C^∞ GENERICITY OF POSITIVE TOPOLOGICAL ENTROPY FOR GEODESIC FLOWS ON S^2

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ABSTRACT. We show that there is a C^∞ open and dense set of positively curved metrics on S^2 whose geodesic flow has positive topological entropy, and thus exhibits chaotic behavior. The geodesic flow for each of these metrics possesses a horseshoe and it follows that these metrics have an exponential growth rate of hyperbolic closed geodesics. The positive curvature hypothesis is required to ensure the existence of a global surface of section for the geodesic flow. Our proof uses a new and general topological criterion for a surface diffeomorphism to exhibit chaotic behavior.

Very shortly after this manuscript was completed, the authors learned about remarkable recent work by Hofer, Wysochi, and Zehnder [HWZ1, HWZ2] on three dimensional Reeb flows. In the special case of geodesic flows on S^2, they show that the geodesic flow for a C^∞ dense set of Riemannian metrics on S^2 possesses either a global surface of section or a heteroclinic connection. It then immediately follows from the proof of our main theorem that there is a C^∞ open and dense set of Riemannian metrics on S^2 whose geodesic flow has positive topological entropy.

This concludes a program to show that every orientable compact surface has a C^∞ open and dense set of Riemannian metrics whose geodesic flow has positive topological entropy.

Introduction

It has long been known that the geodesic flow for a Riemannian metric of negative curvature possesses chaotic dynamics with the strongest possible stochastic behavior. The flow is not only ergodic, but it also has the Bernoulli property. Thus, the geodesic flow has positive Liouville entropy, and of course, positive topological entropy. Topological and metric entropies are among the most important global invariants of smooth dynamical systems. Topological entropy characterizes the total exponential complexity of the orbit structure with a single number. Metric entropy with respect to an invariant measure codes the exponential growth rate of the statistically significant orbits. Geodesic flows in negative curvature are the most interesting example of a uniformly hyperbolic dynamical system, for which there is a highly developed theory.

The dynamics of geodesic flows for positively curved metrics is quite different. The Gauss-Bonnet Theorem tells us that a positively curved surface must be topologically a sphere. The most common examples of positively curved surfaces are the round sphere (and other surfaces of revolution) and the tri-axial ellipsoid. Both of these examples possess simple dynamics (i.e., their geodesic flows are not ergodic and they have zero

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entropies.) One might think that the simple topology of the sphere could be an obstruction for the geodesic flow to have complicated dynamics. This is not the case. Donnay [D1] and Burns and Gerber [BG] have constructed smooth (and real analytic) metrics on the sphere whose geodesic flows are Bernoulli. However, in these and in all later examples, the hyperbolicity induced by the negative curvature is the mechanism that causes the complicated dynamics.

The outstanding open question(s) in the interface of geometry and ergodic theory is whether such examples exist in positive curvature, i.e., does there exist a smooth Riemannian metric on S^2 with positive curvature whose geodesic flow is ergodic or has positive Liouville entropy? At present, this problem seems intractable.

In an earlier work [KW], the authors constructed the first C^∞ examples of positively curved surfaces whose geodesic flows have positive topological entropy. These examples were small perturbations of the tri-axial ellipsoid and could be obtained arbitrarily C^∞ close to the round metric. A major cause of subtlety in this area is that the perturbations are made on the surface while the phase space for the geodesic flow is the unit tangent bundle. Later the second author [W] observed there is a C^∞ open and dense set of positively curved metrics with $1/4-$pinching on S^2 whose geodesic flow has positive topological entropy. The pinching assumption is required to obtain a non-hyperbolic closed geodesic.

In a recent manuscript [CP], Contreras and Paternain prove that the set of C^∞ metrics on S^2 whose geodesic flow has positive topological entropy is open and dense in the C^2 topology. The main contribution in [CP] is to show that for a given metric, *if* there exists a C^2 neighborhood of metrics all of whose closed geodesics are hyperbolic, then the closure of the set of closed geodesics has a dominated splitting [Man] (in the sense of Mañé). They apply techniques developed by Mañé and extended by others, including Contreras, to show that for such metrics the closure of the set of closed geodesics contains a locally maximal hyperbolic set, and as a consequence, the geodesic flow has positive topological entropy. They need to know that every positively curved metric on S^2 has infinitely many closed geodesics, which is a deep result of Franks [F].

Our main result is Theorem 1, which states that the set of positively curved C^∞ metrics on S^2 whose geodesic flow has positive topological entropy is open and dense in the C^∞ topology. These geodesic flows exhibit chaotic behavior. We require positively curved metrics to insure the existence of a global surface of section for the geodesic flow. Our approach is heavily topological, and most of our contribution entails studying topological properties of diffeomorphisms of an annulus. Our proof does not require Franks' theorem. Thus our approach is quite different from that of Contreras and Paternain, and we believe it is conceptually quite a bit simpler.

On the one hand, our proof applies only to positively curved metrics, while the proof by Contreras and Paternain applies to all metrics on S^2. On the other hand, we prove denseness of metrics with positive topological entropy in the C^∞ topology, and not just in the C^2 topology. Furthermore, given a positively curved metric whose geodesic flow has zero topological entropy, we can find an arbitrarily small perturbation of the Riemannian metric supported in an arbitrarily small neighborhood of a point on a closed geodesic, to obtain a metric whose geodesic flow has positive topological entropy. Contreras and Paternain must use non-local perturbations (in this sense) to obtain their C^2 denseness result.

Very shortly after this manuscript was completed, the authors learned about remarkable recent work by Hofer, Wysochi, and Zehnder [HWZ1, HWZ2] on three dimensional Reeb flows. In the special case of geodesic flows on S^2, they show that the geodesic flow for a C^∞ dense set of Riemannian metrics on S^2 possesses either a global surface of section (which is a disk) or a heteroclinic connection. The dense set of metrics consists of those metrics for which no closed geodesic is parabolic and for which all intersections of stable and unstable manifolds for hyperbolic closed geodesics are transversal.

It then immediately follows from the proof of our main theorem that there is a C^∞ open and dense set of Riemannian metrics on S^2 whose geodesic flow has positive topological entropy. More precisely, given a Riemannian metric on S^2 whose geodesic flow has zero topological entropy, one can find an arbitrarily small perturbation of the metric in the C^∞ topology such that the geodesic flow for the perturbed metric has positive topological entropy. Unlike the locally supported perturbations we make to positively curved metrics, these small perturbations need not be locally supported. We state this result at the end of the manuscript as Theorem 7. The possibility of using modern techniques in symplectic topology to obtain a global surface of section was also mentioned to us by G. Contreras.

Our main result is the following theorem.

THEOREM 1. *There is a C^∞ open and dense set of positively curved Riemannian metrics on S^2 whose geodesic flow has positive topological entropy. More precisely, given a positively curved Riemannian metric on S^2 whose geodesic flow has zero topological entropy, one can find an arbitrarily small perturbation of the metric in the C^∞ topology which is supported in an arbitrarily small neighborhood of a point on some closed geodesic, such that the geodesic flow for the perturbed metric has positive topological entropy.*

Proof of the Main Theorem

¿From the structural stability of hyperbolic sets [An] we know that the metrics whose geodesic flow has positive topological entropy form an open set. We now prove such metrics form a dense set in the C^∞ topology.

The starting point of our proof is a theorem of Birkhoff [B], which guarantees the existence of a global surface of section for the geodesic flow for a positively curved metric on S^2. He proved that for every simple closed geodesic γ there exists an symplectic diffeomorphism P_γ of the annulus $A = \gamma \times [0, \pi]$ which is a global Poincaré section for the geodesic flow. By the classical theorem of Lyusternik and Schnirelmann, we know that P_γ must have at least two periodic points. In fact, by a theorem of Franks, there exists even infinitely many periodic orbits, but we do not make use of this deep result.

We can assume that at least one of the periodic points is hyperbolic. If this is not the case, we choose a closed geodesic corresponding to one of the periodic points and perturb the Riemannian metric in a neighborhood of a point on this closed geodesic which is disjoint from γ. By Lemma 2 below this can be done such that the new metric is C^∞ close to the given one and has a hyperbolic closed geodesic. Choosing the support of the perturbation small enough, we can guarantee that γ is not effected by the perturbation, and the global Poincaré section for the perturbed metric is defined on the same set and has a hyperbolic periodic point. The proof, which we provide in the Appendix, combines a perturbation result by Klingenberg and Takens with some standard facts from the theory of twist maps.

LEMMA 2. *Let (M, g) be a Riemannian surface, $c : I \to M$ a closed geodesic, and R a C^∞ neighborhood of Riemannian metrics containing g. Then for any open neighborhood $V \subset M$ of a point $p \in c(I)$, there exists a Riemannian metric $g' \in R$ such that the support* *of $g' - g$ is contained in V, and such that there exists a hyperbolic closed geodesic c' for g' which intersects the set V non-trivially.*

Let us first assume that the global Poincaré section has a hyperbolic periodic point that is part of a heteroclinic connection. Then the corresponding hyperbolic closed geodesic is part of a heteroclinic connection. One can break this heteroclinic connection using a perturbation method of Donnay [D2] (see also [Pe] for a higher dimensional version). Donnay showed how to effect an arbitrarily small local perturbation of the metric to obtain a new metric whose geodesic flow has a hyperbolic closed geodesic for which the stable and unstable manifolds have a transversal intersection. It follows that the geodesic flow for this perturbed metric has positive topological entropy. Donnay states his result in the C^2 topology, but it can be easily extended to the C^∞ topology.

Donnay's idea is to slightly increase the Gaussian curvature in an arbitrarily small neighborhood of some point q along the connecting geodesic, while preserving the connecting geodesic. This increase in curvature changes the unstable manifold but preserves the stable manifold. Properly chosen, the increase in curvature forces a transverse intersection of the new stable and unstable manifolds at a unit vector having a foot point q. The perturbation of the $2-$jet of the metric is explicitly written using Fermi coordinates adapted to the connecting geodesic. one can break the connection using a perturbation method of Donnay with an arbitrarily small local C^∞ perturbation of the metric to obtain a new metric whose geodesic flow has a hyperbolic closed geodesic for which the stable and unstable manifolds have a transversal intersection. It follows that the geodesic flow for this perturbed metric has positive topological entropy.

We can now assume that the global Poincaré section has a hyperbolic periodic point that is not part of any heteroclinic connection. In this case we apply a beautiful result of Mather (Theorem 3), whose proof uses the theory of prime ends. Mather proves that for such a Poincaré map, either the hyperbolic periodic point is part of a heteroclinic connection or all branches of the stable and unstable manifolds have the same closure.

THEOREM 3. *(Mather) Let f be an symplectic C^∞ diffeomorphism of an open subset U of S^2 onto an open subset $f(U)$ of S^2. Assume that f has a hyperbolic fixed point p such that the closure in S^2 of the four stable and unstable branches are contained in U. Then either p is part of a heteroclinic connection or the four branches of the stable and unstable manifold for p have the same closure.*

We apply Mather's theorem to an iterate $f = P_\gamma^k$ of P_γ, chosen such that f has a hyperbolic fixed point p. The positive curvature hypothesis ensures that P_γ has an extension to an symplectic diffeomorphism of an open neighborhood of the closed annulus. By assumption, p is not part of a heteroclinic connection between hyperbolic periodic points. We now show that a stable branch and an unstable branch for the hyperbolic fixed point must intersect with a topological (2-sided) crossing, even without perturbation. It follows from [BW, KW] that the f and the Poincaré map have positive topological entropy and by Abramov's theorem [Av], the geodesic flow for this metric must also have positive topological entropy.

The result will be a consequence of the following general proposition, which also applies to surface diffeomorphisms which are not necessarily symplectic. It says that if the closures of a stable and unstable branch for a hyperbolic periodic point coincide, then the map must exhibit chaotic behavior.

PROPOSITION 4. *Let M be a surface for which the Jordan curve theorem is valid, and let $f\colon M \to M$ be a diffeomorphism. Assume that $p \in M$ is a hyperbolic fixed point such that the closure of an unstable and stable branch coincide. Then the two branches will intersect with a topological (2-sided) crossing. It follows that f has positive topological entropy.*

Proof. Let us assume that $f : M \to M$ be a C^∞ diffeomorphism on a surface M and that f has a hyperbolic fixed point $p \in M$. Choose a coordinate system in a small neighborhood N of p obtained from the Hartman-Grobman C^0 linearization theorem [PM], where f restricted to N is linear and of the form $L\colon (x, y) \to (\mu x, \lambda y)$, where $0 < \lambda < 1 < \mu$ (see Figure 1). The boundary of N has eight sides: four sides, labeled h_1, h_2, h_3, h_4 are segments of hyperbolas and four sides, labeled s_1, s_2, s_3, s_4 are straight line segments parallel to the axes. We denote by R_1, R_2, R_3, R_4 four rectangles, whose boundary consists of sides s_k, $L^{\alpha(k)}(s_k)$, and the two hyperbolic subarcs in h_{k-1} and h_k (where k is considered mod 4, and $\alpha(k)$ is 1 if k is odd and -1 if k is even).

¿From the construction it follows for $i \in \{1, 2\}$

$$f^{-1}(Q_i \setminus R_2) \subset Q_i,$$

and for $i \in \{3, 4\}$

$$f^{-1}(Q_i \setminus R_4) \subset Q_i.$$

Similarly, for $i \in \{1, 4\}$:

$$f(Q_i \setminus R_1) \subset Q_i,$$

and for $i \in \{2, 3\}$

$$f(Q_i \setminus R_4) \subset Q_i.$$

Denote by $u^+_{loc} = \{(x, 0) \in N \mid x \geq 0\}$ and $u^-_{loc} = \{(x, 0) \in N \mid x \geq 0\}$ the local unstable and stable branches and by $s^+_{loc} = \{(0, y) \in N \mid y \geq 0\}$ and $s^-_{loc} = \{(0, y) \in N \mid y \geq 0\}$ the local unstable and stable branches. The corresponding global unstable branches u^+, u^- and global stable branches s^+, s^- are given by

$$u^\pm = \bigcup_{n\in\mathbb{N}_0} f^n(u^\pm_{loc})$$

and

$$s^\pm = \bigcup_{n\in\mathbb{N}_0} f^{-n}(s^\pm_{loc})$$

The following lemma seems to be embedded in [O]. The idea is that the linearization places strong restrictions on the first return of stable and unstable branches of p to N.

LEMMA 5. *With the notation above, suppose that one of the unstable branches u^+ or u^- returns to Q_1 or Q_2. Then the first return must be through R_2. If one of the unstable branches returns to Q_3 or Q_4, the first return must be through R_4.*

Similarly if one of the stable branches s^+ or s^- returns to Q_2 or Q_3 the first return must be through R_3. If one of the stable branches return to Q_1 or Q_4, it must be through R_1.

Proof. Suppose that an unstable branch will intersect Q_1 in r for the first time after it left N. By definition of the unstable branch $f^{-1}(r)$ is strictly contained in the subarc of the stable branch from the fixed point p to r. However, if r is not contained in R_2, then as explained above $f^{-1}(r)$ belongs to Q_1. This contradicts the definition of r. All the other cases are obtained with a similar argument. $\square$

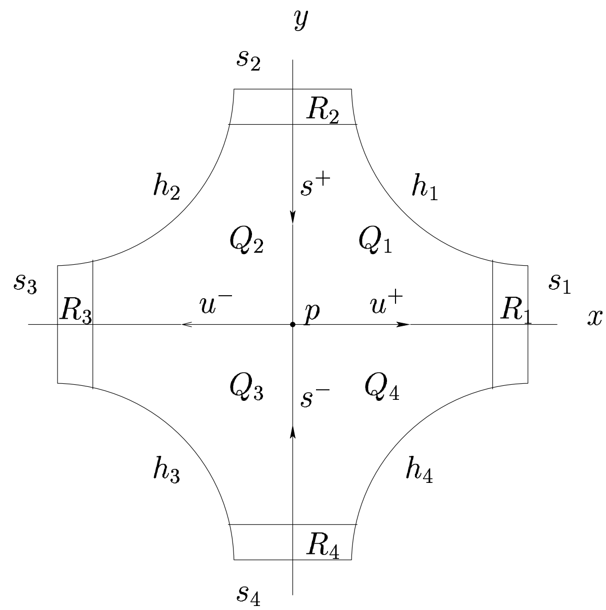

FIGURE 1. Linearization in N

Now we finish the proof of Proposition 4. Fix a coordinate system in a small neighborhood N of p such that f restricted to N is linear. Keeping the previous notation, we assume that the branches u^+ and s^+ have the same closure. Since we assume that u^+ accumulates onto s^+, by Lemma 5, the unstable branch u^+ must enter R_2. Let q be the first point of intersection of this branch with R_2. Consider the simple closed curve formed by the unstable arc from p to q, the horizontal line segment connecting q to the stable boundary of Q_1, and the remainder of the stable boundary to p. By the Jordan curve theorem, which we now use for the first time, this simple closed curve will divide the domain into two connected components (see Figure 2).

Let D be the connected component containing $(R_1 \cap Q_1) \setminus u^+$. We first assume that $q \in Q_1$. Then the image $f(q)$ is contained in D. By hypothesis the stable branch s^+ accumulates onto $f(q)$. After first leaving N the branch s^+ is contained in the connected component complementary to D, and thus the stable branch must intersect and cross the boundary of D (perhaps not with a transversal intersection). By construction of the Jordan boundary, the stable branch s^+ must either cross the unstable branch u^+ or cross the horizontal line segment connecting q and the stable branch. The latter is not possible since by Lemma 5, the first time the stable branch intersects Q_1 it must do so through R_1.

Now assume that $q \in Q_2$ (see Figure 3). This time the image $f(q)$ of q is contained in the component complementary to D. By hypothesis the stable branch s^+ accumulates onto $f(q)$. In this case, when it leaves N, the stable branch s^+ is contained in D and therefore it must intersect and cross the the boundary of D. This implies that s^+ must either cross the unstable branch u^+ or cross the horizontal line segment connecting q. Again by Lemma 5, the latter is not possible since the first time the stable branch intersects Q_2 it must do so through R_3. □

For a smooth flow on a manifold, a natural object of study is the closed orbit counting function, which counts the number of closed orbits with primitive period $\leq T$. It is known

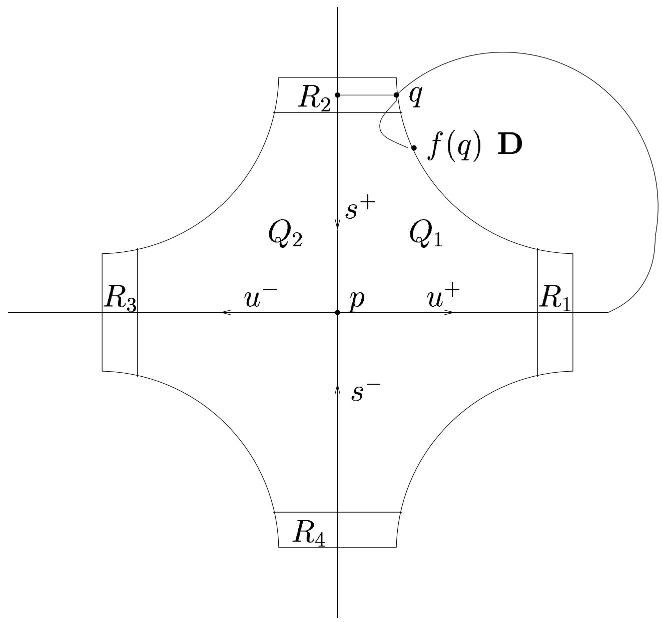

FIGURE 2

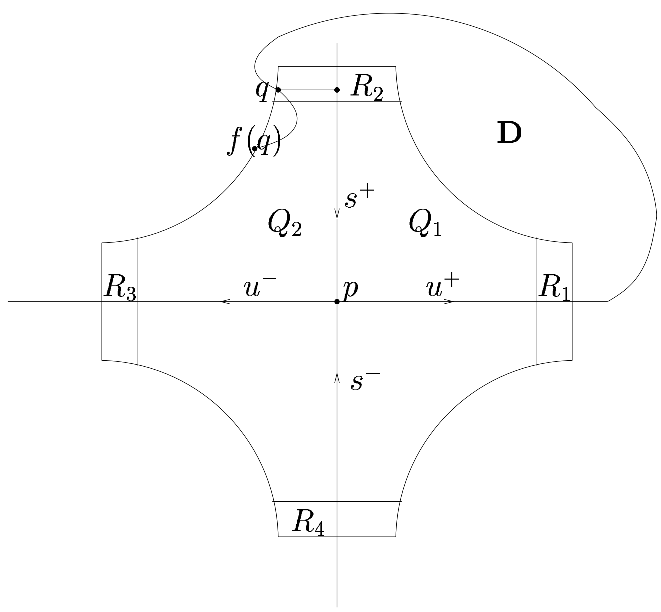

FIGURE 3

that for geodesic flows in negative curvature, the topological entropy of the flow is precisely the exponential growth rate of the closed orbit counting function. It is also known that in dimensions four and greater, there is no general relation between topological entropy and closed orbits. In particular, there exist flows having positive topological entropy and no closed orbits.

However, for smooth flows on compact 3-manifolds, Katok [Ka] showed that topological entropy is always a lower bound for the exponential growth rate of the closed orbit counting function. Furthermore, he showed that if the flow has positive topological entropy, then there exists a horseshoe (locally maximal hyperbolic set). The horseshoe guarantees the existence of infinitely many hyperbolic periodic orbits with an exponential growth rate of the closed orbit counting function. This gives the following corollary of Theorem 1:

COROLLARY 6. *There is a C^∞ open and dense set of positively curved metrics on S^2 having infinitely many hyperbolic closed geodesics, and moreover, having an exponential growth rate of hyperbolic closed geodesics.*

As mentioned in the Introduction, if one combines the recent result of Hofer, Wysochi, and Zehnder on the existence of a global surface of section or heteroclinic connection for a C^∞ dense set of Riemannian metrics on S^2, with the proof of Theorem 1, one obtains that that the typical geodesic flow on S^2 exhibits chaotic behavior.

THEOREM 7. *There is a C^∞ open and dense set of Riemannian metrics on S^2 whose geodesic flow has positive topological entropy. More precisely, given a Riemannian metric on S^2 whose geodesic flow has zero topological entropy, one can find an arbitrarily small perturbation of the metric in the C^∞ topology such that the geodesic flow for the perturbed metric has positive topological entropy.*

Appendix: Proof of Lemma 2

We first formulate a version of the theorem of Klingenberg and Takens which we need to prove Lemma 2. To give a precise statement of their result we begin by developing the following notation.

Let (M,g) be a C^∞ Riemannian manifold and ϕ_g^t be the geodesic flow acting on the unit tangent bundle SM. Let γ be a closed geodesic of length ℓ and $v_0 = \dot\gamma(0)$. Choose a hypersurface $\Sigma \subset SM$ through v_0 which is transversal to the orbit $\phi_{g_0}^t(v_0)$. Then there exist open neighborhoods Σ_0 and Σ_1 of v_0 in Σ and a diffeomorphism $P(\Sigma,g) : \Sigma_0 \to \Sigma_1$ defined by $P(\Sigma,g)(v) = \phi_g^{\delta(v)}(v)$, where $\delta : \Sigma_0 \to \mathbb{R}$ is a differentiable map such that $\delta(v_0) = \ell$.

Note that the smooth mapping $P(\Sigma,g)$ is symplectic with respect to the induced symplectic structure. For Σ sufficiently small, we can define a symplectic coordinate system $\psi : \Sigma \to \mathbb{R}^{2n-2}$ with $\psi(v_0) = 0$, and where $\mathbb{R}^{2n-2}$ the standard symplectic structure. Hence, the map $P(\Sigma,g,\psi) = \psi \circ P(\Sigma,g) \circ \psi^{-1} : U_0 \to U_1$, where $U_i = \psi(\Sigma_i)$, is a smooth symplectic map which preserves the origin. For each $k \in \mathbb{N}$ denote by $P_k(\Sigma,g,\psi) \in J_s^k(n-1)$, the k-jet of $P(\Sigma,g,\psi)$ at 0, where $J_s^k(n-1)$ is the space of k-jets of symplectic automorphisms of $\mathbb{R}^{2n-2}$ which fix the origin.

For an open neighborhood V of $\gamma(0)$, we denote by $G^{k+1}(V,\gamma)$ the set of all C^{k+1} Riemannian metrics g' such that the support of $g'-g$ is contained in V and such that γ is a closed geodesic of length ℓ. Klingenberg and Takens prove the following theorem.

THEOREM 8. *Let γ be a closed geodesic and Σ and ψ as above. Then for each open neighborhood V of $\gamma(0)$ and $m \in \mathbb{N} \cup \{\infty\}$, with $m > k$, the map*

$$P_k(\Sigma,\psi)\colon G^m(V,\gamma) \to J_s^k(n-1)$$

defined by $P_k(\Sigma,\psi)(g') = P_k(\Sigma,g',\psi)$ is an open mapping. In particular, if the closure of a subset $Q \subset J_s^k(n-1)$ contains $P_k(\Sigma,\psi)(g')$, then there exists in each C^∞ neighborhood R of g' a metric $h \in R \cap G^\infty(V,\gamma)$ such that $P_k(\Sigma,\varphi)(h)$ is contained in Q.

We now show how to apply Theorem 8 to prove Lemma 2. Assume first that γ is an elliptic closed geodesic, Σ a transversal section containing $\dot\gamma(0)$, and V a neighborhood of $\gamma(0)$. Let $Q \subset J^3_s(1)$ be the set of non-degenerate symplectic twist maps fixing 0, i.e.,

$$Q = \{\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \phi(r,\theta) = (r, \theta + \alpha_0 + \alpha_1 r) + o(|r|^3))\}$$

such that $\alpha_0 \neq 0, \pm\pi/2, \pm 2\pi/3$ and $\alpha_1 \neq 0$.

Since γ is elliptic, the map $P_3(\Sigma,\psi)(g) = P(\Sigma, g, \psi)$ has the form $(r,\theta) \mapsto (r, \theta + \alpha_0 + \alpha_1 r)$. If this map is a degenerate twist map, one can make an arbitrarily small perturbation of the 4-jet to ensure that $\alpha_0 \neq 0, \pm\pi/2, \pm 2\pi/3$ and $\alpha_1 \neq 0$. Thus the closure of $Q \subset J^3_s(1)$ contains $P(\Sigma, g, \psi)$. Theorem 8 then provides in every C^∞ neighborhood R of g a metric $h \in R \cap G^\infty(V, \gamma)$ such that $P_k(\Sigma, \varphi)(h)$ is contained in Q. MacKay and Stark [MS, Ge] show that *minimal* periodic orbits (minimal with respect to a generating function) of non-degenerate twist maps, which always exist arbitrarily close to the fixed point, must consist of hyperbolic periodic points or parabolic periodic points. Therefore the geodesic flow for h will either have a hyperbolic closed geodesic or a parabolic closed geodesic.

Now suppose γ is a parabolic closed geodesic, Σ a transversal section containing $\dot\gamma(0)$, and V a neighborhood of $\gamma(0)$. Let $Q' \subset J^1_s(1)$ be the set of symplectic maps having 0 as a hyperbolic fixed point, i.e.,

$$Q' = \{\phi : \mathbb{R}^2 \to \mathbb{R}^2 : \phi(0) = 0 \text{ and } \mathrm{spec}(\mathrm{D}\phi(0)) \cap \mathrm{S}^1 = \emptyset\},$$

where $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ and $\mathrm{spec}(\mathrm{D}\phi(0))$ denotes the spectrum of the linear mapping $D\phi(0)$. The map $P_1(\Sigma,\psi)(g) = P(\Sigma, g, \psi)$ has the form $x \mapsto D\phi(0)x$. By assumption $\mathrm{spec}(\mathrm{D}\phi(0)) \cap \{-1, 1\} \neq \emptyset$, and therefore an arbitrarily small perturbation of the 1-jet of ϕ ensures that $\mathrm{spec}(\mathrm{D}\phi(0)) \cap \mathrm{S}^1 = \emptyset$. Thus the closure of $Q' \subset J^1_s(1)$ contains $P(\Sigma, g, \psi)$. Theorem 8 then provides in every C^∞ neighborhood R of g a metric $h \in R \cap G^\infty(V, \gamma)$ such that $P_1(\Sigma, \varphi)(h)$ is contained in Q'. Hence, the geodesic flow for h will have a hyperbolic closed geodesic. This completes the proof of Lemma 2. □

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