C^*-Algebras of Quantum Hamiltonians

by

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1. Introduction

Our aim in these notes is to study spectral properties of quantum mechanical hamiltonians with C^*-algebra techniques. The algebras which will concern us are generated by the hamiltonian operators corresponding to certain types of kinetic and potential energies; for this reason we call them *algebras of hamiltonians* or *algebras of energy observables*. The best way to explain what we have in mind is to begin with an example.

Let us denote by U_x and V_k the unitary operators in $L^2(\mathbb{R}^n)$ corresponding to translation by x and k in position and momentum space respectively; so $(U_x f)(y) = f(x+y)$ and $(V_k f)(y) = \mathrm{e}^{i\langle k,y\rangle} f(y)$. We call *Fréchet filter* on a locally compact space X the family of subsets F such that $X \setminus F$ is relatively compact.

Theorem 1.1 *Let H be a self-adjoint operator in $L^2(\mathbb{R}^n)$ such that:*

$$\lim_{x\to 0} \|(U_x - 1)(H+i)^{-1}\| = 0 \quad \textit{and} \quad \lim_{k\to 0} \|[V_k, (H+i)^{-1}]\| = 0. \tag{1.1}$$

Then, for each ultrafilter $\varkappa$ on $\mathbb{R}^n$ finer than the Fréchet filter, the family of self-adjoint operators $H_x = U_x H U_x^*$, $x \in \mathbb{R}^n$, has a strong limit $H_\varkappa$ when $x \to \infty$ along $\varkappa$, and

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup_\varkappa \boldsymbol{\sigma}(H_\varkappa). \tag{1.2}$$

We shall give later on examples which show that this theorem has interesting consequences even in elementary cases. The convergence involved above has to be understood as follows: there is a self-adjoint (not necessarily densely defined) operator $H_\varkappa$ on $L^2(\mathbb{R}^n)$ such that for each $\varphi \in C_0(\mathbb{R})$, $\varepsilon > 0$, and $f \in L^2(\mathbb{R}^n)$ there is $F \in \varkappa$ such that $\|\varphi(H_x)f - \varphi(H_\varkappa)f\| < \varepsilon$ for all $x \in F$. Note that one can get (quite often, in fact) $H_\varkappa = \infty$, where ∞ is the unique operator with domain $\{0\}$. We say that the operators $H_\varkappa$ are *localizations at infinity* of the hamiltonian H, so the theorem says that *the essential spectrum of H is the union of the spectra of its localizations at infinity*. We should emphasize that we talk about the “infinity” associated to the position observable Q (i.e. we localize at $Q = \infty$). Indeed, the region where some other observable (e.g. the momentum P) is infinite could play a role too, and this actually happens in physically interesting situations, e.g. if there is an external magnetic field $B(x)$ which does not vanish as $|x| \to \infty$.

C^*-algebras do not seem to play a role in Theorem 1.1. However we discovered it by studying a certain C^*-algebra, its proof involves C^*-algebra techniques, and we do not know a proof independent of such techniques. We do not exclude the possibility of proving it by decomposing H with the help of certain partitions of unity, but such a proof would certainly be much more intricate. On the other hand, Theorem 1.1 is only a particular case of a theorem in which $\mathbb{R}^n$ is replaced by an arbitrary locally compact abelian group, where techniques based on partitions of unity would not work.

Now we explain the connection with C^*-algebras. Let $C_\mathrm{b}^\mathrm{u}(\mathbb{R}^n) \rtimes \mathbb{R}^n$ be the crossed product of the C^*-algebra $C_\mathrm{b}^\mathrm{u}(\mathbb{R}^n)$ (bounded uniformly continuous functions) by the natural action of $\mathbb{R}^n$ (translations). This is a C^*-algebra whose exact definition does not matter here (it will be recalled in §2.4). Let $\varphi(Q)$ be the operator in $L^2(\mathbb{R}^n)$ of multiplication by the function φ and $\psi(P) = \mathcal{F}^{-1}\psi(Q)\mathcal{F}$, where $\mathcal{F}$ is the Fourier transformation.

Theorem 1.2 *There is a faithful representation of $C_\mathrm{b}^\mathrm{u}(\mathbb{R}^n) \rtimes \mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ whose range is the norm closed linear space generated by the operators of the form $\varphi(Q)\psi(P)$ with $\varphi \in C_\mathrm{b}^\mathrm{u}(\mathbb{R}^n)$ and $\psi \in C_0(\mathbb{R}^n)$.*

This is a particular case of Theorem 2.17, where the representation is described explicitly. For the moment, the relation with crossed product is irrelevant and one can simply think that $C_\mathrm{b}^\mathrm{u}(\mathbb{R}^n) \rtimes \mathbb{R}^n$ *is* the C^*-algebra of operators on $L^2(\mathbb{R}^n)$ generated by $\varphi(Q)\psi(P)$ with φ and ψ as above. Then we have the following two other equivalent descriptions of it.

Theorem 1.3 *Let h be a real elliptic polynomial of order m on $\mathbb{R}^n$. Then $C_\mathrm{b}^\mathrm{u}(\mathbb{R}^n) \rtimes \mathbb{R}^n$ is equal to the C^*-algebra of operators on $L^2(\mathbb{R}^n)$ generated by the self-adjoint operators of the form $h(P)+V$, where V runs over the set of symmetric differential operators of order $< m$ with coefficients in $C_\mathrm{b}^\infty(\mathbb{R}^n)$.*

We mention that we define the C^*-algebra generated by a family $\mathscr{F}$ of (possibly unbounded) self-adjoint operators on a Hilbert space $\mathscr{H}$, as the smallest C^*-algebra $\mathscr{C}$ of operators on $\mathscr{H}$ such that $\varphi(H) \in \mathscr{C}$ for all $H \in \mathscr{F}$ and $\varphi \in C_0(\mathbb{R})$.

We denoted $C_\mathrm{b}^\infty(\mathbb{R}^n)$ the space of C^∞ functions which are bounded together with all their derivatives. The (easy) proof of Theorem 1.3 can be found in [DG1]. This result justifies our interpretation of $C_\mathrm{b}^\mathrm{u}(\mathbb{R}^n) \rtimes \mathbb{R}^n$ as an "algebra of hamiltonians".

Theorem 1.4 *A bounded operator S on $L^2(\mathbb{R}^n)$ belongs to $C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n$ if and only if* $\lim_{x\to 0} \|(U_x - 1)S^{(*)}\| = 0$ *and* $\lim_{k\to 0} \|V_k^* S V_k - S\| = 0$.

If a symbol like $S^{(*)}$ appears in a relation, this means that the relation holds for S and its adjoint S^*. The preceding theorem has been proved in [DG2] and is a nontrivial description of $C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n$ similar to the Riesz-Kolmogorov characterization of the C^*-algebra $\mathscr{K}(\mathbb{R}^n)$ of compact operators on $L^2(\mathbb{R}^n)$. We mention that $\mathscr{K}(\mathbb{R}^n) = C_0(\mathbb{R}^n) \rtimes \mathbb{R}^n$. Theorem 1.4 allowed us to formulate Theorem 1.1 in a C^*-algebra independent manner and to get general and easy to check assumptions on H.

The assumptions (1.1) of Theorem 1.1 can now be written $(H+i)^{-1} \in C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n$. Then the theorem will be a consequence of the following assertion: *if $S \in C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n$ then for each ultrafilter $\varkappa$ on $\mathbb{R}^n$ finer than the Fréchet filter, the family of operators $S_x = U_x S U_x^*$, $x \in \mathbb{R}^n$, has a strong limit $S_\varkappa$ when $x \to \infty$ along $\varkappa$, and $\boldsymbol{\sigma}_{\text{ess}}(S) = \bigcup_\varkappa \boldsymbol{\sigma}(S_\varkappa)$.*

We now sketch the proof the last assertion. The essential spectrum of S is equal to the spectrum of the image $\widehat{S}$ of S in the quotient C^*-algebra $C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n$ by the ideal $\mathscr{K}(\mathbb{R}^n)$ of compact operators on $L^2(\mathbb{R}^n)$. But $\mathscr{K}(\mathbb{R}^n) = C_0(\mathbb{R}^n) \rtimes \mathbb{R}^n$ and one has a canonical isomorphism

$$C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n \Big/ C_0(\mathbb{R}^n) \rtimes \mathbb{R}^n \cong \left[C_b^u(\mathbb{R}^n) \Big/ C_0(\mathbb{R}^n) \right] \rtimes \mathbb{R}^n .$$

Now $C_b^u(\mathbb{R}^n)/C_0(\mathbb{R}^n) \hookrightarrow C(\gamma\mathbb{R}^n)$ where $\gamma\mathbb{R}^n$ is the boundary of $\mathbb{R}^n$ in the Stone-Čech compactification $\beta\mathbb{R}^n$. One can use this in order to get a natural embedding

$$C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n \Big/ C_0(\mathbb{R}^n) \rtimes \mathbb{R}^n \hookrightarrow \prod_{\varkappa \in \gamma\mathbb{R}^n} C_b^u(\mathbb{R}^n) \rtimes \mathbb{R}^n .$$

To finish the proof it suffices to compute $\widehat{S}$ as element of the right hand side.

The technical details make the complete proof rather involved, but the role of the C^*-algebras and the general strategy of the proof should be clear by now. The main novelty of our approach is the idea of replacing *the* hamiltonian of a physical system by an *algebra* of hamiltonians. Roughly speaking, we proceed as follows. Assume that we are given a quantum physical system (with a Hilbert space $\mathscr{H}$ as state space) subject to a certain type of interactions. The first step is to point out a class of “elementary” admissible hamiltonians. This depends on the class of kinetic and potential energies which are natural in this context; for example, in Theorem 1.3 we fixed the kinetic energy h and considered potentials of the form V described there. But the “real” hamiltonians that we would like to study are much more complicated objects, e.g. they do not admit such a simple decomposition into kinetic and potential parts. To define them, let $\mathscr{C}$ be the C^*-algebra of operators on $\mathscr{H}$ generated by the resolvents of the elementary hamiltonians. It is rather surprising that in many cases $\mathscr{C}$ is a natural mathematical object, for example it is the crossed product $C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R}^n) \rtimes \mathbb{R}^n$ for the class of elementary hamiltonians considered in Theorem 1.3. In several of our examples we have been able to give an “intrinsic” characterization of $\mathscr{C}$, like that of Theorem 1.4. Such a result is important because it allows us to show that the class of hamiltonians affiliated to the algebra is much larger than one would expect and to formulate the final results without explicit reference to C^*-algebras. However, the crucial point is to give a convenient and rather explicit description of the quotient algebra $\widehat{\mathscr{C}} = \mathscr{C}/[\mathscr{C} \cap K(\mathscr{H})]$, where $K(\mathscr{H})$ is the ideal of compact operators on $\mathscr{H}$. Results like those of Theorem 1.1 and other more subtle properties (the proof of the Mourre estimate) depend on this. Thus $\mathscr{C}$ should be considered as the main object of the spectral theory of the given system and the determination of the structure of $\widehat{\mathscr{C}}$ the main problem one has to solve.

To put things in more mathematical terms, assume that you want to compute the essential spectrum of a bounded operator $S \in B(\mathscr{H})$. The quotient C^*-algebra $C(\mathscr{H}) = B(\mathscr{H})/K(\mathscr{H})$ is the Calkin algebra associated to $\mathscr{H}$ and $\boldsymbol{\sigma}_{\text{ess}}(S) = \boldsymbol{\sigma}(\widehat{S})$, where $\widehat{S}$ is the image of S in $C(\mathscr{H})$. It is out of question to compute $\widehat{S}$ as an element of $C(\mathscr{H})$, because this algebra is too complicated. But if one can find a C^*-algebra $\mathscr{C} \subset B(\mathscr{H})$ such that $\widehat{\mathscr{C}}$ can be described relatively explicitly, and if $\widehat{S}$ *as element of* $\widehat{\mathscr{C}}$ can be computed, then one could use this method to compute $\boldsymbol{\sigma}_{\text{ess}}(S)$ (note that $\widehat{\mathscr{C}}$ is a subalgebra of $C(\mathscr{H})$). The main problem now is how to choose $\mathscr{C}$ in nontrivial situations, e.g. if S is (the resolvent of) an N-body hamiltonian, a more general anisotropic hamiltonian on an abelian locally compact group, or in a situation with an infinite number of degrees of freedom like in quantum field theory. All these cases will be treated later on in this lecture.

The idea of considering the quotient algebra $\widehat{\mathscr{C}}$ in connection with the computation of the essential spectrum of an operator appeared quite early in the mathematical literature. What is new here is a kind of experimental observation: we noticed that in several physically interesting and nontrivial situations there is a simple algorithm for constructing $\mathscr{C}$. In particular, we found that crossed products of C^*-algebras by actions of groups play a remarkable, although not exhaustive, role in this context. Moreover, one of our main observations was that taking the quotient of a hamiltonian H with respect to an ideal (an operation which does not make sense if we fix a Hilbert space) gives physically interesting objects, which play an important role in the spectral and scattering theory of H. We explain this below.

The main advantage of the C^*-algebra framework is that one can define an operation on observables which does not make sense in a purely hilbertian setting: that of taking the image through a morphism. For example, if $\mathscr{I}$ is an ideal in a C^*-algebra $\mathscr{C}$ of operators on a Hilbert space $\mathscr{H}$ and if H is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathscr{C}$, then the image $H/\mathscr{I}$ of H through the canonical morphism $\mathscr{C} \to \mathscr{C}/\mathscr{I}$ is a perfectly well defined observable affiliated to the C^*-algebra $\mathscr{C}/\mathscr{I}$ (see §2.7), but in nontrivial cases it cannot be realized as a self-adjoint operator on $\mathscr{H}$. Thus we are forced to work with "abstract" C^*-algebras: although the starting point is a concrete C^*-algebra on the physical Hilbert space, the quotient operation will give an algebra which a priori is not represented. A new Hilbert space is obtained when looking for faithful nondegenerate representations of the quotient algebra, on which $H/\mathscr{I}$ can be realized as a self-adjoint operator with a clear physical interpretation (cf. the N-body and quantum field models).

When $\mathscr{I} = \mathscr{C} \cap K(\mathscr{H})$, we usually denote $\widehat{H} = H/\mathscr{I}$ and call it *localization at infinity of* H. We shall compute it in the second part of this lecture in many models. However, localizing at some region of infinity gives often an object of physical interest, and these "partial" localizations are obtained by taking quotients with respect to larger ideals. The fact that quotients of hamiltonians with respect to ideals other than the compacts are useful and appear naturally in physically interesting situations has first been observed in [BG1] and [BG2]. Indeed, if H is the hamiltonian of a nonrelativistic N-body system and if a is a partition of the system of particles, then two other hamiltonians are defined in the physical literature: H_a, the hamiltonian with interactions between clusters set to zero, and H^a, which is H_a for zero intercluster momentum (see §3.4). In [BG1] it was shown that H_a is the quotient of H with respect to a certain ideal in the "N-body algebra". Then in [BG2] it was proved that the essential spectrum of the internal hamiltonian H^a is the essential spectrum of H with respect to some other ideal (see the last part of §3 in [BG2], or §8.1.5 and §8.2.4 in [ABG]). Similar localizations at "$Q = \pm\infty$" and "$P = \pm\infty$" with the help of ideals appeared in the paper [GI6] concerning one dimensional anisotropic systems (summarized here in §3.5). We met more subtle type of localizations at infinity in our study of the interactions first considered by M. Klaus in [Kla] (§3.2 is devoted to this question; Proposition 2.20 is an abstract example of the kind of quotients which appear in such situations). We mention that in the recent paper [AMP] one can find an interesting observation concerning the relevance of the observables $H/\mathscr{I}$ in relation with the propagation properties of the system.

The idea of using operator algebras in the study of the spectral properties of *one* operator goes back to the origin of the theory of C^*-algebra and many people worked on this topic. We are not really entitled to give references on this question, see however [Dav, Dou, Mur] and references therein. The work of H. O. Cordes on C^*-algebras of pseudodifferential operators has to be especially acknowledged (see [Cor]). However, it seems to us that the goals and methods, as well as the examples studied in these works, are quite far from ours.

Most noteworthy is the work of J. Bellissard on almost periodic and random operators in connection with solid state physics, see [Be1, Be2]. In particular, he considers the C^*-algebra generated by the translates of the hamiltonian of a physical system and shows that under certain conditions it is a crossed product. Although an "algebra of hamiltonians" in our terminology, the resulting algebra is very different from those we consider here, being rather tightly related to *one* hamiltonian H. In the models considered by Bellissard this is an advantage and allows him to get much more precise spectral properties of H and to point out a remarkable connection between K-theory and the quantum Hall effect. The algebras which appear in our work are much larger and, in a certain sense, simpler. So we can treat a larger class of models but we are not able to study finer spectral properties of the hamiltonians.

The first paper in which the algebraic point of view appears in connection with the N-body problem is [BG1]. This paper is devoted to the study of a class of C^*-algebras graded by semilattices (see §3.6) which allow one to formalize in a convenient way the notion of a-connected component of an operator (this approach was motivated by the papers [KPR, Pol, PSS]). It is also shown there that (dispersive) N-body hamiltonians are affiliated to such algebras and that the decomposition of the resolvent according to the homogeneous components of the grading gives exactly the well known Weinberg-Van Winter equation. As a consequence, a purely algebraic proof of the HVZ theorem is obtained (see Theorem 3.25). The algebraic formalism was then extended in [BG2] such as to cover the proof of the Mourre estimate for N-body hamiltonians (not necessarily non-relativistic, the proof being given for a class of hamiltonians abstractly defined). The Mourre estimate for such systems is a highly technical and nontrivial inequality (see the first papers [PSS] and [FrH] devoted to this question) and it seems to us quite remarkable that a purely algebraic statement involving quotients of C^*-algebras is relevant in this context. In fact, it was shown that one can realize a complete decoupling of channels by taking such a quotient (thus eliminating the Simon partition of unity, which gave only an approximative channel decoupling). This paper also contains new examples of graded C^*-algebras associated to symplectic spaces, which allow one to treat N-body systems in constant magnetic fields. These methods were shown to be efficient in the treatment of very singular N-body systems (with hard-core interactions) in [BGS], where the Mourre estimate was proved for such systems, and in [If1], where the scattering theory was treated. A complete and unified presentation of these algebraic techniques can be found in Chapters 8 and 9 of [ABG].

However, the physical meaning of the "N-body algebra" which was in the background of the theory remained obscure, cf. the introduction of the Chapter 8 of [ABG]. The main point of our work on one dimensional anisotropic systems [GI6] was the clarification of this point (although [GI6] was ready in the summer of 1998, we decided not to publish it, because meantime we discovered that the main idea works very easily in much more general situations). We understood that the grading of the algebra was only an accident and we developed a general strategy for the construction of "algebras of energy observables". Here a remark of G. Skandalis played a fundamental role: he noticed that the homogeneous components of the (graded) N-body algebra are crossed products (this was not at all obvious in the presentation of [BG1]). This opened the way to a unified theory of anisotropic hamiltonians along the lines described in §3.1 below, point out view presented in detail in our preprints [GI2, GI3]. These ideas have been applied to (a generalized version of) the N-body problem in [DG1, DG2]. Note that [DG2] also contains the first non-trivial affiliation criterion, which shows that the class of N-body hamiltonians affiliated to the N-body algebra is much larger than what one would think at first sight (Theorem 2.22).

This lecture is mostly based on [GI4] and [GI3] (see also [GI2], which is the preprint version of [GI4]) and consists of two parts. The second one is devoted to examples of C^*-algebras of hamiltonians and to the spectral theory of the operators affiliated to them. Since the audience we have in mind consists of people working in the spectral theory of quantum hamiltonians and having only an elementary knowledge of C^*-algebras, we thought it useful to present in the first part of these notes the necessary background from the theory of C^*-algebras. Thus, although a rough knowledge of the first two chapters of the book [Dix] is necessary (and largely sufficient) to understand what follows, we start in §2.1 by emphasizing some simple but important points. Next we present some elements of the theory of crossed products, which is a less standard subject. Indeed, in these notes we stress the role of crossed products and of algebras of hamiltonians associated to algebras of functions on locally compact abelian groups. This is a nice setting because it gives a unified view of many models of anisotropic interactions in quantum mechanics (and the possibility of their systematic study). Proofs are given only for assertions which are not easy to isolate in the literature. The last part of §2 is devoted to notions and results that are more specific to our subject: observables affiliated to C^*-algebra.

The theory of C^*-algebras developed in strong connection with statistical mechanics and quantum field theory (algebras of local observables and crossed products). We use crossed products, but with quite different purposes. On the other hand, we also apply our ideas in quantum field theory, but our point of view seems to be new, as is the C^*-algebra of hamiltonians of a boson field described in §3.7 and its use in the spectral theory of such hamiltonians. C^*-algebras play a role in scattering theory as algebras of asymptotic observables (see [BW]): this has nothing to do with the algebras of hamiltonians considered here.

Finally, we would like to make some comments concerning the role of C^*-algebras in highly technical regions of mathematical physics, like the N-body problem or the study of quantum field models. Most people working in such domains would probably agree about the “soft” role of C^*-algebras, in the sense that these provide a general and nice framework for the theory. But not many would accept that algebras can be useful for proving estimates. We note, first, that one should not underestimate the impact of the formalism on the technical aspects even in the simplest situations[(1)]. But much more is true: algebraic techniques allow one to get rid of many involved technical arguments and estimates. For example, compare the proof of the Mourre estimate due to Perry, Sigal, and Simon [PSS] (including the improvements due to Froese and Herbst [FrH]) with the algebraic proof from [BG2] (which appears as Theorems 8.4.3 and 9.4.4 in [ABG]), which extends without any effort to a large class of dispersive systems [DG2] and to quantum field models with strictly positive mass (see Theorem 3.32 and [Geo]). Or see the treatment of the relativistic HVZ theorem in [LSV] and compare it with that in [Dam] (cf. Theorem 3.25 here). Theorem 1.1 is another example of the same nature: see the results and treatment from [HeM] (when magnetic fields are absent).

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[(1)] Try to divide $MMMDCCCXCVII$ by $DCCLIX$ without writing the operation as 3897/759, i.e. check $MMMDCCCXCVII = V * DCCLIX + CII$!!!

2. C^*-algebras and observables affiliated to them

2.1. C^*-algebras. We begin with some elementary facts and definitions from the theory of C^*-algebras. A $*$-*algebra* is a complex algebra $\mathscr{A}$ equipped with an involution $S \mapsto S^*$. A map $\mathscr{A} \to \mathscr{B}$ between two $*$-algebras is a *morphism* if it preserves all the operations (i.e. is linear, multiplicative and intertwines the involutions). A C^*-*seminorm* on $\mathscr{A}$ is a seminorm $|\cdot|$ satisfying $|AB| \leq |A| \cdot |B|$ and $|A^*A| = |A|^2$ for all $A, B \in \mathscr{A}$. A C^*-seminorm which is also a norm is called C^*-*norm*.

A $*$-algebra $\mathscr{A}$ is called C^*-*algebra* if there is a complete C^*-norm $\|\cdot\|$ on it. It is remarkable that if such a norm exists, then it is unique. All topological assertions concerning $\mathscr{A}$ refer to this canonical norm. It is also remarkable that if $\mathscr{A}$, $\mathscr{B}$ are C^*-algebras and $\phi : \mathscr{A} \to \mathscr{B}$ is a morphism, then ϕ is a contraction for the (canonical) norms of $\mathscr{A}$ and $\mathscr{B}$ and the range of ϕ is a C^*-algebra. Moreover, ϕ is injective if and only if it is isometric (1.3.7 and 1.8.1 in [Dix]).

We defined C^*-algebras in a way which emphasizes their purely algebraic character :if $\mathscr{A}$ is a $*$-algebra you do not have to give some extra structure to make it a C^*-algebra (as in the case of Banach algebras). In more technical terms, the category of C^*-algebras is a full subcategory of the category of $*$-algebras. Thus, given a $*$-algebra, it makes sense to ask whether it is a C^*-algebra or not.

If two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ are canonically isomorphic we write $\mathscr{A} \cong \mathscr{B}$; in such a situation the canonical morphism is either obvious from the context or we give it explicitly. An algebra with unit element is also called unital; a morphism between two unital C^*-algebras which sends the unit of the first in the unit of the second is called *unital*.

A $*$-subalgebra of a C^*-algebra is a C^*-algebra if and only if it is closed; then we call it C^*-*subalgebra* of $\mathscr{A}$. A closed two-sided (hence self-adjoint) ideal in a C^*-algebra will be simply called *ideal*. If $\mathscr{J}$ is an ideal in a C^*-algebra $\mathscr{A}$, then the quotient $*$-algebra $\mathscr{A}/\mathscr{J}$ is a C^*-algebra, the canonical norm being the quotient norm (1.8.2 in [Dix]).

Let $\mathscr{H}$ be a Hilbert space. Then the space $B(\mathscr{H})$ of bounded linear operators on $\mathscr{H}$ is a C^*-algebra and $K(\mathscr{H})$, the subspace of compact operators, is an ideal in it. $C(\mathscr{H}) = B(\mathscr{H})/K(\mathscr{H})$ is the *Calkin algebra* of $\mathscr{H}$.

A morphism from a $*$-algebra $\mathscr{A}$ into $B(\mathscr{H})$ is called *representation of $\mathscr{A}$ on $\mathscr{H}$*. If the morphism is injective the representation is called *faithful*, and then we say that *$\mathscr{A}$ is realized on $\mathscr{H}$*. The following is nontrivial: *a $*$-algebra is a*

C^*-algebra if and only if it can be realized as a C^*-subalgebra of $B(\mathscr{H})$ for some $\mathscr{H}$ (2.6.1 in [Dix]).

A representation π of $\mathscr{A}$ on $\mathscr{H}$ is *nondegenerate* if the only vector $f \in \mathscr{H}$ such that $\pi(A)f = 0$ for all $A \in \mathscr{A}$ is $f = 0$. If $\mathscr{A}$ is a C^*-algebra of operators on $\mathscr{H}$ and the identity representation is nondegenerate, we say that $\mathscr{A}$ *is nondegenerate on* $\mathscr{H}$. Equivalently, this means that linear subspace generated by the elements of the form Af, with $A \in \mathscr{C}$ and $f \in \mathscr{H}$, is dense in $\mathscr{H}$.

Now some notations. Let T be a topological locally compact space. Then $C_{\mathrm{b}}(T)$ is the C^*-algebra of complex continuous bounded functions on T, $C_\infty(T)$ the C^*-subalgebra of functions which have a limit at infinity, and $C_0(T)$ the C^*-subalgebra of functions convergent to zero at infinity. If the topology of T is associated to a given uniform structure (e.g. if T is a locally compact abelian group) then $C_{\mathrm{b}}^{\mathrm{u}}(T)$ denotes the C^*-algebra of bounded uniformly continuous functions. More generally, if $\mathscr{A}$ is a C^*-algebra then $C_{\mathrm{b}}(T;\mathscr{A})$, $C_\infty(T;\mathscr{A})$, $C_0(T;\mathscr{A})$ and (if T is a uniform space) $C_{\mathrm{b}}^{\mathrm{u}}(T;\mathscr{A})$ are similarly defined C^*-algebras. There is one more useful algebra: $C_{\mathrm{rc}}(T;\mathscr{A})$, the C^*-algebra of continuous maps $T \to \mathscr{A}$ with relatively compact range.

Later on we shall define other interesting algebras in terms of filters on T. If α is a filter on a set T, Y is a topological space, and $\phi : T \to Y$ has a limit y along α (i.e. one has $\phi^{-1}(V) \in \alpha$ for each neighborhood V of y), then we write $\lim_\alpha \phi = y$ or $\lim_{t,\alpha} \phi(t) = y$. If T is a noncompact locally compact topological space, then the *Fréchet filter* on T is the family of subsets V such that $T \setminus V$ is relatively compact; the limit along it is denoted $\lim_{t\to\infty} \phi(t)$.

2.2. Enveloping C^*-algebras. Let $(\mathscr{A}, \|\cdot\|)$ be a Banach $*$-algebra, i.e. a complete norm $\|\cdot\|$ is given on the $*$-algebra $\mathscr{A}$ such that $\|AB\| \le \|A\| \cdot \|B\|$ and $\|A^*\| = \|A\|$. Then *each morphism from $\mathscr{A}$ into a C^*-algebra is a contraction* (1.3.7 in [Dix]). Hence we have $|\cdot| \le \|\cdot\|$ for each C^*-seminorm on $\mathscr{A}$. Since the upper bound of any family of C^*-seminorms is a C^*-seminorm, there is a largest C^*-seminorm on $\mathscr{A}$: we call it *the C^*-seminorm of $\mathscr{A}$* (or *C^*-norm of $\mathscr{A}$*, if it is a norm) and we denote it by $\|\cdot\|_*$ (note that $\|\cdot\|_* \le \|\cdot\|$). The C^*-algebra obtained by separation and completion of $(\mathscr{A}, \|\cdot\|_*)$ is denoted $\mathscr{A}_*$ (and its norm $\|\cdot\|_*$) and is called *the enveloping C^*-algebra of $\mathscr{A}$*.

Let $\theta_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}_*$ be the canonical morphism. Then $\theta_{\mathscr{A}}$ is continuous (contractive) with dense range and it is injective if and only if $\|\cdot\|_*$ is a norm on $\mathscr{A}$, so if and only if there is a C^*-norm on $\mathscr{A}$. In this case we say that $\mathscr{A}$ is an *A^*-algebra* and we identify $\mathscr{A} \subset \mathscr{A}_*$; thus $\mathscr{A}$ becomes a dense $*$-subalgebra of $\mathscr{A}_*$ and $\|A\|_* \le \|A\|$ if $A \in \mathscr{A}$.

The algebra $\mathscr{A}_*$ obviously has the following *universal property*: if $\mathscr{C}$ is a C^*-algebra and $\phi : \mathscr{A} \to \mathscr{C}$ is a morphism then there is a unique morphism $\phi_* : \mathscr{A}_* \to \mathscr{C}$ such that $\phi = \phi_* \circ \theta_{\mathscr{A}}$. As a consequence, we get a (covariant) functor from the category of Banach $*$-algebras to that of C^*-algebras. Indeed, if $\phi : \mathscr{A} \to \mathscr{B}$ is a morphism between two Banach $*$-algebras, then clearly there is a morphism $\phi_* : \mathscr{A}_* \to \mathscr{B}_*$ such that $\phi_* \circ \theta_{\mathscr{A}} = \theta_{\mathscr{B}} \circ \phi_*$.

It is easy to see that *if ϕ has dense range then ϕ_* is surjective*. But we stress that ϕ_* *could be non-injective even if ϕ is injective*.

For example, if there are several distinct C^*-norms on $\mathscr{A}$ (which is the case if $\mathscr{A}$ is the convolution algebra $L^1(G)$ of a non-amenable locally compact group G), then there is a C^*-norm $|\cdot|$ on $\mathscr{A}$ distinct from $\|\cdot\|_*$. So $|\cdot| \le \|\cdot\|_*$ and if $\mathscr{A}_\bullet$ is the algebra obtained by completing $(\mathscr{A}, |\cdot|)$ then there is a canonical morphism $\mathscr{A}_\bullet \to \mathscr{A}_*$ which is surjective but not injective.

For similar reasons it may happen that the inclusion $\mathscr{A} \hookrightarrow \mathscr{B}$ of a closed $*$-subalgebra $\mathscr{A}$ of the Banach $*$-algebra $\mathscr{B}$ induces a morphism $\mathscr{A}_* \to \mathscr{B}_*$ which is not injective. So if $\mathscr{B}$ is an A^*-algebra then $\mathscr{A}_*$ cannot (in general) be identified with the closure $\overline{\mathscr{A}}$ of $\mathscr{A}$ in $\mathscr{B}_*$; but there is a canonical surjection $\mathscr{A}_* \to \overline{\mathscr{A}}$, so $\overline{\mathscr{A}}$ is a quotient of $\mathscr{A}_*$.

Such unpleasant features do not occur, however, in the case of ideals (by *ideal* in a Banach $*$-algebra we mean "two-sided closed $*$-ideal"), as a consequence of the following result.

Theorem 2.1 *Let*

$$0 \longrightarrow \mathscr{J} \xrightarrow{\phi} \mathscr{A} \xrightarrow{\psi} \mathscr{B} \longrightarrow 0$$

be an exact sequence of Banach $*$-algebras. Then

$$0 \longrightarrow \mathscr{J}_* \xrightarrow{\phi_*} \mathscr{A}_* \xrightarrow{\psi_*} \mathscr{B}_* \longrightarrow 0$$

is an exact sequence of C^*-algebras.

Proof: a) We need only one non-trivial result, namely the Theorem VI.19.11 from [FeD], which says that if $\mathscr{I}$ is an ideal in a Banach $*$-algebra $\mathscr{A}$ and π is a nondegenerate representation of $\mathscr{I}$ on a Hilbert space $\mathscr{H}$, then there is a unique representation $\tilde{\boldsymbol{\pi}}$ of $\mathscr{A}$ on $\mathscr{H}$ which extends $\boldsymbol{\pi}$. This implies that the C^*-seminorm $\|\cdot\|_*^{\mathscr{I}}$ of $\mathscr{I}$ is equal to the restriction of the C^*-seminorm $\|\cdot\|_*^{\mathscr{A}}$ of $\mathscr{A}$ to $\mathscr{I}$. Indeed, note that the C^*-norm of $\mathscr{A}$ is also given by:

$$\begin{aligned}\|S\|_*^{\mathscr{A}} &= \sup\{\|\boldsymbol{\pi}(S)\| \mid \boldsymbol{\pi} \text{ is a representation of } \mathscr{A}\}\\ &= \sup\{\|\boldsymbol{\pi}(S)\| \mid \boldsymbol{\pi} \text{ is a nondegenerate representation of } \mathscr{A}\}.\end{aligned}$$

b) We apply these remarks with the choice $\mathscr{I} = \phi(\mathscr{J}) = \ker\psi$. Since ϕ is a bijective morphism of $\mathscr{J}$ onto $\mathscr{I}$, we have $\|\phi(S)\|_*^{\mathscr{I}} = \|S\|_*^{\mathscr{J}}$, hence also $\|S\|_*^{\mathscr{J}} = \|\phi(S)\|_*^{\mathscr{A}}$. So ϕ is an isometry of $(\mathscr{J}, \|\cdot\|_*^{\mathscr{J}})$ into $(\mathscr{A}, \|\cdot\|_*^{\mathscr{A}})$ with range equal to $\mathscr{I}$. This implies that ϕ_* is an isometry of $\mathscr{J}_*$ into $\mathscr{A}_*$ with range equal to $\overline{\mathscr{I}}$, the closure in $\mathscr{A}_*$ of $\theta_{\mathscr{A}}(\mathscr{I})$.
c) Now we compute for $S \in \mathscr{A}$ the norm of $\theta_{\mathscr{B}} \circ \psi(S)$ by noticing that $\boldsymbol{\pi} \mapsto \boldsymbol{\pi} \circ \psi$ realizes a bijective correspondence between the representations of $\mathscr{B}$ and the representations of $\mathscr{A}$ which are zero on $\mathscr{I}$ (denote by $\mathrm{Rep}\,(\mathscr{A}, \mathscr{I})$ the set of unitary equivalence classes of such representations):

$$\begin{aligned}\|\psi(S)\|_*^{\mathscr{B}} &= \sup\{\|\boldsymbol{\pi} \circ \psi(S)\| \mid \boldsymbol{\pi} \text{ is a representation of } \mathscr{B}\}\\ &= \sup\{\|\boldsymbol{\rho}(S)\| \mid \boldsymbol{\rho} \in \mathrm{Rep}\,(\mathscr{A}, \mathscr{I})\}\end{aligned}$$

The map $\boldsymbol{\rho} \mapsto \boldsymbol{\rho}_*$ is a bijective correspondence between the representations of $\mathscr{A}$ and the representations of $\mathscr{A}_*$ (with inverse $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda} \circ \theta_{\mathscr{A}}$) which sends $\mathrm{Rep}\,(\mathscr{A}, \mathscr{I})$ onto the space of representations of $\mathscr{A}_*$ which are zero on $\overline{\mathscr{I}}$. Hence

$$\|\psi(S)\|_*^{\mathscr{B}} = \sup\{\|\boldsymbol{\lambda} \circ \theta_{\mathscr{A}}(S)\| \mid \boldsymbol{\lambda} \in \mathrm{Rep}\,(\mathscr{A}_*, \overline{\mathscr{I}})\}.$$

The right hand side here is equal to the norm of the image of $\theta_{\mathscr{A}}(S)$ into the quotient C^*-algebra $\mathscr{A}_*/\overline{\mathscr{I}}$. So we see that $\psi_* : \mathscr{A}_* \to \mathscr{B}_*$ factorize to a surjective isometry $\mathscr{A}_*/\overline{\mathscr{I}} \to \mathscr{B}_*$. Hence $\ker\psi_* = \overline{\mathscr{I}} = \phi_*(\mathscr{J}_*)$. ■

We shall restate a particular case of the preceding theorem in a form which is particularly useful for us. Let $\mathscr{A}$ be an A^*-algebra and $\mathscr{J}$ an ideal in $\mathscr{A}$ such that the quotient Banach $*$-algebra $\mathscr{B} = \mathscr{A}/\mathscr{J}$ is an A^*-algebra (which is not always the case). Then one has a dense embedding $\mathscr{A} \subset \mathscr{A}_*$ and $\mathscr{J}_*$ is canonically identified with the closure $\overline{\mathscr{J}}$ of $\mathscr{J}$ in $\mathscr{A}_*$ (the C^*-norm of $\mathscr{J}$ being equal to the restriction to $\mathscr{J}$ of the C^*-norm of $\mathscr{A}$).

Moreover, the canonical map $\mathscr{A} \to \mathscr{A}/\mathscr{J}$ induces a surjective morphism $\mathscr{A}_* \to (\mathscr{A}/\mathscr{J})_*$ whose kernel is equal to $\mathscr{J}_* \equiv \overline{\mathscr{J}}$, so we have a canonical identification $(\mathscr{A}/\mathscr{J})_* = \mathscr{A}_*/\mathscr{J}_*$. In other terms, the natural map $\mathscr{A}_* \to \mathscr{A}_*/\mathscr{J}_*$ induces an isomorphism of $(\mathscr{A}/\mathscr{J})_*$ onto $\mathscr{A}_*/\mathscr{J}_*$.

2.3. Tensor products. We shall briefly review some facts concerning the tensor product of two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ (see [DeF] for tensor products of Banach spaces and [Mur], [Tks], and Appendix T in [Weo] for the case of C^*-algebras). The algebraic tensor product $\mathscr{A} \odot \mathscr{B}$ has an obvious structure of $*$-algebra. In general there are many C^*-norms on this $*$-algebra, but it can be shown that there is a smallest one $\|\cdot\|_{\min}$ and a largest one $\|\cdot\|_{\max}$. Hence a C^*-norm satisfies $\|\cdot\|_{\min} \leq \|\cdot\| \leq \|\cdot\|_{\max}$. The completion of $\mathscr{A} \odot \mathscr{B}$ under a C^*-norm is *a C^*-tensor product* of $\mathscr{A}$ and $\mathscr{B}$. The particular cases of $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ give the *minimal* (or spatial) and *maximal* tensor product, respectively. A C^*-algebra $\mathscr{A}$ is called *nuclear* if for any $\mathscr{B}$ one has $\|\cdot\|_{\min} = \|\cdot\|_{\max}$. Abelian algebras are nuclear.

The maximal tensor product is, in a natural sense, the enveloping C^*-algebra of $\mathscr{A} \odot \mathscr{B}$. Indeed, it is easy to see that there is a largest norm on $\mathscr{A} \odot \mathscr{B}$ satisfying $\|A \otimes B\| \leq \|A\| \cdot \|B\|$ and that this is a $*$-algebra norm. The completion of $\mathscr{A} \odot \mathscr{B}$ for this norm is a Banach $*$-algebra and its enveloping C^*-algebra is just the maximal tensor product $\mathscr{A} \otimes_{\max} \mathscr{B}$. Obviously, all the other tensor products are quotients of this one.

For reasons of simplicity we shall consider from now on only the minimal tensor product and we shall denote it $\mathscr{A} \otimes \mathscr{B}$. If $\mathscr{A}$ is realized on a Hilbert space $\mathscr{H}$ and $\mathscr{B}$ on a Hilbert space $\mathscr{K}$, then we have an obvious embedding $\mathscr{A} \odot \mathscr{B} \subset B(\mathscr{H} \otimes \mathscr{K})$ and $\mathscr{A} \otimes \mathscr{B}$ can be defined as the norm closure of $\mathscr{A} \odot \mathscr{B}$. In particular, if $\mathscr{C}$ is a C^*-subalgebra of $\mathscr{A}$, then $\mathscr{C} \otimes \mathscr{B}$ can (and will) be identified with the closure of $\mathscr{C} \odot \mathscr{B}$ in $\mathscr{A} \otimes \mathscr{B}$.

We stress that taking tensor products of continuous linear maps between C^*-algebra does not always make sense. More precisely, let $\mathscr{A}_1, \mathscr{A}_2, \mathscr{B}_1, \mathscr{B}_2$ be C^*-algebras and $\phi : \mathscr{A}_1 \to \mathscr{A}_2, \psi : \mathscr{B}_1 \to \mathscr{B}_2$ be linear continuous maps. Then the algebraic tensor product $\phi \odot \psi$ is a well defined linear map $\phi \odot \psi : \mathscr{A}_1 \odot \mathscr{A}_2 \to \mathscr{B}_1 \odot \mathscr{B}_2$, but in general this map is not continuous for the topologies induced by $\mathscr{A}_1 \otimes \mathscr{A}_2$ and $\mathscr{B}_1 \otimes \mathscr{B}_2$. The next two results are not so easy to prove, see Theorems 6.5.1 and 6.5.2 in [Mur].

Theorem 2.2 *If ϕ, ψ are morphisms, then $\phi \odot \psi$ extends to a morphism $\phi \otimes \psi : \mathscr{A}_1 \otimes \mathscr{A}_2 \to \mathscr{B}_1 \otimes \mathscr{B}_2$. If ϕ, ψ are injective, so is $\phi \otimes \psi$.*

Theorem 2.3 *Let $\mathscr{I}, \mathscr{A}, \mathscr{B}, \mathscr{C}$ be C^*-algebras and let*

$$0 \longrightarrow \mathscr{I} \xrightarrow{\ \phi\ } \mathscr{A} \xrightarrow{\ \psi\ } \mathscr{B} \longrightarrow 0$$

be an exact sequence of morphisms. Assume that $\mathscr{B} \otimes \mathscr{C}$ has a unique C^*-norm. Then, if id *is the identity morphism of $\mathscr{C}$,*

$$0 \longrightarrow \mathscr{I} \otimes \mathscr{C} \xrightarrow{\ \phi \otimes \mathrm{id}\ } \mathscr{A} \otimes \mathscr{C} \xrightarrow{\ \psi \otimes \mathrm{id}\ } \mathscr{B} \otimes \mathscr{C} \longrightarrow 0$$

is an exact sequence.

Corollary 2.4 *Let $\mathscr{A}, \mathscr{B}$ be C^*-algebras and let $\mathscr{I}$ be an ideal in $\mathscr{A}$. Then $\mathscr{I} \otimes \mathscr{B}$ is an ideal in $\mathscr{A} \otimes \mathscr{B}$ and if $\mathscr{A}/\mathscr{I}$ or $\mathscr{B}$ is nuclear, then*

$$[\mathscr{A} \otimes \mathscr{B}] / [\mathscr{I} \otimes \mathscr{B}] = [\mathscr{A}/\mathscr{I}] \otimes \mathscr{B}. \tag{2.1}$$

Let $\mathscr{C}_1, \mathscr{C}_2$ be C^*-algebras and let $\mathscr{J}_1 \subset \mathscr{C}_1$, $\mathscr{J}_2 \subset \mathscr{C}_2$ be ideals. For each $i = 1, 2$ let $\mathcal{P}_i : \mathscr{C}_i \to \mathscr{C}_i/\mathscr{J}_i \equiv \widehat{\mathscr{C}_i}$ be the canonical surjection and let $\mathcal{P}_1' = \mathcal{P}_1 \otimes \mathrm{id}$ and $\mathcal{P}_2' = \mathrm{id} \otimes \mathcal{P}_2$, morphisms of $\mathscr{C}_1 \otimes \mathscr{C}_2$ into $\widehat{\mathscr{C}_1} \otimes \mathscr{C}_2$ and $\mathscr{C}_1 \otimes \widehat{\mathscr{C}_2}$ respectively. The following result from [GI4], quite useful for the study of coupled systems, is another consequence of Theorem 2.3.

Theorem 2.5 *If $\mathscr{C}_1, \mathscr{C}_2$ are nuclear, then the kernel of the morphism*

$$\mathcal{P}_1' \oplus \mathcal{P}_2' : \mathscr{C}_1 \otimes \mathscr{C}_2 \to \left[\widehat{\mathscr{C}_1} \otimes \mathscr{C}_2\right] \oplus \left[\mathscr{C}_1 \otimes \widehat{\mathscr{C}_2}\right]$$

is equal to $\mathscr{J}_1 \otimes \mathscr{J}_2$.

Let $\mathscr{A}$ be a C^*-algebra and T a set. If we think of T as a locally compact topological space equipped with the discrete topology, the C^*-algebras $C_{\mathrm{b}}(T;\mathscr{A})$, $C_{\mathrm{rc}}(T;\mathscr{A})$, and $C_0(T;\mathscr{A})$ are well defined (see §2.1). The first one plays no role in what follows and we adopt special notations for the other two: $\mathscr{A}^{[T]} = C_{\mathrm{rc}}(T;\mathscr{A})$ and $\mathscr{A}^{(T)} = C_0(T;\mathscr{A})$. Thus $\mathscr{A}^{[T]}$ is the set of families $(A_t)_{t\in T}$ such that $\{A_t \mid t \in T\}$ is a relatively compact subset of $\mathscr{A}$ with the natural operations and the sup norm. And $\mathscr{A}^{(T)}$ is the ideal consisting of families $(A_t)_{t\in T}$ such that $\|A_t\| \to 0$ when $t \to \infty$. Below we use the standard (T being discrete) notations $c_0(T), l^\infty(T)$ for the spaces $C_0(T), C_{\mathrm{b}}(T)$.

Lemma 2.6 $\mathscr{A}^{(T)} \cong c_0(T) \otimes \mathscr{A}$ and $\mathscr{A}^{[T]} \cong l^\infty(T) \otimes \mathscr{A}$.

Proof: The first relation is obvious. To prove the second one, assume that $\mathscr{A} \subset B(\mathscr{H})$ and realize $l^\infty(T)$ as a C^*-algebra on $l^2(T)$ in the standard way. Then $l^\infty(T) \odot \mathscr{A}$ is realized on $l^2(T) \otimes \mathscr{H} = l^2(T;\mathscr{H})$ as the set $l^\infty_{\text{fin}}(T;\mathscr{A})$ of operators of multiplication by functions $F : T \to \mathscr{A}$ such that the range of F is included in a finite dimensional subspace of $\mathscr{A}$. Now it suffices to note that $l^\infty_{\text{fin}}(T;\mathscr{A})$ is dense in $\mathscr{A}^{[T]}$. ∎

One more object will appear naturally in our later investigations: the T-*asymptotic algebra* of $\mathscr{A}$. This is the quotient algebra:

$$\mathscr{A}^{\langle T\rangle} = \mathscr{A}^{[T]} \Big/ \mathscr{A}^{(T)} \tag{2.2}$$

From Lemma 2.6 and (2.1) we get $\mathscr{A}^{\langle T\rangle} = [l^\infty(T)/c_0(T)] \otimes \mathscr{A}$. Let δT be the set of ultrafilters $\varkappa$ on T finer than the Fréchet filter (since T is a discrete space, this is the family of sets having finite complements). δT should be thought as the "boundary" of T and it is equipped with a natural topology of compact space (see ch. 2 in [HiS]) such that $[l^\infty(T)/c_0(T)] = C(\delta T)$. Thus

$$\mathscr{A}^{\langle T\rangle} = C(\delta T;\mathscr{A}). \tag{2.3}$$

A detailed presentation of this topic, as well as applications, can be found in [GI2]. Note that one can consider an arbitrary locally compact space T and the algebras $C_0(T;\mathscr{A})$ and $C_{\text{rc}}(T;\mathscr{A})$.

2.4. Crossed products. We now recall the definition of the crossed product of a C^*-algebra by the action of a locally compact abelian group X (with the operation denoted additively). Most of what follows is valid in the non-abelian case too, see [Ped]. We fix a Haar measure $\mathrm{d}x$ on X but note that the crossed products $\mathscr{A} \rtimes X$ defined below will not depend on this choice.

We shall say that a C^*-algebra $\mathscr{A}$ is an X-*algebra* if a homomorphism $\alpha : x \mapsto \alpha_x$ of X into the group of automorphisms of $\mathscr{A}$ is given, such that for each $A \in \mathscr{A}$ the map $x \mapsto \alpha_x(A)$ is continuous. A subalgebra of $\mathscr{A}$ is called *stable* if it is left invariant by all the automorphisms α_x. If $(\mathscr{A},\alpha)$ and $(\mathscr{B},\beta)$ are two X-algebras, a morphism $\phi : \mathscr{A} \to \mathscr{B}$ is called X-*morphism* (or *covariant* morphism) if $\phi[\alpha_x(A)] = \beta_x[\phi(A)]$ for all $x \in X$ and $A \in \mathscr{A}$.

Let $\mathscr{A}$ be an X-algebra and let $L^1(X;\mathscr{A})$ be the Banach $*$-algebra constructed as follows. As a Banach space it is just the space of (Bochner) integrable (equivalence classes of) functions $S : X \to \mathscr{A}$. The product and the involution are defined by:

$$(S\cdot T)(x) = \int_X S(y)\,\alpha_y[T(x-y)]\,\mathrm{d}y, \tag{2.4}$$

$$S^*(x) = \alpha_x[S(-x)^*]. \tag{2.5}$$

Note that $C_c(X;\mathscr{A})$, the space of continuous functions $X \to \mathscr{A}$ with compact support, is a dense $*$-subalgebra of $L^1(X;\mathscr{A})$. Moreover, the algebraic tensor product $\mathscr{A}\odot C_c(X)$ is a dense subspace (identified with the set of elements from $C_c(X;\mathscr{A})$ whose ranges are contained in finite dimensional subspaces of $\mathscr{A}$).

Assume, furthermore, that $\mathscr{A}$ is realized on a Hilbert space $\mathscr{H}$ and let $\mathscr{H}_X = L^2(X;\mathscr{H})$. Then one has a faithful representation of $L^1(X;\mathscr{A})$ on $\mathscr{H}_X$, the so-called *left regular representation*: one defines the action of $S \in L^1(X;\mathscr{A})$ onto $\xi \in \mathscr{H}_X$ by

$$(S \bullet \xi)(x) = \int_X \alpha_{-x}[S(x-y)]\,\xi(y)\,\mathrm{d}y. \tag{2.6}$$

In particular, $L^1(X;\mathscr{A})$ *is an* A^**-algebra* (see §2.2).

Definition 2.7 *If $\mathscr{A}$ is a X-algebra, then the **crossed product** of $\mathscr{A}$ by the action α of X is the enveloping C^*-algebra of the A^*-algebra $L^1(X;\mathscr{A})$. This C^*-algebra is denoted by $\mathscr{A}\rtimes X$.*

Thus $\mathscr{A}\rtimes X$ is the completion of $L^1(X;\mathscr{A})$ under the largest C^*-norm on it, and each representation of $L^1(X;\mathscr{A})$ extends to a representation of $\mathscr{A}\rtimes X$ (see §2.2). Due to the fact that X is abelian (hence amenable) the crossed product defined above coincides with the so-called "reduced crossed product" (cf. Theorems 7.7.5 and 7.7.7 in [Ped]):

Theorem 2.8 *The left regular representation of $L^1(X;\mathscr{A})$ extends to a faithful representation of $\mathscr{A}\rtimes X$. In particular, $\mathscr{A}\rtimes X$ is canonically isomorphic to the closure in $B(\mathscr{H}_X)$ of the set of operators of the form (2.6).*

Heuristically, one should think of $\mathscr{A} \rtimes X$ as a kind of twisted tensor product of the algebras $\mathscr{A}$ and $C_0(X^*)$, where X^* is the group dual to X. In fact, if the action of X on $\mathscr{A}$ is trivial, then $\mathscr{A} \rtimes X = \mathscr{A} \otimes C_0(X^*)$ (if X is not abelian, the crossed product is the maximal tensor product of $\mathscr{A}$ with the group C^*-algebra of X, while the reduced crossed product is the minimal tensor product of $\mathscr{A}$ with the reduced group C^*-algebra of X).

We now point out a certain universal property of the algebra $\mathscr{A} \rtimes X$. We treat this question only as far as we need it (see [Ped] and [Rae] for a more complete discussion). The following notations will be used. If $\mathscr{A}, \mathscr{B}$ are subalgebras of a C^*-algebra $\mathscr{C}$ then $\mathscr{A} \cdot \mathscr{B}$ is the set of finite sums of the form $A_1 B_1 + \cdots + A_n B_n$ with $A_i \in \mathscr{A}, B_i \in \mathscr{B}$. This is a linear subspace of $\mathscr{C}$ and we denote by $[\![\mathscr{A} \cdot \mathscr{B}]\!]$ its norm closure. The notation $\widetilde{u}(P)$ from the next theorem looks strange here, but is suited to our later purposes.

Theorem 2.9 *Assume that $\mathscr{A}$ is a C^*-algebra of operators on a Hilbert space $\mathscr{H}$ and that there is a strongly continuous unitary representation $\{U_x\}_{x \in X}$ of X on $\mathscr{H}$ such that $\alpha_x(A) = U_x A U_x^*$ for all $x \in X, A \in \mathscr{A}$. If $u \in L^1(X)$ we set $\widetilde{u}(P) = \int_X U_x u(x)\,\mathrm{d}x$. Then $\mathscr{B} = \{\widetilde{u}(P) \mid u \in L^1(X)\}$ is a $*$-algebra of bounded operators on $\mathscr{H}$ and $[\![\mathscr{A} \cdot \mathscr{B}]\!]$ is a C^*-subalgebra of $B(\mathscr{H})$. There is a unique morphism $\Phi : \mathscr{A} \rtimes X \to [\![\mathscr{A} \cdot \mathscr{B}]\!]$ such that $\Phi[A \otimes u] = A\widetilde{u}(P)$ for all $A \in \mathscr{A}$ and $u \in L^1(X)$. This morphism is surjective.*

Note that $A \otimes u \in \mathscr{A} \odot L^1(X)$ which is a dense subspace of $L^1(X; \mathscr{A})$, hence of $\mathscr{A} \rtimes X$. The theorem says that $[\![\mathscr{A} \cdot \mathscr{B}]\!]$ is a quotient of $\mathscr{A} \rtimes X$. The morphism Φ is not injective in general (e.g. if X acts trivially on $\mathscr{A}$).

Proof: The map $u \to \widetilde{u}(P)$ is a morphism $L^1(X) \to B(\mathscr{H})$ if we equip $L^1(X)$ with the usual convolution $*$-algebra structure. So $\mathscr{B}$ is a $*$-subalgebra of $B(\mathscr{H})$. From standard properties of the algebraic tensor product it follows that there is a unique linear map $\Phi_0 : \mathscr{A} \odot L^1(X) \to [\![\mathscr{A} \cdot \mathscr{B}]\!]$ such that $\Phi_0[A \otimes u] = A\widetilde{u}(P)$ for all $A \in \mathscr{A}$ and $u \in L^1(X)$. It is clear that the range of Φ_0 is dense in $[\![\mathscr{A} \cdot \mathscr{B}]\!]$. Observe that for $S \in \mathscr{A} \odot L^1(X)$ we have

$$\Phi_0[S] = \int_X S(x) U_x \,\mathrm{d}x. \tag{2.7}$$

The relation is obvious for $S(x) = Au(x)$; the general case is an immediate consequence. From (2.7) we see that Φ_0 extends to a contraction Φ_1 from $L^1(X; \mathscr{A})$ onto a dense subspace of $[\![\mathscr{A} \cdot \mathscr{B}]\!]$. But Φ_1 is a morphism (use (2.4) and (2.5)).

Since $\mathscr{A} \rtimes X$ is the enveloping algebra of $L^1(X;\mathscr{A})$, Φ_1 extends to a morphism $\Phi : \mathscr{A} \rtimes X \to B(\mathscr{H})$. The range of Φ_1 is closed, hence equal to $[\![\mathscr{A} \cdot \mathscr{B}]\!]$, which therefore is a C^*-subalgebra of $B(\mathscr{H})$. ∎

2.5. Functorial properties. The correspondence $\mathscr{A} \mapsto \mathscr{A} \rtimes X$ extends to a covariant functor from the category of X-algebras (with X-morphisms as morphisms) into the category of C^*-algebras. Indeed, if $\phi : \mathscr{A} \to \mathscr{B}$ is an X-morphism, then it clearly induces a morphism $\phi_0 : L^1(X;\mathscr{A}) \to L^1(X;\mathscr{B})$ by the formula $(\phi_0 S)(x) := \phi[S(x)]$. Hence we may define the morphism $\phi_* : \mathscr{A} \rtimes X \to \mathscr{B} \rtimes X$ as the canonical extension of ϕ_0 to the enveloping algebras, i.e. $\phi_* = (\phi_0)_*$.

Theorem 2.10 *Let $\mathscr{J}$, $\mathscr{A}$, $\mathscr{B}$ be X-algebras and let*

$$0 \longrightarrow \mathscr{J} \xrightarrow{\ \phi\ } \mathscr{A} \xrightarrow{\ \psi\ } \mathscr{B} \longrightarrow 0$$

be an exact sequence of X-morphisms. Then

$$0 \longrightarrow \mathscr{J} \rtimes X \xrightarrow{\ \phi_*\ } \mathscr{A} \rtimes X \xrightarrow{\ \psi_*\ } \mathscr{B} \rtimes X \longrightarrow 0$$

is an exact sequence.

Proof: It suffices to prove that

$$0 \longrightarrow L^1(X;\mathscr{J}) \xrightarrow{\ \phi_0\ } L^1(X;\mathscr{A}) \xrightarrow{\ \psi_0\ } L^1(X;\mathscr{B}) \longrightarrow 0$$

is an exact sequence of Banach $*$-algebras; then we use Theorem 2.1. The injectivity of ϕ_0 and the relation $\psi_0 \circ \phi_0 = 0$ are obvious. If $S \in L^1(X;\mathscr{A})$ and $\psi_0(S) = 0$ then $\psi[S(x)] = 0$ for a.e. $x \in X$, i.e. $S(x) \in \phi(\mathscr{J})$ for a.e. x. But ϕ is an isometry, so there is $T \in L^1(X;\mathscr{J})$ such that $S(x) = \phi[T(x)]$ for a.e. x. This proves that $\ker \psi_0 = \phi_0(L^1(X;\mathscr{J}))$.

The surjectivity of ψ_0 is a consequence of the following general property (see §3.5 in [DeF]). Let $\mathbf{E}$ be a Banach space and $F \in L^1(X; \mathbf{E})$. Then for each number $\varepsilon > 0$ there are sequences $f_n \in L^1(X)$ and $e_n \in \mathbf{E}$ such that $F = \sum_{n=1}^{\infty} e_n \otimes f_n$ and

$$\sum_{n=1}^{\infty} \|e_n\| \int |f_n(x)| \, \mathrm{d}x \leq (1+\varepsilon) \int \|F(x)\| \, \mathrm{d}x.$$

Note also that since the map $\mathscr{A} / \ker \psi \to \mathscr{B}$ induced by ψ is an isometric bijection, for each $b \in \mathscr{B}$ there is $a \in \mathscr{A}$ such that $\psi(a) = b$ and $\|a\| = (1+\varepsilon)\|b\|$. ■

Let $\mathscr{J}$ be a stable ideal of an X-algebra $\mathscr{A}$. According to Theorem 2.10, if $j : \mathscr{J} \to \mathscr{A}$ is the inclusion map, then $j_* : \mathscr{J} \rtimes X \to \mathscr{A} \rtimes X$ is an isometric morphism of $\mathscr{J} \rtimes X$ onto an ideal of $\mathscr{A} \rtimes X$. From now on *we shall identify $\mathscr{J} \rtimes X$ with its image under j_**. More explicitly, $\mathscr{J} \rtimes X$ is just the closure in $\mathscr{A} \rtimes X$ of the ideal $L^1(X; \mathscr{J})$ of $L^1(X; \mathscr{A})$.

Now the quotient C^*-algebra $\mathscr{B} = \mathscr{A} / \mathscr{J}$ has a natural structure of X-algebra such that the canonical morphism $\mathscr{A} \to \mathscr{A} / \mathscr{J}$ is an X-morphism. The Theorem 2.10 says also that the morphism $\mathscr{A} \rtimes X \to [\mathscr{A} / \mathscr{J}] \rtimes X$ associated to it has $\mathscr{J} \rtimes X$ as kernel. We thus get the following reformulation of Theorem 2.10:

Theorem 2.11 *If $\mathscr{J}$ is a stable ideal of an X-algebra $\mathscr{A}$ then*

$$\mathscr{A} \rtimes X \Big/ \mathscr{J} \rtimes X \cong [\mathscr{A} / \mathscr{J}] \rtimes X. \tag{2.8}$$

The simplest case of the preceding situation is that when the exact sequence splits, so that $\mathscr{A} / \mathscr{J}$ can be realized as a stable C^*-subalgebra of $\mathscr{A}$. Then we have:

Corollary 2.12 *Let $\mathscr{A}$ be an X-algebra, $\mathscr{J}$ a stable ideal, and $\mathscr{B}$ a stable C^*-subalgebra such that $\mathscr{A} = \mathscr{B} + \mathscr{J}$ direct linear sum. Then $\mathscr{J} \rtimes X$ is an ideal in $\mathscr{A} \rtimes X$, $\mathscr{B} \rtimes X$ is a C^*-subalgebra of $\mathscr{A} \rtimes X$, and $\mathscr{A} \rtimes X = \mathscr{B} \rtimes X + \mathscr{J} \rtimes X$ direct linear sum.*

In particular, if $\mathscr{A}, \mathscr{B}$ are X-algebras and $\mathscr{A} \oplus \mathscr{B}$ is equipped with the natural X-algebra structure, then $(\mathscr{A} \oplus \mathscr{B}) \rtimes X \cong (\mathscr{A} \rtimes X) \oplus (\mathscr{B} \rtimes X)$. We mention one more fact:

Proposition 2.13 *If $\phi : \mathscr{A} \to \mathscr{B}$ is an injective or surjective X-morphism then $\phi_* : \mathscr{A} \rtimes X \to \mathscr{B} \rtimes X$ is injective or surjective respectively. In particular, if $\mathscr{A}$ is a stable C^*-subalgebra of the X-algebra $\mathscr{B}$, then $\mathscr{A} \rtimes X$ can be identified with a C^*-subalgebra of $\mathscr{B} \rtimes X$.*

The assertion is obvious in the surjective case. For the injective case, see Proposition 7.7.9 in [Ped]. So what we proved above for ideals is valid for subalgebras too.

Proposition 2.14 *Let $(\mathscr{A}, \alpha)$ be an X-algebra and $(\mathscr{B}, \beta)$ a Y-algebra. Equip $\mathscr{A} \otimes \mathscr{B}$ with the $X \times Y$-algebra structure defined by $\gamma_{(x,y)}(a \otimes b) = \alpha_x(a) \otimes \beta_y(b)$. Then*

$$(\mathscr{A} \otimes \mathscr{B}) \rtimes (X \times Y) \cong (\mathscr{A} \rtimes X) \otimes (\mathscr{B} \rtimes Y). \tag{2.9}$$

We send to Takai (Proposition 2.4 in [Tak]) for a similar result. Since there is no proof there, we shall sketch here a simple one.

Proof: Assume that $\mathscr{A}$ and $\mathscr{B}$ are realized on the Hilbert spaces $\mathscr{H}$ and $\mathscr{G}$ respectively. Then $\mathscr{A} \otimes \mathscr{B}$ can be identified with the norm closure in $B(\mathscr{H} \otimes \mathscr{G})$ of the algebraic tensor product $\mathscr{A} \odot \mathscr{B}$ (realized on $\mathscr{H} \otimes \mathscr{G}$). The C^*-norm on $L^1(X \times Y; \mathscr{A} \otimes \mathscr{B})$ is obtained by using the left regular representation on $\mathscr{E} \equiv L^2(X \times Y; \mathscr{H} \otimes \mathscr{G}) = L^2(X; \mathscr{H}) \otimes L^2(Y; \mathscr{G}) \equiv \mathscr{H}_X \otimes \mathscr{G}_Y$. If $L \in L^1(X \times Y; \mathscr{A} \otimes \mathscr{B})$ is of the form $L(x,y) = S(x) \otimes T(y)$, with $S \in L^1(X; \mathscr{A})$ and $T \in L^1(Y; \mathscr{B})$, we have

$$\begin{aligned} [L \bullet (\xi \otimes \eta)](x,y) &= \iint_{X \times Y} \gamma_{-(x,y)}[S(x-s) \otimes T(y-t)] \cdot (\xi \otimes \eta)(s,t)\, \mathrm{d}s\, \mathrm{d}t \\ &= \int_X \alpha_{-x}[S(x-s)] \cdot \xi(s)\, \mathrm{d}s \otimes \int_Y \beta_{-y}[T(y-t)] \cdot \eta(t)\, \mathrm{d}t \\ &= [(S \bullet \xi) \otimes (T \bullet \eta)](x,y). \end{aligned}$$

If we denote by $\pi_{\mathscr{A} \otimes \mathscr{B}}$, $\pi_{\mathscr{A}}$ and $\pi_{\mathscr{B}}$ the left regular representations of $L^1(X \times Y; \mathscr{A} \otimes \mathscr{B})$, $L^1(X; \mathscr{A})$ and $L^1(Y; \mathscr{B})$ respectively, we see that $\pi_{\mathscr{A} \otimes \mathscr{B}} = \pi_{\mathscr{A}} \otimes \pi_{\mathscr{B}}$ in $\mathscr{E} = \mathscr{H}_X \otimes \mathscr{G}_Y$, which proves the proposition. ■

The next consequence is sufficient for our purposes.

Corollary 2.15 *Let $\mathscr{A}$ be an X-algebra and let $\mathscr{B}$ be a nuclear (e.g. abelian) C^*-algebra. Equip $\mathscr{A} \otimes \mathscr{B}$ with the X-algebra structure defined by $\alpha_x(a \otimes b) = \alpha_x(a) \otimes b$. Then*

$$(\mathscr{A} \otimes \mathscr{B}) \rtimes X \cong (\mathscr{A} \rtimes X) \otimes \mathscr{B}. \tag{2.10}$$

We close this paragraph with a consequence of (2.2) and of Lemma 2.6.

Corollary 2.16 *If $\mathscr{A}$ is an X-algebra and T a set, then*

$$\mathscr{A}^{(T)} \rtimes X \cong (\mathscr{A} \rtimes X)^{(T)} \text{ and } \mathscr{A}^{[T]} \rtimes X \cong (\mathscr{A} \rtimes X)^{[T]}. \tag{2.11}$$

Moreover, the T-asymptotic algebra $\mathscr{A}^{\langle T \rangle}$ has a canonical X-algebra structure and one has

$$(\mathscr{A} \rtimes X)^{[T]} \Big/ (\mathscr{A} \rtimes X)^{(T)} = (\mathscr{A} \rtimes X)^{\langle T \rangle} \cong C(\delta T; \mathscr{A} \rtimes X). \tag{2.12}$$

This follows from Lemma 2.6 and the definition (2.2).

2.6. Pseudodifferential operators. We introduce several new notations and recall facts concerning the harmonic analysis on X (see [Fol], [Loo], and [Wei] for details). Note that the Hilbert space $L^2(X) = L^2(X, \mathrm{d}x)$ depends on the choice of the Haar measure $\mathrm{d}x$, but the C^*-algebras $\mathscr{B}(X) = B(L^2(X))$, $\mathscr{K}(X) = K(L^2(X))$ do not. We shall embed the C^*-algebras $C_0(X)$, $C_\mathrm{b}^\mathrm{u}(X)$, $C_\mathrm{b}(X)$ in $\mathscr{B}(X)$ by associating to $\varphi \in C_\mathrm{b}(X)$ the operator of multiplication by the function φ. In order to avoid ambiguities we often denote this operator by $\varphi(Q)$ (as in quantum mechanics, where Q is the X-*valued position observable*).

Let X^* be the abelian locally compact group dual to X. The Fourier transform of $u \in L^1(X)$ is the function $\mathcal{F}u \equiv \widehat{u} \in C_0(X^*)$ given by $\widehat{u}(k) = \int_X \overline{k(x)} u(x)\, \mathrm{d}x$. Let us equip X^* with the unique Haar measure $\mathrm{d}k$ such that $\mathcal{F}$ induces a unitary map $\mathcal{F} : L^2(X) \to L^2(X^*)$. For each $\psi \in C_\mathrm{b}(X^*)$ we define the operator $\psi(P) \in \mathscr{B}(X)$ by $\psi(P) = \mathcal{F}^{-1} M_\psi \mathcal{F}$, where

M_ψ is the operator of multiplication by ψ in $L^2(X^*)$ (in the quantum mechanical setting P is interpreted as the X^*-*valued momentum observable*). The injective morphism $\psi \mapsto \psi(P)$ gives us an embedding $C_{\mathrm{b}}(X^*) \subset \mathscr{B}(X)$.

Let U_x be the unitary operator in $L^2(X)$ defined by $(U_x f)(y) = f(x+y)$. We get a strongly continuous unitary representation of X on $L^2(X)$. We also set $\widetilde{u}(k) = \int_X k(x)u(x)\,\mathrm{d}x$ for $u \in L^1(X)$. Then it is easy to check that $\widetilde{u}(P) = \int_X U_x u(x)\,\mathrm{d}x$, which explains the notation used in Theorem 2.9. Since $\mathcal{F}L^1(X)$ is dense in $C_0(X^*)$, we see that the closure of the algebra denoted $\mathscr{B}$ in Theorem 2.9 is just $C_0(X^*)$ (when this C^*-algebra is embedded in $\mathscr{B}(X)$ as a subalgebra of $C_{\mathrm{b}}(X^*)$).

If $x \in X$ we denote by τ_x the operator acting on functions f on X according to $(\tau_x f)(y) = f(y-x)$. We equip $C_{\mathrm{b}}^{\mathrm{u}}(X)$ with the X-algebra structure defined by the action $\alpha_x = \tau_{-x}$. If $\mathscr{A}$ is a C^*-subalgebra of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ stable under translations then $\mathscr{A}$ becomes an X-algebra too. We have $\alpha_x[\varphi(Q)] = U_x\varphi(Q)U_x^*$, so that we are under the conditions of Theorem 2.9. The next result is important in our applications: it says that the representation of the crossed product $\mathscr{A} \rtimes X$ on the Hilbert space $L^2(X)$ described in Theorem 2.9 is faithful (this is Theorem 3.12 in [GI2]).

Theorem 2.17 *Let $\mathscr{A}$ be a C^*-subalgebra of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ stable under translations. Then $[\![\mathscr{A} \cdot C_0(X^*)]\!]$ is a C^*-algebra of operators on $L^2(X)$ and*

$$[\![\mathscr{A} \cdot C_0(X^*)]\!] \cong \mathscr{A} \rtimes X. \tag{2.13}$$

More precisely, there is a unique morphism $\Phi : \mathscr{A} \rtimes X \to [\![\mathscr{A} \cdot C_0(X^*)]\!]$ such that $\Phi[\varphi \otimes u] = \varphi(Q)\widetilde{u}(P)$ for all $\varphi \in \mathscr{A}$ and $u \in L^1(X)$ and Φ is an isomorphism.

Corollary 2.18 $\mathscr{K}(X) = [\![C_0(X) \cdot C_0(X^*)]\!] \cong C_0(X) \rtimes X$.

2.7. Observables affiliated to C^*-algebras. If $\mathscr{C}$ is a C^*-algebra then an *observable H affiliated to $\mathscr{C}$* is a morphism $H : C_0(\mathbb{R}) \to \mathscr{C}$. In order to keep close to standard notations we shall denote by $\varphi(H)$ (*not* $H(\varphi)$) the image of $\varphi \in C_0(\mathbb{R})$ through this morphism. We say that H is *$\mathscr{C}$-nondegenerate*, or that *H is strictly affiliated to $\mathscr{C}$*, if the linear subspace generated by $\{\varphi(H)A \mid \varphi \in C_0(\mathbb{R}),\ A \in \mathscr{C}\}$ is dense in $\mathscr{C}$. The following nontrivial fact follows from the Cohen-Hewitt theorem (see V.9.2 in [FeD]):

Theorem 2.19 *If H is an observable strictly affiliated to $\mathscr{C}$, then for each $A \in \mathscr{C}$ there are $\varphi, \psi \in C_0(\mathbb{R})$ and $B \in \mathscr{C}$ such that $A = \varphi(H)B\psi(H)$.*

In particular, a $\mathscr{C}$-nondegenerate H extends to a morphism from $C_{\mathrm{b}}(\mathbb{R})$ to the multiplier algebra of $\mathscr{C}$, but we shall not use this fact.

The *spectrum* of the observable H is defined by:

$$\boldsymbol{\sigma}(H) = \{\lambda \in \mathbb{R} \mid \varphi \in C_0(\mathbb{R}) \text{ and } \varphi(\lambda) \neq 0 \Longrightarrow \varphi(H) \neq 0\}. \tag{2.14}$$

We define the image of an observable through a morphism as follows. If $\mathscr{C}_1$, $\mathscr{C}_2$ are C^*-algebras, H_1 is an observable affiliated to $\mathscr{C}_1$ and $\mathcal{P} : \mathscr{C}_1 \to \mathscr{C}_2$ is a morphism, then $\varphi \mapsto \mathcal{P}[\varphi(H_1)]$ is a morphism $C_0(\mathbb{R}) \to \mathscr{C}_2$. Thus we get an observable affiliated to $\mathscr{C}_2$ that we shall denote by $H_2 = \mathcal{P}[H_1]$. Obviously $\boldsymbol{\sigma}(H_2) \subset \boldsymbol{\sigma}(H_1)$.

In particular, if $\mathscr{J}$ is an ideal in $\mathscr{C}$, $\widehat{\mathscr{C}} = \mathscr{C}/\mathscr{J}$ is the quotient algebra and H is an observable affiliated to $\mathscr{C}$, we may define the quotient $\widehat{H}$ (denoted $H/\mathscr{J}$ in case of ambiguity) as the observable affiliated to $\widehat{\mathscr{C}}$ given by $\widehat{H} = \pi(H)$, where π is the canonical morphism $\mathscr{C} \to \widehat{\mathscr{C}}$. In this context, it is useful to remark the similarity between:

$$\boldsymbol{\sigma}(\widehat{H}) = \{\lambda \in \mathbb{R} \mid \varphi \in C_0(\mathbb{R}) \text{ and } \varphi(\lambda) \neq 0 \Longrightarrow \varphi(H) \notin \mathscr{J}\}, \tag{2.15}$$

and one of the characterizations of the usual notion of essential spectrum in a Hilbert space setting (see the end of §2.8). It is thus natural to call this set the essential spectrum of H with respect to the ideal $\mathscr{J}$, and denote it $\mathscr{J}\text{-}\boldsymbol{\sigma}_{\text{ess}}(H)$.

We mention a result, important for one of our applications, which also involves the essential spectrum with respect to general ideals. Let $\{\mathscr{C}_t\}_{t\in T}$ be an arbitrary family of C^*-algebras. If for each $t \in T$ an observable H_t affiliated to $\mathscr{C}_t$ is given, we may associate to it an observable $H = \prod_{t\in T} H_t$ affiliated to $\mathscr{C} = \prod_{t\in T} \mathscr{C}_t$ by setting $\varphi(H) = (\varphi(H_t))_{t\in T}$ for each $\varphi \in C_0(\mathbb{R})$. It is easily shown that H is affiliated to the subalgebra $\bigoplus_{t\in T} \mathscr{C}_t$ if and only if $H_t \to \infty$ as $t \to \infty$ in T in the following sense: for each compact real set K there is a finite subset $F \subset T$ such that $\boldsymbol{\sigma}(H_t) \cap K = \emptyset$ if $t \in T \setminus F$. One has

$$\boldsymbol{\sigma}(H) = \overline{\textstyle\bigcup_{t\in T} \boldsymbol{\sigma}(H_t)}$$

and if H is affiliated to $\bigoplus_{t\in T}\mathscr{C}_t$ then the union is already closed. The following generalization of this relation is important for us (see [GI2]).

Proposition 2.20 *For each $t \in T$ let $\mathscr{J}_t$ be an ideal in $\mathscr{C}_t$ and let $\mathscr{J} = \bigoplus_{t\in T}\mathscr{J}_t$, so that $\mathscr{J}$ is an ideal in $\mathscr{C} = \prod_{t\in T}\mathscr{C}_t$. Denote by $\widehat{H_t}$ the quotient of H_t in $\mathscr{C}_t/\mathscr{J}_t$ and let $\widehat{H}$ be the quotient of H in $\mathscr{C}/\mathscr{J}$. Then*

$$\boldsymbol{\sigma}(\widehat{H}) = \bigcap_{\substack{F\subset T\\ F\,\text{finite}}} \left\{ \left(\bigcup_{t\in F}\boldsymbol{\sigma}(\widehat{H_t}) \right) \cup \left(\overline{\bigcup_{s\in T\setminus F}\boldsymbol{\sigma}(H_s)} \right) \right\}. \tag{2.16}$$

Remark that, with the notations introduced above, (2.16) may be written:

$$\mathscr{J}\text{-}\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcap_{\substack{F\subset T\\ F\,\text{finite}}} \left\{ \left(\bigcup_{t\in F}\mathscr{J}_t\text{-}\boldsymbol{\sigma}_{\text{ess}}(H_t) \right) \cup \left(\overline{\bigcup_{s\in T\setminus F}\boldsymbol{\sigma}(H_s)} \right) \right\}.$$

2.8. Affiliation of self-adjoint operators. We consider now the case where the C^*-algebra $\mathscr{C}$ is realized on a Hilbert space $\mathscr{H}$, i.e. $\mathscr{C}$ is a C^*-subalgebra of $B(\mathscr{H})$. A self-adjoint operator H on $\mathscr{H}$ is called *affiliated to* $\mathscr{C}$ if $(H-z)^{-1} \in \mathscr{C}$ for some $z \in \mathbb{C} \setminus \boldsymbol{\sigma}(H)$. This implies $\varphi(H) \in \mathscr{C}$ for all $\varphi \in C_0(\mathbb{R})$, so each self-adjoint operator on $\mathscr{H}$ affiliated to $\mathscr{C}$ defines an observable affiliated to $\mathscr{C}$. The other observables affiliated to $\mathscr{C}$ can be realized as *non-densely* defined operators on $\mathscr{H}$.

If H is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathscr{C}$ and if the corresponding observable is $\mathscr{C}$-nondegenerate we say that *H is strictly affiliated to $\mathscr{C}$*. In this case Theorem 2.19 implies that for each $A \in \mathscr{C}$ there are $\varphi_1, \varphi_2 \in C_0(\mathbb{R})$ and $B \in \mathscr{C}$ such that $A = \varphi_1(H)B\varphi_2(H)$. As a consequence, the operators $\varphi(H)A$ and $A\varphi(H)$ belong to $\mathscr{C}$ for all $\varphi \in C_{\mathrm{b}}(\mathbb{R})$.

It can be shown that *if $\mathscr{C}$ is nondegenerate on $\mathscr{H}$, then the correspondence between self-adjoint operators on $\mathscr{H}$ strictly affiliated to $\mathscr{C}$ and observables strictly affiliated to $\mathscr{C}$ defined above is bijective* (see [DG2]). The following is Proposition 2.3 in [DG2]. Note that if there is a self-adjoint operator on a Hilbert space $\mathscr{H}$ affiliated to a C^*-algebra of operators on $\mathscr{H}$, then this algebra is nondegenerate on $\mathscr{H}$.

Theorem 2.21 *Let $\mathscr{C}_1, \mathscr{C}_2$ be nondegenerate C^*-algebras of bounded operators on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively and let $\mathcal{P} : \mathscr{C}_1 \to \mathscr{C}_2$ be a morphism such that $\mathcal{P}(\mathscr{C}_1)$ acts nondegenerately on $\mathscr{C}_2$ (e.g. assume $\mathcal{P}$ surjective). Then for each self-adjoint operator H_1 on $\mathcal{H}_1$ strictly affiliated to $\mathscr{C}_1$ there is a unique self-adjoint operator H_2 on $\mathcal{H}_2$ such that $\mathcal{P}[\varphi(H_1)] = \varphi(H_2)$ for each $\varphi \in C_0(\mathbb{R})$. H_2 is strictly affiliated to $\mathscr{C}_2$.*

We give an example which clarifies the distinction between affiliation and strict affiliation. Consider the C^*-algebra $C_0(\mathbb{R})$ realized as usual on $L^2(\mathbb{R})$ and let h be the real function given by $h(x) = x + x^{-1}$ for $x \neq 0$ and $h(0) = 0$. Then the operator H of multiplication by h is affiliated to $C_0(\mathbb{R})$, because $(h+i)^{-1}$ is equal outside zero (so almost everywhere) to a function from $C_0(\mathbb{R})$. But if $\varphi \in C_0(\mathbb{R})$ then $\varphi(H)$ is the operator of multiplication by a function from $C_0(\mathbb{R})$ which vanishes at zero, so H cannot be strictly affiliated to $C_0(\mathbb{R})$. Moreover, if φ is a continuous function equal to zero near $-\infty$ and to 1 near $+\infty$, then $\varphi(H)$ is multiplication by a function discontinuous at zero, so $\varphi(H)A$ does not belong to $C_0(\mathbb{R})$ if $A \in C_0(\mathbb{R})$ does not vanishes at zero.

We stress that if $\mathscr{J}$ is an ideal in $\mathscr{C}$ and H is a self-adjoint operator affiliated to $\mathscr{C}$, then the quotient $\widehat{H}$ is a well defined observable affiliated to $\widehat{\mathscr{C}} = \mathscr{C}/\mathscr{J}$, but in most cases $\widehat{H}$ has no meaning as operator on $\mathscr{H}$ because $\widehat{\mathscr{C}}$ has no natural realization on $\mathscr{H}$. If H is strictly affiliated to $\mathscr{C}$, then one can realize $\widehat{H}$ as a self-adjoint operator in each nondegenerate representation of $\mathscr{C}$.

Let us take above $\mathscr{J} = K(\mathscr{H}) \cap \mathscr{C}$. We recall that a real number λ does not belong to the essential spectrum of a self-adjoint operator H if and only if $\varphi(H) \in K(\mathscr{H})$ for some $\varphi \in C_0(\mathbb{R})$ such that $\varphi(\lambda) \neq 0$. Hence if H is affiliated to $\mathscr{C}$ we get $\boldsymbol{\sigma}_{\text{ess}}(H) = \boldsymbol{\sigma}(\widehat{H})$.

2.9. Affiliation criteria. The algebras that we consider have to be rather small, such that the quotient with respect to the ideal of compact operators be computable. On the other hand we would like that the class of self-adjoint operators affiliated to them be large. So we are interested in having efficient affiliation criteria. We present two such criteria below.

We remain in the setting of §2.8 and consider a self-adjoint operator H_0 on $\mathscr{H}$. We say that V is a *standard form perturbation* of H_0 if V a continuous symmetric sesquilinear form on $\mathscr{G} = D(|H_0|^{\frac{1}{2}})$ and if there are numbers $\mu \in [0,1)$ and $\delta \in \mathbb{R}$ such that either $\pm V \leq \mu|H_0| + \delta$ as forms on $\mathscr{G}$, or H_0 is bounded from below and $V \geq -\mu H_0 - \delta$ as forms on

$\mathscr{G}$. Then the form sum $H = H_0 + V$ is a self-adjoint operator on $\mathscr{H}$ with the same form domain as H_0. We give a criterion which ensures the affiliation of H to $\mathscr{C}$ if H_0 is affiliated to $\mathscr{C}$. We introduce

$$\mathcal{M}(\mathscr{C}) \;=\; \{A \in B(\mathscr{H}) \mid B \in \mathscr{C} \Longrightarrow AB,\ BA \in \mathscr{C}\}. \tag{2.17}$$

Clearly $\mathcal{M}(\mathscr{C})$ is a C^*-algebra. It can be identified with the usual multiplier algebra (as defined e.g. in [Lan]) if and only if $\mathscr{C}$ is nondegenerate on $\mathscr{H}$. The next result is a particular case of Theorem 2.8 from [DG2].

Theorem 2.22 (i) *Let H_0 be bounded from below and affiliated to $\mathscr{C}$ and assume moreover that the operator $U \equiv (|H_0| + 1)^{-1/2} V (|H_0| + 1)^{-1/2}$ belongs to $\mathcal{M}(\mathscr{C})$. Then H is affiliated to $\mathscr{C}$.*
(ii) *If H_0 is strictly affiliated to $\mathscr{C}$ then $U \in \mathcal{M}(\mathscr{C})$ if and only if one has $\varphi(H_0)V(|H_0| + 1)^{-1/2} \in \mathscr{C}$ for all $\varphi \in C_c(\mathbb{R})$. In this case H is strictly affiliated to $\mathscr{C}$.*

The preceding theorem is interesting because it does not require that the positive part of V be small with respect to H_0 (as in the criteria from [ABG]). However, the fact that the form domain of V must contain that of H_0 is sometimes annoying: it requires $\pm V \le C(|H_0| + 1)$ for some number C. We mention a second method for checking the affiliation to $\mathscr{C}$ of a formal sum $H = H_0 + V$ which is useful when V is in no sense dominated by H_0.

Theorem 2.23 *Let H_0 and V be self-adjoint operators bounded from below. Assume that $\mathscr{K} = D(H_0) \cap D(V)$ is dense in $\mathscr{H}$ and $H = H_0 + V$ with domain $\mathscr{K}$ is a self-adjoint operator. If $\mathrm{e}^{-tH_0}\mathrm{e}^{-2tV}\mathrm{e}^{-tH_0} \in \mathscr{C}$ for all $t > 0$, then H is affiliated to $\mathscr{C}$.*

Proof: This is a slight improvement of a result on page 369 in [ABG]. According to a theorem of D. L. Rogova concerning the Trotter formula (see also [ITZ]) we have $\lim_{n\to\infty} \left[\mathrm{e}^{-tH_0/n}\mathrm{e}^{-2tV/n}\mathrm{e}^{-tH_0/n}\right]^n = \mathrm{e}^{-2tH}$ in norm sense. Thus $\mathrm{e}^{-tH} \in \mathscr{C}$, so H is affiliated to $\mathscr{C}$ (see page 369 in [ABG]). ■

This criterion, coupled with the fact that the set of observables affiliated to a C^*-algebra is closed under the natural norm convergence (cf. page 367 in [ABG]), is efficient in applications to quantum field theory.

3. C^*-algebras of hamiltonians: examples

3.1. Crossed product techniques. In this subsection we introduce a general class of C^*-algebras which can be interpreted as C^*-algebras of energy observables (or hamiltonians). We study quantum systems having as configuration space an arbitrary abelian locally compact group X. This seems to be a natural setting for the quantum theory of systems with a finite number of degrees of freedom: both position and momentum observables[(2)] are naturally defined. Of course, this also shows the power of the algebraic methods. The basic examples one should have in mind are $X = \mathbb{R}^n$ or $\mathbb{Z}^n$. However, the case of finite dimensional vector spaces over local fields (e.g. p-adic numbers) is very interesting and also of some importance (see [Tai, S-C] for a pseudo-differential calculus on such spaces). Self-adjoint operators with "pathological" spectral properties become then quite natural objects. Many other nontrivial situations can be considered, like the "p-adic torus" (the dual of the compact, totally disconnected, non-discrete group of p-adic integers); see [Fol, Gur, Wei]. However, we stress that even in the simplest situations ($X = \mathbb{R}$ or $\mathbb{Z}$) the algebraic techniques give, rather easily, results which do not seem to be covered by other means.

Definition 3.1 *An **algebra of (internal) interactions** on X is any C^*-algebra $\mathscr{A}$ of functions on X such that*

$$C_\infty(X) \subset \mathscr{A} \subset C_{\mathrm{b}}^{\mathrm{u}}(X) \text{ and } \mathscr{A} \text{ is stable under translations.} \tag{3.1}$$

*The **algebra of hamiltonians** associated to $\mathscr{A}$ is $\mathscr{A} \rtimes X$.*

We call *elementary hamiltonian of class $\mathscr{A}$* any self-adjoint operator on $L^2(X)$ of the form $h(P) + v(Q)$, where h is a real continuous function on X^* such that $\lim_{k\to\infty} |h(k)| = \infty$ and $v \in \mathscr{A}$ (the notations are as in §2.6). Then a *hamiltonian of class $\mathscr{A}$* is an observable affiliated to $\mathscr{A} \rtimes X$. The next result explains the terminology. See [GI2] for a more precise assertion and the proof.

[(2)] Here the term observable has a more general meaning than in §2.7: a T-valued observable is a morphism from $C_0(T)$ into a C^*-algebra, where T is a locally compact space, cf. §8.1.2 in [ABG]. Even more general interpretations of this notion are in fact required in order to treat "non-abelian" observables like the momentum when a magnetic field is present, or the kinetic momentum. In our context, we should call T-valued observable any morphism from the group algebra of a locally compact group T into a C^*-algebra.

Proposition 3.2 *$\mathscr{A} \rtimes X$ is the C^*-algebra generated by the elementary hamiltonians of class $\mathscr{A}$.*

As we explained in the Introduction, our main purpose is to give an "explicit" description of the quotient algebra $\mathscr{A} \rtimes X/\mathscr{K}(X)$. From (3.1) we see that $\mathscr{K}(X) = C_0(X) \rtimes X$ is an ideal in $\mathscr{A} \rtimes X$. The main point here is that the crossed product structure leads to a drastic simplification of the problem. Indeed, Theorem 2.11 gives

$$\mathscr{A} \rtimes X\Big/_{\mathscr{K}(X)} \cong [\mathscr{A}/C_0(X)] \rtimes X. \tag{3.2}$$

This relation reduces the problem of the computation of the quotient of the two noncommutative algebras from the left hand side to an easier abelian problem: that of giving a convenient description of $\mathscr{A}/C_0(X)$.

The preceding formalism also covers systems interacting with a vanishing at infinity external magnetic, i.e. hamiltonians of such systems are affiliated to algebras of the form $\mathscr{A} \rtimes X$. This is not so if the magnetic field does not vanish at infinity. We describe now a class of algebras which are suited to such situations (a more detailed account can be found in [GI3]), but we shall not give concrete applications of this formalism. In fact, the framework of §3.6 covers the case of constant nonzero magnetic fields, but there we do not use crossed product methods.

The formalism we propose here forces us to use crossed products of (abelian) C^*-algebras by actions of non-abelian groups, case not treated in this lecture. We only mention that the definition of crossed products in general is essentially identical to that from the abelian case.

We recall that an *extension of X by an abelian group N* is a locally compact group $G \supset N$ equipped with a continuous surjective group morphism $\pi : G \to X$ such that $\ker \pi = N$ (the assumption of local compacity is not convenient in general, but we keep it here to have the standard definition of crossed products). We denote multiplicatively the operation in G, so the trivial case "$G = X$" corresponds to $N = \{1\}$, $G = N \times X$ and $\pi(1, x) = x$.

We get a transitive action of G on X by setting $g.x = x - \pi(g)$ and then $(\alpha_g \varphi)(x) = \varphi(g^{-1}.x) = \varphi(x + \pi(g))$ gives a continuous action of G on the C^*-algebra $C_{\mathrm{b}}^{\mathrm{u}}(X)$, which thus becomes a G-algebra. The action of G on X looks like that of X on itself by translations, hence a G-stable C^*-subalgebra $\mathscr{A}$ of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ is the same thing as a translation invariant C^*-subalgebra. However, the crossed product $\mathscr{A} \rtimes G$ of $\mathscr{A}$ by the action of G is quite different from the crossed product

$\mathscr{A} \rtimes X$ of $\mathscr{A}$ by the action of X. The fact that this has something to do with magnetic fields will be shown below. We mention that, since $\ker \pi = N$ is a closed abelian normal subgroup and $G/\ker\pi = X$ is abelian, the group G is amenable (see 7.3.5 in [Ped]). Thus $\mathscr{A} \rtimes G$ coincides with the reduced crossed product.

Definition 3.3 *The crossed product $\mathscr{A} \rtimes G$ is the **algebra of hamiltonians** of the system having X as configuration space, subject to internal interactions of type $\mathscr{A}$, and interacting with an external field of type N. An observable affiliated to $\mathscr{A} \rtimes G$ is a **hamiltonian of class ($\mathscr{A}, G$)**.*

In order to justify the definition it is useful to think in terms of the universal property of crossed products, which says that the representations of $\mathscr{A} \rtimes G$ are in bijective correspondence with covariant representations of $(\mathscr{A}, G, \alpha)$ (see Theorem 2.9 and [Ped, Rae]). By *covariant representation* we mean a couple consisting of a non-degenerate representation $\varphi \mapsto \varphi(Q)$ of $\mathscr{A}$ and a strongly continuous unitary representation $g \mapsto U_g$ of G on the same Hilbert space $\mathscr{H}$, such that $U_g\varphi(Q)U_g^* = \varphi(Q + \pi(g))$. Set $U(\theta) = \int_G U_g\theta(g)\,\mathrm{d}g$ for $\theta \in C_\mathrm{c}(G)$ and $\mathrm{d}g$ a Haar measure on G. Then the range of the representation of $\mathscr{A} \rtimes G$ associated to this covariant representation is the closed linear subspace of $B(\mathscr{H})$ generated by the operators $\varphi(Q)U(\theta)$. But, if we denote $\mathscr{A}^\flat$ and $C^*(G)^\flat$ the representations of $\mathscr{A}$ and of the group C^*-algebra $C^*(G)$ on $\mathscr{H}$, then $[\![\mathscr{A}^\flat \cdot C^*(G)^\flat]\!]$ is a representation of the abstract crossed product $\mathscr{A} \rtimes G$ on $\mathscr{H}$. In the trivial case $G = X$ one has $C^*(X) = C_0(X^*)$, hence we get $[\![\mathscr{A} \cdot C_0(X^*)]\!]$ as C^*-algebras of energy observables, which is the correct prescription (see Theorem 2.17).

The extensions of X can be classified in terms of N, actions of X on N by automorphisms, and elements of a second order cohomology group of X with coefficients in N (see [GI3] for details). We now construct certain extensions associated to magnetic fields on X.

Let $U(1) = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and let $C(X; U(1))$ be the group of continuous functions $X \to U(1)$, equipped with the group structure given by usual multiplication of functions and with the topology of uniform convergence on compact subsets of X; we get a topological group (not locally compact). Translations induce a natural action of X on $C(X; U(1))$ by group automorphisms: $(x.u)(y) = u(y + x)$. We choose a closed locally compact subgroup $N \subset C(X; U(1))$ stable under translations. For example, the choice $N = U(1)$ (constant functions) suffices to treat constant magnetic fields. A more interesting choice in the case when X is a finite dimensional real vector space is

$$N = \{\mathrm{e}^{ia} \mid a \text{ is a polynomial of order } \leq m\}$$

where m is a fixed positive integer. This suffices for the treatment of magnetic fields of class C^{m+1} with derivatives of order $m+1$ tending to zero at infinity. A (normalized) *2-cocycle on X with coefficients in N* is a continuous map $b : X \times X \to N$ such that

$$b(x,y)b(x+y,z) = x.b(y,z)b(x,y+z) \text{ and } b(x,0) = b(0,y) = 1.$$

We denote $N \times_b X$ the set $N \times X$ provided with the product

$$(u,x) \cdot (v,y) = (u\, x.v\, b(x,y), x+y).$$

It is easy to see that $N \times_b X$ has a locally compact group structure and that the natural map $N \times_b X \to X$ gives a group extension G of X. We shall explain now the relation between the 2-cocycle b and the magnetic field.

Assume that X is a finite dimensional real vector space. Then *a magnetic field is a 2-form on X*, more precisely, it is a continuous map $B : X \to \wedge^2 X^*$. If x, y, z are points of X, let $T(x,y,z)$ be the oriented (possibly degenerate) triangle determined by the points $z, z+x, z+x+y$. Then we take $b(x,y) = b_{x,y}$ with

$$b_{x,y}(z) = \exp\Big(-i\int_{T(x,y,z)} B\Big). \tag{3.3}$$

The integral from the exponent is the *flux of the magnetic field B through the oriented triangle $T(x,y,z)$*. One can check that b is a 2-cocycle with coefficients in $C(X;U(1))$. We assume that the closed subgroup N generated by these coefficients is locally compact and denote $X_B \equiv N \times_b X$ the extension of X associated to it. Finally, *$\mathscr{A} \rtimes X_B$ is the C^*-algebra of hamiltonians of a system having X as configuration space, subject to internal interactions of type $\mathscr{A}$, and interacting with an external field asymptotically equal to B* (a more detailed justification of this interpretation can be found in [GI3]). We emphasize that this algebra depends only on the magnetic field, not on the magnetic potential.

3.2. Bumps algebras. From now on we assume that X *is not compact* and in the next three subsections we give three nontrivial examples of algebras of interactions. Other examples can be found in [GI2]. See [Ma1, Ma2] for constructions based on compactifications of X (see also [Sim]).

Fix a closed set $L \subset X$ such that $L_\Lambda \equiv L + \Lambda \neq X$ if Λ is compact. Then the family of open sets $\{L^c_\Lambda \mid \Lambda \subset X \text{compact}\}$, where $L^c_\Lambda = X \setminus L_\Lambda$, is the base of a filter $\mathcal{F}_L$ which is finer than the Fréchet filter and translation invariant. We denote by $L\text{-}\lim \varphi$ the limit along the filter $\mathcal{F}_L$ and we define $C_L(X)$ as the C^*-algebra of functions $\varphi \in C^{\mathrm{u}}_{\mathrm{b}}(X)$ such that $L\text{-}\lim \varphi$ exists. It is obvious that $C_L(X)$ is an algebra of interactions. The corresponding algebra of hamiltonians will be denoted $\mathscr{C}_L(X)$. This is the first class of algebras that we consider.

Let $C_{L,0}(X) = \{\varphi \in C^{\mathrm{u}}_{\mathrm{b}}(X) \mid L\text{-}\lim \varphi = 0\}$, this is an ideal in $C_L(X)$ and $C_L(X) = \mathbb{C} + C_{L,0}(X)$. According to Corollary 2.12, we can write $\mathscr{C}_L(X)$ as a linear direct sum

$$\mathscr{C}_L(X) = C_0(X^*) + \mathscr{C}_{L,0}(X). \tag{3.4}$$

The algebra $\mathscr{C}_{L,0}(X) = C_{L,0}(X) \rtimes X$ is an ideal of $\mathscr{C}_L(X)$ and $\mathscr{C}_L(X) \to C_0(X^*)$ is a surjective morphism which gives the pure kinetic energy part of a hamiltonian of class $\mathscr{A}$. On the other hand, $C_0(X)$ being a stable ideal of $C_{L,0}(X)$, the crossed product subalgebra $C_0(X) \rtimes X = \mathscr{K}(X)$ is an ideal of $\mathscr{C}_{L,0}(X)$.

We cannot give a complete description of the quotient $\mathscr{C}_L(X)/\mathscr{K}(X)$ for an arbitrary L. From now on we shall assume that L is *sparse*, which means that it is locally finite and for each compact Λ of X there is a finite set $F \subset L$ such that if $l \in M = L \setminus F$ and $l' \in L \setminus \{l\}$ then $(l + \Lambda) \cap (l' + \Lambda) = \emptyset$. The consideration of these sets was suggested to us by the work of M. Klaus [Kla] on Schrödinger operators with “widely separated bumps”. For this reason we call $\mathscr{C}_L(X)$ the *bumps algebra* when L is sparse.

The bumps algebra fits very nicely in our framework, the quotient algebra having an especially interesting structure. Besides $\mathscr{K}(X)$, we need below the *two-body algebra* $\mathscr{T}(X) = C_0(X^*) + \mathscr{K}(X)$ (see §3.4 for terminology).

Theorem 3.4 *The quotient algebra $\mathscr{C}_{L,0}(X)/\mathscr{K}(X)$ is canonically isomorphic to the L-asymptotic algebra $\mathscr{K}(X)^{\langle L\rangle}$. One has a natural embedding:*

$$\mathscr{C}_L(X)\Big/\mathscr{K}(X) \hookrightarrow \mathscr{T}(X)^{[L]}\Big/\mathscr{K}(X)^{(L)} \tag{3.5}$$

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The notations are those of §2.3. The computations are done at an abelian level using (3.2), but neither the abelian case is trivial here. We send to [GI2] for the complete proof and we quote below a more explicit formulation of Theorem 3.4 which is a byproduct of the proof (see also Lemma 2.6).

Theorem 3.5 *There is a unique morphism $\mathscr{C}_L(X) \to \mathscr{T}(X)^{[L]}/\mathscr{K}(X)^{(L)}$ such that the image of an element of the form $\psi(P) + \sum_{l\in M} U_l^* K U_l$, where $\psi \in C_0(X^*)$, $M \subset L$, and $K \in \mathscr{K}(X)$ is such that $K = \chi_\Lambda(Q)\, K\, \chi_\Lambda(Q)$ for some compact set $\Lambda \subset X$, be the quotient of the element $\chi_M \otimes (\psi(P) + K) \in \mathscr{T}(X)^{[L]}$ with respect to the ideal $\mathscr{K}(X)^{(L)}$. The kernel of this morphism is $\mathscr{K}(X)$ and its restriction to $\mathscr{C}_{L,0}(X)$ induces the canonical isomorphism of $\mathscr{C}_{L,0}(X)/\mathscr{K}(X)$ with the L-asymptotic algebra of compact operators $\mathscr{K}(X)^{\langle L\rangle}$.*

Now we give an application in spectral theory. Let H be an observable affiliated to $\mathscr{C}_L(X)$ and let $\widehat{H}$ be its image in $\mathscr{C}_L(X)/\mathscr{K}(X)$. Then there is a family $(H_l)_{l\in L}$ of observables affiliated to the two-body algebra $\mathscr{T}(X)$ such that the quotient of $\prod_{l\in L} H_l$ with respect to the ideal $\mathscr{K}(X)^{(L)}$ is equal to $\widehat{H}$ (we use the embedding (3.5)). We say that $(H_l)_{l\in L}$ is a *representative* of H. We have $\prod_{l\in L}(H_l - z)^{-1} \in \mathscr{T}(X)^{[L]}$ and the component of $(H_l - z)^{-1}$ in $C_0(X^*)$ is independent of $l \in L$, so $\boldsymbol{\sigma}_{\text{ess}}(H_l)$ is independent of l. Thus the next result is a consequence of the Proposition 2.20.

Theorem 3.6 *If H is an observable affiliated to $\mathscr{C}_L(X)$ and $\{H_l\}_{l\in L}$ is a representative of H, then*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcap_{\substack{F\subset L\\ F\,\text{finite}}} \overline{\bigcup_{l\in L\setminus F} \boldsymbol{\sigma}(H_l)}.$$

The simplest case is already interesting. Assume that H_l itself is an operator H° independent of l; then $\boldsymbol{\sigma}_{\text{ess}}(H) = \boldsymbol{\sigma}(H^\circ)$.

Example: Very general examples can be found in [GI2]), here we consider the easiest nontrivial one. Let $X = \mathbb{R}^n$ and let $h : \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

$$C^{-1}|x|^{2s} \le |h(x)| \le C|x|^{2s} \text{ if } |x| > R,$$

for some constants $s > 0$, $C > 0$ and $R < \infty$. Let $t \in [0, s)$ real, let L be a sparse subset of $\mathbb{R}^n$, and let $W : \mathscr{H}^t \to \mathscr{H}^{-t}$ be a symmetric operator such that $\langle Q\rangle^a W \in B(\mathscr{H}^t, \mathscr{H}^{-t})$ for some number $a > 2n$ ($\mathscr{H}^s$ are usual Sobolev spaces and $\langle Q\rangle$ is the operator of multiplication by $(1 + |x|^2)^{1/2}$). Then the series $\sum_{l\in L} U_l^* W U_l$ converges in the strong topology of $B(\mathscr{H}^t, \mathscr{H}^{-t})$ and its sum is a symmetric operator $V : \mathscr{H}^t \to \mathscr{H}^{-t}$. Let $H = h(P) + V$, $H_l \equiv H^\circ = h(P) + W$ be the self-adjoint operators in $\mathscr{H}$ defined as form sums. Then H is strictly affiliated to $\mathscr{C}_L(X)$, H_l is strictly affiliated to $\mathscr{T}(X)$, and the family $\{H_l\}_{l\in L}$ is a representative of H. In particular

$$\boldsymbol{\sigma}_{\text{ess}}[h(P) + V] = \boldsymbol{\sigma}[h(P) + W].$$

We mentioned in the introduction the problem of obtaining "intrinsic" characterizations of a C^*-algebra of hamiltonians. In most of the cases this is a difficult question. The answer in the present case is as follows (see [GI2] for the proof). For $k \in X^*$ we denote by V_k the operator of translation by k in momentum space: $(V_k f)(x) = k(x)f(x)$. The relations involving $T^{(*)}$ must hold separately for T and its adjoint. L is a sparse set.

Theorem 3.7 *An operator $T \in \mathscr{B}(X)$ belongs to $\mathscr{C}_{L,0}(X)$ if and only if*
(i) $\lim_{x\to 0} \|(U_x - 1)T^{(*)}\| = 0$,
(ii) $\lim_{k\to 0} \|V_k T V_k^* - T\| = 0$,
(iii) $\forall \varepsilon > 0\ \exists \Lambda \subset X$ *compact such that* $\|\chi_{L^c_\Lambda}(Q)T^{(*)}\| < \varepsilon$.

This is of the same nature as Theorem 1.4 but holds for an arbitrary group X. We stress that the following characterization of compact operators, the Riesz-Kolmogorov theorem, is behind all our results of this type.

Theorem 3.8 *If $T \in \mathscr{B}(X)$ then $T \in \mathscr{K}(X) = C_0(X) \rtimes X$ if and only if*

$$\lim_{x\to 0} \|(U_x - 1)T\| = 0 \textit{ and } \lim_{k\to 0} \|(V_k - 1)T\| = 0.$$

In [GI1] there are other applications of this remarkable result.

3.3. Localizations at infinity. The second example of algebra of interactions will be $\mathscr{A} = C_{\mathrm{b}}^{\mathrm{u}}(X)$, which is the largest possible choice. The main results in the case $X = \mathbb{R}^n$ have been stated in the Introduction as Theorem 1.2 and Theorem 1.4, which gives an intrinsic description of the algebra. We now discuss in more detail the proof of an analogue of Theorem 1.2 for arbitrary X and give some applications. This results have been announced in [If2]. The proofs are sketched in [GI2] and will be developed in [GI5].

Lemma 3.9 *Let $\varkappa$ be an ultrafilter on X finer than the Fréchet filter. If $S \in C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X \subset \mathscr{B}(X)$ then the strong limit* $\operatorname{s-lim}_{x,\varkappa} U_x S U_x^* = \mathcal{P}_\varkappa[S]$ *exists and belongs to $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$. The map $\mathcal{P}_\varkappa : C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X \to C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ is a morphism.*

Proof: This is based on the remark: *a function $\varphi \in C_{\mathrm{b}}(X)$ belongs to $C_{\mathrm{b}}^{\mathrm{u}}(X)$ if and only if for some (and hence for all) $\theta \in C_0(X), \theta \neq 0$, the set of functions of the form $\theta\, \tau_x \varphi$, $x \in X$, is relatively compact in $C_0(X)$.* In more technical terms (cf. [Lan]), the set $\{\tau_x \varphi \mid x \in X\}$ is relatively compact in the strict topology of $C_{\mathrm{b}}(X)$. But any ultrafilter on a compact space is convergent, hence if $\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(X)$ the limit $\varphi_\varkappa = \lim_{x,\varkappa} \tau_{-x}\varphi$ exists locally uniformly on X and $\varphi_\varkappa \in C_{\mathrm{b}}^{\mathrm{u}}(X)$. Now the lemma is an immediate consequence of the fact that $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ is the norm closure of the linear space generated by the operators of the form $\varphi(Q)\psi(P)$. ■

We call $\mathcal{P}_\varkappa[S] \equiv S_\varkappa$ *localization of S at $\varkappa$* and the family $\{S_\varkappa\}_{\varkappa \in \delta X}$ is the set of *localizations at infinity of S*. We denoted by δX the set of all ultrafilters on X finer than the Fréchet filter. These notions extend immediately to any observable H affiliated to $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ by setting $H_\varkappa = \mathcal{P}_\varkappa[H]$, which is again an observable affiliated to $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$. We have

Theorem 3.10 *If $S \in C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ then $\{\mathcal{P}_\varkappa[S] \mid \varkappa \in \delta X\}$ is a compact subset of $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$. One has $\mathcal{P}_\varkappa[S] = 0$ for all $\varkappa \in \delta X$ if and only if $S \in \mathscr{K}(X)$.*

In particular, we get an embedding

$$C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X / \mathscr{K}(X) \subset [C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X]^{[\delta X]} \tag{3.6}$$

Corollary 3.11 *If H is an observable affiliated to $C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ then*

$$\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \bigcup_{\varkappa \in \delta X} \boldsymbol{\sigma}(H_\varkappa). \tag{3.7}$$

The fact that the union is closed is not trivial. We also stress that a formula like (3.7) is remarkable because it involves only localizations of H in the region $Q = \infty$. For other hamiltonians of physical interest one must include localizations at $P = \infty$, as it will be shown later on.

The definition of localizations at infinity is practically convenient but not sufficient for the proof of the main results. We need a basic fact concerning the Stone-Čech compactification of X: if F is a continuous map from X to a Hausdorff topological space Y and if the range of F is a relatively compact set, then F has a unique continuous extension to βX (this is the universal property of βX). This is related to the fact that the limit of F along any ultrafilter exists.

Recall also that X is an open dense subset of βX (because X is locally compact) and denote $\gamma X = \beta X \setminus X$ the boundary of X in βX. Thus each $\varphi \in C_{\mathrm{b}}(X)$ extends to an element of $C(\beta X)$ and this gives an identification of the algebras $C_{\mathrm{b}}(X)$ and $C(\beta X)$. The restriction map $\varphi \mapsto \varphi|_{\gamma X}$ induces an isomorphism between $C_{\mathrm{b}}(X)/C_0(X)$ and $C(\gamma X)$. Similarly, for each $S \in C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ the strongly continuous map $x \mapsto U_x S U_x^*$ extends to a strongly continuous map $\beta X \to C_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ whose restriction to γX is related to the localizations at infinity of S (see [GiJ]). In [GI2] there are some further comments on this question, but a complete proof is given only in [GI5], where we also treat the case of non-vanishing at infinity magnetic fields.

We give now examples which show that Theorem 1.2 can be used for concrete computations. We consider $X = \mathbb{R}^n$ and hamiltonians $H = h(P) + V(Q)$, where h, V are real functions on $\mathbb{R}^n$. Assume h of class C^1, polynomially bounded, $h(p) \to \infty$ if $p \to \infty$, and $|\nabla h(p)| \le C(1 + |h(p)|)$. Let V be locally integrable and assume that its negative part is form bounded with respect to $h(P)$ with relative bound < 1. Then H is a well defined self-adjoint operator on $L^2(\mathbb{R}^n)$ affiliated to $C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R}^n) \rtimes \mathbb{R}^n$, so we can use (3.7). We have $U_x H U_x^* = h(P) + V(x + Q)$, so the localizations at infinity of H are determined by the (suitably defined) localizations at infinity of the function V. Thus, for the computation of $\boldsymbol{\sigma}_{\mathrm{ess}}(H)$, we are once again reduced to an abelian situation.

In order to use these facts one has to define and study the localizations at infinity of unbounded functions and even of distributions. We stress that most of these localizations are equal to $+\infty$ almost everywhere, so the corresponding localizations of the hamiltonian are also infinite, hence do not contribute to the union in (3.7) (because $\boldsymbol{\sigma}(\infty)=\emptyset$). Thus we shall have

$$\boldsymbol{\sigma}_{\text{ess}}(h(P)+V)=\bigcup\nolimits_v \boldsymbol{\sigma}(h(P)+v)$$

where the union is performed over all the finite localizations at infinity v of the potential V. Results of this type have been obtained by Helffer and Mohamed in [HeM] but only for $h(p)=p^2$ and under quite restrictive conditions on V (however, they also treat the case of nontrivial magnetic fields). We shall quote one of our results, where the localizations are understood in the sense of local uniform convergence.

Proposition 3.12 *Let $V:\mathbb{R}^n\to\mathbb{R}$ be continuous and bounded from below and let $m\geq 0$ an integer. The localizations of V at infinity are either equal to $+\infty$ almost everywhere or are polynomials of order $\leq m$ if and only if*

$$\lim_{y\to\infty}\sup_{|x|\leq 1}\left|\left[(\tau_x-1)^{m+1}V\right](y)\right|=0. \tag{3.8}$$

Example: if V is a function of class C^{m+1} and if all its derivatives of order $m+1$ tend to zero at infinity, then (3.8) holds.

Finally, we shall give an explicit example in the case $n=1$ (which is not covered by the preceding proposition). Note that if $\varkappa\in\delta\mathbb{R}$ then either $[0,\infty)\in\varkappa$ or $(-\infty,0]\in\varkappa$. Thus there are two contributions $\boldsymbol{\sigma}^{\pm}_{\text{ess}}(H)$ to the union from (3.7) and $\boldsymbol{\sigma}_{\text{ess}}(H)=\boldsymbol{\sigma}^{+}_{\text{ess}}(H)\cup\boldsymbol{\sigma}^{-}_{\text{ess}}(H)$. We take $H=h(P)+V(Q)$ on $L^2(\mathbb{R})$, where h is as before and $V:\mathbb{R}\to\mathbb{R}$ is continuous and bounded from below. Then H is affiliated to $C^{\mathrm{u}}_{\mathrm{b}}(\mathbb{R})\rtimes\mathbb{R}$ and $\boldsymbol{\sigma}^{\pm}_{\text{ess}}(H)$ is determined by the behavior of V at $\pm\infty$.

Proposition 3.13 *Assume that for large positive x we have $V(x)=x^a\omega(x^\theta)$ with $a\geq 0, 0<\theta<1$ and ω a positive continuous periodic function with period 1. Moreover, assume that ω vanishes only at the points of $\mathbb{Z}$ and that there are real numbers $\lambda,\mu>0$ such that $\omega(t)\sim\lambda|t|^\mu$ when $t\to 0$. Then there are three possibilities:*

(1) *If $a<\mu(1-\theta)$ the localizations at $+\infty$ of V are all the non-negative constant functions, thus $\boldsymbol{\sigma}^{+}_{\text{ess}}(H)=[\inf h,+\infty)$.*

(2) *If $a=\mu(1-\theta)$ the localizations at $+\infty$ of V are the functions $v(x)=\lambda|\theta x+c|^\mu$ with $c\in\mathbb{R}$. Thus $\boldsymbol{\sigma}^{+}_{\text{ess}}(H)=\boldsymbol{\sigma}(h(P)+\lambda|\theta Q|^\mu)$, hence it is a discrete not empty set.*

(3) *If $a>\mu(1-\theta)$ the only localization at $+\infty$ of V is $+\infty$, so $\boldsymbol{\sigma}^{+}_{\text{ess}}(H)=\emptyset$.*

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The case when ω has different asymptotics from left and right at zero can also be treated.

3.4. The N-body problem. The third example of algebra of interactions covers the N-body hamiltonians. This should be considered as the first example of a C^*-algebra of hamiltonians: it appears in a disguised form in [BG1]. The treatment in terms of crossed products and the extension to infinite semilattices of subspaces was given in [DG1].

In this subsection X will always be a finite dimensional real vector space (some possible extensions are mentioned in an appendix of [GI3]). For each linear subspace Y we denote π_Y the canonical surjection of X onto the quotient vector space X/Y. We shall embed $C_0(X/Y) \subset C_{\mathrm{b}}^{\mathrm{u}}(X)$ with the help of the map $\varphi \mapsto \varphi \circ \pi_Y$. For $Y = \{0\} \equiv O$ and $Y = X$ we get $C_0(X/O) = C_0(X)$ and $C_0(X/X) = \mathbb{C}$ respectively.

Let $\mathbb{G}(X)$, the *grassmannian of* X, be the set of all linear subspaces of X equipped with the natural order relation. This is clearly a complete lattice. It is not difficult to show that the family of C^*-subalgebras $\{C_0(X/Y)\}_{Y\in\mathbb{G}(X)}$ has the following properties:
(i) if $\mathcal{L} \subset \mathbb{G}(X)$ is finite, then $\sum_{Y\in\mathcal{L}} C_0(X/Y)$ is a closed subspace of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ and the sum is direct in the vector space sense;
(ii) for all $Y, Z \in \mathbb{G}(X)$ one has $C_0(X/Y) \cdot C_0(X/Z) \subset C_0(X/(Y \cap Z))$.

The *algebra of interactions of an N-body system* is of the form $C(\mathcal{L}) \equiv \sum_{Y\in\mathcal{L}} C_0(X/Y)$, where $\mathcal{L} \subset \mathbb{G}(X)$ is finite, stable under intersections, and such that $O, X \in \mathcal{L}$. According to (i) and (ii), $C(\mathcal{L})$ is indeed a C^*-algebra of interactions on X. To understand the meaning of N, note that $\mathcal{L}$, equipped with the order relation induced by $\mathbb{G}(X)$, is a finite lattice; then $N+1$ is the rank of this lattice. For example, the *algebra of two-body interactions* must correspond to $\mathcal{L} = \{O, X\}$, hence it is $\mathbb{C} + C_0(X)$. The corresponding algebra of hamiltonians $[\mathbb{C} + C_0(X)] \rtimes X = C_0(X^*) + \mathscr{K}(X) \equiv \mathscr{T}(X)$ is particularly important and has already been used before.

$C_0(X/Y)$ is a translation invariant C^*-subalgebra of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ and so we may construct the crossed product $\mathscr{C}^X(Y) = C_0(X/Y) \rtimes X$. If $C(\mathcal{L})$ is as above, the corresponding algebra of hamiltonians $\mathscr{C}^X(\mathcal{L}) = C(\mathcal{L}) \rtimes X$ is the *N-body algebra associated to* $\mathcal{L}$. It is easy to see that the structure of $C(\mathcal{L})$ is inherited by $\mathscr{C}^X(\mathcal{L})$, more precisely

$$\mathscr{C}^X(\mathcal{L}) = \sum_{Y\in\mathcal{L}} \mathscr{C}^X(Y) \tag{3.9}$$

and properties similar to (i) and (ii) hold. Note that $\min \mathcal{L} = O$ and $\mathscr{C}^X(O) = \mathscr{K}(X)$. In §3.6 we shall explain how to compute the quotient $\mathscr{C}^X(\mathcal{L})/\mathscr{K}(X)$ in a more general abstract setting. In the present particular case, the result is as follows.

Proposition 3.14 *For each $Y \in \mathcal{L}$ let $\mathscr{C}^X_Y(\mathcal{L})$ be the C^*-subalgebra of $\mathscr{C}^X(\mathcal{L})$ defined by $\sum_{Z\in\mathcal{L}, Z\supset Y} \mathscr{C}^X(Z)$. Let $\mathcal{M}$ be the set of atoms of $\mathcal{L}$. Then there is a canonical embedding*

$$\mathscr{C}^X(\mathcal{L})/\mathscr{K}(X) \hookrightarrow \bigoplus_{Y\in\mathcal{M}} \mathscr{C}^X_Y(\mathcal{L}). \tag{3.10}$$

This implies classical results on the essential spectrum of N-body hamiltonians, like the HVZ theorem. Moreover, this has as a corollary the Mourre estimate for N-body systems, as has first been shown in [BG2] (see §8.4 and §9.4 in [ABG] for a more systematic presentation).

The linear direct sum decomposition (3.9) has other interesting consequences. For example, it was shown in [BG1] that the decomposition of the resolvent of a hamiltonian H affiliated to $\mathscr{C}^X(\mathcal{L})$ according to (3.9) is just the Weinberg-Van Winter equation introduced in the N-body problem in the sixties. Moreover, the decomposition of a function $\varphi(H)$ determined by (3.9) gives the connected components and the truncated parts of $\varphi(H)$, objects defined by rather involved combinatoric arguments in the standard approach to the N-body problem, cf. [PSS] and [Pol].

An element $Y \in \mathcal{L}$ determines the ideal $\mathscr{J}^X_Y(\mathcal{L}) = \sum \mathscr{C}^X(Z)$, where the sum runs over $Z \in \mathcal{L}$ such that $Y \not\subset Z$, such that $\mathscr{C}^X(\mathcal{L}) = \mathscr{C}^X_Y(\mathcal{L}) + \mathscr{J}^X_Y(\mathcal{L})$ linear direct sum. Thus $\mathscr{C}^X(\mathcal{L})/\mathscr{J}^X_Y(\mathcal{L}) = \mathscr{C}^X_Y(\mathcal{L})$. The quotient of H with respect to this ideal, or the projection of H onto the C^*-subalgebra $\mathscr{C}^X_Y(\mathcal{L})$ determined by the preceding linear direct sum decomposition, is the sub-hamiltonian H_Y (denoted H_a in the physical literature) which plays an important role in the spectral and scattering theory of H. The algebra $\mathscr{C}^X_Y(\mathcal{L})$ has a special structure which allows one to define the "internal hamiltonian" H^Y, see [DG2] for this question.

An extension of this formalism to arbitrary (not finite) $\mathcal{L}$, in particular a study of the hamiltonians affiliated to the most natural algebra $\mathscr{C}^X$ obtained as closure of $\sum_{Y\in\mathbb{G}(X)} \mathscr{C}^X(Y)$, can be found in [DG2]. Besides the non-relativistic N-body hamiltonians, this framework covers the dispersive case and the class of pluristratified media first considered by Dermenjian and Iftimie in [DeI]. We mention one result from [DG2] in order to make the connection with the localizations at infinity discussed before.

Theorem 3.15 *If $S \in \mathscr{C}^X$ and $\omega \in X$ then* $\text{s-lim}_{\lambda\to\infty} U_{\lambda\omega} S U^*_{\lambda\omega} = \mathcal{P}_\omega[S]$ *exists. The map $\mathcal{P}_\omega$ so defined is a morphism $\mathscr{C}^X \to \mathscr{C}^X$. Let H be an observable affiliated to $\mathscr{C}^X$ and let us set $H_\omega = \mathcal{P}_\omega[H]$. Then*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup_{\omega \in X\setminus\{0\}} \boldsymbol{\sigma}(H_\omega). \tag{3.11}$$

One has $H_\omega = H_Y$ where Y is a vector subspace generated by ω.

The following "intrinsic" characterization of the algebra $\mathscr{C}^X(Y)$ is obtained in [DG2]. We denote by $Y^\perp$ the polar set of Y in X^* and the limits are in norm sense.

Theorem 3.16 *$\mathscr{C}^X(Y)$ is the set of operators $T \in \mathscr{B}(X)$ satisfying the following conditions: $[T, U_y] = 0\ \forall y \in Y$; $[T, V_k] \to 0$ if $k \to 0$ in X^*; $(U_x - 1)T^{(*)} \to 0$ if $x \to 0$ in X; $(V_k - 1)T^{(*)} \to 0$ if $k \to 0$ in $Y^\perp$.*

3.5. (Q, P)-anisotropy. The rest of these notes is devoted to the description of C^*-algebras of hamiltonians which are not necessarily of the crossed product type. The algebra we study in this subsection is much simpler than all those we treat in this lecture. However, we find that its study is quite instructive: the algebra has a simple "intrinsic" definition and the computation of the quotient with respect to the compacts can be done by direct elementary means. In fact this example lies at the origin of our approach: we found it (in [GI6]) when trying to go beyond the graded C^*-algebra framework suited to the N-body problem. After its study we understood the relevance of crossed products and the fact that the interpretation in terms of algebras of energy observables is the relevant one for further investigations. On the other hand, in spite of their simplicity, the algebras studied in this subsection cover several physically interesting models, for example the hamiltonians studied in [Ben, DDI] are affiliated to algebras of this type, hence can be systematically studied in our framework. This section is a résumé of the first part of [GI6].

We shall work in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}; \mathbf{E}) \equiv L^2(\mathbb{R}) \otimes \mathbf{E}$ where $\mathbf{E}$ is a complex Hilbert space (corresponding to internal or confined motion). The operators Q and P are now given by $(Qf)(x) = xf(x)$ and $Pf = -if'$. If $a \in \mathbb{R}$ the operator e^{iaP} is well defined and $\mathrm{e}^{iaP} = U_a$.

We shall use the notation $\varphi(Q)$ even in the more general case when $\varphi : \mathbb{R} \to B(\mathbf{E})$ is a weakly Borel function; in this case $\varphi(Q)$ is the operator of multiplication in $\mathscr{H}$ by the operator-valued function φ. We also set $\psi(P) = \mathcal{F}^*\psi(Q)\mathcal{F}$, so the operator $\psi(P)$ is well defined even if ψ is a $B(\mathbf{E})$-valued weakly Borel function.

If A is a self-adjoint operator on $\mathscr{H}$ then $\{A\}'$ is the *commutant algebra* of A, i.e. the set of all bounded operators commuting with A. For example, it is well known that $\{Q\}'$ is the set of $\varphi(Q)$ with bounded weakly Borel $\varphi : \mathbb{R} \to B(\mathbf{E})$, and similarly for P instead of Q. Observe that

$$\{Q\}' \cap \{P\}' = 1 \otimes B(\mathbf{E}) \equiv B(\mathbf{E}). \tag{3.12}$$

We denote by $\chi(A > r)$ the spectral projection of A associated to the interval $]r, \infty[$. So, if χ_1 is the characteristic function of $]1, \infty[$, then $\chi(A > r) = \chi_1(A/r)$. The symbols $\chi(A < r)$ or $\chi(|A| < r)$ have a similar meaning.

We denote $C_0^{\mathbf{E}}(\mathbb{R}) = C_0(\mathbb{R}) \otimes K(\mathbf{E}) = C_0(\mathbb{R}; K(\mathbf{E}))$. Other interesting algebras are obtained by taking various compactifications of $\mathbb{R}$, but only one of them will be considered here. Let $\overline{\mathbb{R}} = [-\infty, \infty]$ be the two-point compactification of $\mathbb{R}$ and $C^{\mathbf{E}}(\overline{\mathbb{R}}) = C(\overline{\mathbb{R}}) \otimes K(\mathbf{E})$, or

$$\begin{aligned} C^{\mathbf{E}}(\overline{\mathbb{R}}) = \{\varphi : \mathbb{R} \to K(\mathbf{E}) \quad | \; & \varphi \text{ is norm-continuous and the norm limits} \\ & \lim\nolimits_{x \to +\infty} \varphi(x) \text{ and } \lim\nolimits_{x \to -\infty} \varphi(x) \text{ exist}\}. \end{aligned}$$

We define $C_0^{\mathbf{E}}(\mathbb{R}^*) = \mathcal{F}^* C_0^{\mathbf{E}}(\mathbb{R}) \mathcal{F}$ and similarly $C^{\mathbf{E}}(\overline{\mathbb{R}^*})$. For example, $C_0^{\mathbf{E}}(\mathbb{R}^*)$ is the set of operators $\psi(P)$ with $\psi : \mathbb{R} \to K(\mathbf{E})$ norm continuous and norm convergent to zero at infinity.

It is useful to give a meaning to the notion of limit as $Q \to \pm\infty$ or $P \to \pm\infty$ for some operators $T \in B(\mathscr{H})$. These objects will be denoted $\lim_{Q \to \pm\infty} T$ and $\lim_{P \to \pm\infty} T$ respectively. Since Q and P play a similar role, we present in detail only the case of the Q-limits. Note that the limits $\lim_{Q \to \pm\infty}$ should be "independent of Q" and it is natural to identify the constants with respect to Q with translation invariant operators, i.e. the elements of the commutant algebra $\{P\}'$. If $T_+ = \text{s-}\lim_{a \to \infty} \mathrm{e}^{iaP} T \mathrm{e}^{-iaP}$ exists then $T_+ \in \{P\}'$ and one should think of T_+ as the limit of T as $Q \to +\infty$. But this condition is not strong enough in a C^*-algebra setting: some kind of norm convergence is needed.

Definition 3.17 *We say that* $T \in B(\mathscr{H})$ *has a limit at* $Q = +\infty$ *if there is* $T_+ \in \{P\}'$ *such that* $\lim_{a \to \infty} \|\chi(Q > a)\,(T - T_+)^{(*)}\| = 0$. *Then we set* $\lim_{Q \to +\infty} T \equiv T_+$.

Observe that one will necessarily have $T_+ = \text{s-lim}_{a\to\infty} \mathrm{e}^{iaP} T \mathrm{e}^{-iaP}$. Similarly is defined $\lim_{Q\to-\infty} T = T_-$. As we previously said, the roles of Q and P can be interchanged in Definition 3.17. Hence one may define $\lim_{P\to\pm\infty} T \in \{Q\}'$ for some class of operators $T \in B(\mathscr{H})$. However, since the commutation relation $[Q,P]=i$ gives $[P,-Q]=i$, we have to change in the preceding formulas Q in P and P in $-Q$. For example,

$$\lim_{P\to+\pm\infty} T \equiv T^+ = \text{s-lim}_{a\to\infty} \mathrm{e}^{-iaQ} T \mathrm{e}^{iaQ}.$$

The Riesz-Kolmogorov criterion (Theorem 3.8) can now be stated as: $T \in \mathscr{K}(\mathbb{R})$ *if and only if* $\lim_{Q\to\pm\infty} T = \lim_{P\to\pm\infty} T = 0$.

The simplest examples of operators that have limits at $Q=\pm\infty$ and at $P=\pm\infty$ are those of the form $T = \sum_{j=1}^n \varphi_j(Q)\psi_j(P)$ where $\varphi_j,\ \psi_j : \mathbb{R} \to B(\mathbf{E})$ are continuous functions which have limits at $+\infty$ and $-\infty$. We set $\varphi(\pm\infty) = \lim_{x\to\pm\infty} \varphi(x)$. Then

$$\lim_{Q\to\pm\infty} T = \sum_{j=1}^n \varphi_j(\pm\infty)\psi_j(P) \text{ and } \lim_{P\to\pm\infty} T = \sum_{j=1}^n \varphi_j(Q)\psi_j(\pm\infty).$$

The algebras of main interest for us are defined below. Other descriptions of these objects will be given later.

Definition 3.18 *$\mathscr{B}$ is the set of operators $T \in B(\mathscr{H})$ such that $\lim_{Q\to\pm\infty} T$ exist and belong to $C^{\mathbf{E}}(\overline{\mathbb{R}^*})$ and $\lim_{P\to\pm\infty} T$ exist and belong to $C^{\mathbf{E}}(\overline{\mathbb{R}})$. $\mathscr{C}$ is the subset of $\mathscr{B}$ consisting of operators T such that $\lim_{P\to\pm\infty} T = 0$.*

The elements of $\mathscr{B}$ have an anisotropic behavior at infinity in both variables Q and P (phase-space anisotropy). The operators of the subalgebra $\mathscr{C}$ are characterized by anisotropy in the Q-variable only: thus one may call $\mathscr{C}$ the position-anisotropic algebra (it is easy to prove that if $T \in \mathscr{C}$ then $\lim_{Q\to\pm\infty} T \in C_0^{\mathbf{E}}(\mathbb{R}^*)$). Analogously one can introduce a purely momentum-anisotropic algebra (which is in fact $\mathcal{F}^*\mathscr{C}\mathcal{F}$). On the other hand, taking e.g. $\mathscr{H} = L^2(\mathbb{R}^2) \otimes \mathbf{E}$ one may consider anisotropy in several position (or momentum) one dimensional variables, but the structure of the corresponding algebras does not differ essentially of that of $\mathscr{B}$ and may be treated analogously.

It is easy to show that $\mathscr{B}$ and $\mathscr{C}$ are C^*-algebras. The algebra $\mathscr{B}$ is unital if and only if $\mathbf{E}$ is finite dimensional. $\mathscr{C}$ is never unital. We have $C^{\mathbf{E}}(\overline{\mathbb{R}}) \subset \mathscr{B}$ and $C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \subset \mathscr{B}$. Also, $K(\mathscr{H}) \subset \mathscr{C} \subset \mathscr{B}$.

The next theorem contains the alternative characterization of the C^*-algebras $\mathscr{B}$ and $\mathscr{C}$ standing at the origin of our approach.

Theorem 3.19

$$\mathscr{B} = [\![C^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C^{\mathbf{E}}(\overline{\mathbb{R}^*})]\!] = [\![C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \cdot C^{\mathbf{E}}(\overline{\mathbb{R}})]\!], \tag{3.13}$$

$$\mathscr{C} = [\![C^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C_0^{\mathbf{E}}(\mathbb{R}^*)]\!] = [\![C_0^{\mathbf{E}}(\mathbb{R}^*) \cdot C^{\mathbf{E}}(\overline{\mathbb{R}})]\!]. \tag{3.14}$$

Proof: Functions of class C^∞ with derivatives of compact support are dense in $C(\overline{\mathbb{R}})$. Hence, if $S \in C^{\mathbf{E}}(\overline{\mathbb{R}})$ and $T \in C^{\mathbf{E}}(\overline{\mathbb{R}^*})$ then $[S, T]$ is a compact operator. This implies the second equalities in (3.13) and (3.14). The fact that $\mathscr{B}$ contains $[\![C^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C^{\mathbf{E}}(\overline{\mathbb{R}^*})]\!]$ follows from $C^{\mathbf{E}}(\overline{\mathbb{R}}) \subset \mathscr{B}$, $C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \subset \mathscr{B}$ and the fact that $\mathscr{B}$ is a norm closed algebra. We now prove the inverse inclusion. Let $T \in \mathscr{B}$ and denote $T_\pm = \lim_{Q\to\pm\infty} T$ and $T^\pm = \lim_{P\to\pm\infty} T$. Let $\theta_+ \in C^\infty(\overline{\mathbb{R}})$ such that $\theta_+(x) = 0$ if $x < 1$ and $\theta_+(x) = 1$ if $x > 2$ and let us set $\theta_-(x) = \theta_+(-x)$ and $\theta_0 = 1 - \theta_- - \theta_+$. Denote $\theta_0^\varepsilon = \theta_0(\varepsilon Q)$ and $\theta_\pm^\varepsilon = \theta_\pm(\varepsilon Q)$. Then

$$T = \theta_0^\varepsilon T + \theta_+^\varepsilon T_+ + \theta_-^\varepsilon T_- + \theta_+^\varepsilon (T - T_+) + \theta_-^\varepsilon (T - T_-)$$

The last two terms tend to zero in norm when $\varepsilon \searrow 0$, so it suffices to prove that $\theta_k^\varepsilon T_k$ belongs to r.h.s. of (3.13) if $k = 0, \pm$. Since $T_\pm \in C^{\mathbf{E}}(\overline{\mathbb{R}^*})$, this is clear for $k = \pm$. For $k = 0$ we use a decomposition in the P-variable. Setting $\eta_k^\nu = \theta_k(\nu P)$ for $k = 0, \pm$, we have

$$\theta_0^\varepsilon T = \theta_0^\varepsilon \eta_0^\nu T + \theta_0^\varepsilon \eta_+^\nu T^+ + \theta_0^\varepsilon \eta_-^\nu T^- + \theta_0^\varepsilon \eta_+^\nu (T - T^+) + \theta_0^\varepsilon \eta_-^\nu (T - T^-)$$

As before, the last two terms tend in norm to zero as $\nu \searrow 0$. Also, $\theta_0^\varepsilon \eta_0^\nu T$ is compact, so it belongs to the second member of (3.13). Finally, $\theta_0^\varepsilon \eta_\pm^\nu T^\pm$ belong to the third member of (3.13). The proof of (3.14) is quite similar, but simpler (only a Q-variable decomposition suffices). ■

We remark that one also has

$$\mathscr{B} = [\![C(\overline{\mathbb{R}}) \cdot C(\overline{\mathbb{R}^*})]\!] \otimes K(\mathbf{E}) \quad \text{and} \quad \mathscr{C} = [\![C(\overline{\mathbb{R}}) \cdot C_0(\mathbb{R}^*)]\!] \otimes K(\mathbf{E}).$$

In the next theorem we shall give an explicit description of the quotient algebra $\mathscr{B}/K(\mathscr{H})$. For this purpose we shall introduce four maps $\mathcal{P}_\pm, \mathcal{P}^\pm$ of $\mathscr{B}$ into itself by the formulas

$$\mathcal{P}_\pm[T] = \lim_{Q\to\pm\infty} T \ \text{ and } \ \mathcal{P}^\pm[T] = \lim_{P\to\pm\infty} T.$$

It is clear that $\mathcal{P}_\pm$ are morphisms $\mathscr{B} \to \mathscr{B}$ and also that they are projections (in the vector space sense, i.e. $\mathcal{P}_\pm^2 = \mathcal{P}_\pm$) of $\mathscr{B}$ onto its subalgebra $C^{\mathbf{E}}(\overline{\mathbb{R}^*})$ (in particular they are *expectations*). Similarly, $\mathcal{P}^\pm$ are morphisms and projections of $\mathscr{B}$ onto its subalgebra $C^{\mathbf{E}}(\overline{\mathbb{R}})$.

Now we define

$$\mathscr{P} : \mathscr{B} \to \mathscr{D} \equiv C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}})$$

by $\mathscr{P} = (\mathcal{P}_-, \mathcal{P}_+, \mathcal{P}^-, \mathcal{P}^+)$. The space $\mathscr{D}$ is considered as direct sum of C^*-algebras, so it is a C^*-algebra and $\mathscr{P}$ is a morphism.

Theorem 3.20 *The kernel of the morphism $\mathscr{P}$ is $K(\mathscr{H})$ and its range is the set of operators $\boldsymbol{T} = (T_-, T_+, T^-, T^+) \in \mathscr{D}$ such that the following compatibility conditions are satisfied:*

$$\lim_{P\to\pm\infty} T_- = \lim_{Q\to-\infty} T^\pm \ \text{ and } \ \lim_{P\to\pm\infty} T_+ = \lim_{Q\to+\infty} T^\pm. \tag{3.15}$$

Proof: That $K(\mathscr{H})$ is the kernel of $\mathscr{P}$ follows from the Riesz-Kolmogorov theorem as formulated after Definition 3.17. Note that the relations (3.15) may be written as $\mathcal{P}^\pm[T_-] = \mathcal{P}_-[T^\pm]$ and $\mathcal{P}^\pm[T_+] = \mathcal{P}_+[T^\pm]$. We check that they are satisfied on the range of $\mathscr{P}$. By Theorem 3.19, it suffices to show $\mathcal{P}^\pm\mathcal{P}_-[T] = \mathcal{P}_-\mathcal{P}^\pm[T]$ for T of the form $T = \varphi(Q)\psi(P)$; but this is obvious. Reciprocally, let $\boldsymbol{T}$ be as in the statement of the theorem. We have to construct $T \in \mathscr{B}$ such that $\mathscr{P}[T] = \boldsymbol{T}$. Let $\varphi_\pm, \varphi^\pm \in C^{\mathbf{E}}(\overline{\mathbb{R}})$ such that $T_\pm = \varphi_\pm(P)$ and $T^\pm = \varphi^\pm(Q)$. Then the compatibility relations (3.15) can be written as

$$\varphi_-(\pm\infty) = \varphi^\pm(-\infty) \ \text{ and } \ \varphi_+(\pm\infty) = \varphi^\pm(+\infty).$$

Using the cutoffs $\theta_\pm$ introduced in the proof of Theorem 3.19 we define

$$\psi_\pm(P) = \varphi_\pm(P) - \frac{1}{2}\theta_-(P)\varphi_\pm(-\infty) - \frac{1}{2}\theta_+(P)\varphi_\pm(+\infty)$$

$$\psi^\pm(Q) = \varphi^\pm(Q) - \frac{1}{2}\theta_-(Q)\varphi^\pm(-\infty) - \frac{1}{2}\theta_+(Q)\varphi^\pm(+\infty).$$

Then we construct the desired T as:

$$T = \theta_-(Q)\psi_-(P) + \theta_+(Q)\psi_+(P) + \theta_-(P)\psi^-(Q) + \theta_+(P)\psi^+(Q).$$

The relation (3.13) shows that $T \in \mathscr{B}$ and for example one has:

$$\begin{aligned}
&\lim_{Q\to+\infty} T = \\
&= \theta_-(+\infty)\psi_-(P) + \theta_+(+\infty)\psi_+(P) + \theta_-(P)\psi^-(+\infty) + \theta_+(P)\psi^+(+\infty) \\
&= \ \psi_+(P) + \theta_-(P)\left\{\varphi^-(+\infty) - \tfrac{1}{2}\theta_-(+\infty)\varphi^-(-\infty) - \tfrac{1}{2}\theta_+(+\infty)\varphi^-(+\infty)\right\} \\
&\quad + \theta_+(P)\left\{\varphi^+(+\infty) - \tfrac{1}{2}\theta_-(+\infty)\varphi^+(-\infty) - \tfrac{1}{2}\theta_+(+\infty)\varphi^+(+\infty)\right\} \\
&= \psi_+(P) + \tfrac{1}{2}\theta_-(P)\varphi^-(+\infty) + \tfrac{1}{2}\theta_+(P)\varphi^+(+\infty) = \varphi_+(P) \equiv T_+.
\end{aligned}$$

In the last equality the second compatibility relation has been used. Similarly, one shows that $\lim_{Q\to-\infty} T = T_-$ and $\lim_{P\to\pm\infty} T = T^\pm$. ∎

Remark: The morphism $\mathscr{P}$ induces an isomorphism between the quotient C^*-algebra $\mathscr{B}/K(\mathscr{H})$ and the C^*-subalgebra of $\mathscr{D}$ defined by the compatibility relations (3.15). From now on we shall embed

$$\mathscr{B}/K(\mathscr{H}) \subset \mathscr{D} \equiv C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}^*}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}}) \oplus C^{\mathbf{E}}(\overline{\mathbb{R}}) \tag{3.16}$$

with the help of this isomorphism.

The C^*-algebra $\mathscr{C}$ deserves a separate study because many hamiltonians of interest in physics are affiliated to it.

Theorem 3.21 *The map $T \mapsto (\mathcal{P}_-[T], \mathcal{P}_+[T])$ is a surjective morphism of $\mathscr{C}$ onto the C^*-algebraic direct sum $C_0^{\mathbf{E}}(\mathbb{R}^*) \oplus C_0^{\mathbf{E}}(\mathbb{R}^*)$ with $K(\mathscr{H})$ as kernel.*

Remark: We identify the quotient C^*-algebra $\mathscr{C}/K(\mathscr{H})$ and the C^*-algebra $C_0^{\mathbf{E}}(\mathbb{R}^*) \oplus C_0^{\mathbf{E}}(\mathbb{R}^*)$ with the help of the isomorphism induced by the map $(\mathcal{P}_-, \mathcal{P}_+)$. This is the precise meaning of the equality

$$\mathscr{C}/K(\mathscr{H}) = C_0^{\mathbf{E}}(\mathbb{R}^*) \oplus C_0^{\mathbf{E}}(\mathbb{R}^*). \tag{3.17}$$

Observe that the quotient $\mathscr{C}/K(\mathscr{H})$ has a simpler description than $\mathscr{B}/K(\mathscr{H})$. Indeed, in (3.17) we have equality whereas in (3.16) one has only inclusion.

Proof of the theorem: Remember that by Definition 3.18 the algebra $\mathscr{C}$ is the set of $T \in \mathscr{B}$ such that $\mathcal{P}^{\pm}[T] = 0$ and $\mathcal{P}_{\pm}[T] \in C_0^{\mathbf{E}}(\mathbb{R}^*)$. Also, $K(\mathscr{H})$ is the set of $T \in \mathscr{C}$ such that $\mathcal{P}_{\pm}[T] = 0$. On the other hand, with the notations of Theorem 3.20, $\boldsymbol{T} \in \mathscr{P}[\mathscr{C}]$ if and only if $\boldsymbol{T} = (T_-, T_+, 0, 0)$ and $\lim_{P \to \pm\infty} T_+ = \lim_{P \to \pm\infty} T_- = 0$ (see (3.15)). But this is equivalent to $T_{\pm} \in C_0^{\mathbf{E}}(\mathbb{R}^*)$. ■

We would like now to give a description of the action of the projections $\mathcal{P}_{\pm}$ in $\mathscr{C}$ suggested by the definition of the corresponding morphisms in graded C^*-algebras (see [BG1] and §3.6). For this we observe that the kernels

$$\mathscr{C}_{\pm} := \{T \in \mathscr{C} \mid \mathcal{P}_{\mp}[T] = 0\} = \{T \in \mathscr{C} \mid \lim_{Q \to \mp\infty} T = 0\} \tag{3.18}$$

are closed self-adjoints ideals in $\mathscr{C}$ which, by the proof of Theorem 3.19, can also be written as

$$\mathscr{C}_{\pm} = [\![C_{\pm}^{\mathbf{E}}(\overline{\mathbb{R}}) \cdot C_0(\mathbb{R}^*)]\!]. \tag{3.19}$$

Here

$$C_{\pm}^{\mathbf{E}}(\overline{\mathbb{R}}) = \{\varphi \in C^{\mathbf{E}}(\overline{\mathbb{R}}) \mid \lim_{x \to \mp\infty} \varphi(x) = 0\}$$

are closed ideals in $C^{\mathbf{E}}(\overline{\mathbb{R}})$.

Since the restrictions of $\mathcal{P}_\pm$ to $\mathscr{C}$ are projections of norm one of $\mathscr{C}$ onto its closed subspace $C_0^{\mathbf{E}}(\mathbb{R}^*)$, we see that $\mathscr{C} = \mathscr{C}_\pm + C_0^{\mathbf{E}}(\mathbb{R}^*)$, topological direct sum. This allows us to change our point of view, to forget the preceding meaning of the maps $\mathcal{P}_\pm$, and to see them as the projections of $\mathscr{C}$ onto $C_0^{\mathbf{E}}(\mathbb{R}^*)$ determined by the preceding direct sum decompositions. We emphasize that it is possible to adopt this point of view from the beginning and to develop the theory without using the notion of limit at infinity of an operator. We mention some easy to prove properties of the ideals $\mathscr{C}_\pm$:

$$\mathscr{C} = \mathscr{C}_+ + \mathscr{C}_-, \;\; \mathscr{C}_+ \cap \mathscr{C}_- = K(\mathscr{H}), \;\; \mathscr{C}_\pm \cdot \mathscr{C}_\mp \subset K(\mathscr{H}). \tag{3.20}$$

Observe that the space $\mathscr{C}_\infty$ of $T \in \mathscr{C}$ such that $\mathcal{P}_+ T = \mathcal{P}_- T$ is a C^*-subalgebra of $\mathscr{C}$, a two-body type algebra. Indeed,

$$\mathscr{C}_\infty = [\![C_\infty^{\mathbf{E}}(\mathbb{R}) \cdot C_0(\mathbb{R}^*)]\!] = C_0^{\mathbf{E}}(\mathbb{R}^*) + K(\mathscr{H}). \tag{3.21}$$

We give a third description of the C^*-algebra $\mathscr{C}$ (the proof is straightforward and will not be given). An operator $T \in B(\mathscr{H})$ is called *semi-compact* if for all $\theta \in C_0(\mathbb{R})$ the operators $\theta(Q)T$ and $T\theta(Q)$ are compact.

Proposition 3.22 *$\mathscr{C}$ coincides with the set of semicompact operators such that $\lim_{Q\to\pm\infty} T$ exist and belong to $C_0^{\mathbf{E}}(\mathbb{R}^*)$.*

If H is an observable affiliated to $\mathscr{C}$, then $H_\pm := \mathcal{P}_\pm[H]$ are well defined observables affiliated to $C_0^{\mathbf{E}}(\mathbb{R}^*)$. If H is the hamiltonian of a physical system, then $H_\pm$ will be called *asymptotic hamiltonians*, or localizations of H at $Q = \pm\infty$. One has a purely anisotropic situation if $H_- \neq H_+$.

Theorem 3.23 *Let H be an observable affiliated to $\mathscr{C}$. Then:*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \boldsymbol{\sigma}(H_-) \cup \boldsymbol{\sigma}(H_+).$$

Moreover, there are numbers $-\infty \le a_\pm \le b_\pm \le \infty$ such that $\boldsymbol{\sigma}(H_\pm) = \mathbb{R} \setminus (a_\pm, b_\pm)$. Hence, if $a = \max\{a_-, a_+\}$, $b = \min\{b_-, b_+\}$ then $\boldsymbol{\sigma}_{\text{ess}}(H) = \mathbb{R} \setminus (a, b)$.

Proof: We have to prove only the second assertion. We show that for an arbitrary observable A affiliated to $C_0^{\mathbf{E}}(\mathbb{R}^*)$ one has $\boldsymbol{\sigma}(A) = \mathbb{R} \setminus (\alpha, \beta)$ for some $-\infty \leq \alpha \leq \beta \leq \infty$. We can assume that $0 \notin \boldsymbol{\sigma}(A)$ and so it suffices to prove that the spectrum of $S = A^{-1}$ is an interval containing zero. Let S be an arbitrary symmetric operator in $\mathscr{C}$. After a Fourier transform, S becomes the operator of multiplication by a norm continuous and norm convergent to zero at infinity symmetric operator valued function $p \mapsto S(p) \in B(\mathbf{E})$. Then $\boldsymbol{\sigma}(S)$ is the closure of the union of the spectra of the operators $S(p)$ and the lower and upper bounds of the spectrum of $S(p)$ belong to $\boldsymbol{\sigma}(S(p))$, depend continuously on p and tend to zero as $|p| \to \infty$. ■

We finish with some comments on the self-adjoint operators affiliated to $\mathscr{C}$. A self-adjoint operator H on $\mathscr{H}$ is *locally compact* if for each $\phi \in C_0(\mathbb{R})$ and each $\theta \in C_0(\mathbb{R})$ the operator $\theta(Q)\phi(H)$ is compact. Equivalently, if there is $z \notin \boldsymbol{\sigma}(H)$ such that $\theta(Q)R(z)$ is compact. From Proposition 3.22 we get: *H is affiliated to $\mathscr{C}$ if and only if H is locally compact and for some $z \in \mathbb{C} \setminus \mathbb{R}$ the limits $\lim_{Q \to \pm\infty} R(z)$ exist and belong to $C_0^{\mathbf{E}}(\mathbb{R}^*)$.*

Consider a map h from $\mathbb{R}$ to the set of self-adjoint operators on $\mathbf{E}$. We write $\lim_{p \to \infty} h(p) = \infty$ if $\|(h(p)+i)^{-1}\| \to 0$ as $|p| \to \infty$. This is equivalent to: for each $r > 0$ there is $r_0 > 0$ such that $\boldsymbol{\sigma}(h(p)) \cap [-r, r] = \emptyset$ if $|p| > r_0$. Now assume that: (i) the map $p \mapsto (h(p)+i)^{-1}$ is norm-continuous, (ii) for each $p \in \mathbb{R}$, $h(p)$ has purely discrete spectrum, (iii) $\lim_{p \to \infty} h(p) = \infty$. Then the self-adjoint operator $h(P)$ is affiliated to $C_0^{\mathbf{E}}(\mathbb{R}^*)$. This is obvious.

Finally, we give an explicit class of hamiltonians affiliated to $\mathscr{C}$:

Proposition 3.24 *Let H be a self-adjoint operator in $\mathscr{H}$ and $H_\pm$ a pair of self-adjoint operators affiliated to $C_0^{\mathbf{E}}(\mathbb{R}^*)$ such that $D(H_\pm) = D(H)$. Assume that:*

$$\lim_{r \to \pm\infty} \|\chi(\pm Q \geq \pm r)\,(H - H_\pm)\|_{D(H) \to \mathscr{H}} = 0. \tag{3.22}$$

Then H is affiliated to $\mathscr{C}$ and $\mathcal{P}_\pm[H] = H_\pm$.

It is clear that this covers one-dimensional quantum mechanical models like the Schrödinger or Dirac operator with different spatial asymptotics at left and right. For instance, let $H = Pa(Q)P + v(Q)$ such that: (i) $a : \mathbb{R} \to \mathbb{R}$ is continuous, $\inf a(x) > 0$ and $\lim_{x \to \pm\infty} a(x) = a_\pm$ exist in $\mathbb{R}$, (ii) $v \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\lim_{x \to \pm\infty} v(x) = v_\pm$ exist. Then $H_\pm = h_\pm(P) = a_\pm P^2 + v_\pm$.

An important aspect of the algebra $\mathscr{C}$ is that one can prove the Mourre estimate for a large abstract class of hamiltonians affiliated to it (in fact, this can also be done for $\mathscr{B}$). This class covers Schrödinger operators in domains with cylindrical ends, with different asymptotic at each end, like those studied in [Ben, DDI]. This question is treated in [GI6] and will eventually be published.

The preceding techniques can be used in other types of one-dimensional anisotropy. Here is an example from [Rod] where potentials with different periodic asymptotics at $\pm\infty$ are considered. We mention it because the definition of $\mathscr{C}$ is quite nice, namely

$$\mathscr{C} = \{\varphi \in C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R}) \mid \lim_{n\to\pm\infty} \varphi(x + na_{\pm}) \text{ exist for all } x \in \mathbb{R}\},$$

where $a_{\pm}$ are given strictly positive real numbers. Then $\mathscr{C}$ is a C^*-subalgebra of $C_{\mathrm{b}}^{\mathrm{u}}(\mathbb{R})$ which contains $C_0(\mathbb{R})$ and is stable by translations. Let us denote $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then

$$\mathscr{C}/C_0(\mathbb{R}) \cong C(\mathbb{T}) \oplus C(\mathbb{T}).$$

This allows one to compute the quotient with respect to the compacts of the algebra of hamiltonians associated to this problem:

$$\begin{aligned} \mathscr{C} \rtimes \mathbb{R}\big/\mathscr{K}(\mathbb{R}) &\cong \mathscr{C}\big/C_0(\mathbb{R}) \rtimes \mathbb{R} \cong [C(\mathbb{T}) \rtimes \mathbb{R}] \oplus [C(\mathbb{T}) \rtimes \mathbb{R}] \\ &\cong [C(\mathbb{T}) \otimes \mathscr{K}(\mathbb{T})] \oplus [C(\mathbb{T}) \otimes \mathscr{K}(\mathbb{T})]. \end{aligned}$$

3.6. Graded C^*-algebras. In this subsection we consider C^*-algebras graded by semilattices, a class of algebras useful in the study of N-body systems and their generalizations, which have been introduced and studied in [BG1, BG2] for the case of finite semilattices (see also [ABG]) and then in [DG1, DG2] for arbitrary ones. Their usefulness in the N-body problem has already been discussed in §3.4. The new example considered in this subsection concerns an algebra associated to symplectic spaces studied mainly in [GI3]. N-body hamiltonians in constant magnetic fields (as in [GeL]) belong to this framework.

A family $\{\mathscr{C}_i\}_{i\in I}$ of subalgebras of an algebra $\mathscr{C}$ is *linearly independent* if for each family $\{S_i\}_{i\in I}$ such that $S_i \in \mathscr{C}_i\ \forall i, S_i \neq 0$ for at most a finite number of i and $\sum_{i\in I} S_i = 0$, one has $S_i = 0$ for all $i \in I$. The sums of algebras which appear below are understood in the sense of linear spaces.

Let $\mathcal{L}$ be a *semilattice*, i.e. a partially ordered set in which each pair of elements a, b has a lower bound $a \wedge b$. For $a \in \mathcal{L}$ we set $\mathcal{L}_a = \{b \in \mathcal{L} \mid b \geq a\}$; this is also a semilattice.

We say that a C^*-algebra $\mathscr{C}$ is *$\mathcal{L}$-graded* if a linearly independent family $\{\mathscr{C}(a)\}_{a\in\mathcal{L}}$ of C^*-subalgebras of $\mathscr{C}$ has been given such that:

(i) $\mathscr{C}(a) \cdot \mathscr{C}(b) \subset \mathscr{C}(a \wedge b)$ for all $a, b \in \mathcal{L}$;

(ii) if $\mathcal{E} \subset \mathcal{L}$ is finite then $\sum_{a\in\mathcal{E}} \mathscr{C}(a)$ is a closed subspace of $\mathscr{C}$;

(iii) $\sum_{a\in\mathcal{L}} \mathscr{C}(a)$ is dense in $\mathscr{C}$.

For each $\mathcal{E} \subset \mathcal{L}$ we define $\mathscr{C}(\mathcal{E})$ as the closure of $\mathscr{C}(\mathcal{E})^\circ = \sum_{a\in\mathcal{E}} \mathscr{C}(a)$. If $a \in \mathcal{L}$ we set $\mathscr{C}_a = \mathscr{C}(\mathcal{L}_a)$. It is clear that $\mathscr{C}_a$ is a $\mathcal{L}_a$-graded C^*-subalgebra of $\mathscr{C}$. There is a natural map $\mathcal{P}_a^\circ : \mathscr{C}(\mathcal{L})^\circ \to \mathscr{C}(\mathcal{L}_a)^\circ$ defined by $\mathcal{P}_a^\circ \sum_{b\in\mathcal{L}} T(b) = \sum_{b\in\mathcal{L}_a} T(b)$ if $T(b) \in \mathscr{C}(b)$ and $T(b) \neq 0$ only for a finite numbers of b. This map is clearly a surjective morphism of $*$-algebras. It can be shown that this map is continuous, so it extends to a surjective morphism $\mathcal{P}_a : \mathscr{C} \to \mathscr{C}_a$. Moreover, $\mathcal{P}_a$ is a projection (in the sense of linear spaces) and its kernel is $\mathscr{C}(\mathcal{L}'_a)$, where $\mathcal{L}'_a = \{b \in \mathcal{L} \mid b \not\geq a\}$.

Assume that $\mathcal{L}$ has a least element $\min \mathcal{L}$, denote $\mathcal{M}$ the set of atoms of $\mathcal{L}$ (i.e. minimal elements of $\mathcal{L} \setminus \{\min \mathcal{L}\}$) and assume that $\mathcal{L}$ is atomic (i.e. each $a \neq \min \mathcal{L}$ is minorated by an atom). Observe that $\mathscr{C}(\min \mathcal{L})$ is a closed self-adjoint ideal in $\mathscr{C}$, so the C^*-algebra $\mathscr{C}/\mathscr{C}(\min \mathcal{L})$ is well defined. The important fact is that one can explicitly realize this algebra as follows. There is a natural morphism

$$\mathscr{C} \ni T \longmapsto (\mathcal{P}_a[T])_{a\in\mathcal{M}} \in \prod_{a\in\mathcal{M}} \mathscr{C}_a, \tag{3.23}$$

where the direct product is in the C^*-algebra sense, and the kernel of this morphism is equal to $\mathscr{C}(\min \mathcal{L})$. It is easy to see that for each $T \in \mathscr{C}$ the set $\{\mathcal{P}_a[T] \mid a \in \mathcal{M}\}$ is relatively compact in $\mathscr{C}$. Thus, we get a canonical embedding

$$\mathscr{C}/\mathscr{C}(\min \mathcal{L}) \hookrightarrow \prod^{\mathrm{rc}}_{a\in\mathcal{M}} \mathscr{C}_a \tag{3.24}$$

where the right hand side is the C^*-algebra of families $\{S_a\}_{a\in\mathcal{L}}$ such that $\{S_a \mid a \in \mathcal{L}\}$ is relatively compact in $\mathscr{C}$. As usual, such a result allows one to compute the essential spectrum and to prove the Mourre estimate for the observables affiliated to $\mathscr{C}$. The example below is a general and abstract version of the HVZ (Hunziker, Van Winter, Zihslin) theorem.

Theorem 3.25 *Assume that $\mathscr{C}$ is realized as a C^*-algebra of operators on a Hilbert space $\mathscr{H}$ such that $\mathscr{C}(\min\mathcal{L}) = K(\mathscr{H})$. Let H be an observable affiliated to $\mathscr{C}$ and for each $a \in \mathcal{L}$ let $H_a = \mathcal{P}_a[H]$, which is an observable affiliated to $\mathscr{C}_a$, hence to $\mathscr{C}$. Then $\boldsymbol{\sigma}_{\text{ess}}(H)$ is equal to the closure of $\bigcup_{a\in\mathcal{M}} \boldsymbol{\sigma}(H_a)$. Moreover, if for each $T \in \mathscr{C}$ the set $\{\mathcal{P}_a[T] \mid a \in \mathcal{M}\}$ is compact in $\mathscr{C}$, then one has*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \bigcup_{a\in\mathcal{M}} \boldsymbol{\sigma}(H_a).$$

We mention that the property required in the last part of the theorem holds for the algebra $\mathscr{C}^X$ from §3.4 (where $\mathcal{L} = \mathbb{G}(X)$ hence $\mathcal{M}$ is the projective space $\mathbb{P}(X)$ of X), as well as for the symplectic algebra considered below.

We have already seen examples of graded algebras in §3.4. A new example follows (see [GI3] for details). We consider a system $(\Xi, \sigma; \mathscr{H}, W)$ consisting of a symplectic space (Ξ, σ) and a representation W of it on a Hilbert space $\mathscr{H}$. So Ξ is a real finite dimensional vector space equipped with a real antisymmetric nondegenerate bilinear form σ and W is a strongly continuous map from Ξ to the set of unitary operators on $\mathscr{H}$ satisfying, for all $\xi, \eta \in \Xi$:

$$W(\xi + \eta) = \mathrm{e}^{-\frac{i}{2}\sigma(\xi,\eta)} W(\xi)W(\eta).$$

To each abelian group one can associate a structure of this type, Ξ being only an abelian group in general (see §3.6 in [GI2]). If E is a vector subspace of Ξ we set

$$E^\sigma := \{\xi \in \Xi \mid \sigma(\xi, \eta) = 0, \text{ for all } \eta \in E\}.$$

Then $E^{\sigma\sigma} = E$ and $(E \cap F)^\sigma = E^\sigma + F^\sigma$. Note that E^σ can also be defined as the set of those $\xi \in \Xi$ such that $W(\xi)W(\eta) = W(\eta)W(\xi)$ for all $\eta \in E$. We shall take $\mathcal{L} = \mathbb{G}(\Xi)$ the set of all vector subspaces of Ξ. If $E \in \mathbb{G}(\Xi)$, then we define $\mathscr{C}(E)$ as the set of operators $T \in B(\mathscr{H})$ such that:

(i) $\| [W(\xi), T] \| \to 0$ if $\xi \to 0$ in Ξ,

(ii) $[W(\xi), T] = 0$ if $\xi \in E$,

(iii) $\| (W(\xi) - 1)T^{(*)} \| \to 0$ if $\xi \in E^\sigma$ and $\xi \to 0$.

Let $\mathscr{C}$ be the closure of $\sum_E \mathscr{C}(E)$. Then the family of C^*-subalgebras $\mathscr{C}(E)$ provides $\mathscr{C}$ with a $\mathbb{G}(\Xi)$-graded C^*-algebra structure.

The hamiltonians of N-body systems interacting with an external constant magnetic field (see [GeL]) are affiliated to graded C^*-subalgebras of algebras of the preceding form (this is proved in [GI3]).

We mention only one important theorem: *If W is an irreducible representation, then $\mathscr{C}(E)$ coincides with the closure in $B(\mathscr{H})$ of the set of operators $W(\mu)$ with μ an E^σ-a.c. integrable measure.* Here, an integrable measure μ on Ξ is F-a.c. for some vector subspace F of Ξ if there are a Haar measure λ_F on F and a function $\rho \in L^1(F) = L^1(F, \lambda_F)$ such that $\mu = \rho\lambda_F$. Then we define $W(\mu) = \int_\Xi W(\xi)\mu(\mathrm{d}\xi)$.

On can construct graded C^*-algebras by taking tensor products of graded C^*-algebras. The following results (Lemma 3.26 and Propositions 3.27, 3.28) are part of a joint work of M. Măntoiu and one of us (V. G.). They are useful in the study of quantum field models with a particle number cutoff, e.g. the results of C. Gérard concerning the Mourre estimate for the spin-boson model [Ger] are easy to prove in this framework.

Let $\{\mathscr{C}^k\}_{k\in\mathcal{N}}$ be a finite family of C^*-algebras. Assume that $\mathscr{C}^k$ is $\mathcal{L}^k$-graded, where $\mathcal{L}^k$ is a finite semilattice. Thus linear direct sum decompositions $\mathscr{C}^k = \sum_{a\in\mathcal{L}^k} \mathscr{C}^k(a)$ are given with $\mathscr{C}^k(a) \subset \mathscr{C}^k$ C^*-subalgebras such that $\mathscr{C}^k(a) \cdot \mathscr{C}^k(b) \subset \mathscr{C}^k(a \wedge b)$. Recall that $\mathscr{C}^k_a = \sum_{b\geq a} \mathscr{C}^k(b)$ are C^*-subalgebras of $\mathscr{C}^k$ and that the projections $\mathcal{P}^k_a : \mathscr{C}^k \to \mathscr{C}^k_a$ asociated with the given linear direct sum decomposition of $\mathscr{C}^k$ are morphisms.

The product set $\mathcal{L} = \prod_{k\in\mathcal{N}} \mathcal{L}^k$ will be equipped with the product order relation: if $a = (a^k)$, $b = (b^k)$ are elements of $\mathcal{L}$, then $a \leq b$ if and only if $a^k \leq b^k$ for all k. It is clear that $\mathcal{L}$ becomes a semilattice with $a \wedge b = (a^k \wedge b^k)$. Consider now the tensor products

$$\mathscr{C} = \otimes_k \mathscr{C}^k, \quad \mathscr{C}(a) = \otimes_k \mathscr{C}^k(a^k), \quad \mathscr{C}_a = \otimes_k \mathscr{C}^k_{a^k}.$$

Lemma 3.26 *The family of C^*-subalgebras $\{\mathscr{C}(a)\}_{a\in\mathcal{L}}$ defines an $\mathcal{L}$-grading of the C^*-algebra $\mathscr{C}$. One has $\mathscr{C}_a = \sum_{b\geq a}\mathscr{C}(b)$ and the canonical projection $\mathcal{P}_a : \mathscr{C} \to \mathscr{C}_a$ is given by $\mathcal{P}_a = \otimes_k \mathcal{P}_{a^k}$.*

Proof: The projection $\mathcal{P}^k(a^k)$ of $\mathscr{C}^k$ onto $\mathscr{C}^k(a^k)$ is a linear combination of morphisms $\mathcal{P}^k_{b^k}$ (this is clear by induction or follows from the Möbius inversion formula, see chapter 8 in [ABG]), hence the tensor product $\mathcal{P}(a) \equiv \otimes_k \mathcal{P}^k(a^k)$ is a well defined continuous map $\mathscr{C} \to \mathscr{C}$ (use Theorem 2.2). It is easy to check that $\mathcal{P}(a)$ is a projection of $\mathscr{C}$ onto $\mathscr{C}(a)$ and that $\mathcal{P}(a)\mathcal{P}(b) = 0$ if $a \neq b$, $\sum_a \mathcal{P}(a) = \mathrm{id}$. The lemma follows easily from these facts. ■

The simplest case $\mathcal{L}^k = \{0,1\}$ with $0 < 1$ is already interesting. To cover the spin-boson model as treated in [Ger] (or, more generally, a boson field with a particle number cutoff coupled with a confined system) one has to take one of the lattices equal to $\{0\}$ and the corresponding algebra equal to $K(\boldsymbol{E})$ for some Hilbert space $\boldsymbol{E}$, but this is a mathematically trivial extension of what follows, so will not be treated. Now we have $\mathscr{C}^k = \mathcal{C}^k + \mathcal{K}^k$, with new notations $\mathcal{C}^k \equiv \mathscr{C}^k(1)$ and $\mathcal{K}^k \equiv \mathscr{C}^k(0)$. Thus $\mathcal{C}^k \subset \mathscr{C}^k$ is a C^*-subalgebra, $\mathcal{K}^k \subset \mathscr{C}^k$ is an ideal, and $\mathcal{C}^k \cap \mathcal{K}^k = \{0\}$. Let $\pi_k : \mathscr{C}^k \to \mathcal{C}^k$ be the natural projection, so π_k is a morphism with $\mathcal{K}^k$ as kernel. We have an obvious identification of $\mathcal{L}$ with the lattice of all subsets of $\mathcal{N}$ with the order relation given by inclusion, so $a \in \mathcal{L}$ means $a \subset \mathcal{N}$.

In order to avoid confusions below we shall change notations and write $\mathscr{C}^{\mathcal{N}} \equiv \otimes_{k\in\mathcal{N}}\mathscr{C}^k$ for the algebra denoted $\mathscr{C}$ above. Then $\mathscr{C}^{\mathcal{N}}(a) = \otimes\mathcal{B}^k$ with $\mathcal{B}^k = \mathcal{C}^k$ if $k \in a$ and $\mathcal{B}^k = \mathcal{K}^k$ if not. Similarly, $\mathscr{C}^{\mathcal{N}}_a = \otimes\mathcal{B}^k$ with $\mathcal{B}^k = \mathcal{C}^k$ if $k \in a$ and $\mathcal{B}^k = \mathscr{C}^k$ if not. Clearly, $\mathcal{P}_a \equiv \pi_a$ is a tensor product of morphisms π_k at places $k \in a$ and identity operators in the remaining places.

The algebra $\mathscr{C}^{\mathcal{N}}$ has a remarkable ideal $\mathcal{K}^{\mathcal{N}} \equiv \mathscr{C}^{\mathcal{N}}(\emptyset) = \otimes_k\mathcal{K}^k$ and the quotient $\mathscr{C}^{\mathcal{N}}/\mathcal{K}^{\mathcal{N}}$ can be easily described with the help of the general results (3.23), (3.24). We clearly have:

Proposition 3.27 *The map $\oplus_k\pi_{\{k\}} : \mathscr{C}^{\mathcal{N}} \to \bigoplus_k \mathscr{C}^{\mathcal{N}}_{\{k\}}$ is a morphism with $\mathcal{K}^{\mathcal{N}}$ as kernel, hence it gives an embedding*

$$\mathscr{C}^{\mathcal{N}}/\mathcal{K}^{\mathcal{N}} \subset \bigoplus_k \mathscr{C}^{\mathcal{N}}_{\{k\}}. \tag{3.25}$$

We now take above $\mathscr{C}^k \equiv \mathscr{C} = \mathcal{C} + \mathcal{K}$ independent of k and denote π the projection morphism $\mathscr{C} \to \mathcal{C}$. Let $\mathcal{N} = \{1,\dots,n\}$. The algebra $\mathscr{C}$ is interpreted as the one particle energy observable algebra. Now we would like to consider a system of n identical particles, hence the corresponding algebra of energy observables has to be the symmetric part of $\mathscr{C}^{\mathcal{N}}$. This will destroy the grading, but the quotient is easy to compute. If σ is a permutation of $\mathcal{N}$ then we denote by the same symbol the automorphism of $\mathscr{C}^{\mathcal{N}}$ defined by the condition $\sigma \otimes_k T_k = \otimes_k T_{\sigma^{-1}(k)}$. We define the *algebra of energy observables of a system of n identical particles* by

$$\mathscr{C}^{\vee n} = \{T \in \mathscr{C}^{\mathcal{N}} \mid \sigma T = T,\ \forall \sigma\}. \tag{3.26}$$

We emphasize two things concerning the physical interpretation of what we are doing. It is meaningless to speak here about bosons because this is an algebra of observables and we did not mention any statistics (or superselection sector). And in fact, it is not easy to take statistics into account[3]. Observe that working with (3.26) is not a loss of generality if one is interested only in proving the absence of singularly continuous spectrum, but the point is that the set of thresholds predicted by it is not the physical one. Of course, the essential spectrum too depends on the statistics. On the other hand, when one applies this formalism in the context of [Ger], the particle we are talking about is a *dead or alive* boson. There are exactly n such particles, but the number of alive bosons could be anything between 0 and n. As we said it before, it is trivial to add a spin.

Proposition 3.28 *There is a unique morphism $\mathcal{P} : \mathscr{C}^{\vee n} \to \mathcal{C} \otimes \mathscr{C}^{\vee(n-1)}$ such that $\mathcal{P}[T^{\otimes n}] = \pi(T) \otimes T^{\otimes(n-1)}$ for all $T \in \mathscr{C}$. One has $\ker \mathcal{P} = \mathcal{K}^{\vee n}$, hence we get an embedding*

$$\mathscr{C}^{\vee n}/\mathcal{K}^{\vee n} \subset \mathcal{C} \otimes \mathscr{C}^{\vee(n-1)}. \tag{3.27}$$

Proof: Uniqueness is obvious and to show the existence it suffices to define $\mathcal{P} = \pi_{\{1\}}|_{\mathscr{C}^{\vee n}}$. It is obvious that $\sigma\pi_a = \pi_{\sigma(a)}\sigma$ for all permutations σ and all subsets a of $\mathcal{N}$. Hence if $\mathcal{P}[T] = 0$ then $\pi_{\{k\}}[T] = 0$ for all k, so by Proposition 3.27 we have $T \in \mathcal{K}^{\mathcal{N}} \cap \mathscr{C}^{\vee n} \equiv \mathcal{K}^{\vee n}$. ∎

[3] This is related to a problem discussed in the Comment below. Reduction to a symmetry sector is a difficult question also in the context of the N-body problem. M. Damak studied it during the preparation of his thesis, but the results have not been published.

Comment by V.G.: I have used Proposition 3.28 in order to prove Theorems 1.1 and 1.2 from [Geo]. Unfortunately, I recently found a gap in the argument leading to Theorem 1.2, which I am still unable to fill. So for the moment I do not know if that theorem is true in the degree of generality stated in [Geo], more precisely I have to put more conditions on the algebra of one particle kinetic energies (a condition on $\mathcal{C}$ similar to that imposed on $\mathfrak{C}$ after Corollary 3.30 below suffices). In applications to standard boson models, these conditions are easy to check. On the other hand, I found a new proof which extends with no difficulty to models without particle number cutoff. Some results in this direction are described in the next subsection.

3.7. Quantum fields. In this section we consider a bosonic quantum field and define the C^*-algebra of hamiltonians in the case when the boson mass is strictly positive. Our main purpose is to explain how one can derive a Mourre estimate from a knowledge of this algebra. We assume known basic facts concerning the positive commutator method in the version of [BG2]; see [Geo] for a summary adapted to the present situation, or §8.3 from [ABG] for a complete presentation. We refer to [DeG] for a proof of the Mourre estimate for the $P(\phi)_2$ model and for the second quantization formalism that we use without further explanation. We denote by $\mathfrak{H}$ the one-particle Hilbert space, $a(u)$ and $a^*(u)$ the annihilation and creation operators of a boson in the state $u \in \mathfrak{H}$, and recall that the field operator is $\phi(u) = (a(u) + a^*(u))/\sqrt{2}$.

The Hilbert space generated by the states of the field is the symmetric Fock space $\Gamma(\mathfrak{H})$. We will proceed as in §3.1 and define $\mathscr{C}$ as a kind of crossed product of an algebra of interactions $\mathscr{A}$ with an algebra of kinetic energies $\mathscr{B}$, more precisely we take $\mathscr{C} = [\![\mathscr{A} \cdot \mathscr{B}]\!]$ (cf. Theorem 2.17). To understand this choice and the next definitions of $\mathscr{A}$ and $\mathscr{B}$, one has to prove a version of Proposition 3.2 with resolvents replaced by exponentials e^{-H} (see the proof of Proposition 3.2 given in [GI2] and the discussion after Theorem 3.31). Note also that in the standard case $\mathfrak{H} = L^2(\mathbb{R}^s)$ our purpose is to study models for which the "elementary" hamiltonians are of the form $d\Gamma(\omega) + W$, where ω is affiliated to $C_0(\mathbb{R}^{s*})$ with $\inf \omega \equiv m > 0$ and W is a polynomial in the field operators with a particle number cut-off. We recall that an important point of our approach is to start with a small class of elementary hamiltonians which, however, should generate a C^*-algebra to which the physically realistic hamiltonians are affiliated (see §3.1).

We define the *algebra of interactions* $\mathscr{A}$ as the C^*-algebra generated by the operators $\phi(u)\Gamma(\lambda)$, where $u \in \mathfrak{H}$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Clearly, $\mathscr{A}$ is also the algebra generated by $\phi(u)\varphi(\mathbf{N})$ with $u \in \mathfrak{H}$ and $\varphi \in C_c(\mathbb{R})$, where $\mathbf{N}$ is the particle number operator. We denote $\mathscr{K}(\mathfrak{H}) = K(\Gamma(\mathfrak{H}))$.

Theorem 3.29 *There is a unique morphism $\mathcal{P}_0 : \mathscr{A} \to \mathscr{A}$ satisfying one of the equivalent conditions:*
(i) $\mathcal{P}_0[\phi(u)\Gamma(\lambda)] = \lambda\phi(u)\Gamma(\lambda)$ *if* $\lambda \in \mathbb{C}$ *and* $|\lambda| < 1$*;*
(ii) $\mathcal{P}_0[\phi(u)\varphi(\mathbf{N})] = \phi(u)\varphi(\mathbf{N}+1)$ *if* $u \in \mathfrak{H}$ *and* $\varphi \in C_c(\mathbb{R})$*.*
One has $\mathscr{K}(\mathfrak{H}) \subset \mathscr{A}$ *and* $\mathcal{P}_0$ *is surjective and has* $\mathscr{K}(\mathfrak{H})$ *as kernel, hence*

$$\mathscr{A}/\mathscr{K}(\mathfrak{H}) \cong \mathscr{A}. \tag{3.28}$$

In particular, for each $T \in \mathscr{A}$ we have $\|\mathcal{P}_0^k[T]\| \to 0$ if $k \to \infty$. Thus

Corollary 3.30 *All the operators in* $\mathscr{A}$ *have a countable spectrum.*

We assume that *the algebra of one-particle kinetic energies is an abelian C^*-algebra $\mathfrak{C}$ of operators on $\mathfrak{H}$ such that the Von Neumann algebra generated by it does not contain non-zero finite rank projections*. Then we define the *algebra of kinetic energies of the field* as the C^*-algebra $\mathscr{B}$ generated by the operators $\Gamma(S)$ with $S \in \mathfrak{C}$, $\|S\| \le 1$. Finally, as we said before, $\mathscr{C} = [\![\mathscr{A} \cdot \mathscr{B}]\!]$ will be the *algebra of hamiltonians of the field*.

There is a shorter but rather cryptic definition of $\mathscr{C}$: this is the C^*-algebra generated by the operators $\phi(u)\Gamma(S)$, where $u \in \mathfrak{H}$ and $S \in \widetilde{\mathfrak{C}}$ with $\|S\| < 1$, where $\widetilde{\mathfrak{C}}$ is the unital algebra generated by $\mathfrak{C}$. The main result is:

Theorem 3.31 *There is a unique morphism $\mathcal{P} : \mathscr{C} \to \mathfrak{C} \otimes \mathscr{C}$ such that $\mathcal{P}[\phi(u)\Gamma(S)] = S \otimes [\phi(u)\Gamma(S)]$ if $u \in \mathfrak{H}$ and $S \in \widetilde{\mathfrak{C}}$ with $\|S\| < 1$. One has $\mathscr{K}(\mathfrak{H}) \subset \mathscr{C}$ and the kernel of $\mathcal{P}$ is $\mathscr{K}(\mathfrak{H})$, which gives a canonical embedding*

$$\widehat{\mathscr{C}} \equiv \mathscr{C}/\mathscr{K}(\mathfrak{H}) \hookrightarrow \mathfrak{C} \otimes \mathscr{C}. \tag{3.29}$$

In the present situation the most convenient affiliation criterion is the following: if H is a self-adjoint bounded from below operator on $\Gamma(\mathfrak{H})$, and if $\mathrm{e}^{-H} \in \mathscr{C}$, then H is affiliated to $\mathscr{C}$. To check it, we use Theorem 2.23. For example, if ω is a self-adjoint operator on $\mathfrak{H}$ affiliated to $\mathfrak{C}$ with $\inf \omega > 0$ and the symmetric operator W is a (generalized) polynomial in the field operators, and if $W_n = \chi_n(\mathbf{N})W\chi_n(\mathbf{N})$ (where $n \in \mathbb{N}$ and χ is the characteristic function of $[0, n]$), then it is easy to see that $\mathrm{e}^{-W_n}\Gamma(\mathrm{e}^{-\omega}) \in \mathscr{C}$ and $\mathcal{P}\left[\mathrm{e}^{-W_n}\Gamma(\mathrm{e}^{-\omega})\right] = \mathrm{e}^{-\omega} \otimes \left[\mathrm{e}^{-W_{n-1}}\Gamma(\mathrm{e}^{-\omega})\right]$. Then Theorem 2.23 shows that

$H(n) = d\Gamma(\omega) + W_n$ is affiliated to $\mathscr{C}$ and $\mathcal{P}\left[\mathrm{e}^{-H(n)}\right] = \mathrm{e}^{-\omega} \otimes \mathrm{e}^{-H(n-1)}$. If there is a self-adjoint operator H such that $\mathrm{e}^{-H(n)} \to \mathrm{e}^{-H}$ in norm as $n \to \infty$, we get that H is affiliated to $\mathscr{C}$ and $\widehat{H} = \omega \otimes 1 + 1 \otimes H$. Taking limits in norm resolvent sense (which preserves affiliation) one gets a much larger class of hamiltonians affiliated to $\mathscr{C}$. In this way one can prove, for example, that the hamiltonian of the $P(\phi)_2$ model ($s = 1$) with a spatial cut-off is affiliated to $\mathscr{C}$ (we thank Christian Gérard for help in this context).

We come now to the question of the Mourre estimate for a hamiltonian H affiliated to $\mathscr{C}$. Now we assume $\mathfrak{H} = L^2(\mathbb{R}^s)$ and $\mathfrak{C} = C_0(\mathbb{R}^{s*})$. We consider only conjugate operators of the form $A = d\Gamma(\mathfrak{a})$, where $\mathfrak{a} = F(P)Q + QF(P)$ and F is a vector field of class C_c^∞; such an A will be called *standard*. A self-adjoint operator on $\Gamma(\mathfrak{H})$ which is of class $C_\mathrm{u}^1(A)$ or $C^{1,1}(A)$ for each standard A will be called of class C_u^1 or $C^{1,1}$, respectively.

Theorem 3.32 *Let H be a bounded from below hamiltonian strictly affiliated to $\mathscr{C}$ and such that $\widehat{H} = \omega(P) \otimes 1 + 1 \otimes H$, where $\omega : \mathbb{R}^s \to \mathbb{R}$ (the one-particle kinetic energy) is a function of class C^1, $\inf \omega \equiv m > 0$, and $\omega(p) \to \infty$ if $p \to \infty$. Then $\boldsymbol{\sigma}_\mathrm{ess}(H) = [m + \inf H, \infty)$. Assume that H is of class C_u^1. Denote $\boldsymbol{\kappa}(\omega)$ the set of critical values of the function ω, let $\boldsymbol{\kappa}_n(\omega) = \boldsymbol{\kappa}(\omega) + \cdots + \boldsymbol{\kappa}(\omega)$ (n terms), and define the threshold set of H by*

$$\boldsymbol{\tau}(H) = \textstyle\bigcup_{n=1}^\infty \left[\boldsymbol{\kappa}_n(\omega) + \boldsymbol{\sigma}_\mathrm{p}(H)\right] = \left[\bigcup_{n=1}^\infty \boldsymbol{\kappa}_n(\omega)\right] + \boldsymbol{\sigma}_\mathrm{p}(H) \tag{3.30}$$

where $\boldsymbol{\sigma}_\mathrm{p}(H)$ is the set of eigenvalues of H. Then $\boldsymbol{\tau}(H)$ is a closed set and H admits a standard local conjugate operator at each point not in $\boldsymbol{\tau}(H)$. In particular, the eigenvalues of H which do not belong to $\boldsymbol{\tau}(H)$ are of finite multiplicity and their accumulation points belong to $\boldsymbol{\tau}(H)$. If H is of class $C^{1,1}$, then it has no singular continuous spectrum outside $\boldsymbol{\tau}(H)$. If we also assume that $\boldsymbol{\kappa}(\omega)$ is countable, then $\boldsymbol{\tau}(H)$ is countable too, so H has no singular continuous spectrum.

The preceding result is a rather straightforward consequence of Theorem 3.31, as explained in [Geo]. We note only that the threshold and critical set defined by a standard conjugate operator suggest to consider the set $\boldsymbol{\tau}(H)$ satisfying the relation

$$\boldsymbol{\tau}(H) = \boldsymbol{\kappa}(\omega) + \left[\boldsymbol{\tau}(H) \cup \boldsymbol{\sigma}_\mathrm{p}(H)\right] = \left[\boldsymbol{\kappa}(\omega) + \boldsymbol{\tau}(H)\right] \cup \left[\boldsymbol{\kappa}(\omega) + \boldsymbol{\sigma}_\mathrm{p}(H)\right].$$

where $\boldsymbol{\kappa}(H) = \boldsymbol{\tau}(H) \cup \boldsymbol{\sigma}_\mathrm{p}(H)$. The unique solution is given by (3.30).

Observe that the strict positivity condition $m > 0$ plays an important role above. This is no more necessary if we consider hamiltonians with a particle number cut-off, as in [Geo]. Indeed, if H is given by a formal expression $H = d\Gamma(\omega) + W$, the restrictions $H_n = \chi_n(\mathbf{N}) H \chi_n(\mathbf{N})$ are often well defined self-adjoint operators and they satisfy $\widehat{H_n} = \omega \otimes 1 + 1 \otimes H_{n-1}$. Then the threshold set of H_n is defined by the relation (with $\boldsymbol{\sigma}_{\mathrm{p}}(H_0) = \{0\}$):

$$\boldsymbol{\tau}(H_n) = \bigcup_{i=1}^{n} \left[\boldsymbol{\kappa}_i(\omega) + \boldsymbol{\sigma}_{\mathrm{p}}(H_{n-i})\right].$$

This is the solution of the recursive relation (3.5) from [Geo].

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