TROTTER PRODUCT FORMULA FOR NONSELF-ADJOINT GIBBS SEMIGROUPS

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Abstract

We prove the trace-norm convergence of the Trotter product formula for nonself-adjoint Gibbs semigroups. For any m-sectorial generators A and B such that $e^{-t\mathrm{Re}\,A}$ is in the trace-class for $t > 0$, the Trotter product formula converges in the trace-norm. With smallness conditions on B with respect to A, we give error bound estimates of the convergence rate in this topology.

Keywords : Gibbs semigroups, nonself-adjoint operator, Trotter formula, trace-norm convergence, holomorphic semigroups, sectorial operators.

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1 Motivations and results

Let $\mathfrak{H}$ be a separable Hilbert space with $\mathcal{B}(\mathfrak{H})$ the algebra of all bounded operators on $\mathfrak{H}$. Let $\mathcal{C}_\infty(\mathfrak{H})$ denote the ideal of compact operators in $\mathcal{B}(\mathfrak{H})$. Then the eigenvalues $\{\lambda_k(|C|)\}_{k=1}^\infty$ of the compact self-adjoint operator $|C| = \sqrt{C^*C}$ are known as the singular values $\{s_k(C)\}_{k=1}^\infty$ of the operator C. We denote by $\mathcal{C}_p(\mathfrak{H})$ $(1 \le p < \infty)$ the ideals in $\mathcal{B}(\mathfrak{H})$ consisting of all compact operators C such that

$$\sum_{k=1}^\infty s_k(C)^p < \infty,$$

(see e.g. [8, 16]). Then each ideal $\mathcal{C}_p(\mathfrak{H})$ is a Banach space with the norm:

$$\|C\|_p = \left(\sum_{k=1}^\infty s_k(C)^p\right)^{1/p}.$$

In particular the ideal $\mathcal{C}_1(\mathfrak{H})$ is called the *trace-class* , with the trace-norm $\|C\|_1 = \operatorname{Tr}|C|$.

Definition 1 *A strongly continuous one-parameter semigroup $\{U_t\}_{t\ge 0}$ is called a Gibbs semigroup if $U_t \in \mathcal{C}_1(\mathfrak{H})$ for any $t > 0$ (see e.g. [1, 17, 18, 19]).*

This class of semigroups arises naturally in quantum statistical mechanics, where the thermodynamical expectations are defined by the trace with the density matrix $e^{-\beta H}/\operatorname{Tr} e^{-\beta H}$ for a finite system with Hamiltonian H and temperature β^{-1}.

There are several results on the trace-norm convergence of the Trotter product formula for self-adjoint Gibbs semigroups : [18, 14, 10, 11, 15]. We extend them here to some *nonself-adjoint* semigroups. The first result concerns m-sectorial generators A and B and gives no error bound. The second result involves an m-sectorial or a positive self-adjoint generator A and an m-accretive generator B with some smallness conditions on B which imply error bound estimates of the convergence rate in the trace-norm topology.

Recall that (see e.g. [12, Ch.V,VI,IX]) the operator A in a Hilbert space $\mathfrak{H}$ is accretive if its numerical range $\operatorname{Nr}(A) = \{(u, Au), u \in \operatorname{dom}(A), \|u\| = 1\}$ is a subset of the right closed half-plane $\overline{\mathbb{C}}_+ = \{z \in \mathbb{C}, \operatorname{Re} z \ge 0\}$. If in addition the left open half-plane is a subset of the resolvent set of A, the operator A is said to be m-accretive. An m-accretive operator A is called m-sectorial (with semi-angle $\theta \in [0, \pi/2)$) if its numerical range $\operatorname{Nr}(A) \subseteq \overline{S}_\theta$, i.e., it is a subset of the closure of the sector $S_\theta = \{z \in \mathbb{C}, z \ne 0, |\arg z| < \theta\}$. The set of the m-sectorial operators with semi-angle θ coincides with the set of generators of semigroups, which are holomorphic and contractive in the sector $S_{\pi/2-\theta}$ (see e.g. [12, Ch.IX, Theorem 1.24]).

Theorem 1 *Let A and B be m-sectorial operators in a Hilbert space $\mathfrak{H}$ such that $e^{-t\mathrm{Re}\, A} \in \mathcal{C}_1(\mathfrak{H})$ for $t > 0$. Then the Trotter product formula converges in the trace-norm to $e^{-t(A\dot{+}B)}P_0$, where P_0 is the orthogonal projector on the closure $\overline{\mathrm{dom}(a) \cap \mathrm{dom}(b)}$ of the intersection of the domains of the closed sectorial forms associated with A and B. Here $A\dot{+}B$ is the form-sum of the operators A and B.*

If one adds some smallness conditions on B with respect to A, then we obtain error bound estimates for the Trotter product formula in the trace-norm.

Theorem 2 *Let A be an m-sectorial operator in a Hilbert space $\mathfrak{H}$ such that $e^{-t\mathrm{Re}\, A} \in \mathcal{C}_1(\mathfrak{H})$ for $t > 0$. If B is an m-accretive operator in $\mathfrak{H}$ such that $\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B)$ for some $\alpha \in [0, 1)$ and $\mathrm{dom}(A^*) \subseteq \mathrm{dom}(B^*)$, then the Trotter formula converges in the trace-norm uniformly for $t \geq t_0 > 0$, and we have:*

$$\left\| \left(e^{-tA/n} e^{-tB/n}\right)^n - e^{-t(A+B)} \right\|_1 \leq O\left(\frac{\ln n}{n^{1-\alpha}}\right), \text{ or } O\left(\frac{(\ln n)^2}{n}\right) \text{ if } \alpha = 0. \tag{1}$$

Theorem 3 *Let A be a positive self-adjoint operator in $\mathfrak{H}$, such that $(\lambda + A)^{-1} \in \mathcal{C}_p(\mathfrak{H})$ for some $\lambda \in \mathbb{C}$ and finite $p \geq 1$. If B is an m-accretive operator such that $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ and $\mathrm{dom}(A) \subseteq \mathrm{dom}(B^*)$ with:*

$$\|Bu\| \leq a\|Au\|, \; u \in \mathrm{dom}(A), \; 0 < a < 1, \tag{2}$$

$$\|B^*u\| \leq a_*\|Au\|, \; u \in \mathrm{dom}(A), \; 0 < a_* < 1, \tag{3}$$

then the Trotter formula converges in trace-norm uniformly for $t \geq t_0 > 0$, and we have:

$$\left\| \left(e^{-tA/n} e^{-tB/n}\right)^n - e^{-t(A+B)} \right\|_1 \leq O\left(\frac{\ln n}{n}\right). \tag{4}$$

The following Section contains some indispensable preliminaries. The proof of Theorem 1 is given in Section 3, whereas the two results with estimates (1), (4) are proved in Section 4.

2 Preliminaries

Here we collect some preliminary results which we shall use for proofs of our Theorems.

Proposition 1 (Theorem 1 of [9]) *The maps* $(A,B)\mapsto AB$, $(A,B)\mapsto BA$ *are continuous on* $\mathcal{B}\times\mathcal{C}_p(\mathfrak{H})\to\mathcal{C}_p(\mathfrak{H})$, $\mathcal{B}$ *being some bounded set in* $\mathcal{B}(\mathfrak{H})$ *with the strong operator topology and* $\mathcal{C}_p(\mathfrak{H})$ *carrying its* $\|\cdot\|_p$*-topology,* $1\leq p\leq\infty$.

As a first consequence, one gets that any Gibbs semigroup is continuous in the trace-norm topology for $t>0$. Furthermore, we have the following

Corollary 1 *If* $z\mapsto A(z)\in\mathcal{B}(\mathfrak{H})$ *is a holomorphic function for* $z\in D\subseteq\mathbb{C}$, *and* $z\mapsto B(z)\in\mathcal{C}_p(\mathfrak{H})$ *is* $\|\cdot\|_p$*-holomorphic in* D, *then* $A(z)B(z)$ *and* $B(z)A(z)$ *are* $\|\cdot\|_p$*-holomorphic in* D. *In particular, if* U_t *is a holomorphic semigroup in some open sector* $S_\theta=\{z\in\mathbb{C}:\ z\neq 0,\ |\arg z|<\theta\}$, *such that* $U_{z_0}\in\mathcal{C}_1(\mathfrak{H})$, $z_0\in S_\theta$, *then* U_t *is* $\|\cdot\|_1$*-holomorphic for* $t-z_0\in S_\theta$.

The following Lemma is an extension of [2, Theorem 2] to any m-sectorial generator.

Lemma 1 *Let* A *be an m-sectorial operator, with the real part* $\operatorname{Re}A$, *which is a positive self-adjoint operator. If* $e^{-t\operatorname{Re}A}\in\mathcal{C}_1(\mathfrak{H})$ *for* $t>0$, *then also* $e^{-tA}\in\mathcal{C}_1(\mathfrak{H})$ *and*

$$\|e^{-tA}\|_1\leq\cdots\leq\|e^{-tA/2^p}\|_{2^p}^{2^p}\leq\|e^{-tA/2^{p+1}}\|_{2^{p+1}}^{2^{p+1}}\leq\cdots\leq\lim_{p\to\infty}\|e^{-tA/2^p}\|_{2^p}^{2^p}=\|e^{-t\operatorname{Re}A}\|_1. \quad (5)$$

Moreover if the generator A *is normal, then all these inequalities become equalities.*

Proof Notice that by $(I+\operatorname{Re}A)^{-1}=\int_0^\infty e^{-t(I+\operatorname{Re}A)}dt$ and by [12, Ch.VI, Theorem 3.3] $e^{-t\operatorname{Re}A}\in\mathcal{C}_1(\mathfrak{H})$ implies that $\operatorname{Re}A$, and thus A, have compact resolvents. Then by [3, Theorem 1], the Trotter product formula

$$\|(e^{-tA^*/2n}e^{-tA/2n})^n-e^{-t\operatorname{Re}A}\|\underset{n\to\infty}{\longrightarrow}0 \quad (6)$$

converges in operator-norm, where the form-sum $(A\dot{+}A^*)/2$ is by definition the real part $\operatorname{Re}A$ of the operator A. Since for any compact operators C and D one has [8, Ch.II, Corollary 2.3]

$$|s_k(C)-s_k(D)|\leq\|C-D\|, \quad (7)$$

this convergence implies the convergence of all the singular values of the compact operators involved in the Trotter formula:

$$s_k((e^{-tA^*/2n}e^{-tA/2n})^n) = s_k(e^{-tA/2n})^{2n} \underset{n\to\infty}{\longrightarrow} s_k(e^{-t\mathrm{Re}\,A}), \tag{8}$$

for all $k \geq 1$. Moreover, by virtue of the property [8, Ch.II, Corollary 4.2] of singular values, one has for any integer $k \geq 1$ and $p \geq 0$:

$$S_k(p) = \sum_{j=1}^{k} s_j(e^{-tA/2^p})^{2^p} \leq \sum_{j=1}^{k} s_j(e^{-tA/2^{p+1}})^{2^{p+1}} = S_k(p+1). \tag{9}$$

By (8), the finite sum $S_k(p)$ has the limit

$$S_k = \sum_{j=1}^{k} s_j(e^{-t\mathrm{Re}\,A}) \tag{10}$$

when $p \to \infty$. Thus we get $S_k(p) \leq S_k(p+1) \leq \cdots \leq S_k$ for $k = 1, 2, \ldots$. Since $e^{-t\mathrm{Re}\,A} \in \mathcal{C}_1(\mathfrak{H})$, the sums $\{S_k(p)\}_{k\geq 1, p\geq 0}$ are bounded by $\sup_{k\geq 1} S_k = \|e^{-t\mathrm{Re}\,A}\|_1$. Then $\lim_{k\to\infty} S_k(p=0) = \|e^{-tA}\|_1$. By (9) and by definition of the norm $\|\cdot\|_{2^p}$ one gets $\lim_{k\to\infty} S_k(p) = \|e^{-tA/2^p}\|_{2^p}^{2^p}$. Then taking the limit $k \to \infty$ in $S_k(0) \leq \cdots \leq S_k(p) \leq S_k(p+1) \leq \|e^{-t\mathrm{Re}\,A}\|_1$ we get the announced inequalities (5), and $\lim_{p\to\infty} \|e^{-tA/2^p}\|_{2^p}^{2^p} \leq \|e^{-t\mathrm{Re}\,A}\|_1$.

Notice that $\|e^{-tA/2^p}\|_{2^p}^{2^p} \geq S_k(p)$, cf (9). Then $\lim_{p\to\infty} \|e^{-tA/2^p}\|_{2^p}^{2^p} \geq \lim_{p\to\infty} S_k(p) = S_k$ for all $k \geq 1$. Thus, $\lim_{p\to\infty} \|e^{-tA/2^p}\|_{2^p}^{2^p} \geq \sup_{k\geq 1} S_k = \|e^{-t\mathrm{Re}\,A}\|_1$, and we conclude that $\lim_{p\to\infty} \|e^{-tA/2^p}\|_{2^p}^{2^p} = \|e^{-t\mathrm{Re}\,A}\|_1$ in (5).

If A is normal, then e^{-tA} is also normal. Then we have $s_k(e^{-tA}) = |\lambda_k(e^{-tA})|$ and the spectral representations

$$A = \sum_{k=1}^{\infty} \lambda_k(A) P_k, \quad e^{-tA} = \sum_{k=1}^{\infty} e^{-t\lambda_k(A)} P_k, \quad \mathrm{Re}\,A = \sum_{k=1}^{\infty} \mathrm{Re}\,\lambda_k(A) P_k, \tag{11}$$

where P_k is the (orthogonal) spectral projector associated to the eigenvalue $\lambda_k(A)$. Therefore,

$$s_k(e^{-tA}) = |\lambda_k(e^{-tA})| = |e^{-t\lambda_k(A)}| = e^{-t\mathrm{Re}\,\lambda_k(A)} = e^{-t\lambda_k(\mathrm{Re}\,A)} = s_k(e^{-t\mathrm{Re}\,A}), \tag{12}$$

and we obtain $\|e^{-tA}\|_1 = \|e^{-t\mathrm{Re}\,A}\|_1$ in (5). $\square$

Remark 1 *Recall that if* $K_n \xrightarrow{s} K$ *and* $K_n^* \xrightarrow{s} K^*$ *in the strong operator topology for* $n \to \infty$, *and* $\|K_n\|_p \to \|K\|_p$ *for some* $p \in [1, \infty)$, *then* $\|K_n - K\|_p \to 0$ *(Grümm's convergence theorem [9]). Since by (5) one gets :*

$$\left\| \left(e^{-tA^*/2^p} e^{-tA/2^p}\right)^{2^{p-1}} \right\|_1 = \|e^{-tA/2^p}\|_{2^p}^{2^p} \underset{p\to\infty}{\longrightarrow} \|e^{-t\mathrm{Re}\, A}\|_1, \tag{13}$$

the convergences (6) implies the $\|\cdot\|_1$*-convergence of the Trotter product formula for* $\mathrm{Re}\, A = (A \dot{+} A^*)/2$*:*

$$\left\| \left(e^{-tA^*/2^p} e^{-tA/2^p}\right)^{2^{p-1}} - e^{-t\mathrm{Re}\, A} \right\|_1 \underset{p\to\infty}{\longrightarrow} 0. \tag{14}$$

The next theorem is a generalization of the Ginibre-Gruber inequality [7] to m-sectorial generators and to the trace-norm.

Theorem 4 *Let* A *be an m-sectorial operator such that* $e^{-t\mathrm{Re}\, A} \in \mathcal{C}_1(\mathfrak{H})$, *for* $t > 0$. *Let* $V_1, \ldots, V_n$ *be bounded operators on* $\mathfrak{H}$. *Then for any positive numbers* $t_1, \ldots, t_n$ *one has:*

$$\left\| \prod_{i=1}^n e^{-t_i A} V_i \right\|_1 \leq \mathrm{Tr}\ e^{-(t_1+\cdots+t_n)\mathrm{Re}\, A/4} \prod_{i=1}^n \|V_i\|. \tag{15}$$

Proof First we suppose that $V_1, \ldots, V_n$ are compact operators. We set $t_m = \min\{t_1, \ldots, t_n\}$ and $T = t_1 + \cdots + t_n$. For $i = 1, \ldots, n$ we define $l_i \in \mathbb{N}$ by $2^{l_i} t_m \leq t_i < 2^{l_i+1} t_m$. Hence we have $\sum_{i=1}^n 2^{l_i} t_m > T/2$ and

$$\prod_{i=1}^n e^{-t_i A} V_i = \prod_{i=1}^n e^{-(t_i - 2^{l_i} t_m)A} (e^{-t_m A})^{2^{l_i}} V_i. \tag{16}$$

Then by [8, Ch.II, Corollary 4.1], extended to n operators, we obtain:

$$\begin{aligned} \left\| \prod_{i=1}^n e^{-t_i A} V_i \right\|_1 &= \sum_{k=1}^\infty s_k \left(\prod_{i=1}^n e^{-(t_i - 2^{l_i} t_m)A} (e^{-t_m A})^{2^{l_i}} V_i \right) \\ &\leq \sum_{k=1}^\infty \prod_{i=1}^n s_k(e^{-(t_i - 2^{l_i} t_m)A}) s_k(e^{-t_m A})^{2^{l_i}} s_k(V_i) \\ &\leq \sum_{k=1}^\infty s_k(e^{-t_m A})^{2^{l_1}+\cdots+2^{l_n}} \prod_{i=1}^n \|V_i\|, \end{aligned}$$

where we have used that for any $k \geq 1$, $s_k(e^{-(t_i - 2^{l_i} t_m)A}) \leq \|e^{-(t_i - 2^{l_i} t_m)A}\| \leq 1$ and $s_k(V_i) \leq \|V_i\|$. By setting $N = 2^{l_1} + \cdots + 2^{l_n}$ and $T_m = N t_m > T/2$, we obtain

$$\left\| \prod_{i=1}^n e^{-t_i A} V_i \right\|_1 \leq \|e^{-T_m A/N}\|_N^N \prod_{i=1}^n \|V_i\|. \tag{17}$$

In order to apply Lemma 1, we consider $p \in \mathbb{N}$ such that $2^p \leq N < 2^{p+1}$. Then we have $2^p T_m/N > T_m/2 > T/4$. Hence

$$\|e^{-T_m A/N}\|_N^N = \sum_{k=1}^{\infty} s_k(e^{-T_m A/N})^N \leq \sum_{k=1}^{\infty} s_k(e^{-2^p T_m A/2^p N})^{2^p} \leq \sum_{k=1}^{\infty} s_k(e^{-TA/2^{p+2}})^{2^p}, \quad (18)$$

by $s_k(e^{-T_m A/N}) = s_k(e^{-2^p T_m A/2^p N}) \leq \|e^{-T_m A/N}\| \leq 1$ and $s_k(e^{-(t+\tau)A}) \leq \|e^{-tA}\| s_k(e^{-\tau A}) \leq s_k(e^{-\tau A})$ for any $t, \tau > 0$. Finally by (17), (18) and Lemma 1 one gets the estimate (15).

General case. For $0 < \epsilon < t_m$, we set $\tilde{V}_i = e^{-\epsilon A} V_i$. Then $\tilde{V}_i \in \mathcal{C}_1(\mathfrak{H})$ and $s_k(\tilde{V}_i) \leq \|\tilde{V}_i\| \leq \|V_i\|$. Then we set $\tilde{t}_i = t_i - \epsilon$, and we obtain

$$\left\| \prod_{i=1}^{n} e^{-t_i A} V_i \right\|_1 \leq \mathrm{Tr}\ e^{-(\tilde{t}_1 + \cdots + \tilde{t}_n)\mathrm{Re}\, A/4} \prod_{i=1}^{n} \|V_i\|. \quad (19)$$

By the $\|\cdot\|_1$-continuity of the semigroup $e^{-t\mathrm{Re}\, A}$, we can now take the limit $\epsilon \to 0$, which gives the inequality (15) in the general case. □

3 Trace-norm convergence without error bound

In this section we extend the method of [3] to the case of Gibbs semigroups. Let A and B be m-sectorial operators in a Hilbert space $\mathfrak{H}$, with semi-angles θ_A, θ_B. We denote by a and b the associated closed sectorial forms. Following [3] we consider the forms $a(z) = \mathrm{Re}\, a + z \mathrm{Im}\, a$ and $b(z) = \mathrm{Re}\, b + z \mathrm{Im}\, b$, with $\mathrm{dom}(a(z)) = \mathrm{dom}(a)$, $\mathrm{dom}(b(z)) = \mathrm{dom}(b)$, and the corresponding m-sectorial operators $A(z)$ and $B(z)$. By [3, Corollary 1] these operators form holomorphic families of type (B) respectively in the domains $D_A = \{z \in \mathbb{C}, |\mathrm{Re}\, z| < \cot \theta_A\}$ and $D_B = \{z \in \mathbb{C}, |\mathrm{Re}\, z| < \cot \theta_B\}$.

Lemma 2 *If A is an m-sectorial operator such that $e^{-t\mathrm{Re}\, A} \in \mathcal{C}_1(\mathfrak{H})$ for $t > 0$, then $e^{-tA(z)} \in \mathcal{C}_1(\mathfrak{H})$ for $z \in D_A$. Moreover $e^{-tA(z)}$ is $\|\cdot\|_1$-holomorphic in z, as well as in t in some sector depending on z.*

Proof Lemma 1 implies that e^{-tA} is compact and the operators A and $\mathrm{Re}\, A$ have compact resolvents [12, Ch.V, Theorem 3.3]. Moreover, since $A(z)$ is a holomorphic family of type (B), it has compact resolvent either for no or for any $z \in D_A$ [12, Ch.VII, Theorem 4.3], the same is true for $\mathrm{Re}\, A(z)$.

Since

$$\operatorname{Re} a(z)[u] \geq (1 - |\operatorname{Re} z| \tan\theta_A)\operatorname{Re} a[u], \ u \in \operatorname{dom}(a) \tag{20}$$

(cf (6) in [3]), by the minimax principle for the self-adjoint operator $\operatorname{Re} A(z)$ and by condition $e^{-t\operatorname{Re} A} \in \mathcal{C}_1(\mathfrak{H})$ one gets that $e^{-t\operatorname{Re} A(z)} \in \mathcal{C}_1(\mathfrak{H})$ for any $z \in D_A$. Finally, by Lemma 1 we obtain that $\|e^{-tA(z)}\|_1 \leq \|e^{-t\operatorname{Re} A(z)}\|_1 < \infty$.

On the other hand, for the holomorphic family $\{e^{-tA(z)}\}_{z\in D_A}$ [3, Proposition 3], we have the $\|\cdot\|$-convergent Cauchy integral:

$$e^{-tA(z)} = \frac{1}{2\pi i}\int_\gamma \frac{e^{-tA(w)}}{z-w}dw, \tag{21}$$

where γ is a small circle with center z and radius r inside D_A. Moreover, we have a trace-norm estimate of the derivative:

$$\left\| \frac{d}{dz}\frac{e^{-tA(w)}}{z-w} \right\|_1 \leq \frac{\|e^{-t\operatorname{Re} A(w)}\|_1}{r^2} \leq \|e^{-ct\operatorname{Re} A}\|_1 \frac{1}{r^2}, \tag{22}$$

where $c = \inf_{w\in\gamma}(1 - |\operatorname{Re} w| \tan\theta_A)$ by (20). Thus the Cauchy integral (21) can be differentiated in trace-norm, i.e. $z \mapsto e^{-tA(z)}$ is $\|\cdot\|_1$-holomorphic in D_A. The $\|\cdot\|_1$-holomorphic properties in t follows from $e^{-tA(z)} \in \mathcal{C}_1(\mathfrak{H})$, Corollary 1 and the fact that $A(z)$ is m-sectorial for $z \in D_A$.

□

Proof of Theorem 1

We define for $z \in D = D_A \cap D_B$ the sequence of operator-valued functions:

$$f_n(z) = \left(e^{-tA(z)/n}e^{-tB(z)/n}\right)^n, \ n \geq 1.$$

By Lemma 2 one has $e^{-tA(z)/n} \in \mathcal{C}_1(\mathfrak{H})$, thus $f_n(z) \in \mathcal{C}_1(\mathfrak{H})$. Moreover, since $e^{-tA(z)/n}$ is $\|\cdot\|_1$-holomorphic for $z \in D_A$ and since $e^{-tB(z)/n}$ is $\|\cdot\|$-holomorphic (because $B(z)$ is a holomorphic family of type (B) for $z \in D_B$ [3, Proposition 3]), $z \mapsto f_n(z)$ is a $\|\cdot\|_1$-holomorphic operator-valued function. Notice that by Theorem 4, the sequence $\{f_n(z)\}_{n\geq 1}$ is uniformly bounded in the trace-norm for $z \in D$:

$$\|f_n(z)\|_1 \leq \|e^{-tB(z)/n}\|^n \operatorname{Tr} e^{-t\operatorname{Re} A(z)/4} \leq \|e^{-t\operatorname{Re} A(z)/4}\|_1. \tag{23}$$

By [15, Theorem 3.5], for real $z \in D$ the sequence $\{f_n(z)\}_{n\geq 1}$ converges in trace-norm to $e^{-tC}P_0$ where $C = A(z)\dot{+}B(z)$ is the form-sum. Then by (23) and the Theorem of Vitali we get the $\|\cdot\|_1$-convergence for any $z \in D$. Since $z = i \in D$, we obtain the assertion.

□

4 Trace-norm convergence with error bound

In this section we show that the error bound estimates for the Trotter product formula in operator-norm can be extended to the trace-norm, in those cases when one can use the perturbation results of [19]. Two classes of perturbations were studied there: $\mathcal{P}_0$ and $\mathcal{P}_1$.

Definition 2 *A closed operator B belongs to the class $\mathcal{P}_0(A)$ of a strongly continuous semigroup U_t generator A, if the domain* $\mathrm{dom}(B) \supseteq \bigcup_{t>0} U_t\mathfrak{H}$ *and*

$$\int_0^1 dt\|BU_t\| < \infty. \tag{24}$$

The perturbations $B \in \mathcal{P}_0$ are also called the "class $\mathcal{P}$ perturbations" in [6].

Definition 3 *A closed operator B belongs to the class $\mathcal{P}_1(A)$ of perturbations for the generator A of a strongly continuous semigroup if there are $a \geq 0$ and $0 \leq b < 1$ such that:*

$$\mathrm{dom}(A) \subseteq \mathrm{dom}(B) \ \textit{and} \ \|Bu\| \leq a\|u\| + b\|Au\|, \ u \in \mathrm{dom}(A). \tag{25}$$

Notice that $\mathcal{P}_0 \subseteq \mathcal{P}_1$, by [6, Lemma 3.4].

Following [19, Corollary 2.1], for perturbations $\mathcal{P}_0$ we have:

Proposition 2 *Let A be an m-sectorial operator such that $e^{-t\mathrm{Re}\,A} \in \mathcal{C}_1(\mathfrak{H})$ for $t > 0$ and $B \in \mathcal{P}_0(A)$. Then $A + \lambda B$ is the generator of a Gibbs semigroup, $e^{-t(A+\lambda B)}$, which is a $\|\cdot\|_1$- holomorphic function of $\lambda \in \mathbb{C}$.*

For perturbations of the class $\mathcal{P}_1$, one has [13, 19] :

Proposition 3 *Let A be a positive self-adjoint operator such that $(\zeta + A)^{-1} \in \mathcal{C}_p(\mathfrak{H})$ for some $\zeta \in \mathbb{C}$ and finite $p \geq 1$. If $B \in \mathcal{P}_1(A)$, then for $\lambda \in C_b = \{\lambda \in \mathbb{C}, |\lambda| < b^{-1}\}$, $A+\lambda B$ is the generator of a Gibbs semigroup, $G_\lambda(t) = e^{-t(A+\lambda B)}$, which is a $\|\cdot\|_1$-holomorphic mapping $G_\lambda(t) : S_{\lambda,b} \to \mathcal{C}_1(\mathfrak{H})$. Here $S_{\lambda,b}$ is the sector*

$$S_{\lambda,b} = \left\{ z \in \mathbb{C}, \left|\frac{\mathrm{Im}\, z}{\mathrm{Re}\, z}\right| < \frac{\sqrt{1-|\lambda|^2 b^2}}{|\lambda| b} \right\}.$$

Proposition 4 (Theorem 1 of [4]) *Let e^{-tA} be a holomorphic semigroup on a Hilbert space $\mathfrak{H}$ and $\|e^{-tA}\| \leq 1$ for $t \geq 0$. If B is an m-accretive operator in $\mathfrak{H}$ such that*

$\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B)$ *for some* $\alpha \in [0,1)$ *and* $\mathrm{dom}(A^*) \subseteq \mathrm{dom}(B^*)$*, then the Trotter product formula converges in operator-norm uniformly in* $t \geq 0$ *and:*

$$\left\| \left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-t(A+B)} \right\| \leq O\left(\frac{\ln n}{n^{1-\alpha}}\right), \text{ or } O\left(\frac{(\ln n)^2}{n}\right) \text{ if } \alpha = 0. \tag{26}$$

Proposition 5 (Theorem 3.5 of [5]) *Let A be a self-adjoint positive operator in $\mathfrak{H}$. Let B be an m-accretive operator such that* $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ *and* $\mathrm{dom}(A) \subseteq \mathrm{dom}(B^*)$ *with:*

$$\begin{aligned} \|Bu\| &\leq b\|Au\|,\ u \in \mathrm{dom}(A),\ 0 < b < 1, &(27)\\ \|B^*u\| &\leq b_*\|Au\|,\ u \in \mathrm{dom}(A),\ 0 < b_* < 1. &(28)\end{aligned}$$

Then the Trotter product formula converges in operator-norm uniformly in $t \geq 0$ *and:*

$$\left\| \left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-t(A+B)} \right\| \leq O\left(\frac{\ln n}{n}\right). \tag{29}$$

Lemma 3 *Let A be an m-sectorial operator such that $e^{-t\mathrm{Re}\,A} \in \mathcal{C}_1(\mathfrak{H})$ for $t > 0$. Let B be generator of a contraction semigroup and $\{e^{-tH}\}_{t>0}$ be Gibbs semigroup with $\|e^{-tH}\| \leq 1$. Let*

$$\epsilon(s,t) = \sup_{\frac{2s}{2s+1}t \leq \tau \leq \frac{2s}{2s-1}t} \left\| \left(e^{-\tau A/s}e^{-\tau B/s}\right)^s - e^{-\tau H} \right\|, \tag{30}$$

for $s \in \mathbb{N}$, $s > 1$ and $t > 0$. Then for each $t_0 > 0$ there exist $L(t_0)$ and $L'(t_0)$ such that:

$$\left\| \left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-tH} \right\| \leq L(t_0)\epsilon([n/2], t/2) + L'(t_0)\epsilon([(n+1)/2], t/2) \tag{31}$$

for all $n > 1$ and $t \geq t_0$.

Proof For any $n > 1$, we define two variables $k_n = [n/2]$ and $m_n = [(n+1)/2]$, where $[x]$ denotes the integer part of $x \geq 0$. Then $n = k_n + m_n$, and one has the representation :

$$\begin{aligned}\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-tH} &= \left(\left(e^{-tA/n}e^{-tB/n}\right)^{k_n} - e^{-k_n tH/n}\right)\left(e^{-tA/n}e^{-tB/n}\right)^{m_n}\\ &\quad + e^{-k_n tH/n}\left(\left(e^{-tA/n}e^{-tB/n}\right)^{m_n} - e^{-m_n tH/n}\right).\end{aligned}$$

By Theorem 4 and by $m_n \geq (n-1)/2 > n/2$, $n \in \mathbb{N}$, one gets the estimate :

$$\left\| \left(e^{-tA/n}e^{-tB/n}\right)^{m_n} \right\|_1 \leq \left\|e^{-tB/n}\right\|^{m_n}\ \mathrm{Tr}\, e^{-m_n t\mathrm{Re}\,A/4n} \leq \|e^{-t\mathrm{Re}\,A/8}\|_1, \tag{32}$$

uniformly in $n > 1$. On the other hand, since $e^{-tH}P_0 \in \mathcal{C}_1(\mathfrak{H})$ and $k_n/n > 1/3$, $n \in \mathbb{N}$, we get $\|e^{-k_n tH/n}\|_1 \leq \|e^{-tH/3}\|_1$ for those n. Therefore,

$$\begin{aligned}\left\|\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-tH}\right\|_1 &\leq \left\|\left(\left(e^{-tA/n}e^{-tB/n}\right)^{k_n} - e^{-k_n tH/n}\right)\right\| \|e^{-t\mathrm{Re}\,A/8}\|_1 \\ &+ \|e^{-tH/3}\|_1 \left\|\left(\left(e^{-tA/n}e^{-tB/n}\right)^{m_n} - e^{-m_n tH/n}\right)\right\|,\end{aligned}$$

for $n > 1$. Let $t_n = tk_n/n$ and $s_n = tm_n/n$, i.e. $t_n, s_n \to t/2$ as $n \to \infty$, and let $t_0 > 0$. Then by (30) one gets

$$\begin{aligned}\left\|\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-tH}\right\|_1 &\leq \left\|\left(\left(e^{-t_nA/k_n}e^{-t_nB/k_n}\right)^{k_n} - e^{-t_nH}\right)\right\| \|e^{-t\mathrm{Re}\,A/8}\|_1 \\ &+ \|e^{-tH/3}\|_1 \left\|\left(\left(e^{-s_nA/m_n}e^{-s_nB/m_n}\right)^{m_n} - e^{-s_nH}\right)\right\| \\ &\leq \epsilon(k_n, t/2)\|e^{-t_0\mathrm{Re}\,A/8}\|_1 + \|e^{-t_0H/3}\|_1\epsilon(m_n, t/2)\end{aligned}$$

for $t \geq t_0 > 0$ and $n > 1$. Thus, we obtain (31) by setting $L(t_0) = \|e^{-t_0\mathrm{Re}\,A/8}\|_1$ and $L'(t_0) = \|e^{-t_0H/3}\|_1$.

□

Proof of Theorem 2

The condition $\mathrm{dom}(A^\alpha) \subseteq \mathrm{dom}(B)$ implies that $B \in \mathcal{P}_0(A)$ because [4, Proposition 6]:

$$\int_0^1 dt\|Be^{-tA}\| \leq \int_0^1 dt\|BA^{-\alpha}\|\|A^\alpha e^{-tA}\| \leq \|BA^{-\alpha}\| \int_0^1 dt\frac{M_\alpha}{t^\alpha} < \infty \tag{33}$$

for $0 \leq \alpha < 1$. By Proposition 2 we conclude that $A + B$ generates a Gibbs semigroup. Now by Lemma 3 with the estimate of Proposition 4, we obtain the announced estimate.
□

Proof of Theorem 3

By the hypothesis of the theorem, $B \in \mathcal{P}_1(A)$. Thus one uses Proposition 3, which shows that $A+B$ generates a Gibbs semigroup. Hence by the Lemma 3 and the estimate of Proposition 5, one gets the announced error bound estimate. □

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