COMMENTS ON THE TROTTER PRODUCT FORMULA ERROR-BOUND ESTIMATES FOR NONSELF-ADJOINT SEMIGROUPS

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Let A be a positive self-adjoint operator and let B be an m-accretive operator which is A-small with a relative bound less than one. Let $H = A + B$, then H is well-defined on $\mathrm{dom}(H) = \mathrm{dom}(A)$ and m-accretive. If B is a strictly m-accretive operator obeying

$$\mathrm{dom}((H^*)^\alpha) \subseteq \mathrm{dom}(A^\alpha) \cap \mathrm{dom}((B^*)^\alpha) \neq \{0\} \quad \text{for some} \quad \alpha \in (0,1], \tag{1}$$

then for the Trotter product formula we prove that

$$\left\| \left(e^{-tB/n} e^{-tA/n}\right)^n - e^{-tH} \right\| \wedge \left\| \left(e^{-tA/n} e^{-tB/n}\right)^n - e^{-tH} \right\| = O(\ln n / n^\alpha) \tag{2}$$

(and similar for H^*) as $n \to \infty$, uniformly in $t \geq 0$. We also show that:
(a) the A-smallness of B guarantees the condition (1) for $\alpha \in (0, 1/2)$, i.e. the estimate (2) holds for $\alpha \in (0, 1/2)$;
(b) if B is strictly m-sectorial, then there are sufficient conditions ensuring the relation (1) for $\alpha = 1/2$, that implies (2);
(c) if B is A-small, m-sectorial and such that $\mathrm{dom}(A^{1/2})$ is a subset of the form-domain of B, then again (2) is valid for $\alpha = 1/2$.

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1 Introduction

In our recent paper [2] we have shown the following : Let A be a densely defined self-adjoint operator $A = A^* \geq I$ in a separable Hilbert space $\mathfrak{H}$ and let B a densely defined m-accretive operator B in $\mathfrak{H}$ such that

$$\mathrm{dom}(A) \subseteq \mathrm{dom}(B) \tag{1.1}$$

and

$$\|Bf\| \leq a\|Af\|, \quad f \in \mathrm{dom}(A), \quad \text{for some} \quad a \in (0,1). \tag{1.2}$$

If the adjoint operator B^* satisfies conditions

$$\mathrm{dom}(A) \subseteq \mathrm{dom}(B^*) \tag{1.3}$$

and

$$\|B^* f\| \leq a_* \|Af\|, \quad f \in \mathrm{dom}(A), \quad \text{for some} \quad a_* \in (0,1), \tag{1.4}$$

then the Trotter product formula convergence rate is of the order $O(\ln(n)/n)$. It turns out that the proof crucially depends on conditions (1.3) and (1.4) and, consequently, on the domain $\mathrm{dom}(B^*)$. The aim of the present comments is to show that error-bound estimates can be found even when the conditions (1.3) and (1.4) are not satisfied.

To this end we first construct in Section 2 an example of a self-adjoint operator A and an m-accretive operator B such that conditions (1.1) and (1.2) are satisfied but

$$\mathrm{dom}(A) \cap \mathrm{dom}(B^*) = \{0\}, \tag{1.5}$$

which demonstrates that condition (1.3) is not a consequence of (1.1) and (1.2). In [2] it is proved that under conditions (1.1) and (1.2) $H = A + B$ and $H^* = (A+B)^*$ are well-defined m-accretive operators, but $H^* \neq A + B^*$ in the case of (1.5). This is the reason why the techniques used in [2] should be refined if one would like to relax the conditions (1.3),(1.4) and to indicate operator-norm error bounds à la [2] for the Trotter product formula.

Therefore, in Section 3 we look for a condition which is weaker than (1.3), but which is enough to provide error bound estimates for the Trotter product formula. The *intermediate* condition we propose in the present paper is :

$$\mathrm{dom}((H^*)^\alpha) \subseteq \mathrm{dom}(A^\alpha) \cap \mathrm{dom}((B^*)^\alpha) \neq \{0\}, \quad \text{for some} \quad \alpha \in (0,1]. \tag{1.6}$$

Notice that conditions (1.3) and (1.4) imply (1.6) for $\alpha = 1$. Assuming (1.1), (1.2) and (1.6) for a *strictly* m-accretive operator B we prove in Section 3 that the estimates

$$\left\| \left(e^{-tB/n} e^{-tA/n}\right)^n - e^{-tH} \right\| \leq L_\alpha \frac{\ln n}{n^\alpha}, \qquad n = 3, 4, \ldots, \tag{1.7}$$

and

$$\left\| \left(e^{-tA/n} e^{-tB/n}\right)^n - e^{-tH} \right\| \leq L_\alpha' \frac{\ln n}{n^\alpha}, \qquad n = 3, 4, \ldots, \tag{1.8}$$

hold uniformly in $t \geq 0$ for the corresponding $\alpha \in (0,1)$. Similarly, these estimates are valid for adjoint semigroup generated by $-H^*$, as well as for the symmetrized Trotter product formulae.

In Section 4 we present some applications of the condition (1.6) to the case of m-accretive and m-sectorial operators B. They are related to the famous *Kato's square-root problem* [6, 7] and to the *Miyazaki condition* [10]. First, we prove that our *minimal* conditions (1.1), (1.2) for the m-accretive B imply the *intermediate* condition (1.6) for any $\alpha \in (0, 1/2)$. Consequently, one gets error-bound estimates (1.7),(1.8) for those α's with appropriate L_α, L_α'. Second, we show that Miyazaki's condition (see [10, Theorem 1, (2.1)]) ensures (1.6) for $\alpha = 1/2$, if B is m-sectorial, verifying (1.1), (1.2). In Section 5 we show that to prove (1.7), (1.8) for $\alpha = 1/2$ one can relax the above conditions to: (1.1), (1.2), B is m-sectorial and $\mathrm{dom}(a) \subseteq \mathrm{dom}(b)$. Here a and b are quadratic forms associated to operators A and B. In the last Section 6 we make some remarks and comments. Finally, we note that below we essentially follow terminology of the Kato book [9].

2 Example

The aim of this section is to construct a pair of operators A and B, where A is a positive self-adjoint obeying $A \geq I$ and B is *m-accretive* (i.e. a closed operator with $\|(B+\lambda)^{-1}\| \leq (\mathfrak{Re}\lambda)^{-1}$, for $\mathfrak{Re}\lambda > 0$, see [9], Ch.V), such that in spite of conditions (1.1), (1.2) one has (1.5).

We start by some general remarks. Recall that a closed operator K is m-accretive, if

$$\mathfrak{Re}\,(Kf,f) \geq 0, \quad f \in \mathrm{dom}(K), \tag{2.1}$$

and $\mathrm{ran}(\lambda + K) = \mathfrak{H}$ for $\mathfrak{Re}(\lambda) > 0$, see [9] Ch.V, i.e. the operator $I + K$ is boundedly invertible. The *Cayley transform* of K, see [3], is defined by

$$T := (I - K)(I + K)^{-1}. \tag{2.2}$$

If K is m-accretive, then T is obviously a contraction with $\mathrm{dom}(T) = \mathfrak{H}$. Moreover, one has the following lemma.

Lemma 2.1 *A contraction T is the Cayley transform of an m-accretive operator K if and only if*

$$\ker(I + T) = \{0\}. \tag{2.3}$$

Moreover, the m-accretive operator K corresponding to T is given by

$$\mathrm{dom}(K) = \mathrm{ran}(I + T) \tag{2.4}$$

and

$$Kf = (I - T)(I + T)^{-1}f, \quad f \in \mathrm{ran}(I + T). \tag{2.5}$$

The map $T \longrightarrow K$ is bijective.

For the proof see e.g. [3].

Let R and S be non-negative self-adjoint operators such that

$$\mathrm{dom}(R) \cap \mathrm{dom}(S) = \{0\}. \tag{2.6}$$

We set

$$T := (I + R)^{-1}(I + S)^{-1}. \tag{2.7}$$

The operator T is a contraction. Let us verify for it the condition (2.3). If we choose $f = (I+S)g$, $g \in \mathrm{dom}(S)$, then the condition (2.3) transforms into

$$(I+S)g + (I+R)^{-1}g = 0. \tag{2.8}$$

Since $S \geq 0$ and $(I+R)^{-1} \geq 0$ we get $g = 0$ which yields $f = 0$. Consequently, by Lemma 2.1 the contraction T defined by (2.7) is the *Cayley transform* of an m-accretive operator K which is given by (2.4) and (2.5), and one obviously has :

$$\begin{aligned}\mathrm{dom}(K) = \mathrm{ran}(I+T) = \\ \mathrm{ran}((I+R)^{-1}(I+R+(I+S)^{-1})) = \mathrm{ran}((I+R)^{-1}) = \mathrm{dom}(R).\end{aligned} \tag{2.9}$$

The *Cayley transform* of the m-accretive operator K^* is T^*. Applying again Lemma 2.1 we find

$$\begin{aligned}\mathrm{dom}(K^*) = \mathrm{ran}(I+T^*) = \\ \mathrm{ran}((I+S)^{-1}(I+S+(I+R)^{-1})) = \mathrm{ran}((I+S)^{-1}) = \mathrm{dom}(S).\end{aligned} \tag{2.10}$$

So we have constructed an m-accretive operator K such that $\mathrm{dom}(K) = \mathrm{dom}(R)$ and $\mathrm{dom}(K^*) = \mathrm{dom}(S)$. By condition (2.6) we find

$$\mathrm{dom}(K) \cap \mathrm{dom}(K^*) = \{0\}. \tag{2.11}$$

Now we are going to construct the pair A and B. To this end we set

$$B := aK, \quad \text{for some} \quad a \in (0,1), \tag{2.12}$$

and

$$A := \sqrt{I + K^*K}. \tag{2.13}$$

The operator B is obviously m-accretive with

$$\mathrm{dom}(B) = \mathrm{dom}(K), \tag{2.14}$$

and A is a self-adjoint operator obeying $A \geq I$ and

$$\mathrm{dom}(A) = \mathrm{dom}(\sqrt{K^*K}) = \mathrm{dom}(|K|) = \mathrm{dom}(K). \tag{2.15}$$

Hence $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$. Moreover, we have

$$\|Bf\| = a\|Kf\| \leq a\|\sqrt{I+K^*K}f\| = a\|Af\|, \quad f \in \mathrm{dom}(A). \tag{2.16}$$

Hence $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$. Since $B^* = aK^*$, we finally obtain that

$$\mathrm{dom}(B^*) \cap \mathrm{dom}(A) = \mathrm{dom}(K^*) \cap \mathrm{dom}(K) = \{0\}. \tag{2.17}$$

To finish the construction of our example it remains only to indicate unbounded non-negative self-adjoint operators R and S with the property (2.6). Let $\mathfrak{H} = L^2([0,1])$ and let R be one-dimensional Laplace operator on the interval $[0,1]$:

$$\begin{aligned} \mathrm{dom}(R) &:= \{f \in W^{2,2}([0,1]) : f(0) = f(1) = 0\}, &(2.18)\\ (Rf)(x) &:= -\frac{d^2}{dx^2}f(x), \quad f \in \mathrm{dom}(R). &(2.19) \end{aligned}$$

Then the operator R is self-adjoint and non-negative. Now let $\{r_n\}_{n=1}^{\infty}$ be set of rational numbers of the open interval $(0,1)$. To define S we set

$$q(x) = \sum_{n=1}^{\infty} \frac{c_n}{|x-r_n|^{\alpha}}, \quad \alpha \in (\frac{1}{2},1), \quad c_n > 0, \tag{2.20}$$

such that

$$\sum_{n=1}^{\infty} c_n < \infty. \tag{2.21}$$

Then $q \in L^1([0,1])$. Indeed,

$$\begin{aligned} &\int_0^1 q(x)dx \le \sum_{n=1}^{\infty} c_n \int_0^1 \frac{1}{|x-r_n|^{\alpha}}dx \le &(2.22)\\ &\sum_{n=1}^{\infty} c_n \int_{-2}^{2} \frac{1}{|x|^{\alpha}}dx = \sum_{n=1}^{\infty} 2c_n \int_0^2 \frac{1}{|x|^{\alpha}}dx = \frac{2^{2-\alpha}}{1-\alpha}\sum_{n=1}^{\infty} c_n < +\infty. \end{aligned}$$

We define

$$\begin{aligned} \mathrm{dom}(S) &:= \{f \in L^2([0,1]) : q(x)f(x) \in L^2([0,1])\}, &(2.23)\\ (Sf)(x) &:= q(x)f(x), \quad f \in \mathrm{dom}(S). &(2.24) \end{aligned}$$

The operator S is unbounded, non-negative and self-adjoint.

To verify (2.6) assume that

$$f \in \mathrm{dom}(R) \cap \mathrm{dom}(S) \subseteq W^{2,2,}([0,1]) \cap \mathrm{dom}(S). \tag{2.25}$$

By virtue of (2.25) the function $f(\cdot)$ is continuous and such that

$$q(x)f(x) = \sum_{n=1}^{\infty} \frac{c_n}{|x-r_n|^{\alpha}} f(x) \in L^2([0,1]). \tag{2.26}$$

Then we get

$$\int_0^1 |q(x)f(x)|^2 dx \ge c_n^2 \int_0^1 \frac{1}{|x-r_n|^{2\alpha}}|f(x)|^2 dx \tag{2.27}$$

for any $n = 1, 2, \ldots$. Since $f(\cdot)$ is continuous, function $|f(\cdot)|$ is also continuous. Assume that $f(r_n) \neq 0$. Therefore, there is a sufficiently small neighbourhood $(r_n - \epsilon, r_n + \epsilon) \subset (0, 1)$, $\epsilon > 0$, of r_n and a $\delta > 0$ such that $|f(x)| \geq \delta$ for $x \in (r_n - \epsilon, r_n + \epsilon)$. Hence

$$\int_0^1 |q(x)f(x)|^2 dx \geq c_n^2 \delta^2 \int_{r_n - \epsilon}^{r_n + \epsilon} \frac{1}{|x - r_n|^{2\alpha}} dx = +\infty. \tag{2.28}$$

But this is impossible by (2.26). Therefore, one gets $f(r_n) = 0$, $n = 1, 2, \ldots$. Since $f(\cdot)$ is continuous this yields $f \equiv 0$. Hence $\mathrm{dom}(R) \cap \mathrm{dom}(S) = \{0\}$.

So, we have in the Hilbert space $\mathfrak{H} = L^2([0, 1])$ the self-adjoint operator A (see (2.9),(2.16),(2.17)) and the m-accretive operator B (see (2.9),(2.12),(2.17)) such that

$$\mathrm{dom}(A) \subseteq W^{2,2}([0, 1]), \tag{2.29}$$

but

$$\mathrm{dom}(B^*) \cap W^{2,2}([0, 1]) = \{0\}. \tag{2.30}$$

This finishes construction of desired example.

3 Intermediate condition

The example of the previous section shows that the conditions (1.1) and (1.2) do not guarantee that

$$\mathrm{dom}(H^*) \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(B^*) \neq \{0\}, \tag{3.1}$$

where

$$Hf := Af + Bf, \quad f \in \mathrm{dom}(H) = \mathrm{dom}(A). \tag{3.2}$$

Notice that condition (3.1) is satisfied, for example, if we assume (as in [2]) that in addition to (1.1) and (1.2) conditions (1.3) and (1.4) are satisfied. In this case condition (3.1) takes the form

$$\mathrm{dom}(H^*) = \mathrm{dom}(A) \subseteq \mathrm{dom}(A) \cap \mathrm{dom}(B^*). \tag{3.3}$$

In view of Example of Section 2 the conditions (1.3) and (1.4) sound rather strong. It is known that together with (1.1) and (1.2) they guarantee the error bounds (1.7), (1.8) with $\alpha = 1$, see [2]. In this section we show that one can relax them to the *intermediate* condition (1.6), which is applicable even in the case $\mathrm{dom}(A) \cap \mathrm{dom}(B^*) = \{0\}$, cf. (1.5).

To this end we assume that B is a *strictly* m-accretive operator (see [9],Ch.V), i.e. there is $\epsilon > 0$ such that

$$\mathfrak{Re}(Bf, f) \geq \epsilon \|f\|^2, \quad f \in \mathrm{dom}(B). \tag{3.4}$$

In particular, this yields that the left open half-plane $\mathbb{C}_- = \{\zeta \in \mathbb{C} : \mathfrak{Re}(\zeta) < 0\}$ is contained in the resolvent set $\varrho(B)$ of B. Then our *minimal* conditions (1.1) and (1.2) imply that H defined by (3.2) is also strictly m-accretive. It is known (see [7]-[9]) that m-accretivity of B and H implies that B^* and H^* are also m-accretive and that fractional powers of all of these operators: B^α, H^α, $(B^*)^\alpha$, $(H^*)^\alpha$ are well-defined for $\alpha \in (0,1)$. The *strict* m-accretivity of B guarantees invertibility of these operators. Now we recall some properties of H , see [1, 2]:

Proposition 3.1 *Let $A = A^* \geq I$ and let B be a m-accretive operator such that conditions (1.1) and (1.2) are satisfied. Then for H defined by (3.2) one has:*

(a) The operator $-H$ is a boundedly invertible generator of contraction holomorphic semi-group $\{e^{-tH}\}_{t\geq 0}$ in the sector $S_\omega = \{t \in \mathbb{C} : |\arg t| < \omega = \arccos(a)\}$.

(b) For any $\alpha > 0$ and $t > 0$ the range $\mathrm{ran}(e^{-tH}) \subseteq \mathrm{dom}(H^\alpha)$, and there is $d_\alpha(H) > 0$ such that

$$\|H^\alpha e^{-tH}\| \leq \frac{d_\alpha(H)}{t^\alpha}, \quad t > 0. \tag{3.5}$$

Next we generalize Lemma 5 of [1] :

Lemma 3.2 *Let $A \geq I$ and let B be a strictly m-accretive operator obeying conditions (1.1) and (1.2). If H is given by (3.2), then there is a constant $M > 0$ such that*

$$\left\|\left(e^{-\tau B}e^{-\tau A} - e^{-\tau H}\right)A^{-1}\right\| \leq M\tau \tag{3.6}$$

for $\tau \geq 0$. If in addition the intermediate condition (1.6) is satisfied, then for any $\alpha \in (0,1]$ there is $N_1(\alpha) > 0$ such that

$$\left\|H^{-\alpha}\left(e^{-\tau B}e^{-\tau A} - e^{-\tau H}\right)\right\| \leq N_1(\alpha)\tau^\alpha \tag{3.7}$$

for $\tau \geq 0$.

Proof: The first estimate (3.6) readily follows from the inequality

$$\begin{aligned}
&\left\|\left(e^{-\tau B}e^{-\tau A} - e^{-\tau H}\right)A^{-1}\right\| \\
&\leq \left\|\int_0^\tau ds\, e^{-sB}BA^{-1}e^{-\tau A}\right\| + \left\|\left(I - e^{-\tau A}\right)A^{-1}\right\| + \left\|\int_0^\tau ds\, e^{-sH}HA^{-1}\right\|
\end{aligned}$$

and conditions (1.1), (1.2) which give $M := \|BA^{-1}\| + 1 + \|HA^{-1}\| \leq 2(a+1)$.

To prove the second estimate (3.7) notice that for any strictly m-accretive operator C the fractional powers $C^{-\alpha}$, $0 < \alpha < 1$, are defined by

$$C^{-\alpha} = \frac{\sin\alpha\pi}{\pi}\int_0^\infty \lambda^{-\alpha}(C+\lambda)^{-1}d\lambda,$$

see [7]-[9]. Hence $(C^{-\alpha})^* = (C^*)^{-\alpha}$, which allows us to pass from (3.7) to the norm-estimate of the conjugate operator

$$\left(H^{-\alpha}\left(e^{-\tau B}e^{-\tau A} - e^{-\tau H}\right)\right)^* = \left(e^{-\tau A}e^{-\tau B^*} - e^{-\tau H^*}\right)(H^*)^{-\alpha}.$$

To this end we use the identity

$$
\begin{aligned}
&(e^{-\tau A}e^{-\tau B^*} - e^{-\tau H^*})(H^*)^{-\alpha} \qquad (3.8)\\
&\quad = (e^{-\tau A} - I)(H^*)^{-\alpha} - e^{-\tau A}(I - e^{-\tau B^*})(H^*)^{-\alpha} + (I - e^{-\tau H})(H^*)^{-\alpha}\\
&\quad = (e^{-\tau A} - I)A^{-\alpha}A^{\alpha}(H^*)^{-\alpha} + e^{-\tau A}(e^{-\tau B^*} - I)(B^*)^{-\alpha}(B^*)^{\alpha}(H^*)^{-\alpha} - (e^{-\tau H^*} - I)(H^*)^{-\alpha}.
\end{aligned}
$$

Since for any strictly m-accretive operator C and $0 < \alpha < 1$ one has

$$
(e^{-tC} - I)C^{-\alpha} = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{-\alpha}(C + \lambda)^{-1}(e^{-tC} - I)\, d\lambda, \qquad (3.9)
$$

we get for $\xi > 0$

$$
\begin{aligned}
(e^{-tC} - I)C^{-\alpha} \; = \; & \frac{\sin \alpha\pi}{\pi} \int_0^\xi \lambda^{-\alpha} \left(C(C + \lambda)^{-1}\right) C^{-1} \left(e^{-tC} - I\right) d\lambda\\
& + \frac{\sin \alpha\pi}{\pi} \int_\xi^\infty \lambda^{-\alpha}(C + \lambda)^{-1} \left(e^{-tC} - I\right) d\lambda.
\end{aligned}
$$

Since the above identity is valid for any $\xi > 0$, by the estimates $\|(e^{-tC} - I)C^{-1}\| \leq t$, $\|C(C + \lambda)^{-1}\| \leq 1$ and $\|(C + \lambda)^{-1}\| \leq 1/\lambda$ we obtain

$$
\|(e^{-tC} - I)C^{-\alpha}\| \leq \frac{\sin \alpha\pi}{\pi} \inf_{\xi > 0} \left(\frac{t}{1 - \alpha}\xi^{1-\alpha} + \frac{2}{\alpha}\xi^{-\alpha}\right) = \frac{\sin \alpha\pi}{\pi} \frac{2^{1-\alpha}}{\alpha(1 - \alpha)} t^{\alpha}. \qquad (3.10)
$$

for $\alpha \in (0, 1)$ and $t \geq 0$. Applying the estimate (3.10) to operators A, B^* and H^*, by virtue of identity (3.8) we find the assertion (3.7) with constant

$$
N_1(\alpha) = \frac{2^{1-\alpha} \sin \alpha\pi}{\pi\alpha(1 - \alpha)} \left(\|A^{\alpha}(H^*)^{-\alpha}\| + \|(B^*)^{\alpha}(H^*)^{-\alpha}\| + 1\right), \quad \alpha \in (0, 1).
$$

Here $\|A^{\alpha}(H^*)^{-\alpha}\|$ and $\|(B^*)^{\alpha}(H^*)^{-\alpha}\|$ are bounded due to our condition (1.6). The estimate (3.7) can be extended to $\alpha = 1$ by setting $N_1(\alpha = 1) := \lim_{\alpha\to 1} N_1(\alpha) = \|A(H^*)^{-1}\| + \|B^*(H^*)^{-1}\| + 1$. □

Next we have to modify Lemma 6 of [1].

Lemma 3.3 *Let $A \geq I$ and let B be a strictly m-accretive operator obeying conditions (1.1) and (1.2). If the operator H defined by (3.2) satisfies the intermediate condition (1.6), then for any $\alpha \in (0, 1]$ there is a constant $N_2(\alpha) > 0$ such that*

$$
\left\|H^{-\alpha} \left(e^{-\tau B}e^{-\tau A} - e^{-\tau H}\right) A^{-1}\right\| \leq N_2(\alpha)\tau^{1+\alpha} \qquad (3.11)
$$

for all $\tau \geq 0$.

Proof: We use the following representation

$$
\begin{aligned}
&H^{-\alpha} \left(e^{-\tau B}e^{-\tau A} - e^{-\tau H}\right) A^{-1} = && (3.12)\\
&\quad H^{-\alpha}(I - e^{-\tau B})(I - e^{-\tau A})A^{-1} + && (3.13)\\
&\quad H^{-\alpha}(e^{-\tau B} + e^{-\tau A} - e^{-\tau H} - I)A^{-1}. && (3.14)
\end{aligned}
$$

Let $\alpha \in (0,1)$. By virtue of (3.10) one estimates (3.13) as follows:

$$\begin{aligned}
&\|H^{-\alpha}(I-e^{-\tau B})(I-e^{-\tau A})A^{-1}\| \qquad (3.15)\\
&\leq \ \|(I-e^{-\tau B^*})(H^*)^{-\alpha}\|\|(I-e^{-\tau A})A^{-1}\|\\
&\leq \ \|(I-e^{-\tau B^*})(B^*)^{-\alpha}\|\|(B^*)^{\alpha}(H^*)^{-\alpha}\|\tau \leq \frac{2^{1-\alpha}\sin\alpha\pi}{\pi\alpha(1-\alpha)}\tau^{1+\alpha}\|(B^*)^{\alpha}(H^*)^{-\alpha}\|.
\end{aligned}$$

To estimate (3.14) we use the representation

$$\begin{aligned}
&e^{-\tau B}+e^{-\tau A}-e^{-\tau H}-I= \qquad (3.16)\\
&\qquad (e^{-\tau B}-I+\tau B)+(e^{-\tau A}-I+\tau A)-(e^{-\tau H}-I+\tau H).
\end{aligned}$$

Since

$$e^{-\tau C}-I+\tau C=\int_0^{\tau} ds\,(I-e^{-sC})C. \qquad (3.17)$$

one gets for a strictly m-accretive operator C that

$$\begin{aligned}
&\|H^{-\alpha}(e^{-\tau C}-I+\tau C)A^{-1}\| \leq \left\|\int_0^{\tau} ds\,(I-e^{-sC^*})(H^*)^{-\alpha}\right\| \|CA^{-1}\|\\
&\leq \ \int_0^{\tau} ds\|(I-e^{-sC^*})(C^*)^{-\alpha}\|\,\|(C^*)^{\alpha}(H^*)^{-\alpha}\|\,\|CA^{-1}\|.
\end{aligned}$$

By (3.10) and (3.16), (3.17) this yields for (3.14) the estimate

$$\begin{aligned}
&\|H^{-\alpha}(e^{-\tau B}+e^{-\tau A}-e^{-\tau H}-I)A^{-1}\| \leq \qquad (3.18)\\
&\quad \frac{\tau^{1+\alpha}}{1+\alpha}\frac{2^{1-\alpha}\sin\alpha\pi}{\pi\alpha(1-\alpha)}\left(\|(B^*)^{\alpha}(H^*)^{-\alpha}\|\|BA^{-1}\|+\|A^{\alpha}(H^*)^{-\alpha}\|+\|HA^{-1}\|\right).
\end{aligned}$$

Hence (3.15) and (3.18) verify assertion (3.11) with the constant

$$N_2(\alpha):=\frac{2^{1-\alpha}\sin\alpha\pi}{\pi\alpha(1-\alpha)}\left\{\|(B^*)^{\alpha}(H^*)^{-\alpha}\|+\frac{1}{1+\alpha}\left(a\|(B^*)^{\alpha}(H^*)^{-\alpha}\|+\|A^{\alpha}(H^*)^{-\alpha}\|+1+a\right)\right\}.$$

Setting $N_2(1):=\lim_{\alpha\to 1}N_2(\alpha)$ one easily gets the estimate (3.11) for $\alpha=1$. □

Theorem 3.4 *Let $A \geq I$ and let B be a strictly m-accretive operator obeying conditions (1.1) and (1.2). If the operator H defined by (3.2) satisfies (1.6), then there is a constant $L_\alpha > 0$ such that the estimates*

$$\begin{aligned}
\left\|\left(e^{-tB/n}e^{-tA/n}\right)^n-e^{-tH}\right\| &\leq L_\alpha\frac{\ln n}{n^{\alpha}}, \qquad (3.19)\\
\left\|\left(e^{-tA/n}e^{-tB^*/n}\right)^n-e^{-tH^*}\right\| &\leq L_\alpha\frac{\ln n}{n^{\alpha}}, \qquad (3.20)
\end{aligned}$$

hold for $\alpha \in (0,1]$ and $n=3,4,\ldots$ uniformly in $t \geq 0$.

Proof: Notice that (3.19) implies (3.20) and vice versa. Therefore, we start with (3.20). Let $\tau = t/n$ and $F(\tau) = e^{-\tau B}e^{-\tau A}$, i.e., $F^*(\tau) = e^{-\tau A}e^{-\tau B^*}$. Then we use the operator identity

$$F^*(\tau)^n - e^{-n\tau H^*} = \sum_{m=0}^{n-1} F^*(\tau)^{(n-m-1)} \left(F^*(\tau) - e^{-\tau H^*}\right) e^{-m\tau H^*}, \tag{3.21}$$

and the representation

$$I - F^*(\tau) = (I - e^{-\tau A})(I + X(\tau)),$$

where $X(\tau) = (I - e^{-\tau A})^{-1} e^{-\tau A}(I - e^{-\tau B^*})$. By Lemma 3.2 of [2] and Lemma 2.3 of [11] the operator $I + X(\tau)$ is invertible and one gets

$$\begin{aligned} &\|(I - F^*(\tau))^{-1}u\| \leq \\ &\quad \frac{1}{1-a}\|(I - e^{-\tau A})^{-1}u\| \leq \frac{1}{1-a}(\|u\| + \frac{1}{\tau}\left\|A^{-1}u\right\|). \end{aligned} \tag{3.22}$$

Since $(H^*)^\alpha$ is invertible one can use (3.22) to estimate (3.21) as follows:

$$\begin{aligned} &\|F^*(\tau)^n - e^{-n\tau H^*}\| \leq \|F^*(\tau)^{n-1}(I - F^*(\tau))\|\|(I - F^*(\tau))^{-1}(F^*(\tau) - e^{-\tau H^*})\| + \\ &\sum_{m=1}^{n-1} \|F^*(\tau)^{n-m-1}(I - F^*(\tau))\|\,\|(I - F^*(\tau))^{-1}(F^*(\tau) - e^{-\tau H^*})(H^*)^{-\alpha}\|\,\|(H^*)^\alpha e^{-m\tau H^*}\|. \end{aligned} \tag{3.23}$$

Then Theorem 2.4 of [2] gives

$$\|F^*(\tau)^s(I - F^*(\tau))\| \leq \frac{c}{s+1}, \quad c > 0 \text{ and } s = 0, 1, 2, ... \tag{3.24}$$

¿From the estimate (3.22) we get

$$\|(I - F^*(\tau))^{-1}(F^*(\tau) - e^{-\tau H^*})\| \leq \frac{1}{1-a}\left\{\|F^*(\tau) - e^{-\tau H^*}\| + \frac{1}{\tau}\left\|A^{-1}\left(F^*(\tau) - e^{-\tau H^*}\right)\right\|\right\} \tag{3.25}$$

and

$$\begin{aligned} &\|(I - F^*(\tau))^{-1}(F^*(\tau) - e^{-\tau H^*})(H^*)^{-\alpha}\| \\ &\leq \frac{1}{1-a}\left\{\|(F^*(\tau) - e^{-\tau H^*})(H^*)^{-\alpha}\| + \frac{1}{\tau}\|A^{-1}\left(F^*(\tau) - e^{-\tau H^*}\right)(H^*)^{-\alpha}\|\right\}. \end{aligned} \tag{3.26}$$

Therefore, by Lemmata 3.2 and 3.3 (taking the adjoint) one obtains the estimates

$$\|(I - F^*(\tau))^{-1}(F^*(\tau) - e^{-\tau H^*})\| \leq \frac{1}{1-a}(2 + M), \tag{3.27}$$

$$\|(I - F^*(\tau))^{-1}(F^*(\tau) - e^{-\tau H^*})(H^*)^{-\alpha}\| \leq \frac{\tau^\alpha}{1-a}(N_1(\alpha) + N_2(\alpha)) \tag{3.28}$$

for $\alpha \in (0, \tau]$ and $\tau \geq 0$. By Proposition 3.1 one gets

$$\|(H^*)^\alpha e^{-m\tau H^*}\| \leq \frac{d_\alpha(H^*)}{(m\tau)^\alpha}. \tag{3.29}$$

Then using (3.23) and inequalities (3.24) - (3.29) we find

$$\begin{aligned} &\|F^*(\tau)^n - e^{-n\tau H^*}\| \\ \leq\ & \frac{c}{n(1-a)}(2+M) + \sum_{m=1}^{n-1} \frac{c}{n-m}\frac{1}{1-a}(N_1(\alpha)+N_2(\alpha))\frac{d_\alpha(H^*)}{m^\alpha}, \end{aligned} \tag{3.30}$$

for $n = 3, 4, \ldots$. Since for $0 < \alpha < 1$ and $n \geq 3$ one has

$$\sum_{m=1}^{n-1} \frac{1}{(n-m)m^\alpha} = \frac{1}{n}\sum_{m=1}^{n-1} m^{1-\alpha}\left(\frac{1}{m} + \frac{1}{n-m}\right) \leq \frac{2\ln n}{n^\alpha}, \tag{3.31}$$

the estimate (3.30) leads to (3.20) with

$$L_\alpha := \frac{2c}{1-a}\left\{(2+M)/2 + (N_1(\alpha)+N_2(\alpha))d_\alpha(H^*)\right\}$$

for $\alpha \in (0, 1]$ and $\tau \geq 0$. □

Corollary 3.5 *Let $A = A^* \geq I$ and let B be a strictly m-accretive operator obeying (1.1) and (1.2). If the operator H defined by (3.2) satisfies the intermediate condition (1.6), then we have the following estimates*

$$\begin{aligned} \left\|\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-tH}\right\| &\leq L'_\alpha \frac{\ln n}{n^\alpha}, && (3.32)\\ \left\|\left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n - e^{-tH}\right\| &\leq L''_\alpha \frac{\ln n}{n^\alpha} && (3.33) \end{aligned}$$

for $\alpha \in (0, 1]$ and $n = 3, 4, \ldots$ uniformly in $t \geq 0$.

Proof: Let $T(\tau) = e^{-\tau A}e^{-\tau B}$ and $\tau = t/n$. We use the identity

$$\begin{aligned} &T(\tau)^n - e^{-tH} = \\ &\quad e^{-\tau A}(F(\tau)^{n-1} - e^{-tH})e^{-\tau B} + (e^{-\tau A} - I)e^{-tH}e^{-\tau B} + (e^{-tH}H^\alpha)(H^{-\alpha}(e^{-\tau B} - I)). \end{aligned}$$

Since $F(\tau)^{n-1} - e^{-tH} = F(\tau)^{n-1}(I - F(\tau)) + (F(\tau)^n - e^{-tH})$, by Theorem 2.4 of [2] and by (3.19) one gets the estimate

$$\|e^{-\tau A}(F(\tau)^{n-1} - e^{-tH})e^{-\tau B}\| \leq \|F(\tau)^{n-1} - e^{-tH}\| \leq \frac{c}{n} + L_\alpha\frac{\ln n}{n^\alpha}, \quad n \geq 3. \tag{3.34}$$

Following the line of reasoning developed in proofs of Lemmata 3.2 and 3.3 we obtain

$$\left\|(e^{-\tau A} - I)e^{-tH}e^{-\tau B}\right\| \leq \|(e^{-\tau A} - I)A^{-1}\|\|AH^{-1}\|\|He^{-tH}\| \leq \frac{1}{n}\frac{d_1(H)}{1-a}, \quad n \geq 1, \tag{3.35}$$

and

$$\|H^\alpha e^{-tH} H^{-\alpha}(e^{-\tau B} - I)\| \leq \tag{3.36}$$
$$\|H^\alpha e^{-tH}\| \|(e^{-\tau B^*} - I)(B^*)^{-\alpha}\| \|(B^*)^\alpha (H^*)^{-\alpha}\| \leq \frac{1}{n^\alpha} d_\alpha(H) \frac{2^{1-\alpha} \sin \alpha\pi}{\pi\alpha(1-\alpha)} \|(B^*)^\alpha (H^*)^{-\alpha}\|.$$

Thus, estimates (3.34)-(3.36) yield the first assertion (3.32) with

$$L_\alpha^{'} = L_\alpha + c + \frac{d_1(H)}{1-a} + d_\alpha(H) \frac{2^{1-\alpha} \sin \alpha\pi}{\pi\alpha(1-\alpha)} \|(B^*)^\alpha (H^*)^{-\alpha}\|$$

for $\alpha \in (0,1)$ and $\tau \geq 0$. If $\alpha = 1$, then one has to replace $\{2^{1-\alpha} \sin \alpha\pi / \pi\alpha(1-\alpha)\}$ by 1. The proof of the second assertion is similar. □

4 Applications to accretive and sectorial generators

In this section we indicate sufficient conditions, which give error-bound estimates (1.7), (1.8) for some $0 < \alpha \leq 1$. Since for m-accretive generator B these estimates are guaranteed by (1.6) (see Section 3), we scrutinize, in particular, conditions ensuring (1.6) for some $\alpha \in (0,1)$.

4.1 We start by the following simple observation. Let J be a conjugation on $\mathfrak{H}$, i.e, an antilinear operator on $\mathfrak{H}$ such that

$$(Jf, Jg) = (g, f), \quad f, g \in \mathfrak{H}. \tag{4.1}$$

If J obeys the condition

$$JAJ = A \quad \text{and} \quad JBJ = B^*, \tag{4.2}$$

then conditions (1.1) and (1.2) imply conditions (1.3) and (1.4) with $a_* = a$. Therefore, the assumptions of Theorem 3.5 of [2] are satisfied. That yields the error estimates (1.7), (1.8) with $\alpha = 1$. As an example one can consider the Schrödinger operator. If A is the Laplace operator on $\mathfrak{H} = L^2(\mathbb{R}^n)$ and B is a multiplication operator arising from some complex potential $V(\cdot)$ such that $\mathfrak{Re}\, V \geq 0$ and conditions (1.1) and (1.2) are satisfied, then denoting by J the complex conjugation one easily verifies that conditions (4.1) and (4.2) are valid.

4.2 The problem becomes more difficult if we loose the condition $\mathrm{dom}(A) \subseteq \mathrm{dom}(B^*)$, in particular, if we face up to $\mathrm{dom}(A) \cap \mathrm{dom}(B^*) = \{0\}$. Then it seems to be a delicate problem to satisfy the *intermediate* condition (1.6). But it turns out that our *minimal* conditions (1.1) and (1.2) are sufficient to imply (1.6) for any $0 \leq \alpha < 1/2$. This arises from the following line of reasoning:

(i) Notice that for any m-accretive operators A and B satisfying (1.1) one has $\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B^\alpha)$ for $0 \leq \alpha \leq 1$, see [8]. Since operator H is m-accretive and $\mathrm{dom}(H) \subseteq \mathrm{dom}(A)$ (in fact, $\mathrm{dom}(H) = \mathrm{dom}(A)$, cf.(3.2)), we also obtain that $\mathrm{dom}(H^\alpha) \subseteq \mathrm{dom}(A^\alpha)$ for $0 \leq \alpha \leq 1$.

(ii) Since B and H are m-accretive operators, Theorem 1.1 of [6] yields $\mathrm{dom}(B^\alpha) = \mathrm{dom}((B^*)^\alpha)$ and $\mathrm{dom}(H^\alpha) = \mathrm{dom}((H^*)^\alpha)$ for $0 \le \alpha < 1/2$.

(iii) Summing up the statements in (i) and (ii) one immediately gets that $\mathrm{dom}((H^*)^\alpha) = \mathrm{dom}(H^\alpha) \subseteq \mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(A^\alpha) \cap \mathrm{dom}(B^\alpha) = \mathrm{dom}(A^\alpha) \cap \mathrm{dom}((B^*)^\alpha)$ for $0 \le \alpha < 1/2$. Hence, we obtain the *intermediate* condition (1.6) for $0 \le \alpha < 1/2$.

Thus, taking into account Theorem 3.4 we have proved the following statement.

Theorem 4.1 *Let $A \ge I$ and let B be a strictly m-accretive operator. If the conditions (1.1) and (1.2) are satisfied, then estimates (1.7) and (1.8) hold for any $0 < \alpha < 1/2$ uniformly in $t \in [0, \infty)$.*

Therefore, to cover the case of the error bound with $0 < \alpha < 1/2$ our *minimal* conditions (1.1), (1.2) are sufficient, in spite of the fact that $\mathrm{dom}(A) \cap \mathrm{dom}(B^*) = \{0\}$ can happen.

4.3 Now we are going to demonstrate how we can ensure condition (1.6) in the limit case $\alpha = 1/2$. The considerations below are mainly relied on the paper of Miyazaki [10] which improves some results of Kato [6, 7] and Shimakura [14]. Again we assume our *minimal* conditions (1.1), (1.2), and we suppose that B is *strictly* m-accretive and *sectorial* (in short m-sectorial) with semi-angle $\theta_B \in [0, \pi/2)$, i.e. with *numerical range* in the sector $S_{\theta_B} := \{t \in \mathbb{C} : |\arg t| < \theta_B\}$. For simplicity, we set $\epsilon = 1$ in (3.4). With B we associate a sesquilinear form $\tilde{b}$ defined by

$$\tilde{b}[f, g] = (Bf, g), \quad f, g \in \mathrm{dom}(\tilde{b}) = \mathrm{dom}(B). \tag{4.3}$$

The form $\tilde{b}$ is closable. Its closure is denote by b. This definition implies

$$\mathrm{dom}(B) \subseteq \mathrm{dom}(b). \tag{4.4}$$

One can associate with b its *real part* b_R defined by

$$b_R[f, g] = \frac{1}{2}\{b[f, g] + \overline{b[g, f]}\}, \quad \mathrm{dom}(b_R) = \mathrm{dom}(b). \tag{4.5}$$

The form b_R is symmetric and, by virtue of (3.4), it is semibounded from below with lower bound $\epsilon = 1$. Since B is m-accretive, the form b_R is closed. By representation theorems (see [9],Ch.VI) there is a non-negative self-adjoint operator $B_R \ge I$ such that

$$b_R[f, g] = (B_R^{1/2} f, B_R^{1/2} g), \quad f, g \in \mathrm{dom}(B_R^{1/2}) = \mathrm{dom}(b_R). \tag{4.6}$$

The operator B_R is called the real part of B. Similarly, we can introduce the *imaginary part* b_I of b defined by

$$b_I[f, g] = \frac{1}{2i}\{b[f, g] - \overline{b[g, f]}\}, \quad f, g \in \mathrm{dom}(b_I) = \mathrm{dom}(b). \tag{4.7}$$

Since B is m-sectorial, the associated form b is also m-sectorial. It is known (see [9],Ch.VI) that in this case there is a non-negative constant $C = \tan\theta_B$ such that

$$|b_I[f, g]| \le C b_R[f, f]^{1/2} b_R[g, g]^{1/2}, \quad f, g \in \mathrm{dom}(b) = \mathrm{dom}(b_R), \tag{4.8}$$

which is equivalent to

$$|b_I[f,g]| \leq C\|B_R^{1/2}f\|\|B_R^{1/2}g\|, \quad f,g \in \mathrm{dom}(b_I) = \mathrm{dom}(b_R) = \mathrm{dom}(B_R^{1/2}). \tag{4.9}$$

Following Miyazaki [10] we *assume* that there is a constant M_β such that the condition

$$|b_I[f,g]| \leq M_\beta \|B_R^{\beta/2}f\|\|B_R^{\beta/2}g\|, \quad f,g \in \mathrm{dom}(B_R^{1/2}). \tag{4.10}$$

holds for some $\beta \in [0,1)$. Since $B_R \geq I$, one has

$$\|B_R^{\beta/2}f\| \leq \|B_R^{1/2}f\|, \quad f \in \mathrm{dom}(B_R^{1/2}) \subseteq \mathrm{dom}(B_R^{\beta/2}), \tag{4.11}$$

which shows that the Miyazaki condition (4.10) is *stronger* than (4.9).

Let a be closed quadratic form associated with a positive self-adjoint operator A :

$$a[f,g] = (A^{1/2}f, A^{1/2}g), \quad \mathrm{dom}(a) = \mathrm{dom}(A^{1/2}). \tag{4.12}$$

By condition (1.1) the domain

$$\mathcal{D} = \mathrm{dom}(a) \cap \mathrm{dom}(b) \tag{4.13}$$

is dense in $\mathfrak{H}$. Then the sum of the closed sectorial forms

$$h[f,g] = a[f,g] + b[f,g], \quad f,g \in \mathrm{dom}(h) = \mathcal{D} \tag{4.14}$$

is a well-defined on $\mathcal{D}$ closed sectorial form by [9, Theorem VI-1.31]. By the representation Theorem 2.1 of [9, Ch.VI] there is an m-sectorial operator H denoted by $H = A \dot{+} B$ (the *form-sum* of A and B) such that one has

$$h[f,g] = (Hf,g), \quad f \in \mathrm{dom}(H) \subseteq \mathrm{dom}(h), \quad g \in \mathrm{dom}(h). \tag{4.15}$$

By virtue of conditions (1.1) and (1.2) the operator H coincides with that defined by the *algebraic sum* $A + B$ on $\mathrm{dom}(A)$, see (3.2).

Theorem 4.2 *Let $A \geq I$ and let B be a strictly m-sectorial operator obeying conditions (1.1) and (1.2). If in addition the operator B satisfies conditions (3.4), with $\epsilon = 1$, and (4.10), then there are constants $L_{\alpha=1/2}$ and $L'_{\alpha=1/2}$ such that the estimates (1.7) and (1.8) hold for $\alpha = 1/2$ uniformly in $[0,\infty)$.*

Proof: Since by (3.4), (4.5) and (4.6) one has

$$\mathfrak{Re}(b[f,f]) = \|B_R^{1/2}f\| \geq \|f\|^2 \quad f \in \mathrm{dom}(B_R^{1/2}), \tag{4.16}$$

the sesquilinear form b is *strongly coercive*, see [10]. Then condition (4.10) and Theorem 2 of [10] imply

$$\mathrm{dom}(B^{1/2}) = \mathrm{dom}((B^*)^{1/2}) = \mathrm{dom}(B_R^{1/2}) = \mathrm{dom}(b). \tag{4.17}$$

Next, the real part h_R of h (4.14) is given by

$$h_R[f,g] = a[f,g] + b_R[f,g], \quad f,g \in \mathrm{dom}(h_R) = \mathrm{dom}(h). \tag{4.18}$$

The corresponding self-adjoint operator is denoted by H_R. By the same line of reasoning as above for the form b, the sesquilinear form h is *strongly coercive*. By definition (4.18) we have $B_R \leq H_R$. Then taking into account the Heinz inequality for self-adjoint operators we find

$$B_R^\beta \leq H_R^\beta, \quad \beta \in [0,1). \tag{4.19}$$

By virtue of (4.14) for the imaginary part of h we get :

$$h_I[f,g] = \frac{1}{2i}\{h[f,g] - \overline{h[g,f]}\} = b_I[f,g], \quad f,g \in \mathrm{dom}(h_I) = \mathrm{dom}(h). \tag{4.20}$$

Taking into account (4.10) and (4.19) we find that the Miyazaki condition

$$|h_I[f,g]| \leq M_\beta \|H_R^{\beta/2} f\| \|H_R^{\beta/2} g\|, \quad f,g \in \mathrm{dom}(b_I), \quad \beta \in [0,1), \tag{4.21}$$

is valid for the form h. Then again by Theorem 2 of [10] one gets

$$\mathrm{dom}(H^{1/2}) = \mathrm{dom}((H^*)^{1/2}) = \mathrm{dom}(H_R^{1/2}) = \mathrm{dom}(h). \tag{4.22}$$

By (4.13), (4.14) and [9, Theorem VI-1.31] we have

$$\mathrm{dom}(h) = \mathrm{dom}(a) \cap \mathrm{dom}(b), \tag{4.23}$$

which together with (4.17) yields

$$\mathrm{dom}((H^*)^{1/2}) = \mathrm{dom}(A^{1/2}) \cap \mathrm{dom}((B^*)^{1/2}). \tag{4.24}$$

Hence, our condition (1.6) is satisfied for $\alpha = 1/2$. By virtue of Theorem 3.4 and Corollary 3.5 this gives the estimates (1.7) and (1.8) for $\alpha = 1/2$. □

5 More about the case $\alpha = 1/2$

The aim of this section is to relax the Miyazaki condition (4.10), which is *sufficient* to ensure (1.6) and, hence, to apply our main Theorem 3.4 for the case $\alpha = 1/2$. Here we propose for this case another strategy of the proof essentially based on Theorem 3 of [8]. We recall it for the reader convenience:

Proposition 5.1 *Let R, S be bounded accretive operators in Hilbert spaces $\mathfrak{H}$, $\mathfrak{H}'$,respectively, and let T be a bounded operator from $\mathfrak{H}$ to $\mathfrak{H}'$. Then*

$$\|R^\alpha T S^\alpha\| \leq e^{\pi^2 \alpha(1-\alpha)/2} \|T\|^{1-\alpha} \|RTS\|^\alpha, \quad 0 \leq \alpha \leq 1. \tag{5.1}$$

Below we shall prove the error-bound estimates (1.7) and (1.8) for $\alpha = 1/2$ under assumptions avoiding our *intermediate* condition (1.6), i.e. involving *only* operators A and B without any conditions on the adjoint operator B^*. We begin by setting $\hat{F}(\tau) = e^{-\tau A/2}e^{-\tau B}e^{-\tau A/2}$ and $F(\tau) = e^{-\tau B}e^{-\tau A}$, for $\tau \geq 0$. Then we have

$$\hat{F}(\tau)^n = e^{-\tau A/2}F(\tau)^{n-1}e^{-\tau B}e^{-\tau A/2}. \tag{5.2}$$

Lemma 5.2 *Let $A \geq I$ and let B be an m-accretive operator. If conditions (1.1) and (1.2) are satisfied, then there is a constant $c > 0$ such that for $n \geq 1$ one has*

$$\left\|\hat{F}(\tau)^n \left(I - \hat{F}(\tau)\right)\right\| \leq \frac{c}{n}, \quad \tau > 0. \tag{5.3}$$

Proof: By (5.2) we get

$$\begin{aligned} \hat{F}(\tau)^n \left(I - \hat{F}(\tau)\right) &= e^{-\tau A/2}F(\tau)^{n-1}e^{-\tau B}e^{-\tau A/2}\left(I - \hat{F}(\tau)\right) \\ &= e^{-\tau A/2}F(\tau)^{n-1}\left(I - F(\tau)\right)e^{-\tau B}e^{-\tau A/2}, \end{aligned} \tag{5.4}$$

which yields the estimate

$$\left\|\hat{F}(\tau)^n \left(I - \hat{F}(\tau)\right)\right\| \leq \left\|F(\tau)^{n-1}\left(I - F(\tau)\right)\right\|. \tag{5.5}$$

Then by [2, Theorem 2.4], we obtain (5.3). □

Lemma 5.3 *Let $A \geq I$ and let B be an m-accretive operator. If conditions (1.1) and (1.2) are satisfied, then $(I - \hat{F}(\tau))$ is boundedly invertible for any $\tau > 0$.*

Proof: We use the representation

$$I - \hat{F}(\tau) = I - e^{-\tau A} + e^{-\tau A/2}(I - e^{-\tau B})e^{-\tau A/2}. \tag{5.6}$$

Setting

$$Y(\tau) = e^{-\tau A/2}(I - e^{-\tau B})e^{-\tau A/2}(I - e^{-\tau A})^{-1}, \tag{5.7}$$

we get

$$I - \hat{F}(\tau) = (I + Y(\tau))(I - e^{-\tau A}). \tag{5.8}$$

Since

$$\|\tau A e^{-\tau A/2}(I - e^{-\tau A})^{-1}\| \leq 1, \quad \tau \geq 0, \tag{5.9}$$

and

$$\|(I - e^{-\tau B})(\tau A)^{-1}\| \leq a \tag{5.10}$$

we get $\|Y(\tau)\| \leq a < 1$. Hence, the operator $I + Y(\tau)$ is boundedly invertible. Since for $\tau > 0$ the operator $(I - e^{-\tau A})$ is also boundedly invertible, the same is valid for $(I - \hat{F}(\tau))$, see (5.8). □

Lemma 5.4 *Let $A \geq I$ and let B be an m-accretive operator. If conditions (1.1) and(1.2) are satisfied, then there is a constant $b > 0$ such that for any $\tau > 0$ one has*

$$\left\|(I - \hat{F}(\tau))^{-1/2}h\right\| \leq b\left\{\|h\| + \frac{1}{\sqrt{\tau}}\|A^{-1/2}h\|\right\}, \quad h \in \mathfrak{H}. \tag{5.11}$$

Proof: From (5.8) we find the representation

$$\left(I - \hat{F}(\tau)\right)^{-1}(I + Y(\tau)) = (I - e^{-\tau A})^{-1}. \tag{5.12}$$

Since $\|\hat{F}(\tau)\| \leq 1$, the bounded operators $I - \hat{F}(\tau)$ and $(I - \hat{F}(\tau))^{-1}$are accretive. Then we can apply Proposition 5.1 to $\|(I - \hat{F}(\tau))^{1/2}(I - e^{-\tau A})^{-1/2}\|$ and $T = I$ to obtain the estimate

$$\left\|(I - \hat{F}(\tau))^{-1/2}(I + Y(\tau))f\right\| \leq e^{\pi^2/8}\|I + Y(\tau)\|^{1/2}\|(I - e^{-\tau A})^{-1/2}f\|, \quad f \in \mathfrak{H}. \tag{5.13}$$

Since $\|Y(\tau)\| \leq a < 1$, we get

$$\left\|(I - \hat{F}(\tau))^{-1/2}h\right\| \leq e^{\pi^2/8}\sqrt{1+a}\;\|(I - e^{-\tau A})^{-1/2}(I + Y(\tau))^{-1}h\|, \quad h \in \mathfrak{H}. \tag{5.14}$$

Moreover, we have

$$(I + Y(\tau))(I - e^{-\tau A})^{1/2} = (I - e^{-\tau A})^{1/2}(I + Z(\tau)), \tag{5.15}$$

where $Z(\tau)$ is the bounded operator

$$Z(\tau) = \frac{e^{-\tau A/2}}{(I - e^{-\tau A})^{1/2}}(I - e^{-\tau B})\frac{e^{-\tau A/2}}{(I - e^{-\tau A})^{1/2}}. \tag{5.16}$$

Notice that conditions on B imply that the operator $Z(\tau)$ is m-accretive, thus $\|(I + Z(\tau))^{-1}\| \leq 1$. Then by (5.14), (5.15), and by the spectral representation for operator $(I - e^{-\tau A/2})^{-1/2}$ we get the inequality

$$\begin{aligned}
\left\|(I - \hat{F}(\tau))^{-1/2}h\right\| &\leq e^{\pi^2/8}\sqrt{1+a}\left\|(I - e^{-\tau A/2})^{-1/2}h\right\| \\
&\leq e^{\pi^2/8}\sqrt{1+a}\left(\|h\| + \sqrt{\frac{2}{\tau}}\|A^{-1/2}h\|\right).
\end{aligned} \tag{5.17}$$

Setting $b := \sqrt{2}e^{\pi^2/8}\sqrt{1+a}$, one obtains the estimate (5.11). □

Lemma 5.5 *For any m-sectorial operator C in a Hilbert space $\mathfrak{H}$ with the real part C_R, one has the following estimate:*

$$\left\|C_R^{1/2}e^{-tC}\right\| \leq \sqrt{\frac{d_1(C)}{t}}, \quad t > 0. \tag{5.18}$$

Proof: First we remark that

$$\frac{d}{dt}\|e^{-tC}u\|^2 = -2\Re\mathrm{e}\,(e^{-tC}u, Ce^{-tC}u) = -2\|C_R^{1/2}e^{-tC}u\|^2 \leq 0, \quad h \in \mathfrak{H}. \tag{5.19}$$

Then since $0 \leq \|e^{-tC}u\| - \|e^{-(t+h)C}u\| \leq \|e^{-tC}u - e^{-(t+h)C}u\|$ for $t, h > 0$, one gets (cf.(3.5) :

$$\begin{aligned} -\frac{d}{dt}\|e^{-tC}u\| &= \lim_{h\to 0} h^{-1}(\|e^{-tC}u\| - \|e^{-(t+h)C}u\|) \leq \\ &\lim_{h\to 0} h^{-1}\|e^{-tC}u - e^{-(t+h)C}u\| \leq \left\|\frac{d}{dt}e^{-tC}u\right\| \leq \|Ce^{-tC}u\| \leq \frac{d_1(C)}{t}\|u\|. \end{aligned} \tag{5.20}$$

By (5.19) and (5.20) we find

$$2\|C_R^{1/2}e^{-tC}u\|^2 = -2\|e^{-tC}u\|\frac{d}{dt}\|e^{-tC}u\| \leq 2\frac{d_1(C)}{t}\|u\|^2, \tag{5.21}$$

which leads to the estimate (5.18). □

Lemma 5.6 *Let $A \geq I$ be a self-adjoint operator and let B be an m-sectorial operator with semi-angle $\theta_B \in [0, \pi/2)$. If $\mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(b)$, then we have*

$$\left\|A^{-1/2}(I - e^{-\tau B})\right\| \leq (1 + \tan\theta_B)\|B_R^{1/2}A^{-1/2}\|2\tau^{1/2}d_1(B)^{1/2}, \quad \tau \geq 0. \tag{5.22}$$

Proof: For any $f, g \in \mathfrak{H}$ one has

$$\left(A^{-1/2}(I - e^{-\tau B})f, g\right) = \int_0^\tau ds\,\left(Be^{-sB}f, A^{-1/2}g\right).$$

Then we get

$$\left(A^{-1/2}(I - e^{-\tau B})f, g\right) = \int_0^\tau ds\; b\left[e^{-sB}f, A^{-1/2}g\right], \tag{5.23}$$

where $b[\cdot,\cdot]$ is closed sesquilinear sectorial form corresponding to B. By definitions (4.5), (4.7) of its real and imaginary parts, and by inequality (4.9) we get :

$$|b[u,v]| \leq (1 + \tan\theta_B)b_R[u]^{1/2}b_R[v]^{1/2} = (1 + \tan\theta_B)\|B_R^{1/2}u\|\|B_R^{1/2}v\|. \tag{5.24}$$

Thus, for the integrand in (5.23) we obtain that

$$|b[e^{-sB}f, A^{-1/2}g]| \leq (1 + \tan\theta_B)\|B_R^{1/2}e^{-sB}f\|\|B_R^{1/2}A^{-1/2}g\|.$$

Together with inequality (5.18) this gives :

$$\begin{aligned} |\left(A^{-1/2}(I - e^{-\tau B})f, g\right)| &\leq (1 + \tan\theta_B)\|B_R^{1/2}A^{-1/2}g\|\int_0^\tau ds\sqrt{\frac{d_1(B)}{s}}\|f\| \\ &\leq (1 + \tan\theta_B)\|B_R^{1/2}A^{-1/2}\|2d_1(B)^{1/2}\tau^{1/2}\|f\|\|g\|, \end{aligned}$$

which implies the annonced estimate (5.22). Here we used that by condition of lemma $\mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(b) = \mathrm{dom}(B_R^{1/2})$, cf.(4.17), and thus $\|B_R^{1/2}A^{-1/2}\| < \infty$. □

Lemma 5.7 *Let $A \geq I$ and let B be an m-sectorial operator obeying conditions (1.1) and (1.2). If in addition the condition* $\mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(b)$ *is satisfied, then there are constants* $M_{1/2}, M'_{1/2} > 0$ *such that the estimates*

$$\left\| A^{-1/2} \left(\hat{F}(\tau) - e^{-\tau H} \right) \right\| \leq M_{1/2} \tau^{1/2} \tag{5.25}$$

and

$$\left\| A^{-1/2} \left(\hat{F}(\tau) - e^{-\tau H} \right) H^{-1} \right\| \leq M'_{1/2} \tau^{3/2} \tag{5.26}$$

are valid for $\tau \geq 0$.

Proof: To prove (5.25) we use the representation

$$\begin{aligned} &A^{-1/2}(e^{-\tau A/2} e^{-\tau B} e^{-\tau A/2} - e^{-\tau H}) = \\ &\quad -e^{-\tau A/2} A^{-1/2}(I - e^{-\tau B}) e^{-\tau A} - A^{-1/2}(I - e^{-\tau A}) + A^{-1/2}(I - e^{-\tau H}). \end{aligned}$$

By spectral representation one readily gets the estimate $\|A^{-1/2}(I - e^{-\tau A})\| \leq (2\tau/e)^{1/2}$. Taking into account Lemma 5.6 for estimations of $A^{-1/2}(I - e^{-\tau B})$ and $A^{-1/2}(I - e^{-\tau H})$, we obtain (5.25) with

$$M_{1/2} = (2/e)^{1/2} + 2(1 + \tan\theta_B)(\|B_R^{1/2} A^{-1/2}\| d_1(B)^{1/2} + \|H_R^{1/2} A^{-1/2}\| d_1(H)^{1/2}).$$

To prove (5.26) we use the representation

$$\begin{aligned} A^{-1/2}(\hat{F}(\tau) - e^{-\tau H}) H^{-1} &= A^{-1/2}(I - e^{-\tau A/2})(I - e^{-\tau B}) e^{-\tau A/2} H^{-1} + && (5.27)\\ &\quad A^{-1/2}(I - e^{-\tau B})(I - e^{-\tau A/2}) H^{-1} + && (5.28)\\ &\quad A^{-1/2}(e^{-\tau A} + e^{-\tau B} - e^{-\tau H} - I) H^{-1}. && (5.29) \end{aligned}$$

Since by the spectral representation: $\|A^{-1/2}(I - e^{-\tau A/2})\| \leq (\tau/e)^{1/2}$ and $\|(I - e^{-\tau B}) A^{-1}\| \leq \tau \|BA^{-1}\|$, we get for the term (5.27) the following estimate :

$$\|A^{-1/2}(I - e^{-\tau A/2})(I - e^{-\tau B}) e^{-\tau A/2} H^{-1}\| \leq \frac{1}{\sqrt{e}} \|BA^{-1}\| \|AH^{-1}\| \tau^{3/2}. \tag{5.30}$$

For the second term (5.28) we find by Lemma 5.6 that

$$\|A^{-1/2}(I - e^{-\tau B})(I - e^{-\tau A/2}) A^{-1} A H^{-1}\| \leq (1 + \tan\theta_B) d_1(B)^{1/2} \|B_R^{1/2} A^{-1/2}\| \tau^{3/2} \|AH^{-1}\|. \tag{5.31}$$

Taking into account (3.16) and (3.17), we obtain representation for the third term (5.29):

$$\begin{aligned} &A^{-1/2}(e^{-\tau A} + e^{-\tau B} - e^{-\tau H} - I) H^{-1} = \\ &\quad \int_0^\tau A^{-1/2}(I - e^{-sA}) A H^{-1} \, ds + \int_0^\tau A^{-1/2}(I - e^{-sB}) B A^{-1} A H^{-1} \, ds + \\ &\quad \int_0^\tau A^{-1/2}(e^{-sH} - I) \, ds. \end{aligned}$$

Taking into account that $\|A^{-1/2}(I-e^{-sA})\| \leq (2s/e)^{1/2}$, and applying Lemma 5.6 to $A^{-1/2}(I-e^{-sB})$ and to $A^{-1/2}(e^{-sH}-I)$, one gets the estimate for the last term (5.29):

$$\begin{aligned}
&\|A^{-1/2}(e^{-\tau A}+e^{-\tau B}-e^{-\tau H}-I)H^{-1}\| && (5.32)\\
&\leq \frac{2}{3}\tau^{3/2}\sqrt{\frac{2}{e}}\|AH^{-1}\|+2(1+\tan\theta_B)\left(d_1(B)^{1/2}\|B_R^{1/2}A^{-1/2}\|\|\|BA^{-1}\|\|\|AH^{-1}\|\right.\\
&\left.+d_1(H)^{1/2}\|H_R^{1/2}A^{-1/2}\|\right)\tau^{3/2}.
\end{aligned}$$

Together with (1.1) and (1.2) this gives for (5.32) the estimate :

$$\begin{aligned}
&\|A^{-1/2}(e^{-\tau A}+e^{-\tau B}-e^{-\tau H}-I)H^{-1}\| && (5.33)\\
\leq\;& \frac{2^{3/2}\tau^{3/2}}{3\sqrt{e}(1-a)}+2(1+\tan\theta_B)\left(d_1(B)^{1/2}\|B_R^{1/2}A^{-1/2}\|\frac{a}{1-a}+d_1(H)^{1/2}\|H_R^{1/2}A^{-1/2}\|\right)\tau^{3/2}.
\end{aligned}$$

Applying conditions (1.1), (1.2) to (5.30), (5.31) one gets similar estimates, which together with (5.33) give the proof of (5.26), if we set

$$\begin{aligned}
M_{1/2}' :=\;& \frac{a}{\sqrt{e}(1-a)}+\frac{1+\tan\theta_B}{1-a}\|B_R^{1/2}A^{-1/2}\|d_1(B)^{1/2} && (5.34)\\
&+\frac{2^{3/2}}{3\sqrt{e}(1-a)}+2(1+\tan\theta_B)\left(d_1(B)^{1/2}\|B_R^{1/2}A^{-1/2}\|\frac{a}{1-a}+d_1(H)^{1/2}\|H_R^{1/2}A^{-1/2}\|\right).
\end{aligned}$$

□

Theorem 5.8 *Let $A \geq I$ be a self-adjoint operator. Let B be an m-sectorial operator obeying conditions (1.1) and (1.2) with associated closed sectorial form b. If in addition one has*

$$\operatorname{dom}(A^{1/2}) \subseteq \operatorname{dom}(b), \tag{5.35}$$

then there exists a constant $\hat{L}>0$ such that

$$\left\|\left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n-e^{-tH}\right\| \leq \hat{L}\frac{\ln n}{\sqrt{n}} \tag{5.36}$$

uniformly in $t\in[0,\infty)$.

Proof: We begin by identity

$$\hat{F}(\tau)^n-e^{-\tau nH}=\sum_{m=0}^{n-1}\hat{F}(\tau)^{n-m-1}(\hat{F}(\tau)-e^{-\tau H})e^{-m\tau H}. \tag{5.37}$$

Since the operator $(I-\hat{F}(\tau))^{-1/2}$ exists (Lemma 5.3), we find the representation:

$$\hat{F}(\tau)^n-e^{-\tau nH}=\sum_{m=0}^{n-1}\hat{F}(\tau)^{n-m-1}(I-\hat{F}(\tau))^{1/2}(I-\hat{F}(\tau))^{-1/2}(\hat{F}(\tau)-e^{-\tau H})e^{-m\tau H}, \tag{5.38}$$

which leads to the estimate

$$\|\hat{F}(\tau)^n - e^{-\tau n H}\| \leq \tag{5.39}$$
$$\sum_{m=0}^{n-1} \|\hat{F}(\tau)^{n-m-1}(I-\hat{F}(\tau))^{1/2}\| \|(I-\hat{F}(\tau))^{-1/2}(\hat{F}(\tau)-e^{-\tau H})e^{-m\tau H}\|.$$

Since $\|\hat{F}(\tau)\| \leq 1$, the operator $I - \hat{F}(\tau)$ is m-accretive, and we can use Proposition 5.1 for $R = I$, $T = \hat{F}(\tau)^{n-m-1}$ and $S = (I - \hat{F}(\tau))$. This together with Lemma 5.2 gives :

$$\begin{aligned} \|\hat{F}(\tau)^{n-m-1}(I-\hat{F}(\tau))^{1/2}\| &\leq e^{\pi^2/8}\|\hat{F}(\tau)^{n-m-1}\|^{1/2}\|\hat{F}(\tau)^{n-m-1}(I-\hat{F}(\tau))\|^{1/2} \\ &\leq e^{\pi^2/8}\|\hat{F}(\tau)^{n-m-1}(I-\hat{F}(\tau))\|^{1/2} \leq e^{\pi^2/8}\sqrt{\frac{c}{n-m-1}}. \end{aligned} \tag{5.40}$$

To estimate the term corresponding in (5.37), (5.39) to $m = 0$, we use again Proposition 5.1 as well as Lemmata 5.4 and 5.7. This gives :

$$\begin{aligned} &\|\hat{F}(\tau)^{n-1}(\hat{F}(\tau)-e^{-\tau H})\| \\ &\leq \|\hat{F}(\tau)^{n-1}(I-\hat{F}(\tau))^{1/2}\| \|(I-\hat{F}(\tau))^{-1/2}(\hat{F}(\tau)-e^{-\tau H})\| \\ &\leq e^{\pi^2/8}\sqrt{\frac{c}{n-1}}\, b\left(\|\hat{F}(\tau)-e^{-\tau H}\| + \frac{1}{\sqrt{\tau}}\|A^{-1/2}(\hat{F}(\tau)-e^{-\tau H})\|\right) \\ &\leq e^{\pi^2/8}\sqrt{\frac{c}{n-1}}\, b(2+M_{1/2}). \end{aligned} \tag{5.41}$$

For the terms with $m > 0$ we use Lemmata 5.4, 5.7 to get the estimate

$$\begin{aligned} &\|(I-\hat{F}(\tau))^{-1/2}(\hat{F}(\tau)-e^{-\tau H})H^{-1}\| \\ &\leq b\left(\|(\hat{F}(\tau)-e^{-\tau H})H^{-1}\| + \frac{1}{\sqrt{\tau}}\|A^{-1/2}(\hat{F}(\tau)-e^{-\tau H})H^{-1}\|\right) \\ &\leq b(M'\tau + M'_{1/2}\tau). \end{aligned} \tag{5.42}$$

Notice that the estimate $M'\tau$ one finds in the same way as (5.26). Since $\|He^{-m\tau H}\| \leq d_1(H)/m\tau$, we obtain from (5.42) that

$$\|(I-\hat{F}(\tau))^{-1/2}(\hat{F}(\tau)-e^{-\tau H})e^{-m\tau H}\| \leq bd_1(H)(M'+M'_{1/2})/m. \tag{5.43}$$

By (5.39)-(5.43) one gets the estimate

$$\begin{aligned} &\|\hat{F}(\tau)^n - e^{-\tau n H}\| \leq \\ &\quad e^{\pi^2/8}\sqrt{\frac{c}{n-1}}\, b(2+M_{1/2}) + \sum_{m=1}^{n-1} e^{\pi^2/8}\sqrt{\frac{c}{n-m-1}}\, bd_1(H)(M'+M'_{1/2})\frac{1}{m} \\ &\leq e^{\pi^2/8}\sqrt{\frac{c}{n-1}}\, b(2+M_{1/2}) + e^{\pi^2/8}\sqrt{c}\, bd_1(H)(M'+M'_{1/2})\frac{2\ln n}{\sqrt{n}}, \end{aligned} \tag{5.44}$$

where we have used (3.31). Thus, setting

$$\hat{L} := e^{\pi^2/8}\sqrt{c}\, b\{2(2+M_{1/2}) + 4d_1(H)(M'+M'_{1/2})\}$$

we obtain (5.36). □

Corollary 5.9 *Let $A \geq I$ be a self-adjoint operator. Let B be an m-sectorial operator of semi-angle θ_B, obeying conditions (1.1) and (1.2), with associated closed sectorial form b. If in addition the condition (5.35) is satisfied, then there are $L > 0$ and $\hat{L}' > 0$ such that*

$$\begin{aligned} \left\| \left(e^{-tB/n} e^{-tA/n}\right)^n - e^{-tH} \right\| &\leq L \frac{\ln n}{\sqrt{n}}, && (5.45) \\ \left\| \left(e^{-tA/n} e^{-tB/n}\right)^n - e^{-tH} \right\| &\leq L' \frac{\ln n}{\sqrt{n}}, && (5.46) \end{aligned}$$

uniformly in $t \in [0, \infty)$.

Proof: We use the identity (cf.(5.2)) :

$$\begin{aligned} F(\tau)^n - e^{-tH} &= e^{-\tau B} e^{-\tau A/2} (\hat{F}(\tau)^{n-1} - e^{-tH}) e^{-\tau A/2} + && (5.47) \\ &+ (e^{-\tau B} - I) e^{-\tau A/2} e^{-tH} e^{-\tau A/2} + && (5.48) \\ &+ (e^{-\tau A/2} - I) e^{-tH} e^{-\tau A/2} + && (5.49) \\ &+ e^{-tH} (e^{-\tau A/2} - I). && (5.50) \end{aligned}$$

Since $\hat{F}(\tau)^{n-1} - e^{-tH} = \hat{F}(\tau)^{n-1}(I - \hat{F}(\tau)) + (\hat{F}(\tau)^n - e^{-tH})$, by estimates (5.3) and (5.36) we obtain an estimate for the first term (5.47). For the second term (5.48) we have

$$\begin{aligned} &\|(e^{-\tau B} - I) e^{-\tau A/2} e^{-tH} e^{-\tau A/2}\| \leq && (5.51) \\ &\quad \|(e^{-\tau B} - I) B^{-1}\| \|BA^{-1}\| \|e^{-\tau A/2}\| \|AH^{-1}\| \|He^{-tH}\| \leq \frac{\tau a}{1-a} \frac{d_1(H)}{t} \leq \frac{d_1(H) a}{n(1-a)}. \end{aligned}$$

For the third term (5.49) we find

$$\begin{aligned} &\|(e^{-\tau A/2} - I) e^{-tH} e^{-\tau A/2}\| \leq && (5.52) \\ &\quad \|(e^{-\tau A/2} - I) A^{-1}\| \|AH^{-1}\| \|He^{-tH}\| \leq \frac{\tau d_1(H)}{2(1-a)t} \leq \frac{d_1(H)}{2n(1-a)}. \end{aligned}$$

To estimate the last term (5.50) we use the identity $A^{1/2} A^{-1/2} = I$. Then

$$\begin{aligned} \|e^{-tH} A^{1/2} A^{-1/2} (e^{-\tau A/2} - I)\| &\leq \|e^{-tH} H_R^{1/2}\| \|H_R^{-1/2} A^{1/2}\| \|A^{-1/2}(e^{-\tau A/2} - I)\| \\ &\leq \|H_R e^{-tH^*}\| \|A^{1/2} H_R^{-1/2}\| \sqrt{\tau/e} \\ &\leq \frac{1}{\sqrt{e}} d_1(H^*)^{1/2} \|A^{1/2} H_R^{-1/2}\| n^{-1/2}, && (5.53) \end{aligned}$$

where we used estimate (5.18) and $\|A^{-1/2}(e^{-\tau A/2} - I)\| \leq (\tau/e)^{1/2}$ for the self-adjoint operator $A \geq I$. Notice that $\|A^{1/2} H_R^{-1/2}\| < \infty$ because $\mathrm{dom}(A^{1/2}) = \mathrm{dom}(a) = \mathrm{dom}(h) = \mathrm{dom}(H_R^{1/2})$. Taking into account the estimate of (5.47) and (5.51), (5.52), (5.53), we get

$$\left\| F(\tau)^n - e^{-tH} \right\| \leq \frac{c}{n-1} + \hat{L} \frac{\ln n}{\sqrt{n}} + \frac{d_1(H) a}{n(1-a)} + \frac{d_1(H)}{2n(1-a)} + \frac{1}{\sqrt{e}} \|A^{1/2} H_R^{-1/2}\| \frac{d_1(H^*)^{1/2}}{\sqrt{n}}. \tag{5.54}$$

which verifies the estimate (5.45).

In order to obtain (5.46) for $T(\tau) = e^{-\tau A} e^{-\tau B}$, we use the identity :

$$
\begin{aligned}
T(\tau)^n - e^{-tH} &= e^{-\tau A/2}(\hat{F}(\tau)^{n-1} - e^{-tH}) e^{-\tau A/2} e^{-\tau B} + && (5.55)\\
&\quad + e^{-\tau A/2} e^{-tH} e^{-\tau A/2}(e^{-\tau B} - I) + && (5.56)\\
&\quad + (e^{-\tau A/2} - I) e^{-tH} e^{-\tau A/2} + && (5.57)\\
&\quad + e^{-tH}(e^{-\tau A/2} - I). && (5.58)
\end{aligned}
$$

The last two terms (5.57), (5.58) are identical with those in (5.49), (5.50). To estimate the first term one has to proceed exactly as in the proof of (5.47). For the second term (5.56) we get the estimate :

$$
\begin{aligned}
\|e^{-\tau A/2} e^{-tH} e^{-\tau A/2}(e^{-\tau B} - I)\| &\leq \|e^{-tH} A^{1/2}\| \|A^{-1/2}(e^{-\tau B} - I)\| \\
&\leq \|A^{1/2} e^{-tH^*}\| \|A^{-1/2}(e^{-\tau B} - I)\| \\
&\leq \|A^{1/2} H_R^{-1/2}\| \frac{d_1(H^*)^{1/2}}{\sqrt{n}} (1 + \tan\theta_B) \|B_R^{1/2} A^{-1/2}\| 2 d_1(B)^{1/2},
\end{aligned}
$$

where we have used Lemma 5.6, the estimate of $\|e^{-tH} A^{1/2}\|$ as it is done in (5.53), and the fact that $\mathrm{dom}(A^{1/2}) = \mathrm{dom}(H_R^{1/2}) \subseteq \mathrm{dom}(B_R^{1/2})$. □

6 Remarks

Finally, let us make the following remarks:

(i) Taking into account Theorem 5.5 of [12], our Theorem 3.4 as well as results of the recent papers of [4] and [5], it seems natural to believe that the following conjecture is true: Let A be a self-adjoint operator such that $A \geq I$, let B be an m-accretive operator and let H be generator of a holomorphic semigroup such that the condition

$$
\mathrm{dom}(H^{\alpha'}) \subseteq \mathrm{dom}(A^{\alpha'}) \cap \mathrm{dom}(B^{\alpha'}) \qquad (6.1)
$$

holds for some $\alpha' \in (0, 1]$ and condition (1.6) is valid for some $\alpha \in (0, 1]$. If one has $\alpha' + \alpha > 1$, then the error-bound estimate for the Trotter product formula convergence is of the order $O(1/n^{\alpha'+\alpha-1})$.

(ii) Notice that if B is a self-adjoint operator, then one has $\alpha' = \alpha$. Hence $\alpha' + \alpha > 1$ yields $\alpha' = \alpha > 1/2$. This assumption is used in [12].

(iii) If $\alpha' = \alpha = 1$ and $B = B^*$, then the error-bound estimate for the Trotter product formula should be $O(1/n)$ instead of $O(\ln(n)/n)$ as it was proved in [11]. Indeed, this optimal error-bound is announced in [4] and [5]. We recall that in [12] the optimal error-bound $O(1/n^{2\alpha-1})$ is proved for $\alpha' = \alpha \in (1/2, 1)$ and $B = B^*$ under condition (1.6).

(iv) The conjecture (i) implicitly contains another one : the smallness of $B^{\alpha'}$ with respect $A^{\alpha'}$, which was assumed in [11] and [12], is *not* indispensable. This conjecture relies on recent results obtained in [4] and [5], which show that in the case $B = B^*$ and $\alpha' = \alpha = 1$ the smallness of B with respect to A used in [11] is indeed not necessary. For the case $\alpha' = \alpha \in (1/2, 1)$ studied in [12] this problem remains open.

(v) In application to the present accretive situation, i.e., when B is maximal accretive, this conjecture gives a strong hint that in [2] as well as in the present paper the smallness assumption (1.1) and (1.2) can be replaced by condition (6.1). For the beginning it would be useful to study the case $\alpha' = \alpha = 1$, and after that to make an extension to the general case $\alpha' + \alpha > 1$.

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