singular. However, in this case, the (matrix-valued) mass μ, though constant, becomes complex. Recall that $\mu^2 = m^2c^2/\hbar^2 - \frac{e}{\hbar c}\boldsymbol{\Sigma}\cdot\mathbf{B}$, with:

$$\mathbf{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}; \quad \sigma_1 = \begin{pmatrix} \mathbf{0} & 1 \\ 1 & \mathbf{0} \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} \mathbf{0} & -i \\ i & \mathbf{0} \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}. \tag{42}$$

Hence, we can rewrite μ as

$$\mu = \begin{bmatrix} (\frac{m^2c^2}{\hbar^2} - \frac{e}{\hbar c}B_3)\mathbf{I}_2 & \frac{ie}{\hbar c}(B_2 - iB_1)\mathbf{I}_2 \\ \frac{-ie}{\hbar c}(B_2 - iB_1)\mathbf{I}_2 & (\frac{m^2c^2}{\hbar^2} + \frac{e}{\hbar c}B_3)\mathbf{I}_2 \end{bmatrix}^{1/2}. \tag{43}$$

From properties of Bessel functions we know that, for nonintegral v, we can represent $\mathbf{K}_v[u]$ as

$$(2/\pi)\mathbf{K}_v[u] = \frac{\mathbf{I}_{-v}(u) - \mathbf{I}_v(u)}{\sin \pi v} = \frac{e^{i/2(\pi v)}\mathbf{J}_{-v}(iu) - e^{-i/2(\pi v)}\mathbf{J}_v(iu)}{\sin \pi v}. \tag{44}$$

In the limit as v approaches an integer, equation (44) takes the indeterminate form $0/0$, and is defined via L'Hôpital's rule. However, for our purposes, we assume that v is close to an integer and $u = u_1 + iu_2$, $u_2 \neq 0$. It follows that $\mathbf{K}_v[u]$ acquires some of the oscillatory behavior of $\mathbf{J}_v[z]$. Thus, we can interpret equation (41) as representing a pulsating mass (extended object of variable mass) with mean value $(\hbar/c)\|\mu\|$. The operator still looks (almost) like the identity outside a few Compton wavelengths

VI. Variable Mass Case ($\mathbf{A} = \mathbf{0}$)

In this case, as $\mathbf{A} = \mathbf{0}$, $w = \mu^2$, $\nabla w = 2\mu\nabla\mu$ and $\Delta w = 2\mu\Delta\mu + 2(\nabla\mu)^2$, so we get:

$$\begin{aligned} \mathbf{S}[\psi] = \frac{\hbar^2 c\boldsymbol{\beta}}{2\pi^2}\Bigg\{ &\int_{\mathbf{R}^3} \mu^4 \frac{\mathbf{K}_1\big[\mu\|\mathbf{z}\|\big]}{\mu\|\mathbf{z}\|}\psi d\mathbf{y} + \int_{\mathbf{R}^3} \mu^3 \frac{\mathbf{K}_2\big[\mu\|\mathbf{z}\|\big]}{\mu\|\mathbf{z}\|} 4\pi\delta(\mathbf{z})\psi d\mathbf{y} \\ &- \int_{\mathbf{R}^3} \Delta w \frac{\mathbf{K}_1\big[\mu\|\mathbf{z}\|\big]}{\mu\|\mathbf{z}\|}\psi d\mathbf{y} - \int_{\mathbf{R}^3} w\Big[\mu^2 + \|\mathbf{z}\|^2(\nabla\mu)^2 - 2\mu\big[\nabla\mu\cdot\mathbf{z}\big]\Big]\frac{\mathbf{K}_3\big[\mu\|\mathbf{z}\|\big]}{\mu\|\mathbf{z}\|}\psi d\mathbf{y} \\ &+ \int_{\mathbf{R}^3} w\Big[\|\mathbf{z}\|\Delta\mu - 2\tfrac{[\nabla\mu\cdot\mathbf{z}-\mu]}{\|\mathbf{z}\|}\Big]\frac{\mathbf{K}_2\big[\mu\|\mathbf{z}\|\big]}{\mu\|\mathbf{z}\|}\psi d\mathbf{y} + 2\int_{\mathbf{R}^3}\Big(\|\mathbf{z}\|(\nabla w\cdot\nabla\mu) - \mu\tfrac{\nabla w\cdot\mathbf{z}}{\|\mathbf{z}\|}\Big)\frac{\mathbf{K}_2\big[\mu\|\mathbf{z}\|\big]}{\mu\|\mathbf{z}\|}\psi d\mathbf{y}\Bigg\}. \end{aligned} \tag{45}$$

After some rearrangement and additional computation, using $K_3[u]/u = K_1[u]/u + 4K_2[u]/u^2$, equation (45) becomes

$$\mathbf{S}[\psi]=\frac{\hbar^2 c\boldsymbol{\beta}}{2\pi^2}\Bigg\{\int_{\mathbf{R}^3}\mu^3\frac{\mathbf{K}_2\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}4\pi\delta(\mathbf{z})\psi d\mathbf{y}-\int_{\mathbf{R}^3}[2\mu\Delta\mu+2(\nabla\mu)^2]\frac{\mathbf{K}_1\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\psi d\mathbf{y}$$

$$-4\int_{\mathbf{R}^3}\mu^2\left[\mu^2+\|\mathbf{z}\|^2(\nabla\mu)^2-2\mu\left[\nabla\mu\cdot\mathbf{z}\right]\right]\left\{\frac{\mathbf{K}_2\left[\mu\|\mathbf{z}\|\right]}{\left[\mu\|\mathbf{z}\|\right]^2}\right\}\psi d\mathbf{y}$$

$$-\int_{\mathbf{R}^3}\mu^2\left[\|\mathbf{z}\|^2(\nabla\mu)^2-2\mu\left[\nabla\mu\cdot\mathbf{z}\right]\right]\left\{\frac{\mathbf{K}_1\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\right\}\psi d\mathbf{y} \tag{46}$$

$$+\int_{\mathbf{R}^3}\mu^2\left[\|\mathbf{z}\|\Delta\mu-2\tfrac{[\nabla\mu\cdot\mathbf{z}-\mu]}{\|\mathbf{z}\|}\right]\frac{\mathbf{K}_2\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\psi d\mathbf{y}+4\int_{\mathbf{R}^3}\left[\mu\|\mathbf{z}\|(\nabla\mu)^2-\mu^2\tfrac{\nabla\mu\cdot\mathbf{z}}{\|\mathbf{z}\|}\right]\frac{\mathbf{K}_2\left[\mu\|\mathbf{z}\|\right]}{\mu\|\mathbf{z}\|}\psi d\mathbf{y}\Bigg\}.$$

Combining terms, we have, after cancellations:

$$\mathbf{S}[\psi]=\frac{\hbar^2 c\boldsymbol{\beta}}{2\pi^2}\Bigg\{-2\int_{\mathbf{R}^3}\left[\frac{1}{\|\mathbf{x}-\mathbf{y}\|}-2\pi\delta(\mathbf{x}-\mathbf{y})\right]\left\{\frac{\mu^2\mathbf{K}_0\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}+\frac{2\mu\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|^2}\right\}\psi(\mathbf{y})d\mathbf{y}$$

$$+\int_{\mathbf{R}^3}\left[2\mu^3\nabla\mu\cdot(\mathbf{x}-\mathbf{y})-\mu^2\|\mathbf{x}-\mathbf{y}\|^2(\nabla\mu)^2-2\mu\Delta\mu-2(\nabla\mu)^2\right]\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\mu\|\mathbf{x}-\mathbf{y}\|}\psi(\mathbf{y})d\mathbf{y} \tag{47}$$

$$+\int_{\mathbf{R}^3}\left[\mu^2\|\mathbf{x}-\mathbf{y}\|^2(\nabla\mu)^2+2\mu[\nabla\mu\cdot(\mathbf{x}-\mathbf{y})]\right]\left\{\frac{\mathbf{K}_2\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]^2}\right\}\psi(\mathbf{y})d\mathbf{y},$$

and

$$\mathbf{S}[\psi]=\frac{\hbar^2 c\boldsymbol{\beta}}{\pi^2}\left\{-\int_{\mathbf{R}^3}\mu\left[\frac{1}{\|\mathbf{x}-\mathbf{y}\|}-2\pi\delta(\mathbf{x}-\mathbf{y})\right]\left[\frac{\mu\mathbf{K}_0\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|}+\frac{2\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\|\mathbf{x}-\mathbf{y}\|^2}\right]\psi(\mathbf{y})d\mathbf{y}\right\}$$

$$+\frac{\hbar^2 c\boldsymbol{\beta}}{2\pi^2}\left\{\int_{\mathbf{R}^3}\left[2\mu^3\nabla\mu\cdot(\mathbf{x}-\mathbf{y})-\mu^2\|\mathbf{x}-\mathbf{y}\|^2(\nabla\mu)^2-2\mu\Delta\mu\right]\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\mu\|\mathbf{x}-\mathbf{y}\|}\psi(\mathbf{y})d\mathbf{y}\right\}$$

$$+\frac{2\hbar^2 c\boldsymbol{\beta}}{\pi^2}\int_{\mathbf{R}^3}\left\{\left[\frac{\nabla\mu\cdot(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\right]\frac{\mathbf{K}_1\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]^2}\right\}\psi(\mathbf{y})d\mathbf{y} \tag{48}$$

$$+\frac{\hbar^2 c\boldsymbol{\beta}}{2\pi^2}\int_{\mathbf{R}^3}\left[\frac{2\nabla\mu\cdot(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}+\mu\|\mathbf{x}-\mathbf{y}\|(\nabla\mu)^2\right]\frac{\mathbf{K}_0\left[\mu\|\mathbf{x}-\mathbf{y}\|\right]}{\mu\|\mathbf{x}-\mathbf{y}\|}\psi(\mathbf{y})d\mathbf{y}.$$

For equation (48), physical motivation is required to choose μ. One possibility is $\mu=\sqrt{\left[\boldsymbol{\beta}(mc/\hbar+\boldsymbol{\beta}V)\right]^2+i\hbar c\nabla V}$ (see Section IIIV). This will be discussed in a later paper.

IIV. General Solution

In this section, we construct the general solution to equation (1b). First, rewrite equation (7) as:

$$\mathcal{Q}(\mathbf{x},\mathbf{y};t)=\left(\frac{1}{4\pi t}\right)^{3/2}\exp\left\{-\frac{(\mathbf{x}-\mathbf{y})^2}{4t}-\mu^2 t+\frac{ie}{\hbar c}(\mathbf{x}-\mathbf{y})\cdot\left[\left(\tfrac{\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y})}{2}\right)\right]\right\}. \tag{49}$$

Now, using the theory of fractional powers, we note that if $\mathbf{T}[t,0]$ is the semigroup associated with $-\mathbf{G}+\omega^2$, then the semigroup associated with $\sqrt{-\mathbf{G}+\omega^2}$ is given by: (Appendix, equations A7, A10)

$$\mathbf{T}_{1/2}[t,0]\varphi(\mathbf{x})=$$

$$\int_0^\infty\left\{\int_{\mathbf{R}^3}\left(\tfrac{1}{4\pi s}\right)^{3/2}\exp\left\{\left(-\tfrac{\|\mathbf{x}-\mathbf{y}\|^2}{4s}-\mu^2 s\right)+\left[\tfrac{ie(\mathbf{x}-\mathbf{y})\cdot(\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y}))}{2\hbar c}\right]\right\}\varphi(\mathbf{y})d\mathbf{y}\right\}\left\{\left(\tfrac{ct}{\sqrt{4\pi}}\right)\frac{1}{s^{3/2}}\exp\left(-\tfrac{(ct)^2}{4s}\right)\right\}ds \tag{50}$$

From a table of Laplace transforms, we get that:

$$\int_0^\infty \exp\left(-\tfrac{a}{s}-ps\right)\frac{ds}{s^3}=2\left(\frac{p}{a}\right)K_2\left[2(ap)^{1/2}\right]. \tag{51}$$

With $a=\left[\|\mathbf{x}-\mathbf{y}\|^2+c^2t^2\right]\Big/4,\ p=\mu^2$, we can interchange the order of integration to get that

$$\mathbf{T}_{1/2}[t,0]\varphi(\mathbf{x})=$$

$$\frac{ct}{4\pi^2}\int_{\mathbf{R}^3}\exp\left\{\frac{ie}{2\hbar c}(\mathbf{x}-\mathbf{y})\cdot(\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y}))\right\}\frac{2\mu^2K_2\left[\mu\left(\|\mathbf{x}-\mathbf{y}\|^2+c^2t^2\right)^{1/2}\right]}{\left[\|\mathbf{x}-\mathbf{y}\|^2+c^2t^2\right]}\varphi(\mathbf{y})d\mathbf{y}. \tag{52}$$

We now use the fact that $\mathbf{T}_{1/2}[t,0]$ has a holomorphic extension into the complex plane so that we may compute the limit as $t\to it$. Setting $\mathbf{U}[t,0]=\lim_{\varepsilon\to 0}\boldsymbol{\beta}\mathbf{T}_{1/2}[(i+\varepsilon)t,0]$, if we define:
(note that $\exp\left\{\boldsymbol{\beta}it\sqrt{-\mathbf{G}+\omega^2}\right\}=\boldsymbol{\beta}\exp\left\{it\sqrt{-\mathbf{G}+\omega^2}\right\}$)

$$\mathbf{Z}\left[\mu\left(c^2t^2-\|\mathbf{x}-\mathbf{y}\|^2\right)^{1/2}\right]=\frac{ct\mathrm{b}}{4\pi}\begin{cases}\dfrac{-H_2^{(1)}\left[\mu\left(c^2t^2-\|\mathbf{x}-\mathbf{y}\|^2\right)^{1/2}\right]}{\left[c^2t^2-\|\mathbf{x}-\mathbf{y}\|^2\right]}, & ct<-\|\mathbf{x}\|,\\ \dfrac{-2iK_2\left[\mu\left(\|\mathbf{x}-\mathbf{y}\|^2-c^2t^2\right)^{1/2}\right]}{\pi\left[\|\mathbf{x}-\mathbf{y}\|^2-c^2t^2\right]}, & c|t|<\|\mathbf{x}\|,\\ \dfrac{H_2^{(2)}\left[\mu\left(c^2t^2-\|\mathbf{x}-\mathbf{y}\|^2\right)^{1/2}\right]}{\left[c^2t^2-\|\mathbf{x}-\mathbf{y}\|^2\right]}, & ct>\|\mathbf{x}\|,\end{cases} \tag{53a}$$

where, $H_2^{(1)}, H_2^{(2)}$ are the Hankel functions (see Gradshteyn and Ryzhik[12]). Then it follows that

$$\mathbf{U}[t,0]\varphi(\mathbf{x})=\int_{\mathbf{R}^3}\mu^2\exp\left\{\frac{ie}{2\hbar c}(\mathbf{x}-\mathbf{y})\cdot(\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y})\right\}\mathbf{Z}\left[\mu\left(c^2t^2-\|\mathbf{x}-\mathbf{y}\|^2\right)^{1/2}\right]\varphi(\mathbf{y})d\mathbf{y} \tag{53b}$$

solves

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t)=\left\{\boldsymbol{\beta}\sqrt{c^2\left(\mathbf{p}-\tfrac{e}{c}\mathbf{A}\right)^2-\tfrac{e\hbar}{c}\boldsymbol{\Sigma}\cdot\mathbf{B}+m^2c^4}\right\}\psi(\mathbf{x},t),\quad \psi(\mathbf{x},0)=\varphi(\mathbf{x}), \tag{54}$$

for all time independent $\mathbf{A}$ and m.

From our discussion in the first section, we see that another solution to equation (54) may be obtained by replacing $\frac{1}{2}\left(\mathbf{A}(\mathbf{x})+\mathbf{A}(\mathbf{y})\right)$ by $\mathbf{A}\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)$ in the exponential of equation (53b). In the path-integral representation of the solution, these two terms give the same (unique) result. Thus, in the path-integral formulation, no ambiguities appear. For a general theory of path-integrals, sufficient to handle problems of the above type, see Gill and Zachary[16-17].

IIIV. An Alternate Dirac, Square-root Connection

Before Minkowski's geometric interpretation of the special theory, it was not uncommon to associate the potential energy with the mass system. Since that time, it has been assumed that the potential energy should always be treated as the fourth component of a four-vector. This convention has only been fruitful in classical and quantum theory, in the one-particle case. It was first noted by Pryce[18] that, in the many-particle case, the canonical center-of-mass vector is not the forth component of a four-vector. *This is the only reason that there is not a (satisfactory) relativistic classical or quantum many-particle theory*. A detail review of the problems at the classical level may be found in Gill, Zachary and Lindesay[19].

Returning to equation (1), when the mass is constant, it was shown by Case[20] that a Foldy-Wouthuysen[21] transformation $\left(U_{FW}^{-1}\mathbf{H}_{\mathrm{S}}U_{FW}=\mathbf{H}_{\mathrm{D}}\right)$, may be constructed to map (1) into:

$$D[\Psi]=\mathbf{H}_D\Psi=\left\{c\boldsymbol{\alpha}\cdot\left(\mathbf{p}-\tfrac{e}{c}\mathbf{A}\right)+mc^2\boldsymbol{\beta}\right\}\Psi. \tag{55}$$

In this section we show, that if we give up the Minkowski convention, there is another possible relationship between the square-root and Dirac equation.

For example, when minimal coupling is introduced, the interacting Dirac operator, may also be written as ($\boldsymbol{\beta}^2 = \mathbf{I} \Rightarrow mc^2\boldsymbol{\beta} + V = \boldsymbol{\beta}(mc^2 + \boldsymbol{\beta}V)$):

$$D[\Psi] = \left\{\boldsymbol{\alpha}\cdot\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right) + \boldsymbol{\beta}\left(mc^2 + \boldsymbol{\beta}V\right)\right\}\Psi. \tag{56}$$

We claim that, if we treat the potential energy as a part of the mass term, then (56) can be transformed to:

$$S[\psi] = \left\{\boldsymbol{\beta}\sqrt{c^2\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right)^2 - e\hbar c\boldsymbol{\Sigma}\cdot\mathbf{B} - ie\hbar c\boldsymbol{\alpha}\cdot\mathbf{E} + \left[\boldsymbol{\beta}\left(mc^2 + \boldsymbol{\beta}V\right)\right]^2}\right\}\psi. \tag{57}$$

In order to see this, start with equation (56) in the form (see Schiff[22], page 329):

$$\left\{E - \boldsymbol{\alpha}\cdot\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right) - \boldsymbol{\beta}\left(mc^2 + \boldsymbol{\beta}V\right)\right\}\Psi = \mathbf{0}, \tag{58}$$

Now, multiply on the left by $E + \boldsymbol{\alpha}\cdot\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right) + \boldsymbol{\beta}\left(mc^2 + \boldsymbol{\beta}V\right)$, do the standard computations, $-e\boldsymbol{\alpha}\cdot\left[E\mathbf{A} - \mathbf{A}E\right] = -ie\hbar c\boldsymbol{\alpha}\cdot\left[\partial\mathbf{A}/c\partial t\right]$, $-c\boldsymbol{\beta\alpha}\cdot\left[\boldsymbol{\beta}V\mathbf{p} - \mathbf{p}\boldsymbol{\beta}V\right] = -ie\hbar c\boldsymbol{\alpha}\cdot\nabla\varphi$, and we get:

$$\left\{E^2 - c^2\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right)^2 + e\hbar c\boldsymbol{\Sigma}\cdot\mathbf{B} + ie\hbar c\boldsymbol{\alpha}\cdot\mathbf{E} - \left[\boldsymbol{\beta}\left(mc^2 + \boldsymbol{\beta}V\right)\right]^2\right\}\Psi = \mathbf{0}, \tag{59}$$

where $V = e\varphi$, and $\mathbf{E} = -\partial\mathbf{A}/c\partial t - \nabla\varphi$ is the electric field. It is now easy to see from (59) that:

$$E\Psi = \left\{\boldsymbol{\beta}\sqrt{c^2\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right)^2 - e\hbar c\boldsymbol{\Sigma}\cdot\mathbf{B} - ie\hbar c\boldsymbol{\alpha}\cdot\mathbf{E} + \left[\boldsymbol{\beta}\left(mc^2 + \boldsymbol{\beta}V\right)\right]^2}\right\}\Psi. \tag{60}$$

The Klein-Gordon (type) equation related to (59) can be written as:

$$i\hbar\frac{\partial^2\Psi}{\partial t^2} = \left\{c^2\left(\mathbf{p} - \tfrac{e}{c}\mathbf{A}\right)^2 - e\hbar c\boldsymbol{\Sigma}\cdot\mathbf{B} - ie\hbar c\boldsymbol{\alpha}\cdot\mathbf{E} + \left[\boldsymbol{\beta}\left(mc^2 + \boldsymbol{\beta}V\right)\right]^2\right\}\Psi. \tag{61}$$

A major advantage of equation (61) is that it has the same eigenfunctions as the Dirac equation, while the eigenvalues of (61) are the squares of the corresponding Dirac eigenvalues.

The relationship and implications of equations (56), (60) and (61) will be studied elsewhere. However, we expect that a generalized Foldy-Wouthuysen transformation can be defined which relates equations (56) and (60). To see that this is indeed possible, following Eriksen[23] (see De Vries[24]), define the projection operators for the positive and negative energy solutions of the Dirac equation, $\Gamma_{\pm}$, and the projection operators for the particle and antiparticle wavefunctions, $\mathcal{B}_{\pm}$, by:

$$\Gamma_{\pm} = \frac{1}{2}\left(\mathbf{I} \pm \left[\left(\mathbf{H}_D\right)^2\right]^{-1/2} \mathbf{H}_D\right), \quad \mathcal{B}_{\pm} = \frac{1}{2}\left(\mathbf{I} \pm \boldsymbol{\beta}\right). \tag{62}$$

Assume that a unitary operator U exists such that:

$$U^* U = UU^* = \mathbf{I}; \quad \mathcal{B}_{\pm} = U\Gamma_{\pm}U^*. \tag{63}$$

This implies that with

$$\lambda = \left[\left(\mathbf{H}_D\right)^2\right]^{-1/2} \mathbf{H}_D, \quad U\lambda U^* = \boldsymbol{\beta} \Rightarrow \quad \lambda = U^* \boldsymbol{\beta} U. \tag{64}$$

Eriksen noted that U is not uniquely determined by (64). He assumes that we can impose the additional constraint

$$U^* \boldsymbol{\beta} = \boldsymbol{\beta} U \Rightarrow \quad \lambda = \boldsymbol{\beta} U^2 \Rightarrow \quad U^2 = \boldsymbol{\beta}\lambda. \tag{65}$$

It follows that, formally U is the square-root of the unitary operator $\boldsymbol{\beta}\lambda$. Thus, as expected, we can define a generalized Foldy-Wouthuysen transformation relating equations (56) and (57).

Conclusion

In this paper, we have shown that the square-root operator has a well-defined analytic representation, which is uniquely determined by the Green's function for the corresponding Schrödinger equation. We have constructed the most general solution when the vector potential and the mass term are time-independent. We explored a number of simple cases in order to obtain some insight into the physical meaning of the operator.

In the free case, when the vector potential is zero and the mass is constant, the operator is represented by a nonlocal composite of three singularities. To our knowledge, this is the first example of a physically relevant operator with these properties. In the standard interpretation, the particle component has two negative parts and one (hard core) positive part, while the antiparticle component has two positive parts and one (hard core) negative part. This effect is confined within a Compton wavelength such that, at the point of singularity, they cancel each other providing a finite result. Furthermore, the operator looks (almost) like the identity outside a Compton wavelength, but has a residual instantaneous connection with all other particles in the universe at each point in time.

In addition to the obvious possibility that the square-root operator may be used to represent the inside of hadrons, it is also possible that the residual attractive (particle) part may be the long sought cause for the gravitational interaction in matter. If this view were correct, then we would expect matter and antimatter to be gravitationally attractive among themselves and gravitationally repulsive to each other. This would make physical sense if we take seriously the interpretation of antimatter as matter with its time reversed (as opposed to hole theory). Santilli[25-26] suggested this possibility and identified relevant test experiments. His approach was based on the isodual theory of numbers and their application to the study of particle physics.

When the mass is constant and the vector potential is constant but nonzero, in addition to the confined singularities, we obtain one additional term, which is attractive (repulsive) for matter (antimatter) but nonsingular. The next case we explored corresponds to a constant magnetic field. In this case, in addition to the confined singularities and the additional attractive (repulsive) term, the effective mass of the composite acquires an oscillatory behavior. We then determine the equation for zero vector potential and variable mass. A study of this case is deferred for a later paper, as it requires additional physical analysis.

We then use the method of fractional powers of semigroups of operators to construct the complete propagator for the general case. We then observed that the path-integral formulation eliminates the ambiguity caused by the two possible representations of the Green's function solution to the Schrödinger equation when the vector potential is present. It should also be noted that, in the cases studied in this paper, the vector potential representation ambiguity does not occur. In closing, we showed that the Dirac equation, with minimal coupling, has an alternative relationship with the square-root equation that is much closer than the conventional one.

There are many issues that we have not discussed in this paper. Many writers have used the square-root operator to develop constituent quark models. These models are very accurate in the description of a large part of meson and baryon properties (see Brau[27] and references therein). The work of Sucher[28] suggests that, with minimal coupling, the square-root operator may not be Lorentz invariant, while Smith[29] suggests that the equation has very limited gauge properties. These are complicated problems that require additional study and analysis.

If the square-root operator does represent the inside of a particle, as we suggest, then the question of Lorentz invariance may be mute. First, recall that the special theory does not apply to non-point particles (extended objects). Furthermore, a number of writers have suggested the possible breakdown of Lorentz invariance inside extended (nonlocal) objects. Indeed, sometime ago, Santilli[30-31] called for detailed experimental study to determine if the special theory was still valid inside a hadron and/or hot hyper-dense matter such as a star. Coleman and Glashow[33] have identified possible terms within the perturbative framework of the standard model which would allow small departures from Lorentz invariance. Their interest is in identifying phenomena which could be relevant to both cosmic and neutrino physics. At the cosmic level their hope is to undo the GZK cutoff for high-energy cosmic rays; while at the neutrino level the hope is to identify novel types of neutrino oscillations. The varying speed of light theory (VSL) of Moffat[34] provides an elegant solution to a number of cosmological problems: the horizon, flatness, and lambda problems of big-bang cosmology (see also Magueijo[35]).

Appendix: Semigroups and Fractional Powers of Operators

This Appendix provides a brief survey of the theory of strongly continuous semigroups of operators, which is used to explain the general theory of fractional powers of operators. The definitions and basic results are recorded here for reference so as to make the paper self-contained. Hille and Phillips[36], and Yosida[37] are the general references on semigroups (see also Goldstein[38], Engel and Nagel[39] and Pazy[40]). Butzer and Berens[41] has a very nice (short) introduction to operator semigroups. Tanabe[42] has a good section on fractional powers, but one should also consult Yosida[37].

Definition A.1 Let $\mathbf{T}(t),\ t \geq 0$, be a family of bounded linear operators on a Banach space $\mathbf{B}$. This family is called a strongly continuous semigroup of operators (or a C_0-semigroup) if the following conditions are satisfied:

1. $\mathbf{T}(t+s) = \mathbf{T}(t)\mathbf{T}(s) = \mathbf{T}(s)\mathbf{T}(t),\ \forall t,\ s \geq 0,\ \mathbf{T}(0) = \mathbf{I}$,
2. $\lim_{t\to s} \mathbf{T}(t)\varphi = \mathbf{T}(s)\varphi,\ \forall \varphi \in \mathbf{B}$.

If the family $\mathbf{T}(t)$ is defined for $t \in (-\infty, \infty)$, then it is called a C_0-group and $\mathbf{T}(-t) = \mathbf{T}^{-1}(t)$. By further restriction, we obtain the (well-known) definition of a unitary group.

Theorem A.2 Let $\mathbf{T}(t),\ t \geq 0$, be a C_0-semigroup and let $D = \{\varphi \mid \lim_{h\to 0} h^{-1}[\mathbf{T}(h) - \mathbf{I}]\varphi \text{ exists}\}$. Define $A\varphi = im_{h\to 0} h^{-1}[\mathbf{T}(h) - \mathbf{I}]\varphi$ for $\varphi \in D$; then:

1. D is dense in $\mathbf{B}$,
2. $\varphi \in D \Rightarrow \mathbf{T}(t)\varphi \in D,\ t \geq 0$,
3. $d[\mathbf{T}(t)\varphi]/dt = A\mathbf{T}(t)\varphi = \mathbf{T}(t)A\varphi, \quad \forall \varphi \in D.$ (A1)

Proof: (see Tanabe[42], page 51).

Thus, it follows that $\psi(\mathbf{x}, t) = \mathbf{T}(t)\varphi(\mathbf{x})$ solves the initial value problem:

$$d\psi(\mathbf{x},t)/dt = A\psi(\mathbf{x},t),\ \psi(\mathbf{x},0) = \varphi(\mathbf{x}). \tag{A2}$$

The operator A is called the generator of $\mathbf{T}(t)$, and we can write $\mathbf{T}(t)\varphi(\mathbf{x}) = \exp\{tA\}\varphi(\mathbf{x})$.

Theorem A.3 The generator A of the semigroup $\{\mathbf{T}(t),\ t \geq 0\}$ is a closed linear operator. If $\left\|\mathbf{T}(t)\right\| \leq M\exp\{\beta t\}$, for fixed constants M and β, then the half-plane $\{\lambda \mid \mathrm{Re}(\lambda) > \beta\}$ is contained in the resolvent set $\rho(A)$, and for each such λ we have: (Tanabe[42], page 55)

$$\left(\lambda \mathbf{I} - A\right)^{-1}\varphi = \int_0^\infty e^{-\lambda t}\mathbf{T}(t)\varphi dt = \mathbf{R}(\lambda, A)\varphi. \tag{A3}$$

$\mathbf{R}(\lambda, A)$ is called the resolvent operator of A and

$$\left\|\mathbf{R}(\lambda, A)\right\| \le M\left[\operatorname{Re}(\lambda) - \beta\right]^{-1}. \tag{A4}$$

Definition A.4 Suppose that the operator A generates a C_0-semigroup on $\mathbf{B}$, and there exists a constant M such that: (see Engel and Nagel[39], page 101, Theorem 4.6)

$$\left\|\mathbf{R}(r + is, A)\right\| \le \frac{M}{|s|}, \forall r > 0 \text{ and } 0 \ne s \in \mathbf{R}^1. \tag{A5}$$

Let Σ represent the above region in the complex plane, then the family $\{\mathbf{T}(z),\ z \in \Sigma\}$ is a called a holomorphic C_0-semigroup on $\mathbf{B}$. (See Engel and Nagel[39] for details.)

Introduce the function $f_{t,\alpha}(\lambda)$ defined by: (Yosida[37], page 259)

$$\begin{aligned} f_{t,\alpha}(\lambda) &= \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \exp\{z\lambda - tz^\alpha\}dz,\ \lambda \ge 0, \\ f_{t,\alpha}(\lambda) &= 0,\ \lambda < 0, \end{aligned} \tag{A6}$$

where, $t > 0, 0 < \alpha < 1$ and $\sigma > 0$, and the branch of z^α is taken so that $\operatorname{Re}(z^\alpha) > 0$ when $\operatorname{Re}(z) > 0$. The branch is a single-valued function in the complex plane cut along the negative real axis. The convergence of the integral (A6) is insured by the factor $\exp\{-tz^\alpha\}$. Define $\mathbf{T}_\alpha(t)$ by $\mathbf{T}_\alpha(0)\varphi = \varphi$, and for $t > 0$,

$$\mathbf{T}_\alpha(t)\varphi = \int_0^\infty f_{t,\alpha}(s)\mathbf{T}(s)\varphi ds, \tag{A7}$$

where $\{\mathbf{T}(t),\ t \ge 0\}$ is a C_0-semigroup of operators on $\mathbf{B}$.

Theorem A.5 Suppose that the operator A generates a C_0-semigroup $\{\mathbf{T}(t),\ t \ge 0\}$ on $\mathbf{B}$. Then:
1. The family $\{\mathbf{T}_\alpha(t),\ t \ge 0\}$, is a holomorphic C_0-semigroup on $\mathbf{B}$,

2. The operator A_α, the generator of $\{\mathbf{T}_\alpha(t),\ t \ge 0\}$, is defined by: $A_\alpha\varphi = -(-A)^\alpha\varphi$, and

$$A_\alpha\varphi = \frac{\sin\alpha\pi}{\pi}\int_0^\infty \lambda^{\alpha-1}R(\lambda, A)\left[-A\varphi\right]d\lambda \tag{A8}$$

Proof: (see Yosida[37], page 260).

For our work, we are only interested in the case when $\alpha = 1/2$. Let us deform the path of integration in equation (A6) into a union of two paths, $re^{-i\theta}$, when $-r \in (-\infty, 0)$ and $re^{i\theta}$, when $r \in (0, \infty)$, where $\pi/2 \le \theta \le \pi$. In particular, we need $\theta = \pi$. This case leads to:

$$f_{t,1/2}(s) = \frac{1}{\pi}\int_0^\infty \exp\{-sr\}\sin\{tr^{1/2}\}dr. \tag{A9}$$

Using a table of Laplace transforms, we have

$$f_{t,1/2}(s) = \frac{ts^{-3/2}}{\sqrt{4\pi}}\exp\{-\frac{t^2}{4s}\}. \tag{A10}$$

References

1. E. Schrödinger, *Ann. Physik* **81** (1926) 109.
2. W. Gordon, *Z. Physik* **40** (1926) 117.
3. O. Klein, *Z. Physik* **37** (1926) 895.
4. V. A. Fock, *Z. Physik* **38** (1926) 242.
5. J. Dudor, *Ann. Physik* **81** (1926) 632.
6. Th. De Donder and H. von Dungen, *Compt. Rend.* **183** (1926) 22.
7. P. A. M. Dirac, *Proc. Roy. Soc (London)* **A117** (1928) 610, **A118** (1928) 351.
8. L. L. Foldy, *Quantum Theory of Radiation III and High Energy Physics*, D. R. Bates Ed. (Academic Press, New York, 1962).
9. L. L. Foldy, *Phys. Rev.* **102** (1956) 568.
10. L. S. Schulman, *Techniques and Applications of Path Integration,* (John Wiley & Sons, New York, 1981).
11. T. L. Gill, FermiLab-Pub-82/60-THY.
12. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic Press, New York, 1980).
13. H. Yukawa, *Proc. Phys. Math. Soc. Japan* **17** (1935) 48.
14. C. D. Anderson and S. H. Neddermeyer, *Phys. Rev.* **50** (1936) 263.
15. C. M. Lattes, H. Muirhead, G. P. S. Occhialini and C. F. Powell, *Nature* **159** (1947) 694.
16. T. L. Gill and W. W. Zachary, *J. Math. Phys*., **28 (**1987) 1459.
17. T. L. Gill and W. W. Zachary, *J. Math. Phys*., **43** (2002) 69.

18. M. H. L. Pryce, *Proc. Roy. Soc (London)* **A195** (1948) 400.
19. T. L. Gill, W. W. Zachary and J. Lindesay *Found. Phys*. **31** (2001) 1299.
20. K. M. Case, *Phys. Rev.* **100** (1955) 1513.
21. L. L. Foldy and S. A. Wouthuysen, *Phys. Rev.* **78** (1950) 29.
22. L. I. Schiff, *Space Quantum Mechanics,* (McGraw-Hill, New York, 1955).
23. E. V. Eriksen, *Phys. Rev.* **111** (1958) 1011.
24. E. deVries, *Fortsch. d. Physik* **18** (1970) 149.
25. R. M. Santilli, *Hadronic J.* **17** (1994) 257.
26. R. M. Santilli, *Elements of Hadronic Mechanics, I: Mathematical Foundations,* (Ukraine Academy of Science, Kiev, 1993).
27. F. Brau, *Phys. Rev. D* **62** (2000) 014005.
28. J. Sucher, *J. Math. Phys*. **4 (**1963) 17.
29. J. R. Smith, U-Cal-D-PHY/IIRPA 93-41.
30. R. M. Santilli, *Isotopic Generalization of Galilei's and Einstein's Relativities, vol.I & II,* 2nd ed. (Ukraine Academy of Science, Kiev, 1994).
31. R. M. Santilli, *Elements of Hadronic Mechanics, II: Theoretical Foundations,* (Ukraine Academy of Science, Kiev, 1993).
32. S. Coleman and S. L. Glashow, *Phys. Lett.* ***B405*** (1997) 249.
33. S. Coleman and S. L. Glashow, hep-ph/9812418-V3 (1999).
34. J. Moffat, *Found. Phys.* **23** (1993) 411.
35. J. Magueijo, *Phys. Rev. D* **62** (2000) 103521.
36. E. Hille and R. S. Phillips *Functional Analysis and Semigroups,* Amer. Math. Soc. Coll. Pub. Vol. **3**, Amer. Math. Soc., 1957.
37. K. Yosida, *Functional Analysis,* 2nd ed., (Springer-Verlag, New York, 1968).
38. J. A. Goldstein, *Semigroups of Operators and Applications,* (Oxford University Press, 1985).
39. K. –J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations,* (Grad. Texts in Math., vol. **194**, Springer-Verlag, New York, 2000).
40. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* (Appl. Math. Sci., vol. **44**, Springer-Verlag, New York, 1983).

41. P. L. Butzer and H Berens, *Semi-Groups of Operators and Approximation,* (Springer-Verlag, Band **145** New York, 1967).

42. H. Tanabe, *Equations of Evolution,* (Pitman, London, 1979).