The Nelson Model with Less Than Two Photons

A. Galtbayar[*], A. Jensen[†‡], and K. Yajima[§]

Abstract

We study the spectral and scattering theory of the Nelson model for an atom interacting with a photon field in the subspace with less than two photons. For the free electron-photon system, the spectral property of the reduced Hamiltonian in the center of mass coordinates and the large time dynamics are determined. If the electron is under the influence of the nucleus via spatially decaying potentials, we locate the essential spectrum, prove the absence of singular continuous spectrum and the existence of the ground state, and construct wave operators giving the asymptotic dynamics.

[*]University Street 3, School of Mathematics and Computer Science, National University of Mongolia, P.O.Box 46/145, Ulaanbaatar, Mongolia

[†]Department of Mathematical Sciences, Aalborg University, Fredrik Bajers Vej 7G, DK-9220 Aalborg Ø, Denmark. E-mail: matarne@math.auc.dk

[‡]MaPhySto, Centre for Mathematical Physics and Stochastics, funded by the Danish National Research Foundation

[§]Graduate School of Mathematical Sciences University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan. E-mail: yajima@ms.u-tokyo.ac.jp

1 Introduction

In this paper we study the spectral and scattering theory for the Hamiltonian

$$H_{\mathrm{Nelson}} = h \otimes I + I \otimes \int_{\mathbf{R}^3} |k| a^\dagger(k) a(k) dk + \Phi(x)$$

describing the electron coupled to a (scalar) radiation field in the Nelson model ([15]), a simplified model of nonrelativistic quantum electrodynamics. The Hamiltonian acts on the state space defined by $\mathcal{H}_{\mathrm{Nelson}} = L^2(\mathbf{R}^3_x) \otimes \mathcal{F}$, where

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \otimes_s^n L^2(\mathbf{R}^3_k)$$

is the boson Fock space, $\otimes_s^n$ being the n-fold symmetric tensor product;

$$h = -\tfrac{1}{2}\Delta_x + V(x), \quad \text{in} \quad L^2(\mathbf{R}^3_x),$$

is the electron Hamiltonian, where V is the decaying real potential describing the interaction between the electron and the nucleus; $a(k)$ and $a^\dagger(k)$ are, respectively, the annihilation and the creation operator;

$$\int_{\mathbf{R}^3} |k| a^\dagger(k) a(k) dk$$

is the photon energy operator; and the interaction between the field and the electron is given by

$$\Phi(x) = \mu \int_{\mathbf{R}^3} \frac{\chi(k)}{\sqrt{|k|}} \left\{ e^{-ikx} a^\dagger(k) + e^{ikx} a(k) \right\} dk,$$

where $\mu > 0$ is the coupling constant, and $\chi(k)$ is the ultraviolet cut-off function, on which we impose the following assumption, using the standard notation $\langle k \rangle = (1 + k^2)^{1/2}$.

Assumption 1.1. *Assume that the function $\chi(k)$ is $O(3)$-invariant, strictly positive, smooth, and monotonically decreasing as $|k| \to \infty$. Moreover, $|\chi(k)| \le C\langle k \rangle^{-N}$ for a sufficiently large N.*

In this paper we study the restriction of H_{Nelson} to the subspace with less than two photons. Let P denote the projection onto the subspace $\mathcal{H}$ of $\mathcal{H}_{\mathrm{Nelson}}$ given by

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \quad \mathcal{H}_0 = L^2(\mathbf{R}^3_x), \quad \mathcal{H}_1 = L^2(\mathbf{R}^3_x) \otimes L^2(\mathbf{R}^3_k),$$

which consists of states with less than two photons. Then we consider the Hamiltonian $H = PH_{\mathrm{Nelson}}P$ on this space. With respect to the direct sum decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, H has the following matrix representation

$$H = \begin{pmatrix} -\frac{1}{2}\Delta + V & \mu\langle g| \\ \mu|g\rangle & -\frac{1}{2}\Delta + V + |k| \end{pmatrix}.$$

Here we have defined the operators $|g\rangle\colon \mathcal{H}_0 \to \mathcal{H}_1$ and $\langle g|\colon \mathcal{H}_1 \to \mathcal{H}_0$ by

$$(|g\rangle u_0)(x,k) = g(x,k)u_0(x), \qquad (\langle g|u_1)(x) = \int_{\mathbf{R}^3} \overline{g(x,k)}u_1(x,k)dk,$$

where the function $g(x,k)$ is given by

$$g(x,k) = \frac{\chi(k)e^{-ixk}}{\sqrt{|k|}}. \tag{1.1}$$

We write

$$g_0(k) = |g(x,k)| = \frac{\chi(k)}{\sqrt{|k|}}. \tag{1.2}$$

It is obvious that $|g\rangle$ is bounded from $\mathcal{H}_0$ to $\mathcal{H}_1$, and that $\langle g|$ is its adjoint. We assume that V is $-\frac{1}{2}\Delta$-bounded with relative bound less than one, so that H is a selfadjoint operator with the domain

$$D(H) = H^2(\mathbf{R}^3) \oplus \left(H^2(\mathbf{R}^3) \otimes L^2(\mathbf{R}^3) \cap L^2(\mathbf{R}^3) \otimes L^2_1(\mathbf{R}^3)\right).$$

Here $L^2_1(\mathbf{R}^3)$ denotes the usual weighted L^2-space, given by

$$L^2_1(\mathbf{R}^3) = L^2(\mathbf{R}^3, \langle k\rangle^2 dk),$$

and $H^2(\mathbf{R}^3)$ is the Sobolev space of order 2. Our goal is to describe the dynamics of this model. In what follows $\hat{u}$ is the Fourier transform of u with respect to the x variables, $D_x = -i\partial/\partial x$ and D_y are the gradients with respect to x and y, respectively. Here y is the variable dual to k.

We denote by H_0 the operator H with $V \equiv 0$, the Hamiltonian for the free electron-photon system. H_0 is translation invariant, and it commutes with the total momentum $D_x \oplus (D_x + k)$. Thus, if we introduce the Hilbert space $\mathcal{K} = \mathbf{C} \oplus L^2(\mathbf{R}^3)$ and define the unitary operator

$$U\colon \mathcal{H} \ni \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{u}_0(p) \\ \tilde{u}_1(p,k) \end{pmatrix} = \begin{pmatrix} \hat{u}_0(p) \\ \hat{u}_1(p-k,k) \end{pmatrix} \in L^2(\mathbf{R}^3_p; \mathcal{K}), \tag{1.3}$$

then, with respect to the decomposition $L^2(\mathbf{R}^3_p;\mathcal{K}) = \int^{\oplus}_{\mathbf{R}^3} \mathcal{K}dp$, we have

$$UH_0U^* = \int^{\oplus}_{\mathbf{R}^3} H_0(p)dp, \quad H_0(p) = \begin{pmatrix} \frac{1}{2}p^2 & \mu\langle g_0| \\ \mu|g_0\rangle & \frac{1}{2}(p-k)^2 + |k| \end{pmatrix}, \tag{1.4}$$

where $|g_0\rangle \colon \mathbf{C} \ni c \mapsto cg_0(k) \in L^2(\mathbf{R}^3)$ and $\langle g_0|$ is its adjoint. Our first result is the following theorem on the spectrum of $H_0(p)$. We define for $p \in \mathbf{R}^3$:

$$\lambda_c(p) = \min_{k \in \mathbf{R}^3} \{\tfrac{1}{2}(p-k)^2 + |k|\} = \begin{cases} \frac{1}{2}|p|^2 & \text{for } 0 \le |p| \le 1, \\ |p| - \frac{1}{2} & \text{for } 1 < |p|, \end{cases}$$

and, for (p,λ) in the domain $\Gamma^- = \{(p,\lambda) \colon \lambda < \lambda_c(p)\}$, define

$$F(p,\lambda) = \tfrac{1}{2}p^2 - \lambda - \int \frac{\mu^2 g_0(k)^2 dk}{\frac{1}{2}(p-k)^2 + |k| - \lambda}. \tag{1.5}$$

It will be shown in Section 2 that

- The function $F(p,\lambda)$ is real analytic, and the derivative with respect to λ is negative: $F_\lambda(p,\lambda) < 0$.
- There exists $\rho_c > 1$ such that the equation $F(p,\lambda) = 0$ for λ has a unique solution $\lambda_\circ(p)$, when $|p| \le \rho_c$, and no solution, when $|p| > \rho_c$.
- The function $\lambda_\circ(p)$ is $O(3)$-invariant, real analytic for $|p| < \rho_c$, $\lambda_\circ(0) < 0$, and it is strictly increasing with respect to $\rho = |p|$.

Theorem 1.2. *The reduced operator $H_0(p)$ has the following properties:*
(1) *When $|p| < \rho_c$, the spectrum $\sigma(H_0(p))$ of $H_0(p)$ consists of a simple eigenvalue $\lambda_\circ(p)$ and the absolutely continuous part $[\lambda_c(p), \infty)$. The normalized eigenfunction associated with the eigenvalue $\lambda_\circ(p)$ can be given by*

$$\mathbf{e}_p(k) = \frac{1}{\sqrt{-F_\lambda(p,\lambda_\circ(p))}} \begin{pmatrix} 1 \\ \dfrac{-\mu g_0(k)}{\frac{1}{2}(p-k)^2 + |k| - \lambda_\circ(p)} \end{pmatrix}. \tag{1.6}$$

(2) *When $|p| \ge \rho_c$, $\sigma(H_0(p)) = [\lambda_c(p), \infty)$ and is absolutely continuous.*

It follows from Theorem 1.2 that the spectrum $\sigma(H_0)$ of H_0 is given by

$$\sigma(H_0) = [\Sigma, \infty), \quad \Sigma = \lambda_\circ(0), \tag{1.7}$$

and that it is absolutely continuous.

We write $B(r)$ for the open ball $\{p \colon |p| < r\}$ and define

$$\mathcal{H}_{\text{one}} = \left\{ \frac{1}{(2\pi)^{3/2}} \int e^{ixp} h(p) \mathbf{e}_p(k) dp \colon h \in L^2(B_{\rho_c}) \right\} \subset \mathcal{H}$$

with the obvious Hilbert space structure. The space $\mathcal{H}_{\rm one}$ corresponds to the space of so called one-particle states ([6]). The operator $H_0(p)$ is a rank two perturbation of

$$\begin{pmatrix} \frac{1}{2}p^2 & 0 \\ 0 & \frac{1}{2}(p-k)^2+|k| \end{pmatrix},$$

and the Kato-Birman theorem and Theorem 1.2 yield the following theorem on the asymptotic behavior of e^{-itH_0}.

Theorem 1.3. *For any* $\mathbf{f} \in \mathcal{H}$ *there uniquely exist* $\mathbf{f}_1 = \begin{pmatrix} f_{1,0} \\ f_{1,1} \end{pmatrix} \in \mathcal{H}_{\rm one}$ *and* $f_{2,1,\pm} \in \mathcal{H}_1$ *such that, as* $t \to \pm\infty$,

$$\left\| e^{-itH_0}\mathbf{f} - \begin{pmatrix} e^{-it\lambda_\circ(D_x)}f_{1,0} \\ e^{-ikx}e^{-it\lambda_\circ(D_x)}f_{1,1} \end{pmatrix} - \begin{pmatrix} 0 \\ e^{it\Delta/2-it|k|}f_{2,1,\pm} \end{pmatrix} \right\| \to 0,$$

and the map $\mathbf{f} \mapsto (\mathbf{f}_1, f_{2,1,\pm})$ *is unitary from* $\mathcal{H}$ *onto* $\mathcal{H}_{\rm one} \oplus L^2(\mathbf{R}^6)$.

This result shows, in particular, that an electron with large momentum $|p| > \rho_c$ in the vacuum state does not survive. One might associate this phenomenon to Cherenkov radiation, in the sense that the electron of high speed always carries one photon. However, it is not clear how relevant this description is. Usually Cherenkov radiation is described differently, in a classical electrodynamic context, see for example [12].

When $V \neq 0$, we prove the following results. The following assumption on V is too strong for some of our results, however, we always assume it in what follows without trying to optimize the conditions on V.

Assumption 1.4. *The potential* V *is real valued,* C^2 *outside the origin, and* $V(x)$, $x \cdot \nabla V(x)$ *and* $(x\cdot\nabla)^2V(x)$ *are* $-\Delta$*-compact and converge to* 0 *as* $|x| \to \infty$.

Under this assumption the spectrum of $h = -\frac{1}{2}\Delta + V$ has an absolutely continuous part $[0,\infty)$. If h has no negative eigenvalues, we let $E_0 = 0$. Otherwise, the eigenvalues are denoted $E_0 < E_1 \le \ldots < 0$. They are discrete in $(-\infty, 0)$. Zero may be an eigenvalue, but there are no positive eigenvalues under Assumption 1.4.

Definition 1.5. (1) *The set* $\Theta(H) = \{E_0, E_1, \ldots\} \cup \{\Sigma\} \cup \{0\} \cup \{\lambda_\circ(\rho_c)\}$ *is called the threshold set for* H.
(2) *We say that* V *is short range, if, in addition to* Assumption 1.4, *it satisfies the following condition:* $V \in L^2_{\rm loc}(\mathbf{R}^3)$ *and for any* $0 < c_1 < c_2 < \infty$

$$\int_1^\infty \left(\int_{c_1 \le |x| \le c_2} |V(tx)|^2 dx \right)^{1/2} dt < \infty.$$

Theorem 1.6. *Let V satisfy* Assumption 1.4 *and* $\Sigma_{\mathrm{ess}} = \min\{E_0, \Sigma\}$. *Then:*
(1) *The spectrum $\sigma(H)$ of H consists of the absolute continuous part* $[\Sigma_{\mathrm{ess}}, \infty)$ *and the eigenvalues, which may possibly accumulates at $\Theta(H)$.*
(2) *Assume h has at least one strictly negative eigenvalue, i.e. $E_0 < 0$. Then the bottom of the spectrum* $\inf \sigma(H)$ *is an isolated eigenvalue of H and* $\inf \sigma(H) \leq \Sigma + E_0$ $(< \Sigma_{\mathrm{ess}})$.

For the possible asymptotic profiles of the wave packet $e^{-itH}\mathbf{f}$ as $t \to \pm\infty$, we prove the existence of the following two wave operators.

Theorem 1.7. (1) *Assume that V is short range. Then, for $\mathbf{f} \in \mathcal{H}$, the following limits exist:*

$$W_{0\pm}\mathbf{f} = \lim_{t\to\pm\infty} e^{itH} e^{-itH_0}\mathbf{f}. \tag{1.8}$$

(2) *Let $\phi \in L^2(\mathbf{R}^3)$ be an eigenfunction of h with eigenvalue E: $h\phi = E\phi$. Suppose $|\phi(x)| \leq C\langle x\rangle^{-\beta}$ for some $C > 0$ and $\beta > 2$. Then, for $f \in L^2(\mathbf{R}^3_k)$ the following limits exist:*

$$W_{\pm}^{E,\phi} f = \lim_{t\to\pm\infty} e^{itH} \begin{pmatrix} 0 \\ e^{-itE-it|k|}\phi(x)f(k) \end{pmatrix}. \tag{1.9}$$

We remark that eigenfunctions actually decay exponentially in many cases, but eigenfunctions at a threshold may only decay polynomially.

There is a renewed interest in the Nelson model H_{Nelson} recently, which was first studied in detail in [15], and a large number of papers have appeared (we refer to [5] and [6] and references therein for earlier works and a physical account of the model). In [5] and [6], the spectral and scattering theory of H_{Nelson} ($V = 0$ in [6]) has been studied in detail, when the infrared cutoff is imposed on the interaction, in addition to the ultraviolet cutoff. In particular, the essential spectrum is located, the existence of the ground state is proved, and the asymptotic completeness (AC) of scattering is established in the range of energy below the ionization energy ([5]), and in the range of energy, where the propagation speed of the dressed electron is smaller than 1 ([6]), under the additional condition that the coupling constant μ is sufficiently small. In many papers the atom is modelled by either $-\frac{1}{2}\Delta + V$ with compact resolvent (confining potential), or the atom is replaced by a finite level system (spin-boson Hamiltonian). For the spin-boson Hamiltonian, the spectral and scattering theory has been studied in detail in subspaces with less than three photons in [13], less than four in [19], and with an arbitrary finite number of photons in [8]. For models with confining potentials, after the works for the exactly solvable model with harmonic potentials ([1]) and its perturbation

([18]), [3] extensively studied the model with general confining potentials and proved in particular (AC), when photons are assumed to be massive (see the recent paper by Gérard [9] for the massless case).

In this paper we deal with the model with massless photons, without infrared cutoff, with arbitrary large coupling constant, and with a decaying potential, which allows ionization of the electron. However, the number of photons is restricted to less than two, and the problem with an infinite number of soft photons is avoided. Nonethelss, the model retains the difficulties arising from the singularity of the photon dispersion relation $|k|$, the different dispersion relations of the electron and the photon, and the photon-electron interaction, which is foreign to the mind accustomed to the classical two body interaction. We think, therefore, that the complete understanding of this very simple model is important (and unavoidable) for understanding models of quantum electrodynamics, which are intrinsically more difficult.

Finally, let us outline the contents of this paper. In §2 we study in detail the function $F(p, z)$ and show, in particular, the properties stated before Theorem 1.2. In §3 we prove Theorem 1.2. The existence of the isolated eigenvalue of $H_0(p)$ will be shown by examining its resolvent and identifying the eigenvalues with zeros of $F(p, \lambda)$; the absolute continuity will be shown by establishing the Mourre estimate for $H_0(p)$ and proving the absence of eigenvalues by elementary calculus. In §4 we then prove Theorem 1.3 via the Birman-Kato theorem and study the propagator e^{-itH_0} in the configuration space. In §5 we prove Theorem 1.6. We show $\sigma_{\rm ess}(H) = [\Sigma_{\rm ess}, \infty)$ by adapting the "geometric" proof of the HVZ theorem, the corresponding result for N-body Schrödinger operators; we prove that the singular continuous spectrum is absent from H, and that the eigenvalues of H are discrete in $\mathbf{R} \setminus \Theta(H)$, by applying the Mourre estimate for H with the conjugate operator $A = A_x + A_y$, A_x, A_y being the generator of the dilation. This Mourre estimate, however, is not suitable for proving the so called minimal velocity estimate, an indispensable ingredient for proving (AC) of the wave operators by the now standard methods (cf. e.g. [6]). This is because our $A_y \neq i[|D_y|, y^2]$, and A cannot be directly related to the dynamical variables associated with H. For proving the existence of the ground state it suffices to show $\inf \sigma(H) \leq \Sigma + E_0$, since $\Sigma_{\rm ess} = \min\{\Sigma, E_0\}$, $\Sigma < 0$, and furthermore $E_0 < 0$ by assumption. We prove this by borrowing the argument of [10]. Finally in §6, we prove the existence of the wave operators (1.8) and (1.9). The proof of completeness of the wave operators is still missing, mainly because of the aforementioned lack of the minimal velocity estimate.

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2 Properties of the function $F(p, \lambda)$

In this section we study the function $F(p, z)$, defined for $(p, z) \in \mathbf{R}^3 \times (\mathbf{C} \setminus [0, \infty))$ by (1.5):

$$F(p, z) = \tfrac{1}{2}p^2 - z - \int \frac{\mu^2 g_0(k)^2 dk}{\frac{1}{2}(p-k)^2 + |k| - z}. \tag{2.1}$$

The following Lemma is obvious.

Lemma 2.1. (1) *For each* $z \in \mathbf{C} \setminus [0, \infty)$*,* $F(p, z)$ *is* $O(3)$*-invariant with respect to* $p \in \mathbf{R}^3$*.*
(2) $\mp \operatorname{Im} F(p, z) > 0$*, when* $\pm \operatorname{Im} z > 0$*.*
(3) *Let* $K \subset \mathbf{C} \setminus [0, \infty)$ *be a compact set. Then* $|F(p, z)| \to \infty$ *as* $|p| \to \infty$*, uniformly with respect to* $z \in K$*.*

We will write $F(\rho, z) = F(p, z)$, $\rho = |p|$, and $F_\rho(\rho, z)$ will denote the derivative of $F(\rho, z)$ with respect to ρ. We will use the notation $F(p, z)$ and $F(\rho, z)$ interchangeably. Let $G(p, k) = \frac{1}{2}(p-k)^2 + |k|$. Then elementary computations show that for each fixed p the function $k \to G(p, k)$ has a global minimum, which we denote by $\lambda_c(p)$. Due to the invariance, it is only a function of ρ. Thus we will also denote it by $\lambda_c(\rho)$. We have

$$\lambda_c(\rho) = \begin{cases} \frac{1}{2}\rho^2 & \text{for } 0 \le \rho \le 1, \\ \rho - \frac{1}{2} & \text{for } 1 < \rho. \end{cases} \tag{2.2}$$

Note that this function is only once continuously differentiable. We use the notation Γ^- for the domain $\{(p, \lambda) \in \mathbf{R}^3 \times \mathbf{R} \colon \lambda < \lambda_c(|p|)\}$ of $\mathbf{R}^4$ also for denoting the corresponding two dimensional domain

$$\Gamma^- = \{(\rho, \lambda) \in \mathbf{R}^2 \colon \rho \ge 0, \lambda < \lambda_c(\rho)\}.$$

It is obvious that $F(\rho, \lambda)$ is real analytic on Γ^- with respect to (ρ, λ). Later we will also need the domain

$$\Gamma^+ = \{(\rho, \lambda) \in \mathbf{R}^2 \colon \rho \ge 0, \lambda > \lambda_c(\rho)\}.$$

Lemma 2.2. *The derivatives satisfy $F_\lambda(\rho,\lambda) < 0$ and $F_\rho(\rho,\lambda) > 0$ in Γ^-, and $F(\rho,\lambda)$ is strictly decreasing with respect to λ and is strictly increasing with respect to ρ.*

Proof. We set $\mu = 1$ in the proof. Direct computation shows

$$\frac{\partial F}{\partial \lambda} = -1 - \int \frac{g_0(k)^2 dk}{(\frac{1}{2}(p-k)^2 + |k| - \lambda)^2} < 0. \tag{2.3}$$

To prove $F_\rho > 0$, it suffices to show that $F_{p_1}(p,\lambda) > 0$, when $p_1 \geq 0, p_2 = p_3 = 0$, as F is $O(3)$-invariant. We compute

$$\frac{\partial F}{\partial p_1} = p_1 + \int \frac{g_0(k)^2(p_1-k_1)dk}{(\frac{1}{2}(p-k)^2 + |k| - \lambda)^2} = p_1 - \int \frac{g_0(p+k)^2 k_1 dk}{(\frac{1}{2}k^2 + |p+k| - \lambda)^2}. \tag{2.4}$$

The last integral (including the sign in front) can be written in the form

$$\int_{\mathbf{R}^2} \left\{ \int_0^\infty \left(\frac{g_0(p_1-k_1,k')^2}{(\frac{1}{2}k^2 + |(p_1-k_1,k')| - \lambda)^2} \right.\right. \\ \left.\left. - \frac{g_0(p_1+k_1,k')^2}{(\frac{1}{2}k^2 + |(p_1+k_1,k')| - \lambda)^2} \right) k_1 dk_1 \right\} dk',$$

where $k' = (k_2,k_3) \in \mathbf{R}^2$ and, for $p_1, k_1 > 0$,

$$g_0(p_1-k_1,k')^2 > g_0(p_1+k_1,k')^2,$$
$$\sqrt{(p_1-k_1)^2 + (k')^2} \leq \sqrt{(p_1+k_1)^2 + (k')^2}.$$

The first inequality follows from Assumption 1.1. Thus the integral is positive, and the lemma follows. $\square$

Remark 2.3. A computation via spherical coordinates yields for $(\rho,\lambda) \in \Gamma^-$

$$F(\rho,\lambda) = \tfrac{1}{2}\rho^2 - \lambda \\ - \frac{2\pi\mu^2}{\rho}\int_0^\infty g_0(r)^2 r \log\left(1 + \frac{2\rho r}{\frac{1}{2}(r-\rho)^2 + r - \lambda}\right) dr. \tag{2.5}$$

Lemma 2.4. *There exist a constant $\rho_c > 1$ and a function $\lambda_\circ \colon [0,\rho_c] \to \mathbf{R}$ with the following properties:*
(i) *$\lambda_\circ(0) < 0$, $(\rho_c, \lambda_\circ(\rho_c)) \in \gamma \equiv \{(\rho, \lambda_c(\rho)) : \rho \geq 0\}$, and*

$$\Xi = \{(\rho, \lambda_\circ(\rho)) \colon 0 \leq \rho < \rho_c\} \subset \Gamma^-. \tag{2.6}$$

(ii) *$F(\rho, \lambda_\circ(\rho)) = 0$, $\rho \in [0, \rho_c]$.*
(iii) *$\lambda_\circ$ is real analytic for $0 < \rho < \rho_c$.*
(iv) *$\lambda_{\circ\rho}(\rho) > 0$ for $0 < \rho < \rho_c$.*
(v) *There are no other zeros of $F(\rho,\lambda)$ in Γ^-, than those given by Ξ in (2.6).*

Proof. We examine the behavior of $F(\rho,\lambda)$ on the curve γ. Taking the limit $\lambda \uparrow \lambda_c(\rho)$ in (2.5), we have (recall (2.2), and also $|p| = \rho$)

$$F(\rho,\lambda_c(\rho)) = -\frac{2\pi\mu^2}{\rho}\int_0^\infty |\chi(r)|^2 \log\left(1+\frac{4\rho}{r+2(1-\rho)}\right) dr < 0$$

for $\rho \le 1$, and it is increasing for $\rho > 1$ and diverges to ∞ as $\rho \to \infty$. Indeed, we have, using (2.2) and (2.5),

$$F(\rho,\lambda_c(\rho)) = \tfrac{1}{2}(\rho-1)^2 - \frac{2\pi\mu^2}{\rho}\int_0^\infty |\chi(r)|^2 \log\left(1+\frac{4\rho r}{(r-\rho+1)^2}\right) dr$$

for $\rho > 1$, and it is evident that $\lim_{\rho\to\infty} F(\rho,\lambda_c(\rho)) = \infty$. By a change of variable,

$$\begin{aligned} F(\rho+1,&\lambda_c(\rho+1)) \\ &= \tfrac{1}{2}\rho^2 - \frac{2\pi\mu^2}{\rho+1}\int_0^\infty |\chi(r)|^2 \log\left(1+\frac{4r(\rho+1)}{(r-\rho)^2}\right) dr \\ &= \tfrac{1}{2}\rho^2 - \frac{2\pi\mu^2\rho}{\rho+1}\int_0^\infty |\chi(\rho r)|^2 \log\left(1+\frac{4r(1+\frac{1}{\rho})}{(r-1)^2}\right) dr \\ &= \rho\Big[\tfrac{1}{2}\rho - \frac{2\pi\mu^2}{\rho+1}\int_0^\infty |\chi(\rho r)|^2 \log\left(1+\frac{4r(1+\frac{1}{\rho})}{(r-1)^2}\right) dr\Big]. \end{aligned}$$

This is manifestly increasing for $\rho > 0$. Thus, there exists a unique $\rho_c > 1$ such that $F(\rho,\lambda_c(\rho))$ changes sign from $-$ to $+$ at $\rho = \rho_c$. It follows, since $F(\rho,\lambda)$ in Γ^- is decreasing with respect to λ and $F(\rho,\lambda) \to \infty$ as $\lambda \to -\infty$ that the function $\lambda \to F(\rho,\lambda)$ has a unique zero $\lambda_\circ(\rho)$ for $0 \le \rho \le \rho_c$. It satisfies $(\rho,\lambda_\circ(\rho)) \in \Gamma^-$ for $0 \le \rho < \rho_c$ and also $\lambda_\circ(0) < 0$. By the implicit function theorem, $\lambda_\circ(\rho)$ is real analytic, and $\lambda_{\circ\rho}(\rho) > 0$ for $0 < \rho < \rho_c$.

The last statement follows from the explicit formulae above. $\square$

As above, we will also consider $\lambda_\circ$ as a function of p, through $\rho = |p|$. The Hessian is given by

$$\nabla_p^2 \lambda_\circ(p) = \lambda_{\circ\rho\rho}(\rho)\hat{p}\otimes\hat{p} + \lambda_{\circ\rho}(\rho)\frac{\mathbf{1}-(\hat{p}\otimes\hat{p})}{\rho}.$$

Here we write $\hat{p} = p/|p|$, and $\hat{p}\otimes\hat{p}$ denotes the matrix with entries $\hat{p}_j\hat{p}_k$. A straightforward computation yields

$$\det \nabla_p^2 \lambda_\circ(p) = \frac{1}{\rho^2}\lambda_{\circ\rho\rho}(\rho)(\lambda_{\circ\rho}(\rho))^2. \tag{2.7}$$

Remark 2.5. The second derivative $\lambda_{\circ\rho\rho}(\rho)$ at $\rho = 0$ is called the effective mass of the dressed electron. Due to (2.4), we have $F_\rho(0, \lambda) = 0$, and hence

$$\lambda_{\circ\rho}(0) = -\frac{F_\rho(0, \lambda_\circ(0))}{F_\lambda(0, \lambda_\circ(0))} = 0.$$

Using (2.5), one can compute $F_{\rho\rho}(\rho, \lambda)$, and then take the limit $\rho \downarrow 0$ to get the result

$$F_{\rho\rho}(0, \lambda_\circ(0)) = 1 - \frac{2\pi\mu^2}{3} \int_0^\infty \chi(r)^2 r \frac{r^2 - 6r + 6\lambda_\circ(0)}{(\frac{1}{2}r^2 + r - \lambda_\circ(0))^3}. \tag{2.8}$$

The integral in this expression can be evaluated explicitly in the case $\chi(r) \equiv 1$ for $r \geq 0$. The value is negative for all $\lambda_\circ(0) < 0$. Thus we conjecture (perhaps with an additional assumption on χ) that we always have $F_{\rho\rho}(0, \lambda_\circ(0)) > 0$ (taking the sign in front of the integral into account). For $|\mu|$ small we have this result, without additional conditions on χ.

Now using implicit differentiation and the result $F_\rho(0, \lambda_\circ(0)) = 0$, we find

$$\lambda_{\circ\rho\rho}(0) = -\frac{F_{\rho\rho}(0, \lambda_\circ(0))}{F_\lambda(0, \lambda_\circ(0))}. \tag{2.9}$$

We recall from (2.3) that $F_\lambda(0, \lambda_\circ(0)) \leq -1$. Thus we conjecture that we always have a positive effective mass (perhaps with an additional condition on χ). Let us note that $\lambda_{\circ\rho}(0) = 0$ and the monotonicity of $\lambda_\circ(\rho)$ imply $\lambda_{\circ\rho\rho}(0) \geq 0$.

3 Spectrum of $H_0(p)$ and H_0

In this section we first carry out the separation of mass in detail, and then we prove Theorem 1.2.

3.1 Separation of the center of mass

It is easy to see that the Hamiltonian of the free electron-photon system

$$H_0 = \begin{pmatrix} -\frac{1}{2}\Delta & \mu\langle g| \\ \mu|g\rangle & -\frac{1}{2}\Delta + |k| \end{pmatrix}.$$

commutes with the spatial translations

$$\tau_j(s)\colon \begin{pmatrix} u_0(x) \\ u_1(x,k) \end{pmatrix} \mapsto \begin{pmatrix} u_0(x + s\mathbf{e}_j) \\ e^{isk_j} u_1(x + s\mathbf{e}_j, k) \end{pmatrix}, \quad s \in \mathbf{R}, \quad j = 1, 2, 3.$$

Hence H_0 and the generators

$$P_j = \begin{pmatrix} -i\partial/\partial x_j & 0 \\ 0 & -i\partial/\partial x_j + k_j \end{pmatrix} \tag{3.1}$$

of $\tau_j(s)$ can simultaneously be diagonalized. Thus, if we introduce the Hilbert space $\mathcal{K} = \mathbf{C} \oplus L^2(\mathbf{R}^3)$ and the unitary operator $U\colon \mathcal{H} \to L^2(\mathbf{R}^3_p; \mathcal{K}) = \int_{\mathbf{R}^3}^{\oplus} \mathcal{K} dp$ by (1.3), then we have the direct integral decomposition $UH_0U^* = \int_{\mathbf{R}^3}^{\oplus} H_0(p)dp$ as in (1.4), where

$$H_0(p) = \begin{pmatrix} \frac{1}{2}p^2 & 0 \\ 0 & \frac{1}{2}(p-k)^2 + |k| \end{pmatrix} + \begin{pmatrix} 0 & \mu\langle g_0| \\ \mu|g_0\rangle & 0 \end{pmatrix} \equiv H_{00}(p) + T, \tag{3.2}$$

and T is a rank two perturbation of $H_{00}(p)$. $H_0(p)$ is essentially the operator known as the Friedrichs model. Thus, it is standard to compute its resolvent and, if we write

$$(H_0(p) - z)^{-1}\tilde{\mathbf{f}} = \begin{pmatrix} \tilde{u}_0(p,z), \\ \tilde{u}_1(p,k,z) \end{pmatrix}, \quad \tilde{\mathbf{f}} = \begin{pmatrix} \tilde{f}_0, \\ \tilde{f}_1(k) \end{pmatrix}, \tag{3.3}$$

we have

$$\tilde{u}_0(p,z) = \frac{1}{F(p,z)}\Big(\tilde{f}_0 - \mu \int \frac{g_0(k)\tilde{f}_1(k)dk}{\frac{1}{2}(p-k)^2 + |k| - z}\Big), \tag{3.4}$$

$$\tilde{u}_1(p,k,z) = \frac{\tilde{f}_1(k)}{\frac{1}{2}(p-k)^2 + |k| - z} - \frac{\mu g_0(k)\tilde{u}_0(p,z)}{\frac{1}{2}(p-k)^2 + |k| - z}. \tag{3.5}$$

3.2 Proof of Theorem 1.2

We prove here Theorem 1.2 on the spectrum of a reduced operator $H_0(p)$. In what follows ρ_c is the threshold momentum defined in Lemma 2.4.

As was shown in Lemmas 2.2 and 2.4, $F(p,z)$ is an analytic function of $z \in \mathbf{C} \setminus [\lambda_c(p), \infty)$, it has a simple zero at $\lambda_\circ(p)$, when $|p| < \rho_c$, and has no zero, when $|p| \geq \rho_c$. It follows from (3.3)–(3.5) that $\mathbf{C} \setminus [\lambda_c(p), \infty) \ni z \mapsto (H_0(p) - z)^{-1}$ is meromorphic with a simple pole at $\lambda_\circ(p)$, if $|p| < \rho_c$, and that it is holomorphic, if $|p| \geq \rho_c$. Hence:

1. If $|p| < \rho_c$, $H_0(p)$ has an eigenvalue $\lambda_\circ(p)$, and $(-\infty, \lambda_c(p)) \setminus \{\lambda_\circ(p)\} \subset \rho(H_0(p))$, the resolvent set of $H_0(p)$.

2. If $|p| \geq \rho_c$, $(-\infty, \lambda_c(p)) \subset \rho(H_0(p))$.

By virtue of (3.3)–(3.5), we can compute the eigenprojection E_p for $H_0(p)$ associated with the eigenvalue $\lambda_\circ(p)$ as follows:

$$E_p = -\operatorname*{s\text{-lim}}_{z\to\lambda_\circ(p)}(z-\lambda_\circ(p))(H_0(p)-z)^{-1} = \mathbf{e}_p\otimes\mathbf{e}_p.$$

Thus $\lambda_\circ(p)$ is simple, and $\mathbf{e}_p$ is a normalized eigenvector, see (1.6) and (2.3).

Due to the decomposition (3.2) it is clear that $\sigma_{\mathrm{ess}}(H_0(p)) = [\lambda_c(p),\infty)$. Using Mourre theory [14] we show in Lemma 3.1 that $[\lambda_c(p),\infty)$ is an absolutely continuous component of the spectrum, the singular continuous spectrum is empty, and eigenvalues are discrete in this set. In the following Lemma 3.3 we then show that there are no eigenvalues embedded in $[\lambda_c(p),\infty)$. This concludes the proof of the theorem.

We take $\eta\in C^\infty(\mathbf{R}^3)$ such that $\eta(k)=1$ for $|k|>1$ and $\eta(k)=0$ for $|k|<1/2$ and define a vector field $X_r(k)$ on $\mathbf{R}^3$ by $X_r(k)=\eta(k/r)\nabla_k G(p,k)$ for a small parameter $r>0$. Recall that $G(p,k)=\frac{1}{2}(p-k)^2+|k|$. We then define a one-parameter family of auxiliary operators $\mathcal{A}_r$ by

$$\mathcal{A}_r=\begin{pmatrix}0 & 0\\ 0 & A_r\end{pmatrix},\quad A_r=\frac{i}{2}\left(X_r(k)\cdot\nabla_k+\nabla_k\cdot X_r(k)\right),$$

and let $\mathcal{D}=\mathbf{C}\oplus C_0^\infty(\mathbf{R}^3)$. The vector field $X_r(k)$ is smooth, $X_r(k)=0$ when $|k|<r/2$, and its derivatives are bounded. Hence it generates a flow $k\to\Phi_r(t,k)$ of global diffeomorphisms on $\mathbf{R}^3$, such that $\Phi_r(t,k)=k$ for $|k|<r/2$, and for some $c>0$

$$e^{-c|t|}|k|\le|\Phi_r(t,k)|\le e^{c|t|}|k|,\quad e^{-c|t|}\le\|\nabla_k\Phi_r(t,k)\|\le e^{c|t|}\tag{3.6}$$

for all $k\in\mathbf{R}^3$ and $t\in\mathbf{R}$. It follows that if we define

$$J_r(t)\begin{pmatrix}c\\ u(k)\end{pmatrix}=\begin{pmatrix}c\\ \sqrt{\det(\nabla_k\Phi_r(t,k))}u(\Phi_r(t,k))\end{pmatrix},$$

then $J_r(t)$ is a strongly continuous unitary group on $\mathcal{K}$, such that $J_r(t)\mathcal{D}=\mathcal{D}$, $t\in\mathbf{R}$, and it satisfies

$$-i\frac{d}{dt}J_r(t)\begin{pmatrix}c\\ u(k)\end{pmatrix}\Big|_{t=0}=\begin{pmatrix}0\\ A_r u(k)\end{pmatrix}.$$

Thus $\mathcal{A}_r$ is essentially selfadjoint on $\mathcal{D}$, and we denote its closure again by $\mathcal{A}_r$, such that $J_r(t)=e^{it\mathcal{A}_r}$.

Lemma 3.1. *Let I be a bounded open interval such that $\overline{I}\subset(\lambda_c(p),\infty)\setminus\{p^2/2\}$. Then there exist $r>0$, such that $\mathcal{A}_r$ is a conjugate operator of $H_0(p)$ at $E\in I$ in the sense of Mourre, viz.*

(1) $\mathcal{D}$ *is a core of both* $\mathcal{A}_r$ *and* $H_0(p)$.

(2) *We have that* $e^{it\mathcal{A}_r}D(H_0(p)) \subset D(H_0(p))$, *and* $\sup_{|t|<1} \|H_0(p)e^{it\mathcal{A}_r}u\| < \infty$ *for* $u \in D(H_0(p))$.

(3) *The form* $i[H_0(p), \mathcal{A}_r]$ *on* $\mathcal{D}$ *is bounded from below and closable, and the associated selfadjoint operator* $i[H_0(p), \mathcal{A}_r]^0$ *satisfies* $D(i[H_0(p), \mathcal{A}_r]^0) \supseteq D(H_0(p))$.

(4) *The form, defined on* $D(\mathcal{A}_r) \cap D(H_0(p))$ *by* $[[H_0(p), \mathcal{A}_r]^0, \mathcal{A}_r]$, *is bounded from* $D(H_0(p))$ *to* $D(H_0(p))^*$.

(5) *There exist* $\alpha > 0$, $\delta > 0$, *and a compact operator* K, *such that*

$$P(E,\delta)i[H_0(p), \mathcal{A}_r]^0 P(E,\delta) \geq \alpha P(E,\delta) + P(E,\delta)KP(E,\delta),$$

where $P(E,\delta)$ *is the spectral projection of* $H_0(p)$ *for the interval* $(E-\delta, E+\delta)$.

Thus $(\lambda_c(p), \infty) \subseteq \sigma_{\mathrm{ac}}(H_0(p))$, $\sigma_{\mathrm{sc}}(H_0(p)) \cap (\lambda_c(p), \infty) = \emptyset$, *and the point spectrum of* $H_0(p)$ *is discrete in* $(\lambda_c(p), \infty) \setminus \{p^2/2\}$.

Proof. We first show statements (1) $\sim$ (4) hold for $\mathcal{A}_r$ for any $r > 0$. (1) is obvious. (2) is also evident because $D(H_0(p)) = \mathbf{C} \oplus L^2_2(\mathbf{R}^3)$ and the diffeomorphisms $k \to \Phi_r(t,k)$ satisfy the bound (3.6). On $\mathcal{D}$ we compute the commutator

$$i[H_{00}(p), \mathcal{A}_r] = \begin{pmatrix} 0 & 0 \\ 0 & i[G(p,k), A_r] \end{pmatrix} \equiv L(p). \tag{3.7}$$

Here we have

$$i[G(p,k), A_r] = \eta(k/r)|\nabla_k G(p,k)|^2 = \eta(k/r)|k + \hat{k} - p|^2 \geq 0, \tag{3.8}$$

and it behaves like $(|k|+1)^2$ for large k. Since $g_0(k) = \chi(k)/\sqrt{|k|}$, χ is smooth and decays rapidly at infinity and $X_r(k) = 0$ for $|k| < r/2$, it follows that $A_r g_0$ is C^∞ and rapidly decaying at infinity, such that

$$i[T, \mathcal{A}_r] = \begin{pmatrix} 0 & \mu\langle A_r g_0| \\ \mu|A_r g_0\rangle & 0 \end{pmatrix}$$

has an extension to a bounded rank two operator. Thus $i[H_0(p), \mathcal{A}_r]$ is bounded from below, closable, and the associated selfadjoint operator has the same domain as $H_0(p)$. This proves (3). (4) holds due to (3.8) and the arguments used in establishing (3).

To prove (5), fix $p \in \mathbf{R}^3$ and $E \in (\lambda_c(p), \infty) \setminus \{p^2/2\}$. (Recall $\lambda_c(p)$ is the unique minimum of $k \to G(p,k)$ and $\lambda_c = p^2/2$ is attained by $k = 0$ if $|p| \leq 1$, and $\lambda_c(p) = |p| - 1/2$ is attained by the unique solution $k = k_c(p) = (|p|-1)\hat{p}$ of $\nabla_k G(p,k) = k + \hat{k} - p = 0$, if $|p| \geq 1$, see (2.2).) Then $E \neq G(p,0)$ and $E \neq G(p, k_c(p))$ and, by continuity, there exist $r > 0$ and $\delta > 0$, such that

$$0 < \delta < \tfrac{1}{2} \min\{|E - \tfrac{1}{2}p^2|, |E - \lambda_c(p)|\}, \tag{3.9}$$

and such that

$$|G(p,k) - E| > 2\delta \quad \text{if} \quad |k| < 2r \text{ or } |k - k_c(p)| < 2r. \tag{3.10}$$

Then $|G(p,k) - E| \leq 2\delta$ implies $|k| > 2r$ and $|k - k_c(p)| > 2r$, and hence

$$\eta(k/r)|\nabla_k G(p,k)|^2 = |\nabla_k G(p,k)|^2 \geq r^2. \tag{3.11}$$

Indeed, for $|p| \leq 1 + r$, $|k| > 2r$ implies

$$|\nabla_k G(p,k)| \geq |k + \hat{k}| - |p| \geq 1 + 2r - |p| \geq r$$

and, if $|p| \geq 1 + r$, we have $p = k_c(p) + \hat{k}_c(p)$ and

$$|\nabla_k G(p,k)| = |k + \hat{k} - k_c(p) - \hat{k}_c(p)| \geq |k - k_c(p)| > 2r.$$

Thus, if $\phi_0 \in C_0^\infty(\mathbf{R})$ is such that $\phi_0(\lambda) = 0$ for $|\lambda| > 2\delta$, we have

$$\phi_0(H_{00}(p) - E) = \begin{pmatrix} 0 & 0 \\ 0 & \phi_0(G(p,k) - E) \end{pmatrix}$$

and, by virtue of (3.11),

$$\phi_0(H_{00}(p) - E)L(p)\phi_0(H_{00}(p) - E) \geq r^2 \phi_0(H_{00}(p) - E)^2.$$

Thus, if we choose r and δ as above, statement (5) holds with $\alpha = r^2$ and this δ by virtue of the following lemma. $\square$

Lemma 3.2. *Let $\phi \in C_0^\infty(\mathbf{R})$, and let $L(p)$ be given by* (3.7). *Then the operator $L(p)\{\phi(H_0(p)) - \phi(H_{00}(p))\}$ is a compact operator.*

Proof. Let $\tilde{\phi}$ be a compactly supported almost analytic extension of ϕ. Then writing $R_0(p,z) = (H_{00}(p) - z)^{-1}$ and $R(p,z) = (H_0(p) - z)^{-1}$, we have

$$\phi(H_0(p)) - \phi(H_{00}(p)) = \frac{1}{2\pi i} \int_{\mathbf{C}} \partial_{\bar{z}} \tilde{\phi}(z) R_0(p,z) T R(p,z) dz \wedge d\bar{z}, \tag{3.12}$$

see [4, Theorem 8.1]. Here $R_0(p,z)$ commutes with $L(p)$ and $L(p)T$ is a compact operator, as T is of rank two, and $\operatorname{Ran} T \subset D(L(p))$. Since (3.12) is the norm limit of the Riemann sums, the lemma follows. $\square$

Lemma 3.3. *We have* $\sigma_{\mathrm{pp}}(H_0(p)) \cap [\lambda_c(p), \infty) = \emptyset$.

Proof. We recall from Assumption 1.1 that $g_0(k) > 0$ for all $k \in \mathbf{R}^3 \setminus \{0\}$, g_0 is smooth away from $k = 0$ and rapidly decaying at infinity. If λ is an eigenvalue, then there exists a non-zero vector $(c, f) \in \mathbf{C} \oplus L^2(\mathbf{R}^3)$, such that

$$\mu\langle g_0, f\rangle + \tfrac{1}{2}p^2 c = \lambda c, \tag{3.13}$$

$$\mu g_0(k)c + \{\tfrac{1}{2}(p-k)^2 + |k|\}f(k) = \lambda f(k). \tag{3.14}$$

We show that these equations lead to a contradiction, if $(|p|, \lambda) \in \overline{\Gamma^+}$.

We write $G(p,k) = \frac{1}{2}(p-k)^2 + |k|$ as previously. It is easy to see that

$$S_\lambda = \{k\colon G(p,k) - \lambda = 0\}$$

has Lebesgue measure zero. (This will be checked in what follows.) This result will imply $c \neq 0$, because $\{\frac{1}{2}(p-k)^2 + |k| - \lambda\}f(k) = 0$ otherwise, which implies $f(k) = 0$ almost everywhere, a contradiction.

We divide the proof into a number cases.

Case 1. Assume $|p| > 1$ and $\lambda > \lambda_c(p) = |p| - \frac{1}{2}$.

(i) Assume first $\lambda \neq \frac{1}{2}p^2$. Then $0 \notin S_\lambda$, and the gradient

$$\nabla_k G(p,k) = k - p + \hat{k} \tag{3.15}$$

does not vanish on S_λ (it vanishes only when $\lambda = \lambda_c(p)$). Therefore S_λ is a smooth hypersurface (of Lebesgue measure zero), and we have

$$f(k) = \frac{c\mu g_0(k)}{G(p,k) - \lambda} \notin L^2(\mathbf{R}^3).$$

This is a contradiction.

(ii) Consider now the subcase $\lambda = \frac{1}{2}p^2$. Then besides $k = 0$ the equation

$$G(p,k) - \lambda = \tfrac{1}{2}k^2 - p \cdot k + |k| = 0$$

has a root $k_0 \neq 0$, and around any $0 \neq k_0 \in S_\lambda$, S_λ is a smooth hypersurface, because $\nabla_k G(p, k_0) \neq 0$. Thus f cannot be in $L^2(R^3)$ by the argument given above.

Case 2. Assume $|p| \leq 1$ and $\lambda > \lambda_c(p) = \frac{1}{2}p^2$. Then S_λ does not contain $k = 0$, and the gradient (3.15) does not vanish, since $|k + \hat{k}| > 1 \geq |p|$. Hence S_λ is again a nonempty hypersurface, and $f \notin L^2$.

Case 3. Consider now the threshold case $\lambda = \lambda_c(p)$.

(i) If $|p| > 1$, then $\lambda_c(p)$ is the minimum of $\mathbf{R}^3 \ni k \to G(p,k)$ at the critical point $k = k_c(p)$, where the Hessian satisfies

$$\nabla_k^2 G(p,k) = I + \frac{I - \hat{k} \otimes \hat{k}}{|k|}.$$

Hence $k_c(p)$ is a Morse type critical point, and

$$0 \le G(p,k) - \lambda \le \tfrac{1}{2}|k - k_c(p)|^2$$

near $k = k_c(p)$. Hence $S_\lambda = \{k_c(p)\}$ (obviously of Lebesgue measure zero),

$$|f(k)| = \frac{|c\mu g_0(k)|}{|G(p,k) - \lambda|} \ge C \frac{g_0(k)}{|k - k_c(p)|^2},$$

and f cannot be square integrable.
(ii) If $|p| = 1$, then $\lambda = \lambda_c(p) = \frac{1}{2}p^2 = \frac{1}{2}$ and, if we let θ be the angle between k and p,

$$G(p,k) - \lambda = -p \cdot k + \tfrac{1}{2}k^2 + |k| = \tfrac{1}{2}r^2 + r(1 - \cos\theta) \ge 0, \quad r = |k|.$$

Hence $S_\lambda = \{0\}$ is a single point and, by using polar coordinates, we have

$$\begin{aligned}
\int |f(k)|^2 dk &= \int \frac{|c\mu|^2 g_0(k)^2 dk}{|G(p,k) - \lambda|^2} \\
&= C \int_0^\infty \int_0^\pi \frac{4 g_0(r)^2 r^2 \sin\theta d\theta dr}{r^2(r + 2(1 - \cos\theta))^2} \quad (C = 2\pi|c\mu|^2) \\
&= 4C \int_0^\infty g_0(r)^2 \left(\int_0^2 \frac{1}{(r + 2t)^2} dt \right) dr \\
&= 8C \int_0^\infty \frac{\chi(r)^2 dr}{r^2(r + 4)} = \infty,
\end{aligned}$$

as $\chi(0) > 0$. Thus again λ cannot be an eigenvalue.
(iii) Finally we consider the case $|p| < 1$ and $\lambda = \lambda_c(p) = \frac{1}{2}p^2$. In this case

$$G(p,k) - \lambda = \tfrac{1}{2}k^2 + |k| - p \cdot k \ge |k|(1 - |p|),$$

hence $S_\lambda = \{0\}$, and $|G(p,k) - \lambda| \le C|k|$ for small $|k|$. Then,

$$f(k) = \frac{c\mu g_0(k)}{G(p,k) - \lambda} \notin L^2(\mathbf{R}^3),$$

since $f(k)$ has a singularity $|k|^{-3/2}$ at $k = 0$. $\square$

Remark 3.4. Since the Hamiltonian $H_0(p)$ is essentially a Friedrich Hamiltonian, it is possible to give a different proof of Theorem 1.2, by proving the limiting absorption principle for $H_0(p)$ on $(\lambda_c(p), \infty) \setminus \{\frac{1}{2}p^2\}$. In order to do this one studies the boundary values $\lim_{\varepsilon \downarrow 0} F(\rho, \lambda \pm i\varepsilon)$ for $(\rho, \lambda) \in \Gamma^+$, and then uses the explicit representation for the resolvent given in (3.4) and (3.5). The argument in Lemma 3.3 is then needed only in the cases $\lambda = \lambda_c(p)$ and $\lambda = \frac{1}{2}p^2$.

3.3 Resolvent and spectrum of H_0

From the equations (3.3), (3.4), and (3.5), we derive the formula for the resolvent:

$$(H_0 - z)^{-1} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} G_0(x, z) \\ G_1(x, k, z) \end{pmatrix}. \tag{3.16}$$

Lemma 3.5. *Let $z \notin \mathbf{R}$. Then we have*

$$\hat{G}_0(p, z) = \frac{1}{F(p, z)} \left(\hat{f}_0(p) - \mu \int \frac{g_0(k) \hat{f}_1(p - k, k)}{\frac{1}{2}(p - k)^2 + |k| - z} dk \right), \tag{3.17}$$

$$\hat{G}_1(p, k, z) = \frac{\hat{f}_1(p, k)}{\frac{1}{2}p^2 + |k| - z} - \frac{\mu g_0(k) \hat{G}_0(p + k, z)}{\frac{1}{2}p^2 + |k| - z}, \tag{3.18}$$

where $F(p, z)$ is given by (2.1).

Since $\lambda_{\circ\rho}(\rho) > 0$ for $0 < \rho < \rho_c$, Theorem 1.2 implies the following theorem, by well-known results on the spectrum of an operator with a direct integral representation, see for example [17, Theorem XIII.85].

Theorem 3.6. *The spectrum of H_0 is absolutely continuous and is given by $\sigma(H_0) = [\Sigma, \infty)$, where $\Sigma = \lambda_\circ(0) < 0$.*

Remark 3.7. In quantum field theory one is often interested in the joint spectrum of (H_0, P), where the components of the momentum are given in (3.1). Such results follow immediately from the results in this section. In particular, we see that in our case the eigenvalue $\lambda_\circ(p)$ generates an isolated shell in the joint spectrum.

4 The behavior of e^{-itH_0}

In this section we prove Theorem 1.3 and study the asymptotic behavior as $t \to \pm\infty$ of $e^{-it\lambda_\circ(D_x)}$ in the configuration space.

4.1 Proof of Theorem 1.3

By virtue of Theorem 1.2, $e^{-itH_0(p)}$ can be decomposed as

$$e^{-itH_0(p)} = e^{-it\lambda_\circ(p)} E_p + e^{-itH_0(p)} P_{\rm ac}(H_0(p)),$$

where we set $E_p = 0$, when $|p| \geq \rho_c$. Here $H_0(p)$ is a rank two perturbation of $H_{00}(p)$, $H_{00}(p)$ has a simple isolated eigenvalue $\frac{1}{2}p^2$ and the absolutely continuous spectrum $[\lambda_c(p), \infty)$. The absolutely continuous subspace is $\mathcal{K}_{\rm ac}(H_{00}(p)) = \{0\} \oplus L^2(\mathbf{R}^3)$. It follows by the celebrated Kato-Birman theorem (see for example [16]) that the limits

$$\underset{t\to\pm\infty}{s\text{-}\lim}\, e^{itH_{00}(p)} e^{-itH_0(p)} P_{\rm ac}(H_0(p)) = \Omega_0^{\pm}(p)$$

exist, and furthermore that the wave operators $\Omega_0^{\pm}(p)$ are partial isometries with initial set $\mathcal{K}_{\rm ac}(H_0(p)) = P_{\rm ac}(H_0(p))\mathcal{K}$ and final set $\{0\} \oplus L^2(\mathbf{R}^3)$. Thus, as $t \to \pm\infty$, we have for any $\tilde{\mathbf{f}} \in \mathcal{K}$

$$\left\| e^{-itH_0(p)}\tilde{\mathbf{f}} - e^{-it\lambda_\circ(p)} E_p \tilde{\mathbf{f}} - e^{-itH_{00}(p)} \Omega_0^{\pm}(p)\tilde{\mathbf{f}} \right\|_{\mathcal{K}} \to 0 \tag{4.1}$$

and $\|\tilde{\mathbf{f}}\|^2 = \|E_p\tilde{\mathbf{f}}\|^2 + \|\Omega_0^{\pm}(p)\tilde{\mathbf{f}}\|^2$.

If we write, with $\hat{u}$ denoting the Fourier transform of u with respect to the x variables as previously,

$$\mathbf{f} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in \mathcal{H}, \quad \text{and} \quad E_p U\mathbf{f}(p) = \begin{pmatrix} \hat{f}_{1,0}(p) \\ \hat{f}_{1,1}(p,k) \end{pmatrix} \in \mathcal{K} \quad \text{for} \quad |p| < \rho_c$$

then, with the understanding that $\hat{f}_{1,0}(p) = \hat{f}_{1,1}(p,k) = 0$, when $|p| \geq \rho_c$, we have

$$U^* \left(\int_{\mathbf{R}^3}^{\oplus} e^{-it\lambda_\circ(p)} E_p dp \right) U\mathbf{f} = \begin{pmatrix} e^{-it\lambda_\circ(D_x)} f_{1,0}(x) \\ e^{-ikx} e^{-it\lambda_\circ(D_x)} f_{1,1}(x,k) \end{pmatrix}. \tag{4.2}$$

Since E_p is the one dimensional projection onto the space spanned by $\mathbf{e}_p(k)$, we have

$$\hat{f}_{1,0}(p) = \frac{-1}{F_\lambda(p, \lambda_\circ(p))} \left(\hat{f}_0(p) - \mu \int \frac{g_0(k)\hat{f}_1(p-k,k)dk}{\frac{1}{2}(p-k)^2 + |k| - \lambda_\circ(p)} \right),$$

and

$$\hat{f}_{1,1}(p,k) = \frac{-\mu g_0(k)\hat{f}_{1,0}(p)}{\frac{1}{2}(p-k)^2 + |k| - \lambda_\circ(p)}.$$

The operator $\left(\int_{\mathbf{R}^3}^{\oplus} E_p dp\right)$ is the orthogonal projection onto $\{h(p)\mathbf{e}_p(k)\colon h \in L^2(B_{\rho_c})\}$ and

$$\left\| U^* \left(\int_{\mathbf{R}^3}^{\oplus} E_p dp \right) U\mathbf{f} \right\|_{\mathcal{H}}^2 = \|f_{1,0}\|_{\mathcal{H}_0}^2 + \|f_{1,1}\|_{\mathcal{H}_1}^2. \tag{4.3}$$

If we write

$$Z^{\pm} \equiv U^* \left(\int_{\mathbf{R}^3}^{\oplus} \Omega_0^{\pm}(p) dp \right) U, \quad Z^{\pm}\mathbf{f} = \begin{pmatrix} 0 \\ f_{2,1,\pm} \end{pmatrix},$$

then $Z^{\pm}$ are unitary from $U^* \left(\int_{\mathbf{R}^3}^{\oplus} \mathcal{K}_{\mathrm{ac}}(H_0(p)) dp \right)$ onto $\{0\} \oplus L^2(\mathbf{R}^6)$, and

$$U^* \left[\int_{\mathbf{R}^3}^{\oplus} e^{-itH_{00}} \Omega_0^{\pm}(p) dp \right] U\mathbf{f} = \begin{pmatrix} 0 \\ e^{-it(-\frac{1}{2}\Delta + |k|)} f_{2,1,\pm} \end{pmatrix}. \tag{4.4}$$

We insert the relation (4.1) into the identity $e^{-itH_0} = U^* \big(\int_{\mathbf{R}^3}^{\oplus} e^{-itH_0(p)} dp \big) U$, and use the identities (4.2) and (4.4). Theorem 1.3 follows.

4.2 Behavior in configuration space

As the operator $e^{-it(-\frac{1}{2}\Delta + |k|)}$ has been well studied, we concentrate on the operator $e^{-it\lambda_\circ(D_x)} v(x)$ for the case $t > 0$. When $\hat{v} \in C_0^\infty(B(\rho_c))$, we may apply the method of stationary phase to

$$v(t,x) = \frac{1}{(2\pi)^{3/2}} \int e^{-it\lambda_\circ(p) + ixp} \hat{v}(p) dp.$$

The points of stationary phase are determined by the equation

$$t \nabla \lambda_\circ(p) = x. \tag{4.5}$$

It follows from (2.7) that $\det \nabla_p^2(p)$ can vanish only for p with $|p| = \rho$ satisfying $\lambda_{\circ\rho\rho}(\rho) = 0$. By the real analyticity of $\lambda_\circ(\rho)$ it follows that these zeros are isolated in $(0, \rho_c)$, with 0 and ρ_c as possible accumulation points. Thus

$$\{\rho \in (0, \rho_c) \colon \lambda_{\circ\rho\rho}(\rho) = 0\} = \{\rho_j\}_{j=M,\dots,N},$$

a strictly increasing sequence, or the set is empty. Here we use M and N to distinguish between the following cases. $M = 1$, if zeros do not accumulate at 0. In that case we also use $\rho_0 = 0$ below. $1 \leq N < \infty$, if the zeroes do not accumulate at ρ_c. In that case we introduce $\rho_{N+1} = r < \rho_c$. We take $M = -\infty$ or $N = \infty$, if the zeroes accumulate at 0 or ρ_c, respectively.

We consider only the case $M = -\infty$ and $N = \infty$. The other cases require simple modifications in the arguments below. Define

$$G_j = \{p \in \mathbf{R}^3 \colon \rho_j < |p| < \rho_{j+1}\}, \quad j \in \mathbf{Z}. \tag{4.6}$$

We restrict our considerations to a set G_j, $j \in \mathbf{Z}$. The equation (4.5) has a unique solution of the form $p(x/t) = \hat{x}\rho(|x|/t)$, where $\rho(|x|/t)$ is the solution of $\lambda_{\circ\rho}(\rho) = |x|/t$, when $\frac{x}{t} \in \nabla\lambda_\circ(G_j)$.

Assume now $\hat{v} \in C_0^\infty(G_j)$. Then $v(t,x)$ can be written in the form

$$v(t,x) = \frac{t^{-3/2} e^{i\phi(t,x) - i\frac{3\pi}{4} + i\frac{\pi}{2}s}}{(\det \nabla_p^2 \lambda_\circ(p(x/t)))^{1/2}} \left(\hat{v}(p(x/t)) + t^{-1} v_1(x/t) + \cdots \right), \tag{4.7}$$

where $s = 0$, if $\det \nabla_p^2 \lambda_\circ(p) > 0$ for $p \in G_j$, and $s = 1$, if $\det \nabla_p^2 \lambda_\circ(p) < 0$ for $p \in G_j$. The phase function is defined by

$$\phi(t,x) = x \cdot p(x/t) - t\lambda_\circ(p(x/t)), \tag{4.8}$$

$v_1, v_2, \ldots$ are determined by standard formulae (see [11, Section 7.7]), and for $\frac{x}{t} \notin \nabla\lambda_\circ(G_j)$, $|v(t,x)| \le C_N |t|^{-N} \langle x \rangle^{-N}$ for any N and t large.

Lemma 4.1. *Let G_j, s and $\phi(t,x)$ be defined as above and let $\hat{f}_{1,0} \in C_0^\infty(G_j)$ for some j. Then the functions $e^{-it\lambda_\circ(D_x)} f_{1,0}(x)$ and $e^{-it\lambda_\circ(D_x)} f_{1,1}(x,k)$ have the following asymptotic expansions as $t \to \infty$ for $x \in t\nabla\lambda_\circ(G_j)$:*

$$e^{-it\lambda_\circ(D_x)} f_{1,0}(x) = \frac{t^{-3/2} e^{i\phi(t,x) - i\frac{3\pi}{4} + i\frac{\pi}{2}s}}{(\det \nabla_p^2 \lambda_\circ(p(x/t)))^{1/2}} \Big(\hat{f}_{1,0}(p(x/t)) + t^{-1} g_1(x/t) + \cdots \Big), \tag{4.9}$$

$$e^{-it\lambda_\circ(D_x)} f_{1,1}(x,k) = \frac{t^{-3/2} e^{i\phi(t,x) - i\frac{3\pi}{4} + i\frac{\pi}{2}s}}{(\det \nabla_p^2 \lambda_\circ(p(x/t)))^{1/2}} \Big(\hat{f}_{1,1}(p(x/t),k) + t^{-1} M_1(x/t,k) + \cdots \Big), \tag{4.10}$$

where $g_1(x/t)$,$g_2(x/t), \ldots$, $M_1(x/t,k)$, $M_2(x/t,k), \ldots$, are defined by standard formulae involving the derivatives of $f_{1,0}$ and $f_{1,1}$. In particular, supports of $g_j(\cdot)$ and $M_j(\cdot,k)$ are contained in those of $\hat{f}_{1,0}(p(\cdot))$ and $\hat{f}_{1,1}(p(\cdot),k)$ respectively. For $x \notin t\nabla\lambda_\circ(\overline{G}_j)$, we have for any N,

$$|e^{-it\lambda_\circ(D_x)} f_{1,0}(x)| \le C_N |t|^{-N} \langle x \rangle^{-N}, \tag{4.11}$$

$$|e^{-it\lambda_\circ(D_x)} f_{1,1}(x,k)| \le C_N\, |t|^{-N}\, \langle x\rangle^{-N} \langle k\rangle^{-N}. \tag{4.12}$$

The results (4.10) *and* (4.12) *hold for* $|k| > \varepsilon$, $\varepsilon > 0$ *arbitrary.*

Proof. The formula for $e^{-it\lambda_\circ(D_x)} f_{1,0}(x)$ is an immediate corollary of (4.7). The result (4.10) can be proved similarly, since $k \to \hat{f}_{1,1}(\cdot, k) \in C_0^\infty(B(\rho_c))$ is smooth and rapidly decaying, when $|k| > \varepsilon > 0$. □

Thus we may consider $e^{-ikx} e^{-it\lambda_\circ(D_x)} f_{1,1}(x,k)$ as the part of the wave function, which represents the motion of the electron under the dispersion relation $\lambda_\circ(p)$, which is dragging the cloud of photons (however only one photon).

5 Proof of Theorem 1.6

In this section we prove Theorem 1.6 on the spectral properties of the Hamiltonian H. In what follows we mostly use the configuration space representations for the photons. The variable dual to k is denoted by y. We then have

$$(|g\rangle u_0)(x,y) = \check{g}_0(y-x)u_0(x), \qquad (\langle g|u_1)(x) = \int \check{g}_0(x-y)u_1(x,y)dy,$$

where $\check{g}_0$ is the inverse Fourier transform of the function given by (1.2). Recall that g_0 is $O(3)$-invariant, such that $\check{g}_0(x)$ is also $O(3)$ invariant, $\check{g}_0$ is C^k for a k depending on N in Assumption 1.1, and it has an asymptotic expansion at infinity:

$$\check{g}_0(y) = C_0|y|^{-5/2} + C_1|y|^{-7/2} + \cdots . \tag{5.1}$$

The photon energy is given by the differential operator $|D_y|$. We write K_0 for the operator H, where the electron-photon interaction is switched–off, viz.

$$K_0 = \begin{pmatrix} h & 0 \\ 0 & h + |D_y| \end{pmatrix} \equiv h \oplus (h + |D_y|).$$

The following two sets of partitions of unity, the one of the electron configuration space $\mathbf{R}^3_x$, and the other of the electron-photon configuration space $\mathbf{R}^6_{(x,y)}$, will play an important role in what follows:

$$\chi_{00}(x)^2 + \chi_{01}(x)^2 = 1, \quad x \in \mathbf{R}^3,$$

$$\chi_{10}(x,y)^2 + \chi_{11}(x,y)^2 + \chi_{12}(x,y)^2 = 1, \quad (x,y) \in \mathbf{R}^6,$$

where χ_{ij} satisfy the following properties:

(1) $\chi_{00} \in C_0^\infty(\{|x| < 1\})$, $\chi_{01} \in C^\infty(\mathbf{R}^3)$ and $\chi_{10} \in C_0^\infty(\{|x|^2 + |y|^2 < 1\})$. $\chi_{00}(x) = 1$ for $|x| \le 1/4$ and $\chi_{10}(x,y) = 1$ for $|x|^2 + |y|^2 \le (1/4)^2$.

(2) For $|x|^2 + |y|^2 \ge 1$, $\chi_{11}, \chi_{12} \in C^\infty(\mathbf{R}^6)$ are homogeneous of degree 0, $\chi_{11}(x,y)$ and $\chi_{12}(x,y)$ vanish in small open cones containing $x = 0$ and $x = y$ such that, outside a ball of radius 1, they are equal to 1 in open cones containing $x = y$ and $x = 0$, respectively.

Such a partition of unity exists, since the linear subspaces $\{(x,y) \colon x = 0\}$ and $\{(x,y) \colon x = y\}$ do not intersect on the unit sphere of $\mathbf{R}^6$. We define

$$\chi_0 = \begin{pmatrix} \chi_{00} & 0 \\ 0 & \chi_{10} \end{pmatrix}, \quad \chi_1 = \begin{pmatrix} \chi_{01} & 0 \\ 0 & \chi_{11} \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 & 0 \\ 0 & \chi_{12} \end{pmatrix}, \tag{5.2}$$

so that as operators in $\mathcal{H}$

$$\chi_0^2 + \chi_1^2 + \chi_3^2 = I. \tag{5.3}$$

We denote the commutator $AB - BA$ of operators A and B by $[A, B]$.

Lemma 5.1. *Let $j \in C_0^\infty(\mathbf{R}^3)$ and let $j_\varrho(y) = j(y/\varrho)$ for $\varrho > 0$. Then:*
(1) The operator $[|D_y|\,, j_\varrho(y)]$, defined on $\mathcal{S}(\mathbf{R}^3)$, extends to a bounded operator on $L^2(\mathbf{R}^3)$. We have $\|\,[|D_y|\,, j_\varrho(y)]\,\|_{B(L^2)} \le C\varrho^{-1}$, $\varrho \ge 1$.
(2) For any $\delta > 0$, $[|D_y|\,, j_\varrho(y)]\,(1 + |D_y|)^{-\delta}$ is compact in $L^2(\mathbf{R}^3)$.
(3) Let K be the multiplication by a function $K(x,y)$ such that

$$\lim_{R\to\infty} \sup_{|x|\ge R} \|K(x,\cdot)\|_{L^2(\mathbf{R}^3)} = 0.$$

Then $K(-\Delta_x + 1)^{-1} \colon L^2(\mathbf{R}^3) \to L^2(\mathbf{R}^6)$ is compact.

Proof. (1) In the momentum representation j_ϱ is convolution with $\varrho^3 \hat{j}(\varrho\xi)$. Thus $[|D_y|\,, j_\varrho(y)]$ is an integral operator with the kernel

$$\varrho^3(|\xi| - |\eta|)\hat{j}(\varrho(\xi - \eta)),$$

which is dominated in modulus by the convolution kernel $\varrho^3\,|\xi - \eta|\,|\hat{j}(\varrho(\xi - \eta))|$. Statement (1) follows from Young's inequality.
(2) When $\varepsilon > 0$, it is well-known that $\left[(\varepsilon^2 - \Delta_y)^{1/2}, j_\varrho(y)\right](1 + |D_y|)^{-\delta}$ is compact. We have $|(\varepsilon^2 + |\xi|^2)^{1/2} - |\xi|| \le \varepsilon$ and

$$\|\,\left[(\varepsilon^2 - \Delta_y)^{1/2}, j_\varrho(y)\right] - [|D_y|, j_\varrho(y)]\,\|_{B(L^2)} \le 2\varepsilon\|j\|_{L^\infty}.$$

The compactness of $[|D_y|, j_\varrho(y)]\,(1 + |D_y|)^{-\delta}$ follows.

(3) Let $K_R(x,y) = \chi_{00}(x/R)K(x,y)$ and K_R be the multiplication operator with $K_R(x,y)$. Then, $K_R \to K$ in the operator norm from $L^2(\mathbf{R}^3)$ to $L^2(\mathbf{R}^6)$ as $R \to \infty$ and $K_R(-\Delta_x+1)^{-1}$ is an operator of Hilbert-Schmidt class because it is an integral operator with the square integrable integral kernel

$$\frac{K_R(x,y)e^{-|x-x'|}}{4\pi|x-x'|}.$$

Hence, $K(-\Delta_x+1)^{-1}$ is compact. $\square$

Lemma 5.2. (1) *The following operators are compact in $\mathcal{H}$ for $z \notin \mathbf{R}$:*

$$[(H-z)^{-1}, \chi_j],\ j=1,2, \qquad ((H-z)^{-1}-(H_0-z)^{-1})\chi_1$$
$$((H-z)^{-1}-(K_0-z)^{-1})\chi_2, \qquad [(K_0-z)^{-1}, \chi_j], \quad j=1,2.$$

(2) *Let $f \in C_0^\infty(\mathbf{R})$. Then the following operators are compact in $\mathcal{H}$:*

$$[f(H), \chi_j], \quad (f(H)-f(H_0))\chi_1, \quad (f(H)-f(K_0))\chi_2, \quad [f(K_0), \chi_j],\ j=1,2.$$

Proof. By virtue of the Helffer-Sjöstrand formula (see (3.12) and [4, Theorem 8.1]), it suffices to show the compactness of the operators in (1).
(i) We first prove that $[(H-z)^{-1}, \chi_j],\ j=1,2$ are compact. Since the proof for the case $j=2$ is similar and simpler, we prove it only for $j=1$. We have

$$[(H-z)^{-1}, \chi_1] = (H-z)^{-1}[H, \chi_1](H-z)^{-1}$$

and

$$[H, \chi_1] = \begin{pmatrix} -\frac{1}{2}\Delta\chi_{01} + \frac{1}{2}\chi_{01}\Delta & \langle g|\chi_{11} - \chi_{01}\langle g| \\ |g\rangle\chi_{01} - \chi_{11}|g\rangle & (-\frac{1}{2}\Delta + |D_y|)\chi_{11} - \chi_{11}(-\frac{1}{2}\Delta + |D_y|) \end{pmatrix}.$$

We show that all entries of the matrix on the right are compact operators between appropriate spaces.
(a) By the Rellich theorem $-\frac{1}{2}\Delta\chi_{01} + \frac{1}{2}\chi_{01}\Delta\colon H^2(\mathbf{R}^3) \to L^2(\mathbf{R}^3)$ is compact.
(b) In the configuration space, $|g\rangle\colon u_0(x) \mapsto \check{g}_0(y-x)u_0(x)$. We write

$$|g\rangle\chi_{01} - \chi_{11}|g\rangle = |g\rangle(\chi_{01}-1) - (\chi_{11}-1)|g\rangle.$$

Then, both $|g\rangle(\chi_{01}-1)(-\Delta+1)^{-1}$ and $(\chi_{11}-1)|g\rangle(-\Delta+1)^{-1}$ are compact from $L^2(\mathbf{R}^3)$ to $L^2(\mathbf{R}^6)$, because $|g\rangle(\chi_{01}-1)$ and $(\chi_{11}-1)|g\rangle$ are multiplications by $\check{g}_0(y-x)(\chi_{01}(x)-1)$ and $\check{g}_0(y-x)(\chi_{11}(x,y)-1)$, respectively, and they satisfy the condition of Lemma 5.1(3). Indeed, we have $\|\check{g}_0(y-x)(\chi_{01}(x)-1)\|_{L^2(\mathbf{R}^3_y)} = 0$ for $|x| \geq 1$, because $\check{g}_0 \in L^2(\mathbf{R}^3)$ by virtue of (5.1) and $(\chi_{01}(x)-1) = 0$ for $|x| \geq 1$, and

$$|\check{g}_0(y-x)(\chi_{11}(x,y)-1)| \leq C(1+|x|+|y|)^{-5/2},$$

because $(1-\chi_{11}(x,y))$ vanishes in an open cone containing $x=y$ outside the unit ball of $\mathbf{R}^6$. Thus, $|g\rangle\chi_{01} - \chi_{11}|g\rangle$ is compact from $H^2(\mathbf{R}^3)$ to $L^2(\mathbf{R}^6)$.
(c) $(-\frac{1}{2}\Delta + 1)^{-1}(\langle g|\chi_{11} - \chi_{01}\langle g|)$ is the adjoint of the operator discussed in step (b) and is compact from $L^2(\mathbf{R}^6)$ to $L^2(\mathbf{R}^3)$.
(d) χ_{11} is homogeneous of degree 0 outside the unit ball and $[-\frac{1}{2}\Delta_x, \chi_{11}]$ is a first order differential operator whose coefficients are derivatives of χ_{11}. It follows that $[-\frac{1}{2}\Delta_x, \chi_{11}](-\frac{1}{2}\Delta_x + |D_y| + 1)^{-1}$ is compact in $L^2(\mathbf{R}^6)$ by the Rellich compactness theorem. We approximate $|D_y|$ by $(\varepsilon^2 + |D_y|^2)^{1/2}$ as in the proof of Lemma 5.1. The operator $[(\varepsilon^2 + |D_y|^2)^{1/2}, \chi_{11}]$ is a pseudo-differential operator of order zero in $\mathbf{R}^6$ whose symbol decays at infinity with respect to (x,y). Hence,

$$[(\varepsilon^2 + |D_y|^2)^{1/2}, \chi_{11}](-\tfrac{1}{2}\Delta_x + |D_y| + 1)^{-1}$$

is compact in $L^2(\mathbf{R}^6)$, and so is $[|D_y|, \chi_{11}](-\frac{1}{2}\Delta_x + |D_y| + 1)^{-1}$ by the argument of the proof of Lemma 5.1. Thus, $[-\frac{1}{2}\Delta + |D_y|, \chi_{01}](-\frac{1}{2}\Delta_x + |D_y| + 1)^{-1}$ is compact in $L^2(\mathbf{R}^6)$. Combination of the results (a) $\sim$ (d) implies that the operator $[(H-z)^{-1}, \chi_1]$ is compact in $\mathcal{H}$.

(ii) Next we show $((H-z)^{-1} - (H_0 - z)^{-1})\chi_1$ is compact. We write $\tilde{V} = V \oplus V$. Then

$$\begin{aligned}((H-z)^{-1} - (H_0 - z)^{-1})\chi_1 &= -(H-z)^{-1}\tilde{V}(H_0 - z)^{-1}\chi_1 \\ &= -(H-z)^{-1}\tilde{V}\chi_1(H_0 - z)^{-1} - (H-z)^{-1}\tilde{V}[(H_0 - z)^{-1}, \chi_1]\end{aligned}$$

By assumption $(H-z)^{-1}\tilde{V}$ is bounded and $[(H_0 - z)^{-1}, \chi_1]$ is compact in $\mathcal{H}$, as was shown in (i) above. It follows that the second summand on the right is compact. To see that the same holds for the first summand, we write $\tilde{V}\chi_1 = (V\chi_{01} \oplus 0) + (0 \oplus V\chi_{11})$. Then, $V\chi_{01} \oplus 0$ is H_0-compact because V is $-\Delta$-compact. Since χ_{11} vanishes in an open cone about $x = 0$, $V(x)\chi_{11}(x,y)$ decays as $|x| + |y| \to \infty$. Hence, $V(x)\chi_{11}(x,y)$ is $-\Delta_x + |D_y|$-compact, viz. $0 \oplus V\chi_{11}$ is H_0-compact. It follows that $\tilde{V}\chi_1(H_0 - z)^{-1}$ is compact and hence so is $((H-z)^{-1} - (H_0 - z)^{-1})\chi_1$.

(iii) To show $((H-z)^{-1} - (K_0 - z)^{-1})\chi_2$ is compact, we write it in the form

$$-(H-z)^{-1}\begin{pmatrix} 0 & \langle g|(-\frac{1}{2}\Delta + |D_y| - z)^{-1}\chi_{12} \\ 0 & 0 \end{pmatrix}.$$

It suffices to show that $(-\Delta + 1)^{-1}\langle g|(-\frac{1}{2}\Delta + |D_y| - z)^{-1}\chi_{12}$ is compact from $L^2(\mathbf{R}^6)$ to $L^2(\mathbf{R}^3)$. We write

$$\begin{aligned}\langle g|(-\tfrac{1}{2}\Delta + |D_y| - z)^{-1}\chi_{12} &= \langle g|\chi_{12}(-\tfrac{1}{2}\Delta + |D_y| - z)^{-1} \\ &\quad - \langle g|(-\tfrac{1}{2}\Delta + |D_y| - z)^{-1}[-\tfrac{1}{2}\Delta + |D_y|, \chi_{12}](-\tfrac{1}{2}\Delta + |D_y| - z)^{-1}\end{aligned}$$

The argument of (i) (d) above implies $[-\frac{1}{2}\Delta + |D_y|, \chi_{12}](-\frac{1}{2}\Delta + |D_y| - z)^{-1}$ is compact in $L^2(\mathbf{R}^6)$. Since $\chi_{12}(x, y) = 0$ in an open cone containing $x = y$, (5.1) implies $|\check{g}(x - y)\chi_{12}(x, y)| \le C(1 + |x| + |y|)^{-5/2}$. It follows that $(-\Delta+1)^{-1}\langle g|\chi_{12}$ is compact from $L^2(\mathbf{R}^6)$ to $L^2(\mathbf{R}^3)$ because it is the adjoint of $\chi_{12}|g\rangle(-\Delta+1)^{-1}$ which is compact from$L^2(\mathbf{R}^3)$ to $L^2(\mathbf{R}^6)$ by the argument of (i) (b).

(iv) The compactness of $[(K_0 - z)^{-1}, \chi_j]$ is well known, and we omit the details. □

5.1 Essential spectrum

In this subsection we show that $\sigma_{\rm ess}(H) = [\min\{\Sigma, E_0\}, \infty)$ by proving the following two lemmas. We write $\Sigma_{\rm ess} = \min\{\Sigma, E_0\}$. We recall the definitions, namely $\Sigma = \inf \sigma(H_0)$ and $E_0 = \inf \sigma(h)$. We impose Assumption 1.4 on V.

Lemma 5.3. *We have* $[\Sigma_{\rm ess}, \infty) \subseteq \sigma_{\rm ess}(H)$.

Proof. We prove the lemma by the standard Weyl sequence method. Let us first assume $\lambda > \Sigma$. We can choose an orthonormalized sequence $\mathbf{u}_n = u_{n0} \oplus u_{n1} \in D(H_0)$, such that $\|(H_0 - \lambda)\mathbf{u}_n\| \to 0$, because $\lambda \in \sigma(H_0)$, and $\sigma(H_0)$ is absolutely continuous. Then, for any choice of $R_n \in \mathbf{R}^3$,

$$\tilde{\mathbf{u}}_n = \begin{pmatrix} u_{n0}(x - R_n) \\ e^{-ikR_n}u_{n1}(x - R_n, k) \end{pmatrix}, \quad n = 1, 2, \dots$$

is still orthonormalized and $\|(H_0 - \lambda)\tilde{\mathbf{u}}_n\| \to 0$ as $n \to \infty$, since H_0 is translation invariant. Due to Assumption 1.4 we can choose R_n such that

$$\|Vu_{n0}(x - R_n)\|_{L^2(\mathbf{R}^3_x)} \to 0 \ \text{ and } \ \|Vu_{n1}(x - R_n, k)\|_{L^2(\mathbf{R}^6_{(x,k)})} \to 0$$

as $n \to \infty$. Then we have $\|(H - \lambda)\tilde{\mathbf{u}}_n\| \to 0$, and we conclude $\lambda \in \sigma_{\rm ess}(H)$, and then $[\Sigma, \infty) \subset \sigma_{\rm ess}(H)$. Next suppose that h has an eigenvalue $E < \Sigma$ with a normalized eigenfunction $\phi(x)$, and let $\lambda > E$. We show that $\lambda \in \sigma_{\rm ess}(H)$ and hence $[E_0, \infty) \subset \sigma_{\rm ess}(H)$. Take $k_0 \in \mathbf{R}^3$ such that $|k_0| = \lambda - E$, and take a function $\psi \in C_0^\infty(\mathbf{R}^3_k)$, such that $\|\psi\| = 1$. Set

$$\mathbf{u}_n = \begin{pmatrix} 0 \\ u_{n1}(x, k) \end{pmatrix}, \quad u_{n1}(x, k) = n^{3/2}\phi(x)\psi(n(k - k_0)).$$

Then $\|\mathbf{u}_n\| = 1$, and $\mathbf{u}_n \to 0$ weakly as $n \to \infty$. Moreover, we have

$$\|\langle g|u_{n1}\|_{\mathcal{H}_0} \le n^{-3/2} \int g_0(k_0 + n^{-1}k)|\psi(k)|dk \to 0,$$

$$\begin{aligned}&\|(h+|k|-\lambda)u_{n1}\| \\ &\qquad = n^{3/2}\|(|k|-|k_0|)\psi(n(k-k_0))\| = \|(|k_0+n^{-1}k|-|k_0|)\psi(k)\| \to 0\end{aligned}$$

as $n \to \infty$. Hence $\|(H-\lambda)\mathbf{u}_n\| \to 0$, and $\lambda \in \sigma_{\mathrm{ess}}(H)$. □

Lemma 5.4. *We have* $\sigma_{\mathrm{ess}}(H) \subseteq [\Sigma_{\mathrm{ess}}, \infty)$.

Proof. We prove that $f(H)$ is a compact operator for any $f \in C_0(-\infty, \Sigma_{\mathrm{ess}})$ by adapting the geometric proof of HVZ-theorem ([2]), the corresponding result for the N-body Schrödinger operators. We have $f(H_0) = f(K_0) = 0$ by the definition of Σ_{ess}. We decompose

$$\begin{aligned} f(H) = f(H)&\begin{pmatrix}\chi_{00}^2 & 0 \\ 0 & \chi_{10}^2\end{pmatrix} \\ &+ (f(H)-f(H_0))\begin{pmatrix}\chi_{01}^2 & 0 \\ 0 & \chi_{11}^2\end{pmatrix} + (f(H)-f(K_0))\begin{pmatrix}0 & 0 \\ 0 & \chi_{12}^2\end{pmatrix}\end{aligned}$$

The first summuand on the right is compact in $\mathcal{H}$ by the Rellich theorem, and so are the others by virtue of Lemma 5.2. Thus $f(H)$ is compact and the lemma follows. □

5.2 The Mourre estimate

In this subsection we complete the proof of statement (1) of Theorem 1.6 via the Mourre theory. For this purpose, we first prove the following Mourre estimate for the operator H with the conjugate operator

$$A = \begin{pmatrix}A_x & 0 \\ 0 & A_x + A_y\end{pmatrix},$$

where $A_x = \frac{1}{2}(x\cdot D_x + D_x \cdot x)$ and $A_y = \frac{1}{2}(y\cdot D_y + D_y\cdot y)$. In the momentum representation A_y can be represented by $-A_k = -\frac{1}{2}(k\cdot D_k + D_k \cdot k)$. We write $I_\delta(\lambda) = (\lambda-\delta, \lambda+\delta)$. $P_H(I)$ is the spectral projection of H for the interval I.

Lemma 5.5. *Let $\lambda_0 \notin \Theta(H)$, the threshold set. Then there exist $\varepsilon > 0$, $\delta > 0$ and a compact operator C such that*

$$iP_H(I_\delta(\lambda_0))[H,A]P_H(I_\delta(\lambda_0)) \geq \varepsilon P_H(I_\delta(\lambda_0)) + C.$$

Proof. In this proof we take $\mu = 1$ without loss of generality. We compute as a quadratic form on $\mathcal{S}(\mathbf{R}^3)\oplus\mathcal{S}(\mathbf{R}^6)$

$$i[H,A] = \begin{pmatrix} i[h, A_x] & i(\langle g|(A_x+A_y) - A_x\langle g|) \\ i(|g\rangle A_x - (A_x+A_y)|g\rangle) & i[h+|D_y|, A_x+A_y]\end{pmatrix}.$$

A simple computation using $e^{ikx}(A_x - A_k)e^{-ikx} = A_x - A_k$ yields

$$i[-\tfrac{1}{2}\Delta + V, A_x] = -\Delta - x \cdot \nabla_x V, \tag{5.4}$$
$$i[|D_y|, A_y] = i[|k|, -A_k] = [k\nabla_k, |k|] = |k|, \tag{5.5}$$
$$i(|g\rangle A_x - (A_x + A_y)|g\rangle) = |e^{-ikx}(iA_k g_0)(k)\rangle, \tag{5.6}$$
$$i(\langle g|(A_x + A_y) - A_x\langle g|) = \langle e^{-ikx}(iA_k g_0)(k)|. \tag{5.7}$$

Thus, writing $g_1(k,x) = e^{-ikx}(iA_k g_0)(k)$ and $W(x) = -x \cdot \nabla_x V$, we obtain

$$i[H, A] = \begin{pmatrix} -\Delta_x + W & \langle g_1| \\ |g_1\rangle & -\Delta_x + |k| + W \end{pmatrix}. \tag{5.8}$$

We define $g_{10}(k) = (iA_k g_0)(k)$ so that $g_1(x,k) = e^{-ixk} g_{10}(k)$. Note that

$$iA_k \frac{1}{\sqrt{|k|}} = \left(k\nabla_k + \frac{3}{2}\right)\frac{1}{\sqrt{|k|}} = \frac{1}{\sqrt{|k|}},$$

such that $g_{10}(k)$ has the same property as that of $g_0(k)$, viz. it is a smooth function of $|k| \neq 0$ which decays rapidly at infinity and it has a $|k|^{-1/2}$ singularity at $k = 0$. In the rest of the proof we fix a function $\psi \in C_0^\infty(\mathbf{R})$, such that $\psi(\lambda) = 1$ for $|\lambda| \leq 1/2$ and $\psi(\lambda) = 0$ for $|\lambda| \geq 1$ and define

$$f_{\lambda_0,\delta}(\lambda) = \psi((\lambda - \lambda_0)/\delta)$$

for $\lambda_0 \in \mathbf{R}$ and $\delta > 0$. It follows that $i[H, A]$ can be extended to a selfadjoint operator in $\mathcal{H}$ with the domain $D(H)$ and $f(H)i[H, A]$ has a bounded extension for any $f \in C_0^\infty(\mathbf{R})$.

We decompose $f(H)i[H, A]f(H)$ as

$$f(H)i[H, A]f(H) = \sum_{j=0}^{2} f(H)i[H, A]f(H)\chi_j^2 \tag{5.9}$$

by using the partitions of unity introduced in (5.2) and (5.3). Here $f \in C_0^\infty(\mathbf{R})$.

(i) By the Rellich theorem $f(H)\chi_0$ is compact. Hence $f(H)[H, A]f(H)\chi_0^2$ is a compact operator in $\mathcal{H}$.

(ii) Replacing $f(H)$ by $f(H_0)$, we write

$$f(H)[H, A]f(H)\chi_1^2 = \\ f(H)[H, A]f(H_0)\chi_1^2 + f(H)[H, A](f(H) - f(H_0))\chi_1^2.$$

Here $f(H)i[H,A](f(H)-f(H_0))\chi_1^2$ is compact by virtue of Lemma 5.2. We then rewrite the first term on the right as follows:

$$\begin{aligned} f(H)[H,A]f(H_0)\chi_1^2 = \\ & f(H)[H,A][f(H_0),\chi_1]\chi_1 + f(H)[[H,A],\chi_1]f(H_0)\chi_1 \\ & + (f(H)-f(H_0))\chi_1[H,A]f(H_0)\chi_1 + f(H_0)\chi_1(W\oplus W)f(H_0)\chi_1 \\ & + [f(H_0),\chi_1][H_0,A]f(H_0)\chi_1 + \chi_1 f(H_0)[H_0,A]f(H_0)\chi_1. \end{aligned} \tag{5.10}$$

Here on the right all terms but the last one are compact in $\mathcal{H}$. Indeed, $\chi_1(W\oplus W)f(H_0)$ is compact by the assumption on V, since $\chi_{11}=0$ in an open cone about $x=0$ in $\mathbf{R}^6$; $[f(H_0),\chi_1]$ and $(f(H)-f(H_0))\chi_1$ are compact by virtue of Lemma 5.2; since W and $g_1(k,x)$ satisfy properties similar to those of V and $g(x,k)$, the proof of Lemma 5.2 implies $[i[H,A],\chi_1]f(H_0)$ is compact. Thus,

$$f(H)i[H,A]f(H)\chi_1^2 = \chi_1 f(H_0)i[H_0,A]f(H_0)\chi_1 + C_0, \tag{5.11}$$

where C_0 is a compact operator in $\mathcal{H}$. We show that, for any $\lambda_0 \notin \Theta(H)$, there exist $\varepsilon>0$ and $\delta_0>0$ such that, for $f=f_{\lambda_0,\delta}$ with $0<\delta<\delta_0$,

$$\chi_1 f(H_0)i[H_0,A]f(H_0)\chi_1 \geq \varepsilon\chi_1 f(H_0)^2\chi_1. \tag{5.12}$$

In the direct integral decomposition introduced in (1.4), we have (recall the definition (1.3) of U)

$$\begin{aligned} & Uf(H_0)i[H_0,A]f(H_0)U^* \\ & \quad = \int_{\mathbf{R}^3}^{\oplus} f(H_0(p)) \begin{pmatrix} p^2 & \langle g_{10}| \\ |g_{10}\rangle & (p-k)^2+|k| \end{pmatrix} f(H_0(p))dp. \end{aligned} \tag{5.13}$$

We divide the proof of (5.12) into three cases.
(a) Assume $\Sigma<\lambda_0<0$, $\lambda_0\notin\Theta(H)$, first. Then choose $\delta_0>0$ such that $\lambda_0+2\delta_0<0$ and $\Sigma<\lambda_0-2\delta_0$, and let $f=f_{\lambda_0,\delta}$, $\delta<\delta_0$. Then Theorem 1.2 implies that $f(H_0(p))$ is supported in a compact subset of $\{p\colon 0<|p|<\rho_c\}$, and $f(H_0(p)) = f(\lambda_\circ(p))\mathbf{e}_p\otimes\mathbf{e}_p$. We compute the inner product

$$\left(\mathbf{e}_p, \begin{pmatrix} p^2 & \langle g_{10}| \\ |g_{10}\rangle & (p-k)^2+|k| \end{pmatrix} \mathbf{e}_p\right)_{\mathcal{K}} \tag{5.14}$$

by using the expression (1.6) for $\mathbf{e}_p$. The result is $(-F_\lambda(p,\lambda_\circ(p)))^{-1}$ times

$$p^2 - 2\int \frac{g_0(k)g_{10}(k)dk}{\frac{1}{2}(p-k)^2+|k|-\lambda_\circ(p)} + \int \frac{g_0(k)^2((p-k)^2+|k|)dk}{(\frac{1}{2}(p-k)^2+|k|-\lambda_\circ(p))^2}.$$

We recall that $g_{10}(k) = iA_k g_0(k) = \frac{d}{d\theta} e^{3\theta/2} g_0(e^\theta k)\Big|_{\theta=0}$ and compute

$$\begin{aligned}
\int \frac{g_0(k)g_{10}(k)dk}{\frac{1}{2}(p-k)^2+|k|-\lambda_\circ(p)} &= \frac{d}{d\theta}\int \frac{g_0(k)e^{3\theta/2}g_0(e^\theta k)dk}{\frac{1}{2}(p-k)^2+|k|-\lambda_\circ(p)}\Big|_{\theta=0} \\
&= \frac{d}{d\theta}\int \frac{e^{-3\theta/2}g_0(e^{-\theta}k)g_0(k)dk}{\frac{1}{2}(p-e^{-\theta}k)^2+|e^{-\theta}k|-\lambda_\circ(p)}\Big|_{\theta=0} \\
&= -\int \frac{g_{10}(k)g_0(k)dk}{\frac{1}{2}(p-k)^2+|k|-\lambda_\circ(p)} \\
&\qquad - \int \frac{g_0(k)^2\{(p-k)k-|k|\}dk}{(\frac{1}{2}(p-k)^2+|k|-\lambda_\circ(p))^2}.
\end{aligned}$$

It follows that

$$2\int \frac{g_0(k)g_{10}(k)dk}{\frac{1}{2}(p-k)^2+|k|-\lambda_\circ(p)} = -\int \frac{g_0(k)^2\{(p-k)k-|k|\}dk}{(\frac{1}{2}(p-k)^2+|k|-\lambda_\circ(p))^2},$$

and (5.14) is equal to

$$\frac{1}{-F_\lambda(p,\lambda_\circ(p))}\left(p^2+\int \frac{g_0(k)^2(p-k)pdk}{(\frac{1}{2}(p-k)^2+|k|-\lambda_\circ(p))^2}\right).$$

It is easy to check that the quantity inside the parentheses is equal to $p\cdot(\nabla_p F)(p,\lambda_\circ(p))$, and (5.14) is exactly equal to $\lambda_{\circ\rho}(\rho)$, which is strictly positive in a compact subset of $\{p\colon 0<|p|<\rho_c\}$ by virtue of Lemma 2.4. Thus we conclude that for some positive $\varepsilon>0$

$$\int_{\mathbf{R}^3}^{\oplus} f(H_0(p))\begin{pmatrix} p^2 & \langle g_{10}| \\ |g_{10}\rangle & (p-k)^2+|k| \end{pmatrix} f(H_0(p))dp, \geq \varepsilon U f(H_0)^2 U^*$$

and (5.12) holds in this case.
(b) Next we let $\lambda_0 > E_c \equiv \lambda_\circ(\rho_c) > 0$. Take $\delta_1>0$ such that $\lambda_0 - 3\delta_1 > E_c$ and let $f = f_{\lambda_0,\delta}$ for $0<\delta<\delta_1$. Then there exists a compact set Ξ of $\mathbf{R}^3$ such that $f(H_0(p)) = 0$ for all $p\notin\Xi$ and $0<\delta<\delta_1$. By virtue of (5.13), we have

$$\begin{aligned}
f(H_0)i[H_0,A]f(H_0) &\geq (\lambda_0-\delta)f(H_0)^2 \\
&+ U^*\int_{\Xi}^{\oplus} f(H_0(p))\begin{pmatrix} 0 & \langle(g_{10}-g_0)| \\ |(g_{10}-g_0)\rangle & 0 \end{pmatrix} f(H_0(p))dpU. \quad (5.15)
\end{aligned}$$

Denote by Z the operator represented by the direct integral on the right of (5.15). Since $H_0(p)$ is an analytic family of type A, $f_{\lambda_0,\delta}(H_0(p))$ is norm continuous with respect to p. We have $f_{\lambda_0,\delta}(H_0(p)) \to 0$ strongly as $\delta \to 0$, and also $f_{\lambda_0,\delta}(\lambda) f_{\lambda_0,\delta/4}(\lambda) = f_{\lambda_0,\delta/4}(\lambda)$. Using the compactness of Ξ, we conclude that the operator norm of Z converges to zero as $\delta \to 0$. Thus we conclude that there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$

$$f_{\lambda_0,\delta}(H_0) i[H_0, A] f_{\lambda_0,\delta}(H_0) \geq (\lambda_0 - 2\delta) f_{\lambda_0,\delta}(H_0)^2$$

and (5.12) holds also in this case.
(c) When $0 < \lambda_0 < E_c$, we take $\delta_0 > 0$ such that $0 < \lambda_0 - 2\delta_0 < \lambda_0 + 2\delta_0 < E_c$, and let $f = f_{\lambda_0,\delta}$ for $0 < \delta < \delta_0$. If δ_0 is taken sufficiently small, then, by virtue of Theorem 1.2, $\{p \in \mathbf{R}^3 \colon f(H_0(p)) \neq 0\}$ consists of two disjoint components $\Omega_1 = \{\rho_0 < |p| < \rho_1\}$ and $\Omega_2 = \{\rho_2 < |p| < \rho_3\}$, $0 < \rho_0 < \rho_1 < \rho_2 < \rho_3 < \rho_c$ such that $H_0(p)$ is purely absolutely continuous on $I_{\lambda,2\delta}$, when $p \in \Omega_1$, and $f(H_0(p)) = f(\lambda_\circ(p)) \mathbf{e}_p \otimes \mathbf{e}_p$ for $p \in \Omega_2$. Then, splitting the direct integral (5.8) into two parts, the one over Ω_1 and the other over Ω_2 and applying the arguments of (b) and (a), respectively, we obtain

$$f_{\lambda_0,\delta}(H_0) i[H_0, A] f_{\lambda_0,\delta}(H_0) \geq \varepsilon f_{\lambda_0,\delta}(H_0)^2,$$

where $\varepsilon = \min\{\min_{p \in \Omega_2}\{\lambda_{\circ\rho}(|p|)\}, \lambda_0 - 2\delta\}$. This completes the proof of (5.12).
(iii) We now study $f(H) i[H, A] f(H) \chi_2^2$. As above we write

$$\begin{aligned} f(H) i[H, A] f(H) \chi_2^2 = f(H) i[H, A] (f(H) - f(K_0)) \chi_2^2 + \\ f(H) i[H, A] [f(K_0), \chi_2] \chi_2 + f(H) [i[H, A], \chi_2] f(K_0) \chi_2 \\ + (f(H) - f(K_0)) \chi_2 i[H, A] f(K_0) \chi_2 + [f(K_0), \chi_2] i[H, A] f(K_0) \chi_2 \\ + \chi_2 f(K_0) i[H, A] f(K_0) \chi_2. \end{aligned}$$

We have shown in Lemma 5.2 that $(f(H) - f(K_0))\chi_2$ and $[f(K_0), \chi_2]$ are compact operators in $\mathcal{H}$. Since $i[H, A]$ has the same form as H, the argument for proving the compactness of $f(H)[H, \chi_2] f(H)$ in (i) of the proof of Lemma 5.2 shows that $f(H)[i[H, A], \chi_2] f(K_0)$ is also compact. We further use that

$$\chi_2 f(K_0) \begin{pmatrix} 0 & \langle g_1| \\ |g_1\rangle & 0 \end{pmatrix} f(K_0) \chi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and conclude that

$$\begin{aligned} f(H) i[H, A] f(H) \chi_2^2 &= \chi_2 f(K_0) \begin{pmatrix} i[h, A_x] & 0 \\ 0 & i[h, A_x] + |D_y| \end{pmatrix} f(K_0) \chi_2 + \mathcal{C}_2 \\ &= 0 \oplus \chi_{12} f(h + |D_y|)(i[h, A_x] + |D_y|) f(h + |D_y|) \chi_{12} + \mathcal{C}_2, \quad (5.16) \end{aligned}$$

where $\mathcal{C}_2$ is compact in $\mathcal{H}$. We study $f(h+|D_y|)(i[h,A_x]+|D_y|)f(h+|D_y|)$. The Fourier transform with respect to the variables k, and the direct integral representation, imply that this operator is unitarily equivalent to the operator

$$\begin{aligned} f(h+|k|)&(i[h,A_x]+|k|)f(h+|k|) \\ &= \int_{\mathbf{R}^3}^{\oplus} f(h+|k|)(i[h,A_x]+|k|)f(h+|k|)dk. \end{aligned}$$

We use the following lemma due to [7].

Lemma 5.6. *Let $c = \inf\sigma(h) - 1$ and $s(\lambda)$ be defined by*

$$s(\lambda) = \begin{cases} \sup(\Lambda \cap (-\infty, \lambda]), & \lambda \geq \inf\sigma(h), \\ c, & \lambda < \inf\sigma(h), \end{cases}$$

where $\Lambda = \{E_0, E_1, \ldots\} \cup \{0\}$. Then, for any $\lambda_0 \in \mathbf{R}$ and $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$\begin{aligned} f(h+|D_y|)&(i[h,A_x]+|D_y|)f(h+|D_y|) \\ &\geq (\lambda_0 - s(\lambda_0+\varepsilon) - 2\varepsilon)f(h+|D_y|)^2 \end{aligned}$$

for any $f = f_{\lambda_0,\delta}$ with $0 < \delta < \delta_0$.

Supppose now $\lambda_0 \notin \Theta$ and $\lambda_0 > \inf\sigma(h) - 1/2$. Then, for sufficiently small $\varepsilon > 0$, we have $s(\lambda_0+\varepsilon) = s(\lambda_0) < \lambda_0$ and, hence, $\lambda_0 - s(\lambda_0+\varepsilon) - 2\varepsilon \geq \varepsilon$. It follows from Lemma 5.6 that there exists $\delta_0 > 0$ such that

$$f(h+|D_y|)(i[h,A_x]+|D_y|)f(h+|D_y|) \geq \varepsilon f(h+|D_y|)^2 \tag{5.17}$$

for all $0 < \delta < \delta_0$, $f = f_{\lambda_0,\delta}$. Note that (5.17) holds for $\lambda_0 \leq \inf\sigma(h) - 1/2$ whenever $\delta < 1/4$, since $f(h+|D_y|) = 0$ then. From (5.16) and (5.17) we have

$$f(H)i[H,A]f(H)\chi_2^2 \geq \varepsilon\chi_2 f(K_0)^2\chi_2 + \mathcal{C}_2. \tag{5.18}$$

Combining the results in (i), (ii), and (iii), we see that for any $\lambda_0 \notin \Theta$, there exist $\varepsilon > 0$ and $\delta > 0$ and a compact operator $\mathcal{C}_3$ such that for $f = f_{\lambda_0,\delta}$

$$f(H)i[H,A]f(H) \geq \varepsilon(\chi_1 f(H_0)^2\chi_1 + \chi_2 f(K_0)^2\chi_2) + \mathcal{C}_3. \tag{5.19}$$

Then, using the compactness of $(f(H)-f(H_0))\chi_1$, $[\chi_1, f(H_0)]$, $[f(K_0), \chi_2]$ and $(f(H)-f(K_0))\chi_2$ again, we derive from (5.19)

$$f(H)i[H,A]f(H) \geq \varepsilon f(H) + \mathcal{C}_4. \tag{5.20}$$

with another compact operator $\mathcal{C}_4$. Lemma 5.5 follows from (5.20) immediately. $\square$

Once the Mourre estimate is estabilished, it is easy to check the conditions of Mourre theory as stated in Lemma 3.1 to conclude the proof of statement (1) of Theorem 1.6. We omit the details here.

5.3 Existence of the ground state

We prove here statement (2) of Theorem 1.6, the existence of the ground state. Since $(-\infty, \Sigma_{\text{ess}}) \cap \sigma(H)$ is discrete, $\Sigma_{\text{ess}} = \min\{\Sigma, E_0\}$, $\Sigma < 0$ and $E_0 < 0$ by assumption, it suffices to show

$$E \equiv \inf \sigma(H) \leq \Sigma + E_0, \tag{5.21}$$

since $\Sigma + E_0 < \Sigma_{\text{ess}}$. We prove this by borrowing ideas in [10]. To simplify the notation we assume that $\mu = 1$ below. We denote by $\langle \mathbf{f}, \mathbf{g} \rangle$ the inner product in $\mathcal{H}$. For $\mathbf{f} = (f_0, f_1) \in D(H)$, we have

$$\begin{aligned}\langle \mathbf{f}, H\mathbf{f} \rangle = &\int \left(\tfrac{1}{2}|\nabla_x f_0(x)|^2 + V(x)|f_0(x)|^2\right) dx \\ &+ 2\,\mathrm{Re} \iint f_0(x)\check{g}(y-x)\overline{f_1(x,y)}dxdy \\ &+ \iint \left(\tfrac{1}{2}|\nabla_x f_1(x,y)|^2 + V(x)|f_1(x,y)|^2 + ||D_y|^{1/2} f_1(x,y)|^2\right) dxdy.\end{aligned}$$

For any $\varepsilon > 0$ there exists $\mathbf{f} \in C_0^\infty(\mathbf{R}^3) \oplus C_0^\infty(\mathbf{R}^6)$ such that

$$\langle \mathbf{f}, H_0\mathbf{f} \rangle < \Sigma + \varepsilon, \quad \|\mathbf{f}\| = 1.$$

We let $\phi(x)$ be the *real-valued* normalized ground state of $-\frac{1}{2}\Delta + V$ (it exists due to our assumption $E_0 < 0$), viz. $(-\frac{1}{2}\Delta + V)\phi = E_0\phi$, and compute $\langle \phi\mathbf{f}, H\phi\mathbf{f} \rangle$, using the fact that

$$(\phi f_0, (-\tfrac{1}{2}\Delta)(\phi f_0))_x = (\phi f_0, (-\tfrac{1}{2}\Delta\phi) f_0)_x + (\phi, \tfrac{1}{2} \left|\nabla f_0\right|^2 \phi)_x,$$

where $(\cdot,\cdot)_x$ denotes the inner product in the $x \in \mathbf{R}^3$ variable. We find that $\langle \phi\mathbf{f}, H\phi\mathbf{f} \rangle$ is equal to

$$\begin{aligned}&\int \Big[\tfrac{1}{2}|\nabla f_0(x)|^2 + 2\,\mathrm{Re}(|g\rangle f_0, f_1)_y + \tfrac{1}{2}\|\nabla_x f_1(x,\cdot)\|_y^2 \\ &\qquad\qquad + E_0\|\mathbf{f}(x,\cdot)\|_y^2 + \||D_y|^{1/2} f_1(x,\cdot)\|_y^2\Big] |\phi(x)|^2 dx \\ &\qquad\qquad\qquad = \int \left[(H_0\mathbf{f}, \mathbf{f})_y + E_0\|\mathbf{f}(x,\cdot)\|_y^2\right] |\phi(x)|^2 dx,\end{aligned}$$

where $(\cdot,\cdot)_y$ and $\|\cdot\|_y$ denote the inner product and the norm with respect to the y-variable. We now replace $\mathbf{f}(x,y)$ by $\mathbf{f}_z(x,y) = \mathbf{f}(x-z, y-z)$ and change the variables $(x,y) \to (x+z, y+z)$, in order to get

$$\langle \phi\mathbf{f}_z, H\phi\mathbf{f}_z \rangle = \int \left[(H_0\mathbf{f}, \mathbf{f})_y + E_0\|\mathbf{f}(x,\cdot)\|_y^2\right] |\phi(x+z)|^2 dx.$$

Here we used the translation invariance of H_0. Now integrate both sides of this equation with respect to z. We then find

$$\int \langle \phi \mathbf{f}_z, (H - E_0 - \Sigma - \varepsilon)\phi \mathbf{f}_z \rangle \, dz = \langle \mathbf{f}, (H_0 - \Sigma - \varepsilon)\mathbf{f} \rangle < 0.$$

It follows that there exists $z \in \mathbf{R}^3$ such that $\langle \phi \mathbf{f}_z, (H - E_0 - \Sigma - \varepsilon)\phi \mathbf{f}_z \rangle < 0$. Since ε is arbitrary, we obtain (5.21).

6 Proof of Theorem 1.7

In this section we prove the existence of the wave operators. In what follows we take $\mu = 1$ to simplify the notation. We begin with

Proof of Existence of the Limits (1.9)**.** Since e^{itH} and $L^2(\mathbf{R}^3_k) \ni f \mapsto e^{-itE-it|k|}\phi \otimes f \in \mathcal{H}$ are isometric operators, it suffices to show that the limits exist for every $f \in C_0^\infty(\mathbf{R}^3 \setminus \{0\})$. For such f the map

$$t \mapsto F_t = e^{itH} \begin{pmatrix} 0 \\ e^{-itE-it|k|}\phi(x) f(k) \end{pmatrix}$$

is strongly differentiable, and we can easily compute to obtain

$$\frac{d}{dt} F_t = \begin{pmatrix} f_t \\ 0 \end{pmatrix}, \quad f_t = ie^{-itE}\phi(x) \int_{\mathbf{R}^3} e^{ixk-it|k|} g_0(k) f(k) dk.$$

It suffices to show that $\|f_t\|$ is integrable with respect to t. We estimate the integral with respect to k. Since $|\nabla_k(xk - t|k|)| = |x - t\hat{k}| \geq ||x| - |t||$, it follows by integration by parts that for any positive N

$$\left| \int_{\mathbf{R}^3} e^{ixk-it|k|} g_0(k) f(k) dk \right| \leq C_N (1 + ||x| - |t||)^{-N}.$$

It follows, by choosing ε such that $0 < \varepsilon < \beta - 2$, that

$$\begin{aligned} \int |f_t(x)|^2 dx &\leq C_N |t|^{-N\varepsilon} \int_{\{x\colon ||x|-|t||>|t|^\varepsilon\}} |\phi(x)|^2 dx \\ &\quad + C_N \int_{\{x\colon ||x|-|t||\leq|t|^\varepsilon\}} |\phi(x)|^2 dx \\ &\leq C_N (|t|^{-N\varepsilon} + \langle t \rangle^{2+\varepsilon-2\beta}) \leq C_N (|t|^{-2-2\varepsilon} + \langle t \rangle^{-\beta}), \end{aligned}$$

for N sufficiently large. Thus $\|f_t\|$ is integrable.

Proof of Existence of the Limits (1.8) By virtue of Theorem 1.2, it suffices to prove that the following two limits exist in the strong topology of $\mathcal{H}$:

$$\lim_{t\to\pm\infty} e^{itH} \begin{pmatrix} 0 \\ e^{it\frac{1}{2}\Delta - it|k|} f(x,k) \end{pmatrix}, \quad f \in L^2(\mathbf{R}^3_x \times \mathbf{R}^3_k). \tag{6.1}$$

$$\lim_{t\to\pm\infty} e^{itH} \begin{pmatrix} e^{-it\lambda_\circ(D_x)} f_{1,0} \\ e^{-ikx} e^{-it\lambda_\circ(D_x)} f_{1,1} \end{pmatrix}, \quad \mathbf{f}_1 \in \mathcal{H}_{\rm one}. \tag{6.2}$$

Proof of Existence of the Limit (6.1)**.** Functions of the form

$$\sum_{j=1}^N u_j(x) v_j(k),$$

with $\hat{u}_j \in C_0^\infty(\mathbf{R}^3_\xi \setminus \{0\})$ and $v_j \in C_0^\infty(\mathbf{R}^3_k)$, are dense in $L^2(\mathbf{R}^3_x \times \mathbf{R}^3_k)$. Thus it suffices to consider $f(x,k) = u(x)v(k)$ with u and v as above. We write again

$$F_t = e^{itH} \begin{pmatrix} 0 \\ e^{it\frac{1}{2}\Delta - it|k|} f(x,k) \end{pmatrix}.$$

We compute the strong derivative with respect to t.

$$\frac{d}{dt} F_t = i \begin{pmatrix} \langle g| e^{it\frac{1}{2}\Delta - it|k|} f \\ V e^{it\frac{1}{2}\Delta - it|k|} f \end{pmatrix} = \begin{pmatrix} g_{0t}(x) \\ g_{1t}(x,k) \end{pmatrix}.$$

We estimate $g_{1t}(x,k)$ first. We have

$$g_{1t}(x,k) = iV(x)(e^{it\frac{1}{2}\Delta} u)(x) e^{-it|k|} v(k),$$

such that $\|g_{1t}\|_{\mathcal{H}_1} = \|V e^{it\frac{1}{2}\Delta} u\|_2 \|v\|_2$. It follows by the well known estimate for the existence of the wave operator for the two body short potentials (see for example [16]) that $\|g_{1t}\|_{\mathcal{H}_1}$ is integrable with respect to t. The function $g_{0t}(x)$ can be written in the form

$$g_{0t}(x) = i \int g_0(k) e^{ikx - it|k|} v(k) dk \cdot e^{it\frac{1}{2}\Delta} u(x).$$

By Assumption 1.1 and $v \in C_0^\infty$, it follows that $w_t(k) = g_0(k) v(k) e^{-it|k|}$ belongs to L^2 with $\|w_t\|_2 = c_0$ independent of t. Thus we can estimate g_{0t} as follows, using the fact that the integral term is the inverse Fourier transform of w_t (up to a constant),

$$\|g_{0t}\|_2 \le (2\pi)^{3/2} \|\check{w}_t\|_2 \|e^{it\frac{1}{2}\Delta} u\|_\infty \le C c_0 |t|^{-3/2} \|u\|_1.$$

Here we have used the estimate $\|e^{it\frac{1}{2}\Delta}\|_{L^1(\mathbf{R}^3)\to L^\infty(\mathbf{R}^3)} \le c|t|^{-3/2}$. This estimate shows that $\|g_{0t}\|_2$ is integrable with respect to t, such that the limits exist.

Proof of Existence of the Limit (6.2). Since $\cup_j C_0^\infty(G_j)$ is dense in $L^2(B(\rho_c))$ (see (4.6)),

$$\{\phi(p)\mathbf{e}_p(k)\colon \phi \in \cup_j C_0^\infty(G_j)\}$$

is dense in $\{h(p)\mathbf{e}_p(k)\colon h \in L^2(B(\rho_c))\}$. $F_\lambda(p, \lambda_\circ(p))$ is smooth and strictly negative in $B(r)$ for any $r < \rho_c$, it follows that it suffices to prove the existence of the limits, when $\hat{f}_{1,0} \in C_0^\infty(G_j)$ for some j. Using the fact that $\lambda_\circ(p)$ is the eigenvalue of $H_{00}(p)$, it is easy to see that

$$\frac{d}{dt} e^{itH} \begin{pmatrix} e^{-it\lambda_\circ(D_x)} f_{1,0} \\ e^{-ikx} e^{-it\lambda_\circ(D_x)} f_{1,1} \end{pmatrix} = ie^{itH} \begin{pmatrix} Ve^{-it\lambda_\circ(D_x)} f_{1,0} \\ Ve^{-ikx} e^{-it\lambda_\circ(D_x)} f_{1,1} \end{pmatrix}.$$

Thus, it suffices to show that both $\|Ve^{-it\lambda_\circ(D_x)} f_{1,0}\|$ and $\|Ve^{-it\lambda_\circ(D_x)} f_{1,1}\|$ are integrable functions of $|t| \geq 1$. But we have seen in Lemma 4.1 that,

$$|e^{-i\lambda_\circ(D_x)} f_{1,0}(x)| \leq C_1 \, |t|^{-3/2} \quad \text{and} \quad |e^{-i\lambda_\circ(D_x)} f_{1,1}(x,k)| \leq C_2(k) \, |t|^{-3/2}.$$

and, moreover, if $|x|/t \notin [\alpha, \beta]$,

$$|e^{-i\lambda_\circ(D_x)} f_{1,0}(x)| \leq C_1 \, |t|^{-N} \, \langle x \rangle^{-N}, \;\; |e^{-i\lambda_\circ(D_x)} f_{1,1}(x,k)| \leq C_2(k) \, |t|^{-N} \, \langle x \rangle^{-N},$$

where $C_2(k)$ is square integrable over $\mathbf{R}^3$. Thus, we have

$$\begin{aligned} &\left(\|Ve^{-i\lambda_\circ(D_x)} f_{1,0}\|^2\right)^{1/2} \\ &\qquad \leq \left(\int_{\alpha<|x|<\beta} |V(tx)|^2 dx \right)^{1/2} + Ct^{-N} \left(\int_{\mathbf{R}^3} |V(x)|^2 \langle x \rangle^{-2N} dx \right)^{1/2} \end{aligned}$$

and the right hand side is integrable by the short range assumption on V. The integrability of $\|Ve^{-i\lambda_\circ(D_x)} f_{1,1}\|$ may be proved similarly. This completes the proof of the Theorem.

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