EXISTENCE OF THE DENSITY OF STATES FOR ONE-DIMENSIONAL ALLOY-TYPE POTENTIALS WITH SMALL SUPPORT

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Dedicated to the Memory of G. A. Mezincescu **1943—2001**

Abstract. We study spectral properties of Schrödinger operators with a random potential of alloy type on $L^2(\mathbb{R})$ and their restrictions to finite intervals. A Wegner estimates for non-negative single site potentials with small support is proven. It implies the existence and local uniform boundedness of the density of states. Our estimate is valid for all bounded energy intervals. Wegner estimates play a key role in an existence proof of pure point spectrum.

1. Model and results

We study spectral properties of families of Schrödinger operators on $L^2(\mathbb{R})$. The considered operators consist of a non-random periodic Schrödinger operator plus a random potential of *Anderson* or *alloy type*:

$$H_\omega := H_0 + V_\omega, \quad H_0 := -\Delta + V_{per}. \tag{1}$$

Here Δ is the Laplace operator on $\mathbb{R}$ and $V_{per} \in L^\infty(\mathbb{R})$ is a $\mathbb{Z}$-periodic potential. The random potential V_ω is a stochastic process of the following form

$$V_\omega(x) = \sum_{k\in\mathbb{Z}} \omega_k\, u(x-k). \tag{2}$$

where $\{\omega_k\}_{k\in\mathbb{Z}}$ is a collection of independent identically distributed random variables, called *coupling constants*. Their distribution has a bounded density f with support equal to a bounded interval. The non-negative *single site potential* $u \in L^\infty(\mathbb{R})$ has compact support and an uniform positive lower bound on some open subset of $\mathbb{R}$. The potential $V := V_{per} + V_\omega$ is (uniformly in ω) bounded, hence H_ω is a selfadjoint, lower semibounded operator on the Sobolev space $W^{2,2}(\mathbb{R})$. For any interval $\Lambda_l := \Lambda_l(x) := [-l/2, l/2] + x$ we can restrict H_ω to $L^2(\Lambda_l)$ with Dirichlet boundary conditions. We denote the restriction by H_ω^l. It is again selfadjoint and lower semibounded and has discrete spectrum.

For $\mu(A) := \int_A f(x)dx$ define the product measure $\mathbb{P} := \otimes_{k\in\mathbb{Z}}\, \mu$. We consider the collection $\{\omega_k\}_{k\in\mathbb{Z}}$ as an element of the probability space $(\Omega = \times_{k\in\mathbb{Z}}\mathbb{R}, \mathbb{P})$. The expectation w.r.t. $\mathbb{P}$ is denoted by $\mathbb{E}$.

By the general theory of ergodic random Schrödinger operators [3, 16] we know that there exist a ω-independent set Σ such that $\sigma(H_\omega) = \Sigma$ for $\mathbb{P}$-almost all ω. In the same way the spectral components σ_{ac}, σ_{sc} and σ_{pp} are non-random subsets of

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the real line. Moreover, there exist non-random distribution function N called the *integrated density of states* (IDS) which can be obtained by a macroscopic limit: Denote by

$$N^l_\omega(E) = l^{-d}\#\{i|\ \lambda_i(H^l_\omega) < E\} = l^{-d}\,\mathrm{Tr}\,P^l_\omega(]-\infty, E[) \tag{3}$$

the *finite volume IDS* or *normalized eigenvalue counting function* of H^l_ω. Here $P^l_\omega(]-\infty, E[)$ denotes the spectral projection of H^l_ω on the energy interval $]-\infty, E[$. Then for all continuity points E of N:

$$N(E) = \lim_{l\to\infty} N^l_\omega(E) \quad \mathbb{P}\text{-almost surely}\ . \tag{4}$$

Our main result reads:

Theorem 1 (Wegner estimate)**.** *Let H_ω be as above. For any $E \in \mathbb{R}$ there exist a constant C such that*

$$\mathbb{E}\left[\mathrm{Tr}\,P^l_\omega([E-\epsilon, E])\right] \le C\,\epsilon\,l, \quad \forall\,\epsilon \ge 0. \tag{5}$$

Estimates of this type on the expectation value of the number of eigenvalues in a given energy interval go back to Wegner's paper [23] where he considers the discrete analog of H_ω on $l^2(\mathbb{Z}^d)$. Theorem 1 proves the Lipschitz-continuity of the averaged finite volume IDS. So this function has a derivative at almost all energy values $E \in \mathbb{R}$. Using (4) this implies

Corollary 2. *Under the assumptions of Theorem 1 the IDS is Lipschitz continuous. Thus its derivative, the* density of states dN/dE *exists for almost all $E \in \mathbb{R}$ and it is uniformly bounded on any interval $]-\infty, E]$.*

By the Čebyšev inequality (5) implies

$$\mathbb{P}\left\{\omega\in\Omega|\,\sigma(H^l_\omega)\cap[E-\epsilon, E] \neq \emptyset\right\} \le C\epsilon\ l, \quad \forall\,\epsilon \ge 0. \tag{6}$$

For the application to the proof of localization only this form of the estimate is needed.

Theorem 3 (Localization)**.** *Let H_ω be as in Theorem 1 and $E \in \partial\sigma(H_\omega)$ any lower spectral edge. Then there exists an $\epsilon > 0$ such that H_ω has $\mathbb{P}$-almost surely pure point spectrum in $[E, E+\epsilon]$ with exponentially decaying eigenfunctions.*

We thus recover a result from [14], whose proof uses different methods. In the present note we prove only Theorem 1, while the proof of Theorem 3 can be found in the Diploma thesis [21]. It uses the multiscale analysis of Fröhlich and Spencer [6] in the version of von Dreifus and Klein [22]. Apart from the Wegner estimate (5), a key ingredient is the Lifshitz asymptotics of the IDS derived by Mezincescu in [15] as well as a finite–infinite volume comparison lemma for the IDS [15, Lem. 4.2]. A detailed discussion of localization can be found e.g. in [18] and the literature cited there. For one-dimensional random operators there are several other techniques at disposal to prove localization, cf. e.g [8, 3, 16, 2] and the references therein.

Remark 4. By assumption we have some open set O and $\kappa > 0$ such that $u \ge \kappa\chi_O$. For some $s > 0$ the set $O \subset \mathbb{R}$ contains a cube Λ_s of sidelength s. By shifting the origin of $\mathbb{R}$ we can therefore assume that Λ_s has its center at 0 and thus $u \ge \kappa\chi_{\Lambda_s(0)}$. Moreover, by rescaling f and u we assume $\kappa = 1$. Note that by adding a part of the periodic potential to V_ω we may assume without loss of generality that the support of f starts at 0, i.e. $\mathrm{supp}\, f = [0, \omega_+]$ for some $\omega_+ > 0$.

Recently there has been increased interest in Wegner estimates for single site potentials u that change sign [13, 19, 20, 9]. However even for nonnegative u with small support the situation is not clearly understood. By "small support" we mean that

$$\sum_{k\in\mathbb{Z}} u(x-k) \tag{7}$$

is not bounded away from zero by a positive constant. Such potentials in arbitrary space dimension are considered e.g. in [13, 10, 1, 12, 5]. However, the derived Wegner estimates are valid only at spectral boundaries.[1]

Let us finish this section by mentioning that so far there is no Wegner estimate proven without the uniform positivity of $|u|$ on some open set. Namely, nothing is known in the case

$$0 \le u \in L^2_c(\mathbb{R}), \quad |\{x| \, u(x) > 0\}| > 0 \tag{8}$$

where $|\cdot|$ denotes Lebesgue measure. Property (8) implies the existence of some $\kappa > 0$ such that $U_\kappa := \{x| \, u(x) > \kappa\}$ has positive measure. However U_κ need not contain an open set.

2. Proof of Theorem 1

Let us denote with $\rho : \mathbb{R} \to [0,1]$ a smooth, monotone function with $\rho = 1$ on $[\epsilon, \infty[$ and $\rho = 0$ on $]-\infty, -\epsilon]$. The n-th eigenvalue of the operator H^l_ω is denoted by $E^l_n(\omega)$. We estimate similarly as in [10, page 509]

$$\mathbb{E}\left[\mathrm{Tr}\, P^l_\omega([E-\epsilon, E+\epsilon])\right] \le \int \dots \int \prod_{k\in\Lambda^+} f(\omega_k)\, d\omega_k \sum_{n\in\mathbb{N}} \int_{-2\epsilon}^{2\epsilon} d\theta \rho'(E^l_n(\omega) - E + \theta).$$

Here $\Lambda^+ := \{k \in \mathbb{Z}| \text{ supp } u(\cdot - k) \text{ intersects } \Lambda_l\}$ while $\tilde\Lambda := \mathbb{Z} \cap \Lambda_l$. The above line can be bounded using the estimates from Section 3 by

$$C_1 \int \dots \int \prod_{k\in\Lambda^+} f(\omega_k)\, d\omega_k \sum_{n\in\mathbb{N}} \int_{-2\epsilon}^{2\epsilon} d\theta \sum_{j\in\tilde\Lambda} \frac{\partial \rho(E^l_n(\omega) - E + \theta)}{\partial \omega_j} \tag{9}$$

$$= C_1 \sum_{j\in\tilde\Lambda} \int \dots \int \prod_{k\in\Lambda^+\setminus j} f(\omega_k)\, d\omega_k \sum_{n\in\mathbb{N}} \int_{-2\epsilon}^{2\epsilon} d\theta \int d\omega_j\, f(\omega_j) \frac{\partial \rho(E^l_n(\omega) - E + \theta)}{\partial \omega_j}.$$

using Beppo Levi's theorem. Denote now by $H^l_\omega(j, \omega_+)$ the operator H^l_ω where the random variable ω_j has been set to its maximum value ω_+, and similarly $H^l_\omega(j,0)$. The eigenvalues of the two above operators are abbreviated by $E^l_n(\omega, \omega_j = \omega_+)$, resp. $E^l_n(\omega, \omega_j = \omega_+)$. Using monotonicity we estimate from above the sum over n in the second line of (9) by

$$\|f\|_\infty \sum_{n\in\mathbb{N}} \int_{-2\epsilon}^{2\epsilon} d\theta \left[\rho(E^l_n(\omega, \omega_j = \omega_+) - E + \theta) - \rho(E^l_n(\omega, \omega_j = 0) - E + \theta)\right] \tag{10}$$

$$\le \|f\|_\infty \int_{-2\epsilon}^{2\epsilon} d\theta\, \mathrm{Tr}\left[\chi_{[E-3\epsilon,\infty[}(H^l_\omega(j,\omega_+)) - \chi_{]E+3\epsilon,\infty[}(H^l_\omega(j,0))\right]$$

[1] See however the recent [4].

Let the single site potential u be supported in $[-R, R]$. Now one introduces b.c. at the points $j-R$ and $j+R$ and uses Dirichlet-Neumann bracketing.

$$\begin{aligned}\operatorname{Tr}&\left[\chi_{[E-3\epsilon,\infty[}(H^l_\omega(j,\omega_+)) - \chi_{]E+3\epsilon,\infty[}(H^l_\omega(j,0))\right]\\ &\qquad\le \operatorname{Tr}\left[\chi_{[E-3\epsilon,\infty[}(H^{l,D}_\omega(j,\omega_+)) - \chi_{]E+3\epsilon,\infty[}(H^{l,N}_\omega(j,0))\right]\end{aligned}$$

The superscripts D and N denote whether the operator has Dirichlet, respectively Neumann b.c. at $\{j-R, j+R\}$. Since b.c. of one-dimensional operators are finite rank perturbations in resolvent sense, see e.g. [17], we infer

$$\begin{aligned}\operatorname{Tr}&\left[\chi_{[E-3\epsilon,\infty[}(H^{l,D}_\omega(j,\omega_+)) - \chi_{]E+3\epsilon,\infty[}(H^{l,N}_\omega(j,0))\right]\\ &\le 4 + \operatorname{Tr}\left[\chi_{[E-3\epsilon,\infty[}(H^{l,D}_\omega(j,0)+\|u_j\|_\infty) - \chi_{]E+3\epsilon,\infty[}(H^{l,D}_\omega(j,0))\right] \le C,\end{aligned}$$

where the constant depends only on $E+3\epsilon, V_{per}, \omega_+$ and u, cf. [11]. The proof of Theorem 1 is finished by the the upper bound on (9):

$$C_1C_2\|f\|_\infty \sum_{j\in\tilde\Lambda}\int\dots\int\prod_{k\in\Lambda^+\setminus j} f(\omega_k)\, d\omega_k\, \epsilon\, l = C_1C_2\|f\|_\infty\, \epsilon\, l.$$

3. Single site potentials of small support

In this section we prove the uniform lower bound

$$\sum_{k\in\tilde\Lambda}\frac{\partial E^l_n(\omega)}{\partial\omega_k} \ge C_4(I) > 0 \tag{11}$$

for all eigenvalues E^l_n of H^l_ω inside a bounded energy interval I. The bound $C(I)$ does not depend on the sidelength $l\in\mathbb{N}$ and on the eigenvalue index $n\in\mathbb{N}$. By the chain rule

$$\sum_{j\in\tilde\Lambda_l}\frac{\partial\rho(E^l_n(\omega)-E+\theta)}{\partial\omega_k} = \rho'(E^l_n(\omega)-E+\theta)\sum_{k\in\tilde\Lambda_l}\frac{\partial E^l_n(\omega)}{\partial\omega_k}$$

(11) implies the estimate needed in Section 2:

$$\rho'(E^l_n(\omega)-E+\theta) \le C_4(I)^{-1}\sum_{j\in\tilde\Lambda}\frac{\partial\rho(E^l_n(\omega)-E+\theta)}{\partial\omega_j}.$$

To infer the lower bound (11) set $S=\bigcup_{k\in\tilde\Lambda}(\Lambda_s(k))$ and apply the Hellman-Feynman theorem. For a normalized eigenfunction ψ_n corresponding to $E^l_n(\omega)$:

$$\sum_{k\in\tilde\Lambda}\frac{\partial E^l_n(\omega)}{\partial\omega_k} = \sum_{k\in\tilde\Lambda}\langle\psi_n, u(\cdot-k)\psi_n\rangle \ge \kappa\int_S|\psi_n|^2.$$

If the integral on the rightern side would extend over the whole of Λ_l it would be equal to 1 due to the normalization of ψ_n. A priori the integral over S could be much smaller, but the following Lemma shows that this is not the case.

Lemma 5. *Let I be a bounded interval and $s>0$. There exists a constant $c>0$ such that*

$$\int_{\Lambda_s(k)}|\psi|^2 \ge c\int_{\Lambda_1(k)}|\psi|^2$$

for all $l\in\mathbb{N}$, all $k\in\Lambda_l$ and for any eigenfunction ψ corresponding to an eigenvalue $E\in I$ of H^l_ω.

Thus $\int_S |\psi|^2 \geq c \int_{\Lambda_l} |\psi|^2$ with the same constant as in Lemma 5.

Proof. For

$$\phi(y) := \int_{\Lambda_s(k+y)} dx\, |\psi(x)|^2 = \int_{\Lambda_s(k)} dx\, |\psi(x-y)|^2 \tag{12}$$

one has

$$\begin{aligned} \left|\frac{\partial}{\partial y}\phi(y)\right| &= \left|\int_{\Lambda_s(k)} dx\, \left[\frac{\partial}{\partial y}\psi(x-y)\right] \overline{\psi(x-y)} + \int_{\Lambda_s(k)} dx\, \psi(x-y)\, \frac{\partial}{\partial y}\overline{\psi(x-y)}\right| \\ &\leq 2\, \|\psi\|_{L^2(\Lambda_s(k+y))}\, \|\psi'\|_{L^2(\Lambda_s(k+y))}\,. \end{aligned}$$

Sobolev norm estimates (e.g. Theorems 7.25 and 7.27 in [7]) imply

$$\|\psi'\|_{L^2(\Lambda_s(k+y))} \leq C_5\, \|\psi\|_{L^2(\Lambda_s(k+y))} + \|\psi''\|_{L^2(\Lambda_s(k+y))}.$$

By the eigenvalue equation we have

$$\left|\frac{\partial}{\partial y}\phi(y)\right| \leq C_6\, \|\psi\|^2_{L^2(\Lambda_s(k+y))} = C_6\, \phi(y), \qquad C_6 = C_6(\|V-E\|_\infty). \tag{13}$$

Gronwall's Lemma implies $\phi(y) \leq \exp(C_6 |y|)\, \phi(0)$ and thus

$$\int_{\Lambda_1(k)} |\psi|^2 \leq e^{C_6}\ s^{-1} \int_{\Lambda_s(k)} |\psi|^2.$$

□

At the Conference in Taxco we learned from J.M. Combes that there is a article in preparation together with P. Hislop and F. Klopp with slightly weaker results as in this work applying also to the higher dimensional case [4].

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