ON SUM RULES OF SPECIAL FORM FOR JACOBI MATRICES

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Abstract. The purpose of this short communication is to give a sketch of the proof of a result. Its complete proof is to appear elsewhere.

We use sum rules of a special form to study spectral properties of Jacobi matrices. As a consequence of the main theorem, we obtain a discrete counterpart of a result by Molchanov-Novitskii-Vainberg [7].

Introduction

The intent of this short communication is to give a brief sketch of the proof of a theorem. Its complete version is to appear elsewhere.

Recently, the Case sum rules [1, 2] were efficiently used to relate properties of elements of a Jacobi matrix of certain class with its spectral properties and vice versa. For instance, spectral data of Jacobi matrices being a Hilbert-Schmidt perturbation of the free Jacobi matrix (see (1)) were characterized in [4]. Different classes of Jacobi matrices were studied in [5, 6].

However, the sum rules become more and more complex with increasing order. In this note, we suggest a modification of the method that permits us to work with higher order sum rules. In particular, we obtain sufficient conditions for a Jacobi matrix to satisfy certain constraints on its spectral measure (see Theorem 1).

We consider a Jacobi matrix

$$J = J(a,b) = \begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix},$$

where $a = \{a_k\}, a_k > 0$, and $b = \{b_k\}, b_k \in \mathbb{R}$. We assume that J is a compact perturbation of the free (or Chebyshev) Jacobi matrix J_0,

$$J_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}. \tag{1}$$

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A scalar spectral measure $\sigma = \sigma(J)$ of the matrix is defined by the formula

$$((J-z)e_0, e_0) = \int_{\mathbb{R}} \frac{d\sigma(x)}{x-z}$$

with $z \in \mathbb{C}\backslash\mathbb{R}$. In our situation, the absolutely continuous spectrum $\sigma_{ac}(J)$ of J fills in $[-2,2]$, and the discrete spectrum consists of two sequences $\{x_j^\pm\}$ with properties $x_j^- < -2, x_j^- \to -2$, and $x_j^+ > 2, x_j^+ \to 2$.

Let $\partial a = \{a_k - a_{k-1}\}$. For a given a and a $k \in \mathbb{N}$, we construct a sequence $\gamma_k(a)$ by formula

$$(\gamma_k(a))_j = \alpha_j^k - \alpha_j \dots \alpha_{j+k-1},$$

where $\alpha = a - 1$ and 1 is a sequence of units.

Theorem 1. *Let $J = J(a,b)$ be a Jacobi matrix described above. If*

$$\begin{aligned} &i) \quad a-1, b \in l^{m+1}, \quad \partial a, \partial b \in l^2, \\ &ii) \quad \gamma_k(a) \in l^1, \ k = 3, [(m+1)/2], \end{aligned} \tag{2}$$

then

$$i') \int_{-2}^{2} \log \sigma'(x) \cdot (4-x^2)^{m-1/2}\, dx > -\infty, \quad ii') \sum_j (x_j^{\pm 2} - 4)^{m+1/2} < \infty. \tag{3}$$

When $m = 1$, the theorem gives a half of [4], Theorem 1.

It is easy to give simple conditions sufficient for $\gamma_k(a) \in l^1$. For instance, put

$$(A_k(a))_j = \alpha_{j+1} + \dots + \alpha_{j+k-1} - (k-1)\alpha_j.$$

Then relations $a - 1 \in l^{m+1}, \partial a \in l^2$, and $A_k(a) \in l^{q(k,m)}$, $q(k,m) = (m+1)/(m+2-k)$, imply that $\gamma_k(a) \in l^1$. In particular, we have the following corollary.

Corollary 1. *Theorem 1 holds if condition* (2) *is replaced with*

$$A_k(a) \in l^{q(k,m)}, \ q(k,m) = (m+1)/(m+2-k),$$

where $k = 3, [\frac{m+1}{2}]$.

We observe that relation (2) is trivially true in the case of a discrete Schrödinger operator, i.e., when $J = J(1,b)$.

Corollary 2. *Let $J = J(1,b)$. If $b \in l^{m+1}, \partial b \in l^2$, then inequalities* (3) *hold.*

Note that assumptions of Theorem 1 may be slightly weakened in this setting. Namely, the corollary is still true if $b \in l^{m+2}$, m being even. The corollary is a direct counterpart of a result from [7] for a "continuous" Schrödinger operator on a half-line.

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1. Proof of Theorem 1

The main tool used in the proof is a sum rule of a special type, see [4, 6, 9, 10] in this connection. First, we obtain it assuming rank $(J - J_0) < \infty$. The passage to the limit is carried out later.

Applying methods of [10], we see that

$$\frac{1}{2\pi}\int_{-2}^{2}\log\frac{1}{\sigma'(x)}\cdot(4-x^2)^{m-1/2}\,dx+\sum_j G_m(x_j^{\pm})=\Psi_m(J),$$

where $\Psi_m(J) = \Psi_m(a, b)$, and

$$G_m(x) = (-1)^{m+1}C_0(x^2-4)^{m+1/2} + O((x^2-4)^{m+3/2})$$

with $x \in \mathbb{R}\backslash[-2, 2]$, C_0 being a positive constant. An elementary, but long and tedious computation gives that

$$\Psi_m(J) = \operatorname{tr}\left\{\sum_{k=1}^{m}\frac{(-1)^{k+1}}{2^{2k+1}k}\tilde{C}_{2m-1}^{2k-1}(J^{2k}-J_0^{2k}) - \frac{(2m-1)!!}{(2m)!!}\log A\right\}, \tag{4}$$

where $A = \operatorname{diag}\{a_k\}$ and $\tilde{C}_m^k = \frac{m!!}{(m-k)!!k!!}$. Notation $k!!$ is used for "even" or "odd" factorials.

The following lemma plays a central role in the whole proof.

Main Lemma. *Let $J = J(a, b)$. We have*

$$|\Psi_m(J)| \le C_1\big(||a-1||_{m+1} + ||b||_{m+1} + ||\partial a||_2 + ||\partial b||_2 + \sum_{k=3}^{[(m+1)/2]}||\gamma_k(a)||_1\big), \tag{5}$$

where C_1 depends on $||J||$ only.

Above, norms $||.||_p$ refer to the standard l^p-space norms.

With exception of the lemma, the proof of Theorem 1 goes along standard lines (see [4, 5, 6, 9]). We quote only its main steps.

Proof of Theorem 1. Define $\Phi_m(J)$ as

$$\Phi_m(J) = \Phi_m(\sigma) = \Phi_{m,1}(\sigma)+\Phi_{m,2}(\sigma) = \frac{1}{2\pi}\int_{-2}^{2}\log\frac{1}{\sigma'(x)}\cdot(4-x^2)^{m-1/2}\,dx+\sum_j G_m(x_j^{\pm}).$$

We have to show that $\Phi_m(J) < \infty$.

We put $a_N = \{(a_N)_k\}$ and $a'_N = \{(a'_N)_k\}$, where

$$(a_N)_k = \begin{cases} a_k, & k \le N,\\ 1, & k > N,\end{cases} \qquad (a'_N)_k = \begin{cases} 1, & k \le N,\\ a_k, & k > N.\end{cases}$$

Define sequences b_N, b'_N in the same way (of course, with 1's replaced by 0's). Let $J_N = J(a_N, b_N)$. As we readily see, $a'_N - 1, b_N \to 0$, $\partial a'_N, \partial b'_N \to 0$, and

$\gamma_k(a'_N) \to 0$ in corresponding norms, as $N \to \infty$. By the Main Lemma, we have for $N' = N - m$

$$\begin{aligned} |\Psi_m(J) - \Psi_m(J_N)| &\leq \Psi_m(a'_{N'}, b_{N'}) \leq C_1(||a'_{N'} - 1||_{m+1} + ||b_{N'}||_{m+1} \\ &+ ||\partial a_{N'}||_2 + ||\partial b_{N'}||_2 + \textstyle\sum_k ||\gamma_k(a'_{N'})||_1), \end{aligned}$$

or, $\Psi_m(J_N) \to \Psi_m(J)$, as $N \to \infty$. On the other hand, $(J_N - z)^{-1} \to (J - z)^{-1}$, for $z \in \mathbb{C}\backslash\mathbb{R}$, and, consequently, $\sigma_N \to \sigma$ weakly. Looking at [4], Corollary 5.3 and Theorem 6.2, we get

$$\Phi_{m,1}(\sigma) \leq \liminf_N \Phi_{m,1}(\sigma_N)$$

and

$$\lim_{N\to\infty} \Phi_{m,2}(\sigma_N) = \Phi_{m,2}(\sigma).$$

We bound the latter quantity recalling [3], Theorem 2

$$|\Psi_{m,2}(J)| = \sum_j |G_m(x_j^\pm)| \leq C_2(||a-1||_{m+1}^{m+1} + ||b||_{m+1}^{m+1})$$

with some constant C_2. Summing up, we obtain

$$\Phi(\sigma) \leq \limsup_N \Phi(\sigma_N) = \limsup_N \Psi(J_N) = \lim_{N\to\infty} \Psi(J_N) = \Psi(J).$$

The proof is complete. □

Remark 1. *The theorem gives one more proof of* [3]*, Theorem 2, when m is odd.*

2. Sketch of the proof of the Main Lemma

We begin with considering expressions $\operatorname{tr}(J^{2k} - J_0^{2k})$, arising in (4). Defining $V = J - J_0 = J(a-1, b)$, we have

$$\operatorname{tr}(J^{2k} - J_0^{2k}) = \operatorname{tr} \sum_{p=1}^{2k} \sum_{i_1+\ldots+i_p=2k-p} VJ_0^{i_1}\ldots VJ_0^{i_p}.$$

We prove the Main Lemma in two steps. First, we reduce the situation to a commutative one. To do this, we bound expressions $|\operatorname{tr}(VJ_0^{i_1}\ldots VJ_0^{i_p} - V^pJ_0^{2k-p})|$ using properties of the commutator $[V, J_0] = VJ_0 - J_0V$. On the second stage, we exploit specifics of $\Psi_m(J)$ to get straightforward estimates of terms obtained after the "commutation".

Lemma 1. *Let $\mathbf{i} = (i_1, \ldots, i_p)$ and $\sum_s i_s = n$. Then*

$$\begin{aligned} VJ_0^{i_1}\ldots VJ_0^{i_p} &= V^pJ_0^n + \sum_{\substack{l_1+l_2+l_3 = p, \\ p_1+p_2+p_3 = n}} C_{\mathbf{l},\mathbf{p}}\, J_0^{p_1}V^{l_1}[V^{l_2}, J_0^{p_2}]V^{l_3}J_0^{p_3} \\ &+ \textstyle\sum_i^{M_{\mathbf{i},p}} A_k[V, J_0]B_k[V, J_0]C_k, \end{aligned}$$

where $\mathbf{p} = (p_1, p_2, p_3), \mathbf{l} = (l_1, l_2, l_3)$*, and* A_k, B_k, C_k *are some bounded operators.*

This proposition leads to the following lemma.

Lemma 2. *Let* $\sum_s i_s = 2k - p$*. We have*

$$|\mathrm{tr}\,(VJ_0^{i_1} \dots VJ_0^{i_p} - V^p J_0^{2k-p})| \le C_3(||\partial a||_2 + ||\partial b||_2)$$

with C_3 *depending on* $||V||$ *only.*

The lemma exactly says that, modulo bounded terms, we may assume operators V and J_0 to commute. Turning back to (4), we see that the problem is reduced to estimating $\Psi'_m(J)$,

$$\Psi'_m(J) = \mathrm{tr}\left\{ \sum_{p=1}^{2m} V^p F_p(J_0) - \frac{(2m-1)!!}{(2m)!!} \log(I + \tilde{\alpha}) \right\}, \tag{6}$$

where $\tilde{\alpha} = \mathrm{diag}\,\{\alpha_k\} = A - I$, and

$$F_p(J_0) = \sum_{k=[(p+1)/2]}^{m} \frac{(-1)^{k+1}}{2^{2k+1}k} \tilde{C}_{2m-1}^{2k-1} C_{2k}^{p}\, J_0^{2k-p}.$$

Here, C_k^p is a usual binomial coefficient.

Observe that for $p \ge m+1$ we have

$$|\mathrm{tr}\,(V^p F_p(J_0))| \le ||F_p(J_0)||\, ||V^p||_{S_1} \le C_4(||a-1||_{m+1}^{m+1} + ||b||_{m+1}^{m+1}),$$

where $||.||_{S_1}$ is the norm in the class of nuclear operators. Hence, it remains to bound the first m terms in (6). Of course, we have

$$\log(I + \tilde{\alpha}) = \sum_{p=1}^{2m} \frac{(-1)^{p+1}}{p}\, \tilde{\alpha}^p + O(\tilde{\alpha}^{2m+1}).$$

Set $J_{0,p}$ to be a symmetric matrix with 1's on p-th auxiliary diagonals and 0's elsewhere. Surprisingly, the following lemma holds.

Lemma 3. *We have*

$$F_p(J_0) = (-1)^{p+1} \frac{(2m-1)!!}{2p(2m)!!} J_{0,p}.$$

Combining this with explicit form of V^p and the series expansion for $\log(I + \tilde{\alpha})$, we get the required bound (5).

References

[1] K. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Mathematical Phys., **15** (1974), 2166–2174.

[2] K. Case, *Orthogonal polynomials, II*, J. Mathematical Phys., **16** (1975), 1435–1440.

[3] D. Hundertmark, B. Simon, *Lieb-Thirring inequalities for Jacobi matrices*, J. Approx. Theory, **118** (2002), 106–130.

[4] R. Killip, B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, to appear in Annals of Math.
[5] S. Kupin, *On a spectral property of Jacobi matrices*, submitted.
[6] A. Laptev, S. Naboko, O. Safronov, *On new relations between spectral properties of Jacobi matrices and their coefficients*, to appear.
[7] S. Molchanov, M. Novitskii, B. Vainberg, *First KdV integrals and absolutely continuous spectrum for 1-D Schrödinger operator*, Comm. Math. Phys., **216** (2001), no. 1, 195–213.
[8] O. Safronov, *The spectral measure of a Jacobi matrix in terms of the Fourier transform of the perturbation*, submitted.
[9] B. Simon, A. Zlatos, *Sum rules and the Szego condition for orthogonal polynomials on the real line*, submitted.
[10] P. Yuditskii, private communication.

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