Renormalization Group and the Melnikov Problem for PDE's

Jean Bricmont[1,†], **Antti Kupiainen**[2,‡], **Alain Schenkel**[2]

[1] UCL, FYMA, 2 chemin du Cyclotron, B-1348 Louvain-la-Neuve, Belgium
[2] Department of Mathematics, Helsinki University, P.O. Box 4, 00014 Helsinki, Finland

Abstract. We give a new proof of persistence of quasi-periodic, low dimensional elliptic tori in infinite dimensional systems. The proof is based on a renormalization group iteration that was developed recently in [BGK] to address the standard KAM problem, namely, persistence of invariant tori of maximal dimension in finite dimensional, near integrable systems. Our result covers situations in which the so called normal frequencies are multiple. In particular, it provides a new proof of the existence of small-amplitude, quasi-periodic solutions of nonlinear wave equations with periodic boundary conditions.

1. Introduction

In this paper, we address the persistence problem of quasi-periodic, low dimensional, elliptic tori in infinite dimensional systems. A typical example that we will consider is the nonlinear wave equation (NLW) on a bounded interval,

$$\partial_t u = \partial_x^2 u - Vu + f(u), \tag{1.1}$$

with Dirichlet or periodic boundary conditions and $f(u) = \mathcal{O}(u^3)$. The first results concerning the existence of quasi-periodic solutions of (1.1) were obtained independently by Kuksin, Pöschel and Wayne, [K, P1, W]. They extended to infinite dimensional Hamiltonian systems Eliasson's proof, [E], of the so called Melnikov problem, i.e., the persistence of elliptic invariant tori of dimension lower than the number of degrees of freedom. Based on the Kolmogorov-Arnold-Moser (KAM) approach, these results were restricted, however, to Dirichlet or Neumann boundary conditions and to specific classes of potential V excluding, in particular, the case $V = Const.$ In [P2], Pöschel covered the case of constant potentials by exploiting the existence of a Birkhoff normal form for

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the Hamiltonian of (1.1). The normal form allowed him to control the torus frequencies via amplitude-frequency modulation, and therefore to dispense with outer parameters provided by an adjustable potential $V(x)$. This approach was applied in [KP] to the persistence of quasi-periodic solutions for the nonlinear Schrödinger equation (NLS) subject to Dirichlet (or Neumann) boundary conditions.

The case of periodic boundary conditions is more delicate due to the fact that the eigenvalues of the Sturm-Liouville operator $L = -d^2/dx^2 + V$ are degenerate. This leads to resonances between pairs of frequencies corresponding to motion in directions normal to the torus (the so called normal frequencies). These additional resonances prevents one from controlling quadratic terms in the Hamiltonian of the system and do not seem to be addressable by KAM techniques. (This difficulty also appears in finite-dimensional Melnikov situations.) Developing new techniques based on the Lyapunov-Schmidt method, Craig and Wayne proved in [CW] persistence of periodic solutions of the NLW with periodic boundary conditions. Later, their approach was significantly improved by Bourgain in [B1-2] who constructed quasi-periodic solutions of the NLW and NLS with periodic boundary conditions. Most notably, it is shown in [B2] that solutions of this type can be constructed, in particular, for the NLS on two-dimensional domains. The usual Melnikov nonresonance condition reads, with $\omega \in \mathbb{R}^d$ and $\mu \in \mathbb{R}^n$ denoting the torus and, respectively, the normal frequencies (n is possibly infinite),

$$\langle k, \omega\rangle + \langle l, \mu\rangle \neq 0, \quad k \in \mathbb{Z}^d,\, l \in \mathbb{Z}^n \text{ with } |k| + |l| \neq 0,\, |l| \leq 2. \tag{1.2}$$

In Bourgain's approach and at the price of a considerable technical effort, condition (1.2) is reduced to

$$\langle k, \omega\rangle + \mu_s \neq 0, \quad k \in \mathbb{Z}^d,\ s = 1, \ldots, n,$$

i.e., all nonresonance conditions on pairs of normal frequencies are absent. More recently, Chierchia and You, see [Y,CY], showed that persistence of quasi-periodic solutions of the NLW with periodic boundary conditions is tractable by KAM techniques. Their nonresonance condition,

$$\langle k, \omega\rangle + \langle l, \mu\rangle \neq 0, \quad k \in \mathbb{Z}^d \setminus \{0\},\, l \in \mathbb{Z}^n \text{ with } |l| \leq 2, \tag{1.3}$$

is weaker than (1.2), but stronger than Bourgain's condition. However, for reasons related to the availability of a normal form mentioned above, they are unable to cover the case of constant potential V. In the present paper, we give a new proof of Bourgain's result for the NLW with periodic boundary conditions. To this end, we will use a renormalization group procedure recently developed in [BGK] for standard KAM problems. The nonresonance condition that we will impose is the same as Chierchia and You's condition, but our technique could in principle accommodate Bourgain's conditon.

In order to describe our result further, we start by specifying the infinite dimensional Hamiltonians we will consider. For d_k, $k \geq 1$, a sequence of strictly positive integers uniformly bounded by some $\bar{d} < \infty$, let $\mathcal{R}^\infty$ denote the set of infinite sequences $x = (x_1, x_2, \ldots)$ with $x_k \in \mathbb{R}^{d_k}$. For an integer $d \geq 1$, let $\mathcal{P} = \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{R}^\infty \times \mathcal{R}^\infty$ where $\mathbb{T}^d$

is the torus $\mathbb{R}^d/(2\pi\mathbb{Z}^d)$. Denoting the coordinates in $\mathcal{P}$ by (ϕ, I, x, y) and endowing $\mathcal{P}$ with the symplectic structure $d\phi \wedge dI + dx \wedge dy$, we consider perturbations of integrable Hamiltonians of the form

$$H(\phi, I, x, y) = \omega \cdot I + \tfrac{1}{2} I \cdot gI + \tfrac{1}{2} \sum_{k\geq 1} \left(\mu_k^2 |x_k|^2 + |y_k|^2\right) + \lambda U(\phi, I, x), \tag{1.4}$$

where $\mu_k \in \mathbb{R}$, $k \geq 1$, $\omega \in \mathbb{R}^d$, and g is a real symmetric, invertible $d \times d$ matrix. Above, $|v|^2$ for $v \in \mathbb{R}^m$ denotes $\sum_{i=1}^m v_i^2$. The Hamiltonian flow generated by (1.4) is given by the equations of motion

$$\dot{I} = -\lambda \partial_\phi U\,, \quad \dot{\phi} = \omega + gI + \lambda \partial_I U, \tag{1.5}$$

and

$$\ddot{x}_k = -\mu_k^2 x_k - \lambda \partial_{x_k} U \tag{1.6}$$

For $\lambda = 0$ and the initial condition $I^0 = \phi^0 = x^0 = y^0 = 0$, the flow $\phi(t) = \omega t$, $I(t) = 0$, and $x(t) = 0$, is quasi-periodic and spans a d-dimensional torus in $\mathbb{T}^d \times \mathbb{R}^d \times \mathcal{R}^\infty \times \mathcal{R}^\infty$. In order to study the case for which the perturbation is turned on, we consider a quasi-periodic solution of the form

$$(\phi(t), I(t), x(t)) = (\omega t + \Phi(\omega t), J(\omega t), Z(\omega t)).$$

Then, (1.5) and (1.6) require that $\mathcal{T} \equiv (\Phi, J, Z) : \mathbb{T}^d \to \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{R}^\infty$ satisfies the equation

$$\mathcal{D}\mathcal{T}(\varphi) = -\lambda \partial U(\varphi + \Phi(\varphi), J(\varphi), Z(\varphi)), \tag{1.7}$$

where $\partial = (\partial_\phi, \partial_I, \partial_x)$ and, setting

$$\mu \equiv \mathrm{diag}(\mu_1 \mathbb{1}_{d_1}, \mu_2 \mathbb{1}_{d_2}, \ldots), \tag{1.8}$$

together with $D \equiv \omega \cdot \partial_\phi$,

$$\mathcal{D} = \begin{pmatrix} 0 & D & 0 \\ -D & g & 0 \\ 0 & 0 & D^2 + \mu^2 \end{pmatrix}. \tag{1.9}$$

Note that if $\mathcal{T}$ is a solution of equation (1.7), then so is $\mathcal{T}_\beta$ for $\beta \in \mathbb{R}^d$, where

$$\mathcal{T}_\beta(\varphi) = \mathcal{T}(\varphi - \beta) - (\beta, 0, 0). \tag{1.10}$$

We now state the two hypothesis under which we shall prove existence of a solution $\mathcal{T}$ of equation (1.7), first introducing the following family of Banach spaces $\mathcal{R}_s^\infty$, $s \in \mathbb{R}$,

$$\mathcal{R}_s^\infty = \{Z \in \mathcal{R}^\infty \mid |Z|_s \equiv \sum_{k\geq 1} k^s |Z_k|_{\mathbb{R}^{d_k}} < \infty\}. \tag{1.11}$$

(H1) Asymptotics of eigenvalues. The sequence $\{\mu_k\}_{k\geq 1}$ satisfies $\mu_k > 0$ and $\mu_k \neq \mu_l$ for all $k \neq l \geq 1$, and there exist $\gamma \geq 1$ and $c > 0$ such that

$$\mu_k \geq ck^\gamma \quad \text{for all} \quad k \geq 1. \tag{1.12}$$

Furthermore, if $\gamma > 1$ then

$$\mu_{k'} - \mu_k \geq c(k'^\gamma - k^\gamma) \quad \text{for all} \quad k' > k \geq 1. \tag{1.13}$$

If $\gamma = 1$, then there exist constants $\xi > 0$ and $c_l > 0$ such that

$$\mu_{k'} - \mu_k = c_l(1 + \mathcal{O}(k^{-\xi})) \quad \text{for all} \quad k' - k = l \geq 1. \tag{1.14}$$

(H2) Regularity of the perturbation. The map $(\phi, I, x) \mapsto U(\phi, I, x)$ is assumed to be real analytic in $\phi \in \mathbb{T}^d$ and real analytic in I and x in a neighborhood of the origin of $\mathbb{R}^d$ and $\mathcal{R}_0^\infty$. In addition, we assume that there exist an $s > 0$ and a $\xi > 0$ such that for some $\mathcal{O}_I \subset \mathbb{R}^d$ and $\mathcal{O}_x \subset \mathcal{R}_s^\infty$ neighborhoods of the origin, the gradient $\partial_x U$ is bounded as a map from $\mathbb{T}^d \times \mathcal{O}_I \times \mathcal{O}_x$ to $\mathcal{R}_{s+\xi-\gamma}^\infty$. In the sequel, we will often use the short notation $s' \equiv s + \xi - \gamma$.

Theorem 1.1. *Let $\{\mu_k\}$ satisfy (H1) and U satisfy (H2). Then, there exists a set $\Omega^* = \Omega^*(U, \mu) \subset \mathbb{R}^d$ such that for $\omega \in \Omega^*$, equation (1.7) has a unique solution (up to translations (1.10)) which is real analytic in λ and ϕ provided that $|\lambda|$ is small enough. Furthermore, for all bounded $\Omega \subset \mathbb{R}^d$ the set Ω^* of admissible frequencies satisfies* $\text{meas}(\Omega \setminus \Omega^*) \to 0$ *as $\lambda \to 0$.*

The proof of Theorem 1.1 is based on an inductive procedure developed in [BGK] for standard KAM problems. This renormalization group iteration can be viewed as an iterative resummation of the Lindsedt series, as is explained in more details in [BGK], and was directly inspired by the quantum field theory analogy with KAM problems forcefully emphasized by Gallavotti *et al.* [G, GGM]. Melnikov type problems require to deal with the additional resonances arising from the normal frequencies μ_k, and the goal of the present paper is to explain how the procedure of [BGK] can be applied in such cases. In contrast to standard KAM problems, the set Ω^* of admissible frequencies depends for Melnikov type problems on the perturbation U. In our approach, this dependence expresses itself by the fact that under iteration, the normal frequencies are renormalized in a U-dependent way and that the set Ω^* is defined according to the renormalized normal frequencies. As usual, the set Ω^* is constructed in such a way that nonresonance conditions are fulfilled in order for the inductive scheme to converge. Our scheme is technically simplified if one imposes nonresonance condition of the form (1.3), i.e., conditions involving pairs of normal frequencies. Hypothesis (H1) ensures that Ω^* has large measure under these conditions, and hypothesis (H2) ensures that the asymptotic properties of the normal frequencies stated in (H1) are preserved under renormalization. The requirement $\xi > 0$ is needed both in (H1) when $\gamma = 1$, and, for

$\gamma > 1$, in (H2) in order to cover the case of degenerate normal frequencies (more precisely the case where $d_k > 1$ for infinitely many k). In Section 2, we show how Theorem 1.1 provides a proof of the existence of quasi-periodic solutions of the 1D NLW with periodic boundary conditions. In particular, $\gamma = 1$ in (H1) and we will see that (H2) is satisfied with $\xi = 1$. In contrast, one has for the 1D NLS $\gamma = 2$ and $\xi = 0$. Thus, the scheme presented here only applies to NLS with Dirichlet boundary conditions (namely $d_k = 1$ for all k) or to the persistence of periodic solutions of NLS (namely $d = 1$). In order to cover the other situations, one must be able to dispense with nonresonance conditions involving certain pairs of normal frequencies.

The remainder of the paper is organized as follows. Section 2 is devoted to the NLW. In Section 3 we explain the renormalization group scheme that will be used to prove Theorem 1.1. Section 4 is devoted to the definition of the spaces we will consider. In Section 5, we state some crucial inductive bounds, which will be shown to hold in Section 6. Section 7 is concerned with the measure estimate of Ω^*, whereas the proof of Theorem 1.1 is carried out in Section 8. Finally, we have collected in the appendix some technical and intermediary results.

2. The 1D Wave Equation

In this section, we show how Theorem 1.1 implies the existence of small amplitude quasi-periodic solutions of nonlinear 1D wave equations of the form

$$\partial_t^2 u = \partial_x^2 u - mu - f(u), \tag{2.1}$$

$t > 0, x \in [0, 2\pi]$, with periodic boundary conditions $u(0,t) = u(2\pi, t)$, $\partial_t u(0,t) = \partial_t u(2\pi, t)$. Here, $m > 0$ is a real parameter and f is a real analytic function of the form $f(u) = u^3 + \mathcal{O}(u^4)$. For $f \equiv 0$, equation (2.1) becomes

$$\partial_t^2 u = \partial_x^2 u - mu \equiv -Lu. \tag{2.2}$$

The operator L with periodic boundary conditions admits a complete orthonormal basis of eigenfunctions $\psi_n \in L^2([0, 2\pi])$, $n \in \mathbb{Z}$, with corresponding eigenvalues

$$\zeta_n = n^2 + m, \tag{2.3}$$

if one sets $\psi_0 = 1/\sqrt{2\pi}$ and for $n \geq 1$,

$$\psi_n(x) = \frac{1}{\sqrt{\pi}} \cos(nx), \quad \psi_{-n}(x) = \frac{1}{\sqrt{\pi}} \sin(nx). \tag{2.4}$$

Every solution of the linear wave equation (2.2) can be written as a superposition of the basic modes ψ_n, namely, for $\mathcal{I}$ any subset of $\mathbb{Z}$ and $\mu_n \equiv \sqrt{\zeta_n}$,

$$u(x,t) = \sum_{n \in \mathcal{I}} a_n \cos(\mu_n t + \theta_n) \psi_n(x), \tag{2.5}$$

with amplitudes $a_n > 0$ and initial phases θ_n. Regarding existence of solutions for the nonlinear wave equation (2.1), we will prove the

Theorem 2.1. *Let $1 \le d < \infty$ and $\mathcal{I} = \{n_1, \ldots, n_d\} \subset \mathbb{Z}$ satisfying $|n_i| \ne |n_j|$ for $i \ne j$. Then, for $\lambda > 0$ small enough there is a set $\mathcal{A} \subset \{a = (a_1, \ldots, a_d) \,|\, 0 < a_i < \lambda\}$ of positive measure such that for $a \in \mathcal{A}$ equation (2.1) has a solution*

$$u(x,t) = \sum_{i=1}^{d} a_i \cos(\mu'_{n_i} t + \theta_i)\psi_{n_i}(x) + \mathcal{O}(|a|^3), \tag{2.6}$$

with frequencies $\mu'_{n_i} = \mu_{n_i} + \mathcal{O}(|a|^2)$. Furthermore, the set $\mathcal{A}$ is of asymptotically full measure as $|a| \to 0$.

As is well known, the nonlinear wave equation (2.1) can be studied as an infinite dimensional Hamiltonian system by taking the phase space to be the product of the Sobolev spaces $H_0^1([0,2\pi]) \times L^2([0,2\pi])$ with coordinates u and $v = \partial_t u$. The Hamiltonian for (2.1) is then

$$H = \tfrac{1}{2}(v,v) + \tfrac{1}{2}(Lu,u) + \int_0^{2\pi} g(u)\, dx, \tag{2.7}$$

where $L = -d^2/dx^2 + m$, $g = \int f ds$, and $(\cdot,\cdot)$ denotes the usual scalar product in $L^2([0,2\pi])$. In order to prove existence of solutions of the type (2.6) by means of Theorem 1.1, we would like to write (2.7) in the form (1.4). This turns out to be possible, through amplitude-frequency modulation, due to the availability of a (partial) normal form theory for (2.7). As we shall see, the requirement for the parameter m to be non zero is crucial for this part of the argument. In the sequel, we will closely follow the exposition of Pöschel in [P2]. Introducing the coordinates $\mathsf{q} = (\mathsf{q}_0, \mathsf{q}_1, \mathsf{q}_{-1}, \ldots)$ and $\mathsf{p} = (\mathsf{p}_0, \mathsf{p}_1, \mathsf{p}_{-1}, \ldots)$ by setting

$$u(x) = \sum_{n\in\mathbb{Z}} \mathsf{q}_n \psi_n(x), \quad v(x) = \sum_{n\in\mathbb{Z}} \mathsf{p}_n \psi_n(x), \tag{2.8}$$

one rewrites the Hamiltonian (2.7) in the coordinates (q, p),

$$H = \frac{1}{2}\sum_{n\in\mathbb{Z}} \left(\mu_n^2 \mathsf{q}_n^2 + \mathsf{p}_n^2\right) + G(\mathsf{q}), \tag{2.9}$$

where

$$G(\mathsf{q}) = \int_0^{2\pi} g\Big(\sum_{n\in\mathbb{Z}} \mathsf{q}_n \psi_n(x)\Big) dx. \tag{2.10}$$

The Hamiltonian flow generated by (2.9) is given by the equations of motion

$$\ddot{\mathsf{q}}_n = -\mu_n^2 \mathsf{q}_n - \partial_{\mathsf{q}_n} G(\mathsf{q}), \tag{2.11}$$

and one can show that a solution q of (2.11) yields a solution of the nonlinear wave equation (2.1) if q has some decaying properties. More precisely, defining l^s_b to be the Banach space of all real valued bi-infinite sequences $w = (w_0, w_1, w_{-1}, \ldots)$ with norm

$$||w||_s = \sum_{n\in\mathbb{Z}} [n]^s |w_n|,$$

where $[n] = \max(1, |n|)$, one has the

Lemma 2.2. *Let $s \geq 2$. If a curve $I \to l^s_b$, $t \mapsto \mathsf{q}(t)$, is a solution of (2.11), then*

$$u(x,t) = \sum_{n\in\mathbb{Z}} \mathsf{q}_n(t)\psi_n(x)$$

is a classical solution of (2.1).

For the proof of Lemma 2.2, see [CY]. Before turning to the normal form analysis of the Hamiltonian (2.9), we state a result concerning the regularity of the gradient $\partial_{\mathsf{q}} G$.

Lemma 2.3. *For all $s > 0$, the gradient $\partial_{\mathsf{q}} G$ is real analytic as a map from some neighborhood of the origin in l^s_b into l^s_b, with*

$$||\partial_{\mathsf{q}} G(\mathsf{q})||_s = \mathcal{O}(||\mathsf{q}||^3_s). \tag{2.12}$$

Proof. We first note that l^s_b is a Banach algebra with respect to convolution of sequences, with

$$||q * p||_s \leq \sum_{i,j\in\mathbb{Z}} [i]^s |q_{j-i}||p_j| \leq \sup_{i,j\in\mathbb{Z}} \left(\frac{[i]}{[j-i][j]}\right)^s ||q||_s ||p||_s \leq 2^s ||q||_s ||p||_s. \tag{2.13}$$

Therefore, using the analyticity of $f(u) = u^3 + \mathcal{O}(u^4)$, one computes that in a sufficiently small neighborhood of the origin,

$$||f(u)||_s \leq C||\mathsf{q}||^3_s. \tag{2.14}$$

On the other hand, since

$$\partial_{\mathsf{q}_n} G(\mathsf{q}) = \int_0^{2\pi} f(u)\psi_n(x)dx,$$

the components of $\partial_{\mathsf{q}} G(\mathsf{q})$ are the Fourier components of $f(u)$ and (2.12) follows from the estimate (2.14). The regularity of $\partial_{\mathsf{q}} G$ follows from the regularity of its components and its local boundedness, cf. [PT] p. 138.

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We now turn to the normal form analysis of (2.9). First, since $g(u) = \frac{1}{4}u^4 + \mathcal{O}(u^5)$, we find that

$$G(\mathsf{q}) = \frac{1}{4} \sum_{i,j,k,l} g_{ijkl}\, \mathsf{q}_i \mathsf{q}_j \mathsf{q}_k \mathsf{q}_l + \mathcal{O}(|\mathsf{q}|^5),$$

where

$$g_{ijkl} = \int_0^{2\pi} \psi_i \psi_j \psi_k \psi_l dx. \tag{2.15}$$

An easy computation shows that $g_{ijkl} = 0$ unless $i \pm j \pm k \pm l = 0$ for at least one combination of plus and minus signs. This will play an important role later on. Next, given a finite subset of indices $\mathcal{I}_d = \{n_1, \ldots, n_d\} \subset \mathbb{Z}$ with $|n_i| \neq |n_j|$ if $i \neq j$, we decompose the Hamiltonian (2.9) as

$$H = H_d + H_\infty,$$

where

$$H_d(\mathsf{q}, \mathsf{p}) = \frac{1}{2} \sum_{n \in \mathcal{I}_d} (\mu_n^2 \mathsf{q}_n^2 + \mathsf{p}_n^2) + \frac{1}{4} \sum_{i,j,k,l \in \mathcal{I}_d} g_{ijkl}\, \mathsf{q}_i \mathsf{q}_j \mathsf{q}_k \mathsf{q}_l \equiv \Lambda_d(\mathsf{q}, \mathsf{p}) + G_d(\mathsf{q}), \tag{2.16}$$

$$H_\infty(\mathsf{q}, \mathsf{p}) = \frac{1}{2} \sum_{n \notin \mathcal{I}_d} (\mu_n^2 \mathsf{q}_n^2 + \mathsf{p}_n^2) + G(\mathsf{q}) - G_d(\mathsf{q}) \equiv \Lambda_\infty(\mathsf{q}, \mathsf{p}) + G_\infty(\mathsf{q}). \tag{2.17}$$

Introducing the complex coordinates z_j, $j = 1, \ldots, d$, by

$$z_j = \frac{1}{\sqrt{2\mu_{n_j}}} (\mu_{n_j} \mathsf{q}_{n_j} + \mathrm{i}\, \mathsf{p}_{n_j}),$$

one obtains the Hamiltonian $H_d(z, \bar{z}) = \sum_j \mu_{n_j} |z_j|^2 + G_d(z, \bar{z})$ on $\mathbb{C}^d$ with symplectic structure $\mathrm{i} \sum_j dz_j \wedge d\bar{z}_j$. For the remaining coordinates, one introduces the notation, for $k \geq 1$,

$$x_k = \begin{cases} (\mathsf{q}_k, \mathsf{q}_{-k}) \in \mathbb{R}^2 & \text{if} \quad k, -k \notin \mathcal{I}_d\,, \\ \mathsf{q}_{-\tilde{k}} \in \mathbb{R} & \text{if} \quad k = |\tilde{k}| \text{ for some } \tilde{k} \in \mathcal{I}_d\,, \end{cases}$$

and similarly for p_n, $n \notin \mathcal{I}_d$, denoted in terms of $y_k \in \mathbb{R}^{d_k}$, $k \geq 1$, with d_k as above, namely, $d_k = 2$ if both $k, -k \notin \mathcal{I}_d$ and $d_k = 1$ otherwise. Clearly, for $\mathsf{q}, \mathsf{p} \in l_b^s$ one has $x, y \in \mathcal{R}_s^\infty$, where $\mathcal{R}_s^\infty$ is defined in (1.11), and H_∞ reads in these notations

$$H_\infty(z, \bar{z}, x, y) = \frac{1}{2} \sum_{k \geq 1} (\mu_k^2 |x_k|^2 + |y_k|^2) + G_\infty(z, \bar{z}, x),$$

with $|G_\infty| = \mathcal{O}\big(\sum_{l=0}^{3} |z|^l \|x\|_s^{4-l}\big)$. The next proposition establishes the existence of a symplectic change of coordinates that transforms the Hamiltonian H_d into a Birkhoff normal form. As it will be clear from the proof, this normal form is *not* available for

$H = H_d + H_\infty$, since most frequencies in H_∞ are degenerate. This is the main difference with [P2] in the present discussion.

Proposition 2.4. *For each $m > 0$ and each subset $\mathcal{I}_d$, $d < \infty$, satisfying $|n_i| \neq |n_j|$ when $i \neq j$, there exists a near identity, real analytic, symplectic change of coordinates Γ_d in some neighborhood of the origin in $\mathbb{C}^d$ that takes the Hamiltonian (2.16) into*

$$H_d \circ \Gamma_d = \Lambda_d + \bar{G}_d + K_d,$$

where $|K_d| = \mathcal{O}(|z|^5)$ and

$$\bar{G}_d(z, \bar{z}) = \frac{1}{2} \sum_{i,j=1}^{d} \bar{g}_{ij} |z_i|^2 |z_j|^2 \quad \text{with} \quad \bar{g}_{ij} = \frac{3}{\pi} \frac{4 - \delta_{ij}}{\mu_{n_i} \mu_{n_j}} . \tag{2.18}$$

Furthermore, setting $\Gamma_\infty = \Gamma_d \oplus \mathbb{1}_{\mathcal{R}_s^\infty \times \mathcal{R}_s^\infty}$, one has $H_\infty \circ \Gamma_\infty = \Lambda_\infty + K_\infty$ with $|K_\infty| = \mathcal{O}\big(\sum_{l=0}^{3} |z|^l ||x||_s^{4-l}\big)$.

Proof. Modulo straightforward modifications, the proof is carried out in [P2] and we restrict ourselves here to a quick overview. The possibility to eliminate all terms in $G_d(z, \bar{z})$ that are not of the form $|z_i|^2 |z_j|^2$ follows from the fact that for integers $i, j, k, l \in \mathcal{I}_d$ satisfying $i \pm j \pm k \pm l = 0$ and $\{i, j, k, l\} \neq \{n, n, n', n'\}$ one has, as shown in [P2],

$$|\mu_i \pm \mu_j \pm \mu_k \pm \mu_l| \geq c \, \frac{m}{(N^2 + m)^{3/2}} > 0 \, , \tag{2.19}$$

with c some absolute constant and $N = \min\{|i|, \ldots, |l|\}$. To see this, it is convenient to adopt the notation $z_j = w_j$ and $\bar{z}_j = w_{-j}$ in which G_d reads

$$G_d = \sum_{i,j,k,l}{}' \tilde{g}_{ijkl} w_i w_j w_k w_l + \mathcal{O}(|z|^5), \quad \tilde{g}_{ijkl} = \frac{g_{n_{|i|} \ldots n_{|l|}}}{\sqrt{\mu_{n_{|i|}} \cdots \mu_{n_{|l|}}}} \, ,$$

where the prime symbol in the summation sign indicates that the sum runs over all indices $i, j, k, l \in \{1, -1, \ldots, d, -d\}$ with $n_{|i|} \pm n_{|j|} \pm n_{|k|} \pm n_{|l|} = 0$ for at least one combination of plus and minus signs. Defining the transformation Γ_d as the time-1 map of the flow of the vector field X_F given by a Hamiltonian $F(z, \bar{z})$ of order four, namely, $\Gamma_d = X_F^t|_{t=1}$ and $F = \sum{}' F_{ijkl} w_i w_j w_k w_l$, one obtains using Taylor's formula $H_d \circ \Gamma_d = \Lambda_d + G_d + \{\Lambda_d, F\} + \mathcal{O}(|z|^6)$ with

$$\{\Lambda_d, F\} = -\mathrm{i} \sum_{i,j,k,l}{}' (\hat{\mu}_i + \hat{\mu}_j + \hat{\mu}_k + \hat{\mu}_l) F_{ijkl} w_i w_j w_k w_l,$$

where $\hat{\mu}_i \equiv \mathrm{sign}(i) \mu_{n_{|i|}}$. Therefore, (2.19) allows to choose F_{ijkl} in such a way that

$$G_d + \{\Lambda_d, F\} = \sum_{i,j=1}^{d} \tilde{g}_{iijj} |z_i|^2 |z_j|^2 + \mathcal{O}(|z|^5) \equiv \bar{G}_d + \mathcal{O}(|z|^5) \, .$$

For the rest of the proof, we refer the reader to [P2].

■

The Hamiltonian $\Lambda_d + \bar{G}_d$ is integrable with integrals $|z_i|^2, i = 1, \ldots, d$. Furthermore, the matrix $\bar{g} = (\bar{g}_{ij})_{i,j}$ is non degenerate, as can be checked from the explicit formula (2.18). Hence, introducing the standard action-angle variables $(I, \phi) \in \mathbb{R}^d \times \mathbb{T}^d$ and linearizing H around a given value for the action, namely, by setting for some $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$,

$$z_i \bar{z}_i = I_i + a_i^2,$$

one finally obtains

$$H_a = \omega \cdot I + \tfrac{1}{2} I \cdot \bar{g} I + \sum_{k \geq 1} (\mu_k^2 x_k^2 + y_k^2) + U_a(I, \phi, x), \tag{2.20}$$

where U_a is just $K_d + K_\infty$ with the variables $z_i, \bar{z}_i, i = 1, \ldots, d$, expressed in terms of I, ϕ, and where $\omega = (\omega_1, \ldots, \omega_d)$ is given by

$$\omega_i = \mu_{n_i} + \sum_{j=1}^{d} \bar{g}_{ij} a_j^2 \,,$$

and covers a cone at $(\mu_{n_1}, \ldots, \mu_{n_d})$ as a varies in a neighborhood of the origin of $\mathbb{R}^d$. Furthermore, U_a is real analytic in $\phi \in \mathbb{T}^d$ and real analytic in I in a sufficiently small neighborhood $\mathcal{O}_I$ of the origin of $\mathbb{R}^d$. As a function of x, U_a is real analytic in a neighborhood $\mathcal{O}_x \subset \mathcal{R}_s^\infty$ and by Lemma 2.3, its gradient $\partial_x U_a$ is bounded as a map from $\mathbb{T}^d \times \mathcal{O}_I \times \mathcal{O}_x$ to $\mathcal{R}_s^\infty$. Therefore, since hypothesis (H1) is satisfied with $\gamma = 1$, U_a satisfies (H2) with $\xi = 1$. Finally, the small parameter λ is given in terms of $|a| = \delta$. In the Hamilton's equations for H_a, rescaling a by δ, x and y by δ^2, and I by δ^4, one obtains an Hamiltonian system given by the rescaled Hamiltonian

$$\begin{aligned} \tilde{H}_a(\phi, I, x, y) &= \delta^{-4} H_{\delta a}(\phi, \delta^4 I, \delta^2 x, \delta^2 y) \\ &= \omega \cdot I + \frac{\delta^4}{2} I \cdot \bar{g} I + \sum_{k \geq 1} (\mu_k^2 x_k^2 + y_k^2) + \tilde{U}_a(I, \phi, x), \end{aligned}$$

with $\tilde{U}_a$ analytic in δ and, as a function of I,

$$\tilde{U}_a = \mathcal{O}(\delta) + \mathcal{O}(\delta^3 |I|) + \mathcal{O}(\delta^5 |I|^2).$$

Hence, Theorem 1.1 implies the existence of quasi-periodic solutions I, x and y of period ω, real analytic in ϕ and λ. Tracing the coordinate transformations back to the original variables $\mathsf{q}_n(t)$ in the expression (2.8) for $u(x,t)$ completes the proof of Theorem 2.1 with $u(x,t)$ given by (2.6).

3. The Renormalization Group Scheme

Equation (1.7) consists in a system of equations for the variables (Φ, J) and Z which are coupled through the perturbation U only. Adopting the notation

$$V(\Phi, J, Z)(\varphi) = \lambda \begin{pmatrix} \partial_\phi U \\ \partial_I U \end{pmatrix} (\varphi + \Phi(\varphi), J(\varphi), Z(\varphi)), \tag{3.1}$$

$$W(\Phi, J, Z)(\varphi) = \lambda \partial_x U(\varphi + \Phi(\varphi), J(\varphi), Z(\varphi)), \tag{3.2}$$

one rewrites equation (1.7) as

$$\begin{pmatrix} 0 & D \\ -D & g \end{pmatrix} \begin{pmatrix} \Phi \\ J \end{pmatrix} = -V(\Phi, J, Z), \tag{3.3}$$

$$(D^2 + \mu^2) Z = -W(\Phi, J, Z). \tag{3.4}$$

Our strategy will be to consider (3.3) and (3.4) separately, treating the functions Z and (Φ, J), respectively, as parameters. As we will see in Section 8, existence of a (unique) solution of the original equation (1.7) can then be proved by using the implicit function theorem. Note that (3.3) involves only the torus frequencies ω and is equivalent to a standard KAM problem. Existence of solution for such equations is well known and has been established by various means. One important feature we will use is the regular dependence of the solution (Φ, J) on the function Z. A precise result about the solution of (3.3) will be stated in Section 4, Theorem 4.1, once the required Banach spaces of functions have been introduced.

We now focus our attention on equation (3.4), and will suppress from the notation the dependence of the vector field W on the parameters Φ and J. Most of our analysis will be conducted in Fourier space, and we will denote by lower case letters the Fourier transforms of functions of φ, the latter being denoted by capital letters, namely,

$$F(\varphi) = \sum_{q \in \mathbb{Z}^d} e^{-iq\cdot\varphi} f(q), \quad \text{where} \quad f(q) = \int_{\mathbb{T}^d} e^{iq\cdot\varphi} F(\varphi) d\varphi,$$

where $d\varphi$ stands for the normalized Lebesgue measure on $\mathbb{T}^d$. For $Z(\varphi) \in \mathcal{R}^\infty$, note that $z(q) \in \hat{\mathcal{R}}^\infty$ with $z_{k_i}(q) = \overline{z_{k_i}(-q)}$, where $\hat{\mathcal{R}}^\infty$ stands for $\bigoplus_{k\geq 1} \mathbb{C}^{d_k}$ and k_i refers to the i^{th} component of $\mathbb{C}^{d_k}$. Similarly, $\hat{\mathcal{R}}_s^\infty$ will denote the complexification of the Banach space $\mathcal{R}_s^\infty$ defined in (1.11). Finally, we will denote the vector space of functions $z(q) \in \hat{\mathcal{R}}^\infty$ by h,

$$h = \{z = (z(q)) \mid z(q) \in \hat{\mathcal{R}}^\infty, q \in \mathbb{Z}^d\}.$$

In terms of the Fourier transform of W, namely,

$$w_0(z)(q) \equiv \lambda \int_{\mathbb{T}^d} e^{iq\cdot\varphi} \partial_x U(\varphi + \Phi(\varphi), J(\varphi), Z(\varphi)) d\varphi, \tag{3.5}$$

equation (3.4) becomes,

$$\mathcal{K}_0 z = w_0(z), \tag{3.6}$$

where the operator $\mathcal{K}_0$ is given by the diagonal kernel

$$\mathcal{K}_0(q, q') = \big(|\omega \cdot q|^2 - \mu^2\big)\delta_{qq'}. \tag{3.7}$$

Solving equation (3.6) requires to invert the operator $\mathcal{K}_0$. Although the inverse of $\mathcal{K}_0$ is unbounded for generic frequencies, restricting ω to a set of admissible frequencies gives sufficient control on the inverse of $\mathcal{K}_0$ to prove existence of a solution. As is well known for Melnikov problems, this set depends on the perturbation U.

In order to prove existence of a solution to equation (3.6), we will follow a strategy developed in [BGK] for standard KAM problems, namely, for equations of the type (3.3). This strategy basically consists in inductively reducing (3.6) to a sequence of effective equations involving denominators of decreasing size. One inductive step, say the n^{th} step, consists in splitting the effective equation obtained at the previous step into two equations involving only large and, respectively, small denominators, where large and small are defined with respect to a scale of order η^n for some fixed $\eta < 1$. This splitting is done in such a way that the nonlinear operator involved in the large denominators equation is a contraction, and this equation can thus be solved by a simple application of the contraction mapping principle. This, in turn, allows to map the small denominators equation into a new effective equation of the type (3.6), with a new right hand side w_n and (eventually) a new linear operator $\mathcal{K}_n$. In [BGK], it was shown that for equations of the type (3.3), the above mentioned contraction property follows naturally from symmetries specific to this case. In contrast, equation (3.4) involves in addition the normal frequencies μ_k and does not possess such symmetry. In order to obtain the required contraction, we must make at every inductive step an additional preparation step. As we shall see below, this amounts to renormalizing the linear operator $\mathcal{K}_{n-1}$ obtained at the previous step into a new operator $\mathcal{K}_n$, which, in effect, corresponds to renormalizing the normal frequencies. Furthermore, we will see that the renormalized normal frequencies converge to a U-dependent set $\{\mu_\alpha^*\}$, $\alpha \geq 1$, as $n \to \infty$. Therefore, since the Diophantine conditions imposed on ω will eventually be defined relatively to this set, one obtains in a constructive way the dependence of the set of admissible frequencies on the perturbation U.

We now describe how the renormalization group approach is implemented in practice for Melnikov type problems. First, we proceed with the above mentioned preparation step by decomposing w_0 as

$$w_0(z) = \tilde{w}_0(z) + A_0 z,$$

where the linear operator A_0 is the dominant part of $Dw_0(z)$ evaluated at $z = 0$. With $\mathcal{K}_1 \equiv \mathcal{K}_0 - A_0$, equation (3.6) now reads

$$\mathcal{K}_1 z = \tilde{w}_0(z). \tag{3.8}$$

As explained in more details below, A_0 can be chosen in such a way that $\mathcal{K}_1$ is of the same form as $\mathcal{K}_0$, cf. (3.7), but now given in terms of a new set of frequencies $\tilde{\mu}_{k_i} \in \mathbb{R}$ which are perturbation of order λ of the original normal frequencies μ_k. The notation $\tilde{\mu}_{k_i}$ reflects the fact that the perturbation A_0 may lift some of the degeneracies. Therefore, when inverting $\mathcal{K}_1$, denominators smaller than $\mathcal{O}(\eta)$ occur for q such that $||\omega \cdot q| - \tilde{\mu}_{k_i}| \leq \mathcal{O}(\eta)$ for some k_i. Furthermore, these small denominators only occur, for such q, in a specific subspace $h^q_{k_i}$ of $\mathbb{C}^{d_k}$ depending on which $\tilde{\mu}_{k_j}$, if any, has been separated from $\tilde{\mu}_{k_i}$ by more than $\mathcal{O}(\eta)$. Introducing P_1 as the projection of h onto $h^q_{k_i}$ for q such that $||\omega \cdot q| - \tilde{\mu}_{k_i}| \leq \mathcal{O}(\eta)$ and defining $Q_1 \equiv \mathbb{1} - P_1$, one thus expects that the restriction of $\mathcal{K}_1$ to $Q_1 h$ is invertible with an inverse of order $\mathcal{O}(\eta^{-1})$. Multiplying (3.8) by Q_1 and P_1 leads to the small and large denominators equations for $\tilde{z}_1 \equiv Q_1 z$ and $z_1 \equiv P_1 z$,

$$\mathcal{K}_1 \tilde{z}_1 = Q_1 \tilde{w}_0(\tilde{z}_1 + z_1), \tag{3.9}$$
$$\mathcal{K}_1 z_1 = P_1 \tilde{w}_0(\tilde{z}_1 + z_1), \tag{3.10}$$

and by definition of Q_1, the first equation can be rewritten as a fixed point equation for the functional R_1 defined as $R_1(z_1) \equiv \tilde{z}_1$, namely,

$$R_1(z) = \mathcal{K}_1^{-1} Q_1 \tilde{w}_0(z + R_1(z)). \tag{3.11}$$

By choice of A_0, the nonlinear operator $\mathcal{K}_1^{-1} Q_1 \tilde{w}_0$ is a contraction and one can solve equation (3.11) for R_1 using the Banach fixed point theorem. (See point (a) of Theorem 5.1 for this part of the inductive step.) Next, with w_1 defined as

$$w_1(z) \equiv \tilde{w}_0(z + R_1(z)),$$

equation (3.10) reads

$$\mathcal{K}_1 z_1 = P_1 w_1(z_1), \tag{3.12}$$

and the solution $z = z_1 + \tilde{z}_1$ of the original equation (3.6) is now given by

$$z = z_1 + R_1(z_1) \equiv F_1(z_1).$$

Hence, the problem of solving (3.6) is reduced to solving the effective equation (3.12). To solve this equation one proceeds similarly, starting with our preparation step. After n steps of this inductive process, the solution of (3.6) is given by

$$z = F_{n-1}(z_n + R_n(z_n)) \equiv F_n(z_n), \tag{3.13}$$

where R_n solves the functional equation

$$R_n(z) = \Gamma_n \tilde{w}_{n-1}(z + R_n(z)), \tag{3.14}$$

with

$$\Gamma_n \equiv \mathcal{K}_n^{-1} Q_n P_{n-1}, \tag{3.15}$$

and, for some linear operator A_{n-1},

$$\tilde{w}_{n-1}(z) \equiv w_{n-1}(z) - A_{n-1}z, \tag{3.16}$$
$$\mathcal{K}_n \equiv \mathcal{K}_{n-1} - P_{n-1}A_{n-1}, \tag{3.17}$$

whereas z_n solves the effective equation

$$\mathcal{K}_n z_n = P_n w_n(z_n), \tag{3.18}$$

with w_n defined as

$$w_n(z) \equiv \tilde{w}_{n-1}(z + R_n(z)). \tag{3.19}$$

Remark 3.1. The point of this inductive procedure is that $P_n w_n(z)$ becomes effectively linear in z for large n. More precisely, we will show, cf. Theorem 5.1 below, that the rescaled maps w_n^r defined by $w_n^r(z) = \eta^{-n} r^{-n} w_n(r^n z)$ satisfy for $r < \eta$,

$$P_n w_n^r(z) = P_n D w_n^r(0) z + \mathcal{O}(\lambda r^{2n} \eta^{-n}) \quad \text{with} \quad P_n D w_n^r(0) = \mathcal{O}(\lambda),$$

in some appropriate Banach space. Thus, $z_n = 0$ becomes a better and better approximation to the solution of (3.18), and we shall construct the solution z of the original equation (3.6) as the limit of the approximate solutions

$$z = \lim_{n\to\infty} F_n(0). \tag{3.20}$$

We now give a precise description of the operators P_n. Note that in order to obtain (3.14) and (3.18), we have tacitly assumed that $P_n P_{n-1} = P_n$. The possibility to define P_n satisfying such a property follows from the convergence of the normal frequencies under renormalization. Recall that renormalization occurs because at every inductive step one turns the nonlinear map w_n of the effective functional equation (3.18) into a contraction by substracting some linear operator A_n. Delaying to subsequent sections the discussion of the appropriate choice for the family A_m, $m \geq 0$, it suffices to point here to the properties of A_m that will ensure convergence of the renormalized normal frequencies. As will be shown, cf. point (c) of Theorem 5.1 for a precise statement, A_m is a perturbation of order $\lambda \eta^m$ and is given by a constant kernel $A_m(q, q') = a_m \delta_{qq'}$ with $a_m : \hat{\mathcal{R}}^\infty \to \hat{\mathcal{R}}^\infty$ linear and hermitian. As a consequence, the operator $\mathcal{K}_n = \mathcal{K}_0 - \sum_{m=0}^{n-1} P_m A_m$ has a kernel of the form (3.7) with μ^2 essentially replaced by the positive definite matrix

$$\tilde{\mu}_n^2 \equiv \mu^2 + \sum_{m=0}^{n-1} a_m, \tag{3.21}$$

with $\tilde{\mu}_n$ having a discrete spectrum $\sigma(\tilde{\mu}_n) \subset \mathbb{R}^+$. One easily checks that the singularities of $\mathcal{K}_n^{-1}$ are given by the eigenvalues of $\tilde{\mu}_n$, which therefore correspond to renormalized normal frequencies. Since a_m is of order $\lambda\eta^m$, one expects the eigenvalues of $\tilde{\mu}_n$ to converge as $n \to \infty$ with $|\nu_{n+1} - \nu_n| \le \mathcal{O}(\lambda\eta^n)$ for $\nu_{n+1} \in \sigma(\tilde{\mu}_{n+1})$ and $\nu_n \in \sigma(\tilde{\mu}_n)$. This, in turn, allows us to define scales of denominators in a consistent way by carefully keeping track of the separation properties of $\sigma(\tilde{\mu}_n)$ as n increases. To this end, one groups the normal frequencies into a hierarchy of clusters satisfying gap conditions that are preserved by the renormalization procedure. We first introduce some notation. For $x \in \mathbb{R}$ and $\mathcal{C}$ a finite collection of points in $\mathbb{R}$, let $d(x, \mathcal{C})$ denote the distance between x and the smallest interval containing all points in $\mathcal{C}$, and for two finite collections $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}$, let

$$d(\mathcal{C}_1, \mathcal{C}_2) \equiv \inf_{x \in \mathcal{C}_1} d(x, \mathcal{C}_2).$$

Then, one can uniquely decompose $\sigma(\tilde{\mu}_n)$ into a maximal number of disjoint clusters $\mathcal{C}^n_{k,i}$, $k \ge 1$, $i = 1, \ldots, M^n_k$, satisfying $d(\mu_k, \mathcal{C}^n_{k,i}) = \mathcal{O}(\lambda)$ and the gap condition

$$d(\mathcal{C}^n_{k,i}, \mathcal{C}^n_{k,j}) > \eta^n \quad \text{if } i \ne j. \tag{3.22}$$

Note that $M^n_k \le d_k$, where d_k denotes the multiplicity of the original normal frequency μ_k, and that by requiring M^n_k to be maximal, the decomposition

$$\sigma(\tilde{\mu}_n) = \bigcup_{k \ge 1} \bigcup_{i=1}^{M^n_k} \mathcal{C}^n_{k,i} \tag{3.23}$$

is unique. The above observation about the rate of convergence of $\sigma(\tilde{\mu}_n)$ as $n \to \infty$ ensures that eigenvalues belonging to different clusters will remain separated. Generically, one expects all degeneracies to be lifted eventually, so that $M^n_k = d_k$ for n sufficiently large and each cluster $\mathcal{C}^n_{k,i}$ contains a single eigenvalue. Next, defining $S_n \subset \mathbb{Z}^d$ as

$$S_n = \bigcup_{k \ge 1} \bigcup_{i=1}^{M^n_k} S^n_{k,i}, \tag{3.24}$$

where

$$S^n_{k,i} = \{q \in \mathbb{Z}^d \mid d(|\omega \cdot q|, \mathcal{C}^n_{k,i}) < \tfrac{1}{4}\eta^n\}, \tag{3.25}$$

one is ensured that all $q \in \mathbb{Z}^d \setminus S_n$ satisfy $d(|\omega \cdot q|, \sigma(\tilde{\mu}_{n'})) \ge \mathcal{O}(\eta^n)$ for $n' \ge n$. Hence, such q can be safely "integrated out" in the large denominators equation. Remark that due to (3.22), the sets $S^n_{k,i}$ are pairwise disjoint. In order to achieve the construction of P_n, one must isolate for every $q \in S_n$ the subspace of $\mathcal{R}^\infty$ in which small denominators will occur. For $q \in S^n_{k,i}$, the latter is given by the eigenspace of $\tilde{\mu}_n$ associated with the eigenvalues belonging to $\mathcal{C}^n_{k,i}$. This eigenspace will be denoted by $\mathcal{J}^n_{k,i}$, whereas the projector onto $\mathcal{J}^n_{k,i}$ will be denoted by $\mathcal{P}^n_{k,i}$. Thus, one defines P_n to be the diagonal operator acting on h given by the kernel

$$P_n(q) = \sum_{k \ge 1} \sum_{i=1}^{M^n_k} \chi^n_{k,i}(\omega \cdot q) \mathcal{P}^n_{k,i}, \tag{3.26}$$

where $\chi^n_{k,i}$ denotes a function in $\in \mathcal{C}^1(\mathbb{R})$ which satisfies

$$\chi^n_{k,i}(\kappa) = \begin{cases} 1 & \text{if } d(|\kappa|, \mathcal{C}^n_{k,i}) \leq \frac{1}{8}\eta^n, \\ 0 & \text{if } d(|\kappa|, \mathcal{C}^n_{k,i}) \geq \frac{1}{4}\eta^n, \end{cases}$$

and interpolates monotonically between 0 and 1 otherwise, with

$$\sup_{\kappa\in\mathbb{R}} |{\chi^n_{k,i}}'(\kappa)| \leq C\eta^{-n}, \tag{3.27}$$

whereas Q_n is defined as

$$Q_n = \mathbb{1} - P_n\,. \tag{3.28}$$

Note that P_n and Q_n are not projectors. The smooth functions $\chi^n_{k,i}$ have been introduced in order to ensure the continuity of the diagonal kernels $\Gamma_n(q,q)$, cf. the discussion preceding Lemma 5.3 below. However, we will make use later of the projector

$$\hat{P}_n(q) = \sum_{k\geq 1}\sum_{i=1}^{M^n_k} \mathrm{I}_{S^n_{k,i}}(q)\mathcal{P}^n_{k,i}, \tag{3.29}$$

where I_Σ denotes the indicator function of a set Σ. Note that $P_n\hat{P}_n = P_n$, whereas $Q_n\hat{P}_n \neq 0$.

We conclude this section by a few remarks related to the convergence of the inductive scheme. First, setting $I^n_{k,i} \subset \mathbb{R}$ to be the smallest interval covering $\mathcal{C}^n_{k,i}$, one easily checks that $|I^n_{k,i}| \leq (d_k - 1)\eta^n$. Hence, since the multiplicities of the normal frequencies μ_k were assumed to be uniformly bounded in k, i.e., $d_k \leq \bar{d}$ for all $k \geq 1$, one obtains for all $n \geq 1$, $k \geq 1$, and $i = 1, \ldots, M^n_k$,

$$|I^n_{k,i}| \leq \bar{d}\eta^n. \tag{3.30}$$

Next, it follows from the gap condition (3.22) being preserved that for all $m < n$ the eigenvalues in a given cluster $\mathcal{C}^n_{k,i}$ are perturbation of all or some eigenvalues belonging to a single cluster $\mathcal{C}^m_{k,j}$, denoted by $\mathcal{C}^m_{k,j^n_i}$. Furthermore, $\mathcal{C}^n_{k,i}$ remains close to $\mathcal{C}^m_{k,j^n_i}$. More precisely, we will show that

$$\sup_{x\in I^n_{k,i}} \inf_{y\in I^m_{k,j^n_i}} d(x,y) \leq \eta^{m+1} \quad \text{for} \quad 1 \leq m < n. \tag{3.31}$$

Finally, we consider the properties of the eigenspaces $\mathcal{J}^n_{k,i}$. One has by construction $\mathcal{P}^n_{k,i}\mathcal{P}^n_{l,j} = \delta_{kl}\delta_{ij}\mathcal{P}^n_{k,i}$. However, it will be possible to chose a_m in (3.21) in such a way that each $\mathcal{J}^m_{k,i}$ is an invariant subspace for a_m. Hence, by definition of $\tilde{\mu}_n$ and $\mathcal{J}^n_{k,i}$, every $\mathcal{J}^n_{k,i}$ is a subspace of some $\mathcal{J}^{n-1}_{k,j}$, and by recursion, of some $\mathcal{J}^m_{k,j}$ for all $m < n$. The (unique) eigenspace $\mathcal{J}^m_{k,j}$ containing $\mathcal{J}^n_{k,i}$ will be denoted by $\mathcal{J}^m_{k,j^n_i}$. Therefore, one has for all $1 \leq m \leq n$, $k \geq 1$, and $i = 1, \ldots, M^n_k$,

$$\mathcal{P}^n_{k,i}\mathcal{P}^m_{l,j} = \delta_{kl}\delta_{jj^n_i}\mathcal{P}^n_{k,i}, \tag{3.32}$$

which, in particular, implies that

$$P_n P_{n-1} = P_{n-1} P_n = P_n. \tag{3.33}$$

Notations. For most of the subsequent analysis, it will not be necessary to distinguish between indices (k, i) and (l, j) with $k = l$ or $k \neq l$. This intervenes only in the description of the asymptotic behavior of the spectrum $\sigma(\tilde{\mu}_n)$ and the measure estimate of Ω^*. For notational convenience, we thus introduce the index sets

$$\mathcal{I}^n = \{(k, i) \mid k \geq 1,\ i = 1, \ldots, M_k^n\}, \quad n \geq 1, \tag{3.34}$$

and will reserve bold letters for indices in $\mathcal{I}^n$. With this convention, $\{\mathcal{C}_{\mathbf{k}}^n \mid \mathbf{k} \in \mathcal{I}^n\}$, denotes for instance the collection of all clusters $\mathcal{C}_{k,i}^n$, $k \geq 1$, $i = 1, \ldots, M_k^n$.

4. Spaces

For the Fourier transform z of the solution Z of our original equation (3.4), we consider the Banach space h_s, $s \in \mathbb{R}$, defined by

$$h_s = \{z = (z(q)) \mid z(q) \in \hat{\mathcal{R}}_s^\infty,\ ||z||_s \equiv \sum_{q \in \mathbb{Z}^d} |z(q)|_s < \infty\}. \tag{4.1}$$

For $s \geq t$, one has the natural embedding $h_s \to h_t$ with $||\cdot||_t \leq ||\cdot||_s$. We will denote by h_s^n the subspace $\hat{P}_n h_s$. In particular, one has for $z \in h_s^n$,

$$||z||_s = \sum_{\mathbf{k} \in \mathcal{I}^n} \sum_{q \in S_{\mathbf{k}}^n} |\mathcal{P}_{\mathbf{k}}^n z(q)|_s. \tag{4.2}$$

The operator norm in $\mathcal{L}(h_s^n, h_t^m)$ will be denoted by $||\cdot||_{s,t}^{(n,m)}$, and by $||\cdot||_s^{(n)}$ when $n = m$ and $s = t$.

Let us now turn to the spaces we will consider for the functions w_n. Recall that in our analysis of (3.4), the functions Φ and J only appear as parameters. In the sequel, we consider Φ, $J : \mathbb{T}^d \to \mathbb{R}^d$ as (fixed) real analytic maps belonging to a small neighborhood of the origin $\mathcal{O}_{\mathcal{B}}$ in the Banach space

$$\mathcal{B} = \{(F, G) : \mathbb{T}^d \to \mathbb{R}^d \times \mathbb{R}^d \mid ||(F, G)||_{\mathcal{B}} \equiv \sum_{q \in \mathbb{Z}^d} |f(q)| + |g(q)| < \infty\}. \tag{4.3}$$

Next, it follows from assumption (H2) that the gradient $\partial_x U$ is real analytic as a map from $\mathbb{T}^d \times \mathcal{O}_I \times \mathcal{O}_x$ to $\mathcal{R}_{s'}^\infty$, cf. [PT] p. 138. (Recall that $\mathcal{O}_I \subset \mathbb{R}^d$ and $\mathcal{O}_x \subset \mathcal{R}_s^\infty$ are neighborhoods of the origin and that $s' \equiv s + \xi - a$.) This implies that for $(\Phi, J) \in \mathcal{O}_{\mathcal{B}}$

small enough, one can write the Taylor expansion of $\partial_x U(\varphi+\Phi(\varphi), J(\varphi), Z) = \partial_x U((\varphi+\Phi(\varphi), J(\varphi), 0) + (0,0,Z))$ as

$$\partial_x U((\varphi + \Phi(\varphi), J(\varphi), 0) + (0,0,Z)) = \sum_{m=0}^{\infty} \frac{1}{m!} U_{m+1}^{\Phi,J}(\varphi)(Z,\ldots,Z), \tag{4.4}$$

where the coefficients $U_{m+1}^{\Phi,J}(\varphi)$ belong to the space of m-linear maps $\mathcal{L}(\mathcal{R}_s,\ldots,\mathcal{R}_s;\mathcal{R}_{s'})$, are real analytic in $\varphi \in \mathbb{T}^d$ and analytic in $(\Phi, J) \in \mathcal{O}_{\mathcal{B}}$. Hence, there exist $\rho > 0$, $\alpha > 0$ and $b < \infty$ such that the Fourier transforms of $U_{m+1}^{\Phi,J}(\varphi)$ satisfy

$$\sum_{q\in\mathbb{Z}^d} e^{\alpha|q|}\, ||u_{m+1}^{\Phi,J}(q)||_{\mathcal{L}(\hat{\mathcal{R}}_s,\ldots,\hat{\mathcal{R}}_s;\hat{\mathcal{R}}_{s'})} \le b\, m!\, \rho^{-m}. \tag{4.5}$$

Inserting the Fourier series for Z into (4.4), one obtains the expansion for w_0 as defined in (3.5),

$$\begin{aligned} w_0(z)(q) &= \lambda \sum_{m=0}^{\infty} \sum_{\mathbf{q}} \frac{1}{m!} u_{m+1}^{\Phi,J}\Big(q - \sum_{i=1}^{m} q_i\Big)(z(q_1),\ldots,z(q_m)) \\ &\equiv \sum_{m=0}^{\infty} \sum_{\mathbf{q}} w_0^{(m)}(q;q_1,\ldots,q_m)(z(q_1),\ldots,z(q_m)), \end{aligned} \tag{4.6}$$

where $\mathbf{q} = (q_1,\ldots,q_m) \in \mathbb{Z}^{md}$. This formula suggests to consider w_0 as an analytic functions of $z \in h_s$. Let $B(r_0)$ be the open ball of radius r_0 in h_s centered at the origin and let $H^\infty(B(r_0), h_{s'})$ denote the Banach space of analytic function $w : B(r_0) \to H_{s'}$ equipped with the supremum norm, which we shall denote by $|||w|||$. Then, bound (4.5) implies that $w_0 \in H^\infty(B(r_0), h_{s'})$ for r_0 small enough.

It will be convenient to encode the decay property of the kernels $w_0^{(m)}$ inherited from the estimate (4.5) as a property of the functional w_0. Let τ_β denote the translation by $\beta \in \mathbb{R}^d$, i.e., $(\tau_\beta Z)(\varphi) = Z(\varphi - \beta)$. On h_s, τ_β is realized by $(\tau_\beta z)(q) = e^{i\beta\cdot q} z(q)$, and it induces a map $w \mapsto w_\beta$ from $H^\infty(B(r_0), h_{s'})$ to itself if we define

$$w_\beta(z) = \tau_\beta w(\tau_{-\beta} z). \tag{4.7}$$

On the kernels $w_0^{(m)}$, this is given by

$$w_\beta^{(m)}(q;q_1,\ldots,q_m) = e^{i\beta\cdot(q-\sum q_i)} w^{(m)}(q;q_1,\ldots,q_m),$$

and makes sense also for $\beta \in \mathbb{C}^d$. Since

$$|||w_{0\beta}||| \le \sum_{m=0}^{\infty} r_0^m \sup_{\mathbf{q}} \sum_{q\in\mathbb{Z}^d} e^{-\mathrm{Im}\beta\cdot((q-\sum q_i)} ||w_0^{(m)}(q;q_1,\ldots,q_m)||_{\mathcal{L}(\hat{\mathcal{R}}_s,\ldots,\hat{\mathcal{R}}_s;\hat{\mathcal{R}}_{s'})},$$

it thus follows from (4.5) that there exist $r_0 > 0$, $\alpha > 0$, and $D < \infty$, such that $w_{0\beta}$ belongs to $H^\infty(B(r_0), h_{s'})$ and extends to an analytic function of β in the strip $|\operatorname{Im}\beta| < \alpha$ with values in $H^\infty(B(r_0), h_{s'})$ satisfying the bound

$$|||w_{0\beta}||| \leq D|\lambda|. \tag{4.8}$$

Let us now come back to the existence of a solution for equation (3.3), namely for the standard KAM problem. One has the classical result (see for instance [BGK]):

Theorem 4.1. *Let U satisfy hypothesis (H2) and let g be an invertible matrix. Then, there is a $\lambda_1 > 0$ small enough such that for $|\lambda| < \lambda_1$ and ω satisfying a Diophantine condition of the form*

$$|\omega \cdot q| > K|q|^{-\nu} \quad \text{for} \quad q \in \mathbb{Z}^d,\ q \neq 0,$$

(3.3) has a solution $(\Phi, J) \in \mathcal{B}$ which is real analytic in φ, analytic in λ, and vanishes for $\lambda = 0$. Furthermore, this solution is unique up to translations $(\Phi, J)(\varphi) \mapsto (\Phi - \beta, J)(\varphi - \beta)$ and depends analytically on Z, for Z in a small ball centered at the origin of the Banach space h_s.

To conclude this section, we list some standard properties of bounded analytic functions defined on open balls in Banach spaces. Let h, h', h'' be Banach spaces, $B(r) \subset h$, $B(r') \subset h'$, and $w_i \in H^\infty(B(r), h')$, $w \in H^\infty(B(r'), h'')$. First, one has the composition property: If $|||w_i||| < r'$ then $w \circ w_i \in H^\infty(B(r), h'')$ and

$$|||w \circ w_i||| < |||w|||. \tag{4.9}$$

Next, one deduces from the Cauchy estimate that for $r_1 < r'$,

$$\sup_{||x||<r_1} ||Dw(x)||_{\mathcal{L}(h',h'')} \leq (r' - r_1)^{-1}|||w|||. \tag{4.10}$$

Taking $r_1 = \frac{1}{2}r'$, we infer that if $|||w_i||| \leq \frac{1}{2}r'$ then

$$|||w \circ w_1 - w \circ w_2||| \leq \frac{2}{r'}|||w|||\,|||w_1 - w_2|||. \tag{4.11}$$

Moreover, if $\delta_k w(x) \equiv w(x) - \sum_{l=0}^{k-1} \frac{1}{l!} D^l w(0)(x)$, then

$$\sup_{||x||\leq\gamma r'} ||\delta_k w(x)|| \leq \frac{\gamma^k}{1-\gamma}|||w|||, \tag{4.12}$$

for $0 \leq \gamma < 1$.

5. Inductive Bounds

We now turn to the inductive bounds that will be used to prove Theorem 1.1. We first note that since in (3.14) and (3.16), Γ_n and A_n are diagonal operators, applying τ_β to equation (3.14) leads to

$$R_{n\beta}(z) = \Gamma_n \tilde{w}_{(n-1)\beta}(z + R_{n\beta}(z)), \tag{5.1}$$

where $\tilde{w}_{(n-1)\beta} = w_{(n-1)\beta} - A_{n-1}$, and $w_{n\beta}$ is now recursively defined by

$$w_{n\beta}(z) \equiv \tilde{w}_{(n-1)\beta}(z + R_{n\beta}(z)). \tag{5.2}$$

For $r < 1$ a parameter to be chosen later, let B_n denote the open ball of radius r^{n+1} in h_s^n centered at the origin. Then, we will show that $w_{n\beta}$ belongs to $H^\infty(B_n, h_{s'})$, the Banach space of analytic functions $w : B_n \to h_{s'}$, provided $|\lambda|$ is taken small enough (uniformly in n) and provided the analyticity strip in β is restricted slightly. In the sequel, we will denote $H^\infty(B_n, h_{s'})$ by $\mathcal{A}_n$. As mentioned in Remark 3.1, the main ingredient in proving Theorem 1.1 is to show that in addition, $\hat{P}_n w_{n\beta}$ becomes essentially linear as $n \to \infty$. Before stating this result, one introduces the following frequency subsets, setting for $K > 0$ and $\{\mathcal{C}^n_{\mathbf{k}}\}_{\mathbf{k}\in\mathcal{I}^n}$ the clusters described in the previous section,

$$\begin{aligned}\Omega_n(K) = \big\{\omega \in \mathbb{R}^d \;\mid\; & d(|\omega\cdot q|, \mathcal{C}^n_{\mathbf{k}}),\; d(|\omega\cdot q|, |\mathcal{C}^n_{\mathbf{k}} \pm \mathcal{C}^n_{\mathbf{k}'}|) > K|q|^{-\nu} \\ & \forall\, |q| < K\eta^{-n/\nu}, q \neq 0, \text{ and } \mathbf{k}, \mathbf{k}' \in \mathcal{I}^n\big\},\end{aligned} \tag{5.3}$$

where $\mathcal{C}^n_{\mathbf{k}} \pm \mathcal{C}^n_{\mathbf{k}'}$ denotes the set $\{\nu \pm \nu' \,|\, \nu \in \mathcal{C}^n_{\mathbf{k}}, \nu' \in \mathcal{C}^n_{\mathbf{k}'}\}$. Note that $\Omega_n(K) \subset \Omega_n(K')$ whenever $K > K'$. Furthermore, one introduces for $\omega \in \mathbb{R}^d$ the subsets of $\mathbb{Z}^d$

$$\mathcal{Q}^+_\omega = \{q \in \mathbb{Z}^d \,|\, \omega\cdot q > 0\}, \quad \mathcal{Q}^-_\omega = \{q \in \mathbb{Z}^d \,|\, \omega\cdot q < 0\}. \tag{5.4}$$

Proposition 5.1. *There exist positive constants r and λ_0 small enough such that the following is true for $|\lambda| < \lambda_0$, $n \geq 1$, and $|\operatorname{Im}\beta| < \alpha_n$, where $\alpha_1 = \alpha$ and, for $n \geq 2$,*

$$\alpha_n = (1 - n^{-2})\alpha_{n-1}. \tag{5.5}$$

There exists $K_\lambda > 0$ satisfying $K_\lambda \to 0$ as $\lambda \to 0$ such that one has for $\omega \in \Omega_n(K_\lambda)$ arbitrary but fixed,

(a) *Equation (5.1) has a solution $R_{n\beta}$ in $H^\infty(B_n, h_s^{n-1})$ analytic in $|\lambda| < \lambda_0$ and $(\Phi, J) \in \mathcal{O}_{\mathcal{B}}$.*

(b) *Defining $w_{n\beta}$ according to (5.2), one has $w_{n\beta} \in \mathcal{A}_n$ and, writing $w_{n\beta}(z) \equiv w_n(z) = w_n(0) + Dw_n(0)z + \delta_2 w_n(z)$,*

$$||\hat{P}_n w_n(0)||_{s'} \leq \varepsilon r^{2n}, \tag{5.6}$$

$$|||\hat{P}_n \delta_2 w_n|||_{\mathcal{A}_n} \leq \varepsilon r^{2n}, \tag{5.7}$$

where $\varepsilon \to 0$ as $\lambda \to 0$.

(c) *There exists $A_n \in \mathcal{L}(h_s, h_{s'})$ such that $\tilde{w}_n \equiv w_n - A_n$ obeys for all $z \in B_n$,*

$$||\hat{P}_n D\tilde{w}_n(z)||_{s,s'}^{(n)} \leq \varepsilon\eta^n. \tag{5.8}$$

Furthermore,

$$||A_n||_{s,s'} \leq 3\varepsilon\eta^{n-1}, \tag{5.9}$$

$A_n(q,q') = 0$ if $q \neq q'$ and

$$A_n(q,q) = a_n \mathrm{I}_{\mathcal{Q}_\omega^+}(q) + \overline{a_n}\, \mathrm{I}_{\mathcal{Q}_\omega^-}(q), \tag{5.10}$$

where $a_n \in \mathcal{L}(\hat{\mathcal{R}}_s^\infty, \hat{\mathcal{R}}_{s'}^\infty)$ is hermitian, i.e., $a_n = \overline{a_n}^{\mathrm{T}}$, and satisfies for all $\mathbf{k} \in \mathcal{I}^n$,

$$a_n \mathcal{J}_{\mathbf{k}}^n = \mathcal{J}_{\mathbf{k}}^n\,. \tag{5.11}$$

(d) *The matrix $\tilde{\mu}_{n+1}^2 \equiv \mu^2 + \sum_{m=0}^n a_m$ is positive definite and the spectrum of $\tilde{\mu}_{n+1}$ can be uniquely decomposed into a maximal family of pairwise disjoint clusters $\mathcal{C}_{k,i}^{n+1}$, $k \geq 1$, $i = 1, \ldots, M_k^{n+1}$, with $M_k^{n+1} \geq M_k^n$, satisfying for all $k \geq 1$ the gap condition*

$$d(\mathcal{C}_{k,i}^{n+1}, \mathcal{C}_{k,j}^{n+1}) > \eta^{n+1} \quad \text{if} \quad i \neq j, \tag{5.12}$$

and

$$\nu = \mu_k + \mathcal{O}(\varepsilon k^{-\xi}) \quad \text{for all} \quad \nu \in \mathcal{C}_{k,i}^{n+1}\,,\ i = 1, \ldots, M_k^{n+1}. \tag{5.13}$$

Furthermore, the sets $S_{k,i}^{n+1}$ defined according to (3.25) are pairwise disjoint, and (3.31), (3.32) and (3.33) hold with n replaced by $n+1$.

Let us briefly comment on Proposition 5.1, whose proof will be carried out in Section 6. First, we note that point (d) ensures, in particular, that the new set of clusters $\mathcal{C}_{k,i}^{n+1}$ enjoy the properties required for proceeding to the next step of the induction, cf. the discussion at the end of Section 3. The asymptotic behavior (5.13) concerns the measure estimate of the set Ω^* of admissible frequencies in Theorem 1.1. Such an asymptotic behavior is required in order to obtain a set of large measure because one imposes Diophantine conditions with respect to differences of the normal frequencies. We will show in Section 7 that (5.13) implies the

Proposition 5.2. *For $\nu = \nu(d,\xi)$ sufficiently large, the set*

$$\Omega^*(K) \equiv \bigcap_{n\geq 1} \Omega_n(K) \tag{5.14}$$

satisfies for all bounded $\Omega \subset \mathbb{R}^d$, $\text{meas}(\Omega \setminus \Omega^*(K)) \to 0$ *as* $K \to 0$.

Note that $\omega \in \Omega^*$ assume a Diophantine condition with respect to zero. Therefore, one has for such ω, $\mathbb{Z}^d \setminus \{0\} = \mathcal{Q}_\omega^+ \cup \mathcal{Q}_\omega^-$. Next, we turn to bound (5.8), the most delicate estimate to establish. To treat the off-diagonal part $Dw_n(q, q')$, $q \neq q'$, we will rely on the fact that the exponential decay of the kernel $Dw_0(q, q')$ in the size of $|q - q'|$ is preserved due to the introduction of the parameter β. We note that imposing Diophantine conditions on ω with respect to the differences $\mathcal{C}_{\mathbf{k}}^n \pm \mathcal{C}_{\mathbf{k}'}^n$ ensures that $|q - q'|$ is of order $\mathcal{O}(\eta^{-n/\nu})$ for $q \neq q' \in S_n$. To treat the diagonal part, we will use that $Dw_n(q, q)$ depends on q through $\omega \cdot q$ only, and is, in some sense, continuous in this variable. More precisely, defining $t_p : \mathcal{L}(h_s, h_{s'}) \to \mathcal{L}(h_s, h_{s'})$, $p \in \mathbb{Z}^d$, by

$$(t_p L)(q, q') = L(q + p, q' + p), \tag{5.15}$$

and setting

$$\Delta_p \equiv t_p - \mathbb{1}, \tag{5.16}$$

we will show that $\Delta_p Dw_n$ is of order $\mathcal{O}(\varepsilon|\omega \cdot p|)$ on the diagonal. Therefore, since $p = q - q'$ satisfies $|\omega \cdot p| \leq \eta^n$ for $q, q' \in S_{\mathbf{k}}^n$ such that $\text{sign}(\omega \cdot q) = \text{sign}(\omega \cdot q')$, one has for $q \in S_{\mathbf{k}}^n$,

$$\hat{P}_n Dw_n(q, q) \hat{P}_n = \hat{a}_{\mathbf{k}} + \mathcal{O}(\varepsilon \eta^n),$$

where $\hat{a}_{\mathbf{k}} : \mathcal{J}_{\mathbf{k}}^n \to \mathcal{J}_{\mathbf{k}}^n$ dependents only on the sign of $\omega \cdot q$. The continuity of $Dw_n(q, q)$ ultimately follows from the fact that $\Gamma_n(q)$ is continuous in $\omega \cdot q$, as stated in the following lemma, whose proof can be found in the Appendix.

Lemma 5.3. *Let* $\sigma \in \mathbb{R}$ *and* $p \in \mathbb{Z}^d$. *Then the operator* $\Gamma_n = \mathcal{K}_n^{-1} Q_n P_{n-1}$ *obeys*

$$||\Gamma_n||_{\sigma, \sigma+\gamma} \leq C\eta^{-n}, \tag{5.17}$$

$$||\Delta_p \Gamma_n||_{\sigma, \sigma+\gamma} \leq C\eta^{-2n} |\omega \cdot p|. \tag{5.18}$$

Finally, the perturbation a_n being hermitian will essentially follow from the reality of the original equation (3.4). More precisely, the derivative Dw_n satisfies

$$Dw_n^{ij}(q, q') = \overline{Dw_n^{ij}(-q, -q')}, \tag{5.19}$$

$$Dw_n^{ij}(q, q') = Dw_n^{ji}(-q', -q). \tag{5.20}$$

Thus, the diagonal element $Dw_n(q, q) : \hat{\mathcal{R}}^\infty \to \hat{\mathcal{R}}^\infty$ is given by an hermitian matrix for all q, and a_n hermitian will follow since, as was mentioned above, a_n will be chosen in such a way that its action on each $\mathcal{J}_{\mathbf{k}}^n$ is the constant approximation of $Dw_n(q, q)$ for $q \in S_{\mathbf{k}}^n$. Note that due to (5.19), one expects $Dw_n(-q, -q)$ to be approximated by $\overline{a_n}$, which explains the decomposition in formula (5.10). Identities (5.19) and (5.20) are easily checked to hold for $n = 0$. Indeed, the perturbation U in the Hamiltonian (1.4) being real analytic ensures (5.19), whereas (5.20) follows from the fact that Dw_0 is the symmetric second derivative of the functional $Z \mapsto \lambda \int U(\varphi + \Phi(\varphi), J(\varphi), Z(\varphi))d\varphi$, cf. (3.5). Using the recursive relations (3.19) and (3.16), one obtains (5.19) and (5.20) for $n \geq 1$ by iteration.

Remark 5.4. The choice of constants is as follows. We first fix η small enough according, essentially, to the constants entering the asymptotics of the frequencies μ_k in (H1), cf. Section 6.4. Given η, ε and r are chosen small enough, and λ_0 is chosen in turn according to ε. The latter choice plays a role only in ensuring that the inductive hypothesis of Proposition 5.1 are satisfied for $n = 0$, cf. the introduction in Section 6. Finally, K_λ is chosen large enough in order for the estimate

$$Ce^{-Cn^{-2}K_\lambda^{1/\nu}\eta^{-n/\nu}} \leq r^{2n}, \tag{5.21}$$

to hold for all $n \geq 1$. This will be needed in order to iterate the bound (5.6) in Section 6.2. Note that due to the double exponential, the dependence of K_λ on η and r is given by the behavior at small n of the expressions entering (5.21). That K_λ can be taken smaller as λ goes to zero will follow from the fact that r and ε, and thus ultimately η, can be taken smaller. Finally, we denote by C a generic constant, independent on n, r, and ε, which may vary from place to place.

6. Proof of Proposition 5.1

We proceed by induction and assume that Proposition 5.1 holds up to $n - 1 \geq 1$. Regarding the inductive hypothesis in the case $n = 1$, we simply choose $A_0 \equiv 0$, so that the bounds for w_0 in points (b) and (c) of Proposition 5.1 are a simple consequence of (4.8). Furthermore, $\tilde{\mu}_1 = \mu$ and point (d) follows immediately from (H1). We note that in Section 6.1 below, point (a) is established for $n = 1$ by taking ε, namely λ, small enough. At some point in the induction, however, one is forced to consider nontrivial A_n in order for the inductive bounds to hold uniformly in n for a given λ.

In the sequel, we adopt the convention, for B a ball of radius r centered at the origin, to denote by γB the ball of radius γr centered at the origin.

6.1. Existence of the Functional $R_{n\beta}$

With the notations $R = R_{n\beta}$, $\Gamma = \Gamma_n$ and $\tilde{w} = \tilde{w}_{(n-1)\beta}$, equation (5.1) reads

$$R(z) = \Gamma\tilde{w}(z + R(z)). \tag{6.1}$$

To prove existence in $H^\infty(B_n, h_s^{n-1})$ of a solution R to equation (6.1), one starts, using the identities $\tilde{w}(0) = w(0)$ and $\delta_2\tilde{w} = \delta_2 w$, by decomposing $\tilde{w}$ as

$$\tilde{w}(z) = w(0) + D\tilde{w}(0)z + \delta_2 w(z), \tag{6.2}$$

to obtain from (6.1),

$$R(z) = \Gamma w(0) + \Gamma D\tilde{w}(0)(z + R(z)) + \Gamma\delta_2 w(z + R(z)). \tag{6.3}$$

Defining

$$H = \big(1 - \Gamma D\tilde{w}(0)\big)^{-1}, \tag{6.4}$$

and using the identity $1 + H\Gamma D\tilde{w}(0) = H$, one rewrites (6.3) as

$$R(z) = H\Gamma w(0) + H\Gamma D\tilde{w}(0)z + u(z), \tag{6.5}$$

where

$$u(z) = H\Gamma\delta_2 w(\tilde{z}) \equiv \mathcal{G}(u)(z), \tag{6.6}$$

and

$$\tilde{z} \equiv z + R(z) = H\big(z + \Gamma w(0)\big) + u(z). \tag{6.7}$$

Since $\Gamma = \Gamma\hat{P}_{n-1} = \hat{P}_{n-1}\Gamma$, (5.17) (with $\sigma = s + \xi - \gamma$) and the recursive bound (5.8) (with n replaced by $n-1$) imply

$$||\Gamma D\tilde{w}(0)||_s^{(n-1)} \leq ||\Gamma D\tilde{w}(0)||_{s,s+\xi}^{(n-1)} \leq C\varepsilon\eta^{-1}. \tag{6.8}$$

Hence,

$$||H||_s^{(n-1)} \leq 2, \tag{6.9}$$

for $\varepsilon = \varepsilon(\eta)$ small enough. Since $B_n \subset B_{n-1}$, $\tilde{w}(0) = w(0)$, and since bounds (5.6) (with n replaced by $n-1$), (5.17) and (6.8) hold, the existence of R in $H^\infty(B_n, h_s^{n-1})$ follows from the existence of u in $H^\infty(B_n, h_s^{n-1})$. For reasons that will become clear in the next section, we actually show that (6.6) has a solution u in the ball

$$\mathcal{B} = \Big\{u \in H^\infty(\tfrac{1}{8}B_{n-1}, h_s^{n-1}) \mid |||u||| \leq \sqrt{\varepsilon}\eta^{-n}r^{2(n-1)}\Big\}. \tag{6.10}$$

This result is stronger, since $B_n \subset \frac{1}{8}B_{n-1}$ for r small enough. Let us first check that $\mathcal{G}$ maps $\mathcal{B}$ into itself. From (6.9) and the recursive bound (5.6), it follows that for all $z \in \frac{1}{8}B_{n-1}$ and $u \in \mathcal{B}$, $\tilde{z} \in h_s^{n-1}$ with

$$||\tilde{z}||_s \leq 2(\tfrac{1}{8}r^n + C\varepsilon\eta^{-n}r^{2(n-1)}) + \sqrt{\varepsilon}\eta^{-n}r^{2(n-1)} \leq \tfrac{1}{2}r^n,$$

for $\varepsilon = \varepsilon(r, \eta)$ and $r = r(\eta)$ small enough. Hence,

$$\tilde{z} \in \tfrac{1}{2}B_{n-1} \subset B_{n-1} \quad \text{for all } z \in \tfrac{1}{8}B_{n-1}, \tag{6.11}$$

and one uses the bound (5.7) to conclude that for all $u \in \mathcal{B}$,

$$|||\mathcal{G}(u)||| \leq 2C\eta^{-n}\varepsilon r^{2(n-1)} \leq \sqrt{\varepsilon}\eta^{-n}r^{2(n-1)},$$

for ε small enough. To show that $\mathcal{G}$ is a contraction in $\mathcal{B}$, we apply the estimate (4.11) to the functions $\tilde{z}_i$ given by (6.7) in terms of $u_i \in \mathcal{B}$, $i = 1, 2$. Noting that $|||\tilde{z}_i||| \le \frac{1}{2} r^n$, which follows from (6.11), and using in addition (5.7), one obtains,

$$\begin{aligned}
|||\mathcal{G}(u_1) - \mathcal{G}(u_2)||| &\le 2C\eta^{-n} \sup_{z \in \frac{1}{8} B_{n-1}} ||\hat{P}_{n-1}\delta_2 w(\tilde{z}_1) - \hat{P}_{n-1}\delta_2 w(\tilde{z}_2)||_{s'} \\
&\le 4C\eta^{-n} r^{-n} |||\hat{P}_{n-1}\delta_2 w|||_{\mathcal{A}_{n-1}} \sup_{z \in \frac{1}{8} B_{n-1}} ||\tilde{z}_1 - \tilde{z}_2||_s \\
&\le 4C\varepsilon\eta^{-n} r^{-n} r^{2(n-1)} \sup_{z \in \frac{1}{8} B_{n-1}} ||u_1(z) - u_2(z)||_s \\
&\le \tfrac{1}{2} |||u_1 - u_2|||,
\end{aligned}$$

for $r = r(\eta)$ and $\varepsilon = \varepsilon(r, \eta)$ small enough.

Before turning to part (b) of Proposition 5.1, we make some remarks that shall be useful later. First note that (6.11) means

$$z + R_n(z) \in \tfrac{1}{2} B_{n-1} \quad \text{for all} \quad z \in \tfrac{1}{8} B_{n-1}. \tag{6.12}$$

Therefore, with

$$\tilde{R}_m(z) \equiv z + R_m(z), \tag{6.13}$$

and

$$F_n^m(z) \equiv \tilde{R}_m \circ \tilde{R}_{m+1} \circ \ldots \circ \tilde{R}_n(z), \tag{6.14}$$

it follows recursively that for $m = 1, \ldots, n$,

$$F_n^m(z) \in \tfrac{1}{2} B_{m-1} \quad \text{for all} \quad z \in B_n. \tag{6.15}$$

Furthermore, since $F_n^1 = F_n$, where F_n is defined in (3.13), one has $F_n \in \mathcal{A}_n$, together with the uniform bound

$$|||F_n|||_{\mathcal{A}_n} \le |||\tilde{R}_1|||_{\mathcal{A}_1} \le \varepsilon. \tag{6.16}$$

6.2. Bounds on the Functional w_n

According to (5.2), one defines

$$w_{n\beta}(z) = \tilde{w}_{(n-1)\beta}(z + R_{n\beta}(z)).$$

Since $R_{n\beta} \in H^\infty(B_n, h_s^{n-1})$, it follows from (6.12) and the inductive bounds that for all β with $|\operatorname{Im}\beta| < \alpha_{n-1}$, $w_{n\beta}$ is well defined as a map from B_n to $h_{s'}$, with $w_{n\beta} \in \mathcal{A}_n$. In the sequel, we adopt the simplified notation $R = R_{n\beta}$, $w = w_{(n-1)\beta}$ and $w' = w_{n\beta}$. We proceed with proving (5.6). Using the decomposition (6.2) at $z = 0$, one may write

$$w'(0) = w(0) + D\tilde{w}(0)R(0) + \delta_2 w(R(0)).$$

Since (6.12) implies that $R(0) \in \frac{1}{2}B_{n-1}$, one obtains using the bounds (5.6), (5.7) and (5.8),

$$\begin{aligned} ||\hat{P}_n w'(0)||_{s'} &\leq \varepsilon r^{2(n-1)} + \tfrac{1}{2}\varepsilon \eta^{n-1} r^n + \varepsilon r^{2(n-1)} \\ &\leq 3\varepsilon. \end{aligned} \tag{6.17}$$

This leads to

$$|\mathcal{P}^n_{\mathbf{k}} w'(0)(q)|_{s'} \leq 3\varepsilon, \tag{6.18}$$

for all $\mathbf{k} \in \mathcal{I}^n$ and $q \in S^n_{\mathbf{k}}$. The latter is valid for all β with $|\operatorname{Im}\beta| < \alpha_{n-1}$. Let now β' with $|\operatorname{Im}\beta'| < \alpha_n$. Then, shifting β' to $\beta = \beta' - i(\alpha_{n-1} - \alpha_n)q/|q|$ and using the recursive relation (5.5) for α_n, one obtains

$$w'_{\beta'}(0)(q) = e^{i(\beta'-\beta)\cdot q} w'_\beta(0)(q) = e^{-n^{-2}\alpha_{n-1}|q|} w'_\beta(0)(q). \tag{6.19}$$

Since for such β' one has $|\operatorname{Im}\beta| < \alpha_{n-1}$, it follows from (6.18) and (6.19) that

$$||\hat{P}_n w'(0)||_{s'} \leq 3\varepsilon \sum_{\mathbf{k}\in\mathcal{I}^n} \sum_{q \in S^n_{\mathbf{k}}} e^{-n^{-2}\alpha_{n-1}|q|} \ . \tag{6.20}$$

From the Diophantine conditions satisfied by $\omega \in \Omega_n(K)$, one infers for $q \in S^n_{\mathbf{k}}$ that $|q| > \min(K\eta^{-n/\nu}, (4K)^{1/\nu}\eta^{-n/\nu})$, cf. (3.25) and (5.3). Therefore, Bound (5.6) finally follows by choosing K appropriately, cf. (5.21).

We now iterate bound (5.7). Using again the decomposition (6.2), one has

$$\delta_2 w'(z) = D\tilde{w}(0)\delta_2 R(z) + \delta_2\delta_2 w(z + R(z)).$$

The first term on the right hand side is estimated by using $\delta_2 R(z) = \delta_2 u(z)$ together with (4.12) applied to $u \in \mathcal{B}$ with $\gamma = 8r$, since $B_n \subset \frac{1}{8}B_{n-1}$, to obtain

$$\begin{aligned} |||\hat{P}_n D\tilde{w}(0)\delta_2 R|||_{\mathcal{A}_n} &\leq ||\hat{P}_{n-1} D\tilde{w}(0)||^{n-1}_{s,s'} \sup_{z\in B_n} ||\delta_2 u(z)||_s \\ &\leq \varepsilon \eta^n \frac{(8r)^2}{1-(8r)^2} |||u||| \\ &\leq \varepsilon r^{2n} \frac{\sqrt{\varepsilon}\, 8^2}{1-(8r)^2} \\ &\leq \tfrac{1}{2}\varepsilon r^{2n}, \end{aligned}$$

for ε small enough. In a similar way, one estimates, using (6.12), that

$$\sup_{z\in B_n} ||\hat{P}_n \delta_2\delta_2 w(z + R(z))||_{s'} \leq \tfrac{1}{2}\varepsilon r^{2n},$$

which finally leads to (5.7).

6.3. Bounds on the Derivative

In this section, we prove the estimates stated in part (c) of Proposition 5.1. The main difficulty consists in controlling the diagonal part of the kernel of the derivative Dw_n evaluated at zero, namely $Dw_n(0)(q,q)$, $q \in \mathbb{Z}^d$. To address this problem, as mentioned in the end of Section 5, we will use the fact that $Dw_n(0)(q,q)$ depends on q through $\omega \cdot q$ only, and satisfies some continuity property when viewed as a function of $\omega \cdot q$.

We start by deriving an a priori bound on the norm of Dw_n. From (3.14), one infers that

$$DR_n(z) = H_n(\tilde{z})\Gamma_n D\tilde{w}_{n-1}(\tilde{z}), \tag{6.21}$$

where

$$H_n(\tilde{z}) = \big(1 - \Gamma_n D\tilde{w}_{n-1}(\tilde{z})\big)^{-1}, \tag{6.22}$$
$$\tilde{z} = z + R_n(z). \tag{6.23}$$

Since by definition, cf. (3.19), one has

$$Dw_n(z) = D\tilde{w}_{n-1}(\tilde{z})\big(1 + DR_n(z)\big),$$

(6.21) and the identity $H_n(\tilde{z}) = 1 + H_n(\tilde{z})\Gamma_n D\tilde{w}_{n-1}(\tilde{z})$, imply the recursive relation

$$Dw_n(z) = D\tilde{w}_{n-1}(\tilde{z})H_n(\tilde{z}). \tag{6.24}$$

As in the previous section, it follows from (5.17), (6.12), and the inductive bounds, that $||H_n(\tilde{z})||_s^{(n-1)} \leq 2$ for all $\tilde{z} \in B_{n-1}$. Therefore, one obtains for all $z \in \frac{1}{8}B_{n-1}$, using again the inductive bound (5.8),

$$||\hat{P}_n Dw_n(z)||_{s,s'}^{(n)} \leq ||\hat{P}_{n-1} Dw_n(z)||_{s,s'}^{(n-1)} \leq 2\varepsilon\eta^{n-1}. \tag{6.25}$$

In order to iterate bounds (5.8), we decompose $Dw_n(z)$ as follows

$$Dw_n(z) = \sigma_n + \tau_n + \delta_1 Dw_n(z), \tag{6.26}$$

where $\sigma_n + \tau_n = Dw_n(0)$ and $\sigma_n(q,q') = Dw_n(0)(q,q')\delta_{qq'}$. Let us consider first the last two terms on the right hand side of (6.26). One has the

Lemma 6.1. *Let r and ε be the positive constants of Proposition 5.1. Then, one has for all $n \geq 0$ and all $z \in B_n$,*

$$||\hat{P}_n \delta_1 Dw_n(z)||_{s,s'}^{(n)} \leq \tfrac{1}{2}\varepsilon r^{\frac{n}{2}}, \tag{6.27}$$
$$||\hat{P}_n \tau_n||_{s,s'}^{(n)} \leq \varepsilon r^n. \tag{6.28}$$

Proof. Proceeding by induction, we suppose that Proposition 5.1 and Lemma 6.1 are true up to some $n-1$, $n \geq 1$. We start with (6.27) and compute from $\delta_1 Dw_n(z) = Dw_n(z) - Dw_n(0)$ and the recursive relation (6.24) that

$$\delta_1 Dw_n(z) = \tilde{H}_n(\tilde{z}_0)\big(D\tilde{w}_{n-1}(\tilde{z}) - D\tilde{w}_{n-1}(\tilde{z}_0)\big)H_n(\tilde{z}),$$

where $\tilde{z}_0 = R_n(0)$ and $\tilde{H}_n(\tilde{z}_0) = 1 + Dw_{n-1}(\tilde{z}_0)H_n(\tilde{z}_0)\Gamma_n$. As previously, the inductive bound (5.8) implies $||\hat{P}_n\tilde{H}_n(\tilde{z}_0)||_{s'}^{(n-1)} \leq 2$. Using (6.12) and $\hat{P}_n\tilde{H}_n = \hat{P}_n\tilde{H}_n\hat{P}_{n-1}$, one infers from the identity $D\tilde{w}_{n-1}(\tilde{z}) - D\tilde{w}_{n-1}(\tilde{z}_0) = \delta_1 D\tilde{w}_{n-1}(\tilde{z}) - \delta_1 D\tilde{w}_{n-1}(\tilde{z}_0)$ that for all $z \in \frac{1}{8}B_{n-1}$,

$$||\hat{P}_n\delta_1 Dw_n(z)||_{s,s'}^{(n-1,n)} \leq C \sup_{z' \in \frac{1}{2}B_{n-1}} ||\hat{P}_{n-1}\delta_1 D\tilde{w}_{n-1}(z')||_{s,s'}^{(n-1)}.$$

Since $\delta_1 D\tilde{w}_{n-1} = \delta_1 Dw_{n-1}$, the recursive bound (6.27) leads to

$$||\hat{P}_n\delta_1 Dw_n(z)||_{s,s'}^{(n-1,n)} \leq C\varepsilon r^{\frac{n-1}{2}},$$

for all $z \in \frac{1}{8}B_{n-1}$. Finally, iterating bound (6.27) is completed by restricting z to $B_n \subset \frac{1}{8}B_{n-1}$ and using (4.12) with $\gamma = 8r$.

Next, we turn to (6.28), the estimate for the off-diagonal part of $Dw_n(0)$. The norm of τ_n reads

$$||\hat{P}_n\tau_n||_{s,s'}^{(n)} = \sup_{\mathbf{k}' \in \mathcal{I}^n} \sup_{q' \in S_{\mathbf{k}'}^n} \sum_{\mathbf{k} \in \mathcal{I}^n} \sum_{q \in S_{\mathbf{k}}^n} |\mathcal{P}_{\mathbf{k}}^n \tau_n(q,q')\mathcal{P}_{\mathbf{k}'}^n|_{s,s'}\,,$$

and one infers from (6.27) and the a priori bound (6.25) that

$$|\mathcal{P}_{\mathbf{k}}^n \tau_n(q,q')\mathcal{P}_{\mathbf{k}'}^n|_{s,s'} \leq 2\varepsilon\eta^{n-1} + \tfrac{1}{2}\varepsilon r^{\frac{n}{2}} \leq 3\varepsilon. \tag{6.29}$$

The latter is valid for all β with $|\operatorname{Im}\beta| < \alpha_{n-1}$. Let now β' with $|\operatorname{Im}\beta'| < \alpha_n$. Then, shifting β' to $\beta = \beta' - i(\alpha_{n-1} - \alpha_n)(q-q')/|q-q'|$, one obtains

$$\tau_{n\beta'}(q,q') = e^{i(\beta'-\beta)\cdot(q-q')}\tau_{n\beta}(q,q') = e^{-n^{-2}\alpha_{n-1}|q-q'|}\tau_{n\beta}(q,q'). \tag{6.30}$$

Hence, since $|\operatorname{Im}\beta| < \alpha_{n-1}$ for such β', (6.29) and (6.30) lead to

$$||\hat{P}_n\tau_n||_{s,s'}^{n} \leq 3\varepsilon \sup_{\mathbf{k}' \in \mathcal{I}^n} \sup_{q' \in S_{\mathbf{k}'}^n} \sum_{\mathbf{k} \in \mathcal{I}^n} \sum_{\substack{q \in S_{\mathbf{k}}^n \\ q \neq q'}} e^{-n^{-2}\alpha_{n-1}|q-q'|}\,. \tag{6.31}$$

We now show that every term in the previous sum yields a super-exponentially small factor. Let $q \in S_{\mathbf{k}}^n$ and $q' \in S_{\mathbf{k}'}^n$ for some $\mathbf{k} \in \mathcal{I}^n$, $\mathbf{k}' \in \mathcal{I}^n$. Then, one estimates using (3.25) and (3.30) that if $\operatorname{sign}(\omega\cdot q) = \operatorname{sign}(\omega\cdot q')$,

$$d\big(|\omega\cdot(q-q')|, \mathcal{C}_{\mathbf{k}}^n + \mathcal{C}_{\mathbf{k}'}^n\big) \leq\ \tfrac{1}{2}\eta^n + |I_{\mathbf{k}}^n| + |I_{\mathbf{k}'}^n| \leq 3\bar{d}\eta^n,$$

and that otherwise

$$d\big(|\omega\cdot(q-q')|, |\mathcal{C}^n_{\mathbf{k}}-\mathcal{C}^n_{\mathbf{k}'}|\big) \le \tfrac{1}{2}\eta^n + |I^n_{\mathbf{k}}| + |I^n_{\mathbf{k}'}| \le 3\bar{d}\eta^n.$$

Therefore, since $q\neq q'$, it follows from (5.3) and $\omega\in\Omega_n(K)$ that

$$|q-q'| \ge \min\Big(\Big(\frac{K}{3\bar{d}}\Big)^{1/\nu}, K\Big)\eta^{-n/\nu}.$$

Hence, the contribution of each term in (6.31) is super-exponentially small, and (6.28) follows for some $r\ll\eta<1$.

■

Finally, we turn to σ_n, the diagonal part of $Dw_n(0)$ in the decomposition (6.26). We first state a result about the continuity properties of the kernel $\sigma_n(q,q)$, namely that $\Delta_p\sigma_n = t_p\sigma_n - \sigma_n$ is of order $|\omega\cdot p|$. More precisely, one has the

Proposition 6.2. *Suppose that Proposition 5.1 is valid up to $n-1$ for some $n\ge 1$. Then, the diagonal part $\sigma_n(z)$ of $Dw_n(z)$ satisfies for all $z\in B_n$ and all p such that $|\omega\cdot p| < \frac{1}{16}\eta^{n-1}$,*

$$||\hat{P}_n\Delta_p\sigma_n(z)||^n_{s,s'} \le \varepsilon^{\frac{3}{2}}|\omega\cdot p|. \tag{6.32}$$

Delaying the proof of the above proposition to the end of this section, we now construct a diagonal operator $A_n\in\mathcal{L}(h_s,h_{s'})$ such that $\tilde{\sigma}_n\equiv\sigma_n - A_n$ obeys

$$||\hat{P}_n\tilde{\sigma}_n||^n_{s,s'} = \sup_{\mathbf{k}\in\mathcal{I}^n}\sup_{q\in S^n_{\mathbf{k}}} |\mathcal{P}^n_{\mathbf{k}}\tilde{\sigma}_n(q,q)\mathcal{P}^n_{\mathbf{k}}|_{s,s'} \le \tfrac{1}{2}\varepsilon\eta^n. \tag{6.33}$$

The equality above follows from the sets $S^n_{\mathbf{k}}$ being pairwise disjoint. This will conclude the proof of iterating (5.8), since (6.27), (6.28) and (6.33) imply that the derivative of $\tilde{w}_n\equiv w_n - A_n$ satisfies the required bound for $r=r(\eta)$ small enough. In order to obtain bound (6.33) by using the continuity property (6.32), we would like to construct A_n as an approximation of $\sigma_n(q,q)$ for $\omega\cdot q$ close to the normal frequencies in $\mathcal{C}^n_{\mathbf{k}}$, $\mathbf{k}\in\mathcal{I}^n$. To this end, we set $\bar{\mu}_{\mathbf{k}}$ to be the center of the interval $I^n_{\mathbf{k}}$ and, using that $\{\omega\cdot q\,|\,q\in\mathbb{Z}^d\}$ is dense in $\mathbb{R}$, we choose a sequence $\{q_{l,\mathbf{k}}\}_{l\ge1}\subset S^n_{\mathbf{k}}$ such that $\omega\cdot q_{l,\mathbf{k}}>0$ for all $l\ge 1$ and

$$\lim_{l\to\infty}\omega\cdot q_{l,\mathbf{k}} = \bar{\mu}_{\mathbf{k}}.$$

Next, one defines the matrix $\hat{a}_{n,\mathbf{k}}\in\mathcal{L}(\mathcal{J}^n_{\mathbf{k}})$ by

$$\hat{a}_{n,\mathbf{k}} \equiv \lim_{l\to\infty}\mathcal{P}^n_{\mathbf{k}}\sigma_n(q_{l,\mathbf{k}},q_{l,\mathbf{k}})\mathcal{P}^n_{\mathbf{k}}. \tag{6.34}$$

Due to (6.32), the limit in (6.34) exists and does not depend on the particular choice of the sequence $\{q_{l,\mathbf{k}}\}_{l\geq 1}$. Finally, setting

$$a_n \equiv \bigoplus_{\mathbf{k}\in\mathcal{I}^n} \hat{a}_{n,\mathbf{k}}, \tag{6.35}$$

we define the operator $A_n : h \to h$ as given by the diagonal kernel

$$A_n(q,q) = a_n \mathrm{I}_{\mathcal{Q}_\omega^+}(q) + \overline{a_n}\ \mathrm{I}_{\mathcal{Q}_\omega^-}(q) \tag{6.36}$$

for all $q \in \mathbb{Z}^d$. We note that by construction, (5.11) is clearly satisfied. Furthermore, it follows from (5.19) and (5.20) that a_n is indeed hermitian. Let us check that definition (6.36) leads to the required bound (6.33). By construction, one has for all $\mathbf{k} \in \mathcal{I}^n$,

$$\lim_{l\to\infty} \mathcal{P}_{\mathbf{k}}^n \tilde{\sigma}_n(q_{l,\mathbf{k}}, q_{l,\mathbf{k}})\mathcal{P}_{\mathbf{k}}^n = 0. \tag{6.37}$$

On the other hand, since $\Delta_p A_n = 0$, bound (6.32) is also satisfied by $\tilde{\sigma}_n$, which by definition of the norm implies that

$$|\mathcal{P}_{\mathbf{k}}^n \Delta_p \tilde{\sigma}_n(q,q)\mathcal{P}_{\mathbf{k}}^n|_{s,s'} \leq \varepsilon^{\frac{3}{2}}|\omega\cdot p|, \tag{6.38}$$

for all $q \in S_{\mathbf{k}}^n$, $\mathbf{k} \in \mathcal{I}^n$, and $p \in \mathbb{Z}^d$ with $|\omega\cdot p| < \frac{1}{16}\eta^{n-1}$. The definition of $S_{\mathbf{k}}^n$ together with (3.30) implies that $|\omega\cdot(q-q')| \leq 2\bar{d}\eta^n \leq \frac{1}{16}\eta^{n-1}$ for all $q, q' \in S_{\mathbf{k}}^n$ with $\mathrm{sign}\,(\omega\cdot q) = \mathrm{sign}\,(\omega\cdot q')$ and η small enough. Therefore, using

$$\tilde{\sigma}_n(q,q) = \tilde{\sigma}_n(q',q') + \Delta_{q-q'}\tilde{\sigma}_n(q',q'),$$

one infers from (6.38) that for all $q_{l,\mathbf{k}}$ and $q \in S_{\mathbf{k}}^n$ with $\omega\cdot q > 0$,

$$|\mathcal{P}_{\mathbf{k}}^n \tilde{\sigma}_n(q,q)\mathcal{P}_{\mathbf{k}}^n|_{s,s'} \leq |\mathcal{P}_{\mathbf{k}}^n \tilde{\sigma}_n(q_{l,\mathbf{k}}, q_{l,\mathbf{k}})\mathcal{P}_{\mathbf{k}}^n|_{s,s'} + \varepsilon^{\frac{3}{2}}|\omega\cdot(q-q_{l,\mathbf{k}})|, \tag{6.39}$$

which, with (6.37), leads to

$$|\mathcal{P}_{\mathbf{k}}^n \tilde{\sigma}_n(q,q)\mathcal{P}_{\mathbf{k}}^n|_{s,s'} \leq 2\bar{d}\varepsilon^{\frac{3}{2}}\eta^n. \tag{6.40}$$

For $q \in S_{\mathbf{k}}^n$ with $\omega\cdot q < 0$, we note that (6.39) is also valid if one replaces $q_{l,\mathbf{k}}$ by $-q_{l,\mathbf{k}}$, and, due to (5.19), that the same is true of (6.37). Therefore, (6.40) holds for all $q \in S_{\mathbf{k}}^n$, $\mathbf{k} \in \mathcal{I}^n$, and bound (6.33) follows by taking ε small enough. Finally, we check that A_n obeys (5.9). The a priori bound (6.25) together with (6.33) imply that $||\hat{P}_n A_n||_{s,s'}^{(n)} \leq 3\varepsilon\eta^{n-1}$, which, with (5.11) and definition (6.36), leads to (5.9).

To complete the proof of part (c) of Proposition 5.1, we are left with the

Proof of Proposition 6.2. Denoting

$$Dw_n(z) = \sigma_n(z) + \tau_n(z),$$

with $\sigma_n(z)(q,q') = Dw_n(z)(q,q')\delta_{qq'}$, one computes from (6.24) the recursive relation

$$\sigma_n(z) = \big(\tilde{\sigma}_{n-1}(\tilde{z}) + T_n(z)\big)H_n(\tilde{z}), \tag{6.41}$$

where

$$H_n(\tilde{z}) = \big(1 - \Gamma_n\tilde{\sigma}_{n-1}(\tilde{z})\big)^{-1},$$
$$T_n(z)(q,q') = \big[\tau_n(z)\Gamma_n\tau_{n-1}(\tilde{z})\big](q,q')\delta_{qq'}.$$

Setting

$$\mathcal{R}_n(z) \equiv \tilde{\sigma}_{n-1}\big(H_n(\tilde{z}) - 1\big), \quad \mathcal{T}_n(z) \equiv T_n(z)H_n(\tilde{z}),$$

and using $\Delta_p\tilde{\sigma}_{n-1} = \Delta_p\sigma_{n-1}$ together with the identity $\Delta_p\sigma_0 = 0$, one applies (6.41) recursively to obtain

$$\Delta_p\sigma_n(z) = \sum_{m=1}^{n} \Delta_p\big(\mathcal{R}_m(z_m) + \mathcal{T}_m(z_m)\big), \tag{6.42}$$

where $z_m = F_n^{m+1}(z)$, cf (6.14), with $F_n^{n+1} \equiv \mathbb{1}$. Note that $\mathcal{R}_m(z)$ is diagonal and can be rewritten as

$$\mathcal{R}_m(z) = \tilde{\sigma}_{m-1}(\tilde{z})\Gamma_m\tilde{\sigma}_{m-1}(\tilde{z})H_m(\tilde{z}). \tag{6.43}$$

As shown below, each term in (6.42) is easily seen to be of order $\varepsilon^2|\omega\cdot p|$. Thus, the main issue in obtaining (6.32) is to ensure that taking the sum will deteriorate the bound only slightly. Let us first consider the terms involving the quantities $\Delta_p\mathcal{T}_m$. They are higher order terms, since $\mathcal{T}_m$ is quadratic in the off-diagonal part τ_m which, as shown in Lemma 6.1, are bounded by powers of r. Indeed, as carried out in the Appendix, one has for all $m = 1,\ldots,n$ and $z \in B_m$,

$$||\hat{P}_m\Delta_p\mathcal{T}_m(z)||_{s,s'}^{(m)} \le \varepsilon^2\eta^m|\omega\cdot p|, \tag{6.44}$$

so that

$$||\sum_{m=1}^{n}\hat{P}_n\Delta_p\mathcal{T}_m(z)||_{s,s'}^{(n)} \le \sum_{m=1}^{n}||\hat{P}_m\Delta_p\mathcal{T}_m(z)||_{s,s'}^{(m)} \le \varepsilon^2|\omega\cdot p|. \tag{6.45}$$

On the other hand, the terms involving $\Delta_p\mathcal{R}_m$ are not higher order terms. Since

$$\Delta_p H_m(\tilde{z}) = t_p H_m(\tilde{z})\Big(\Delta_p\Gamma_m t_p\tilde{\sigma}_{m-1}(\tilde{z}) + \Gamma_m\Delta_p\tilde{\sigma}_{m-1}(\tilde{z})\Big)H_m(\tilde{z}),$$

(5.18) with $\sigma = s + \xi - \gamma$ and n replaced by m yields with the recursive bound (6.32)

$$||\Delta_p H_m(\tilde{z})||_s^{(m-2)} \le \eta^{-m}|\omega\cdot p|. \tag{6.46}$$

Thus, using in addition the recursive bounds (5.8) and (6.32), together with

$$||H_m(\tilde{z}) - 1||_s^{(m-1)} = ||\Gamma_m\tilde{\sigma}_{m-1}(\tilde{y})H_m(\tilde{z})||_s^{(m-1)} \le C\varepsilon,$$

one obtains for all $m = 1, \ldots, n$ and $z \in B_m$,

$$||\hat{P}_n \Delta_p \mathcal{R}_m(z)||_{s,s'}^{(n)} \leq ||\hat{P}_m \Delta_p \mathcal{R}_m(z)||_{s,s'}^{(m)} \leq C\varepsilon^2 |\omega \cdot p|, \tag{6.47}$$

to be compared with (6.44). However, one can actually show that

$$\Big\| \sum_{m=1}^{n} \hat{P}_n \Delta_p \mathcal{R}_m(z) \Big\|_{s,s'}^{(n)} \leq \sup_{\mathbf{k} \in \mathcal{I}^n} \sup_{q \in S_{\mathbf{k}}^n} \sum_{m=1}^{n} |\Delta_p \mathcal{R}_m(z)(q)|_{s,s'} \tag{6.48}$$

$$\leq C\varepsilon^2 |\omega \cdot p|, \tag{6.49}$$

with another n-independent constant C. Although (6.47) yields the a priori bound $|\Delta_p \mathcal{R}_m(z)(q)|_{s,s'} \leq C\varepsilon^2 |\omega \cdot p|$ for all $q \in S_{\mathbf{k}}^n$ and $\mathbf{k} \in \mathcal{I}^n$, (6.49) will follow from the fact that all but a finite number of terms in (6.48) are identically zero. More precisely, there is for all $\mathbf{k} \in \mathcal{I}^n$ a set $\mathcal{Z}_{\mathbf{k}}^n \subset \{1, \ldots, n\}$ with $\#\mathcal{Z}_{\mathbf{k}}^n$ uniformly bounded in n and $\mathbf{k}$ such that for all $q \in S_{\mathbf{k}}^n$,

$$|\Delta_p \mathcal{R}_m(z)(q)|_{s,s'} \equiv 0 \quad \text{if} \quad m \notin \mathcal{Z}_{\mathbf{k}}^n. \tag{6.50}$$

This leads to (6.49) and concludes the proof of bound (6.32), since (6.42), (6.45) and (6.49) lead to (6.32) by taking ε small enough and by noting that $z_m \in B_m$ for all $z \in B_n$, cf. (6.15). Identity (6.50) for some finite set $\mathcal{Z}_{\mathbf{k}}^n$ follows from the expression (6.43) for $\mathcal{R}_m$ since by localization of scales $\Gamma_m(q) = (A^{-1} Q_m P_{m-1})(q) = 0$ for most $m \leq n$ if $q \in S_{\mathbf{k}}^n$. More precisely, one computes that

$$Q_m(q) P_{m-1}(q) = \sum_{\tilde{\mathbf{k}} \in \mathcal{I}^m} \big(1 - \chi_{\tilde{\mathbf{k}}}^m(q)\big) \chi_{\tilde{\mathbf{k}}_{m-1}}^{m-1}(q) \mathcal{P}_{\tilde{\mathbf{k}}}^m,$$

where the index $\tilde{\mathbf{k}}_{m-1}$ serves to denote the (unique) subspace $\mathcal{J}_{\tilde{\mathbf{k}}_{m-1}}^{m-1}$ containing $\mathcal{J}_{\tilde{\mathbf{k}}}^m$. Fix now some $\mathbf{k} \in \mathcal{I}^n$. Then one has for all $q \in S_{\mathbf{k}}^n$ and all $m < n$,

$$Q_m(q) P_{m-1}(q) = \sum_{\substack{\tilde{\mathbf{k}} \in \mathcal{I}^m \\ \tilde{\mathbf{k}} \neq \mathbf{k}_m}} \chi_{\tilde{\mathbf{k}}_{m-1}}^{m-1}(q) \mathcal{P}_{\tilde{\mathbf{k}}}^m = \mathcal{P}_{\mathcal{J}_{\mathbf{k}_{m-1}}^{m-1} \setminus \mathcal{J}_{\mathbf{k}_m}^m},$$

since by construction $\chi_{\mathbf{k}_m}^m(q) = 1$ for such m and q. Therefore, $Q_m(q) P_{m-1}(q) = 0$ for all $q \in S_{\mathbf{k}}^n$ if $m < n$ is such that $\mathcal{J}_{\mathbf{k}_m}^m = \mathcal{J}_{\mathbf{k}_{m-1}}^{m-1}$. On the other hand, $\mathcal{J}_{\mathbf{k}_m}^m$ is a strict subspace of $\mathcal{J}_{\mathbf{k}_{m-1}}^{m-1}$ only if $\#\mathcal{C}_{\mathbf{k}_m}^m < \#\mathcal{C}_{\mathbf{k}_{m-1}}^{m-1}$, i.e., if the eigenvalues contained in $\mathcal{C}_{\mathbf{k}_{m-1}}^{m-1}$ have been divided after perturbation by a_{m-1} into two (or more) clusters. But this can be true only for finitely many m since the original eigenvalues μ_k are finitely many times degenerate. Hence, there is an $L < \infty$ such that for all $n \geq 1$ and all $1 \leq m \leq n$, one has $\hat{P}_n \mathcal{R}_m(q) = 0$, except for some $m_1, \ldots, m_L$. Since the same is true of $\hat{P}_n t_p \mathcal{R}_m(q)$ provided that p satisfies $|\omega \cdot p| < \eta^{n-1}/16$, (6.50) follows.

■

6.4. The Cluster Decomposition

We now check that point (d) of Proposition 5.1 holds. First, (5.9), (5.10) and (5.11) lead to, for $\mathbf{k} = (k, \cdot) \in \mathcal{I}^n$,

$$|a_n \mathcal{P}_{\mathbf{k}}^n|_{\mathcal{L}(\mathcal{J}_{\mathbf{k}}^n)} \leq 3k^{\gamma-\xi}\varepsilon\eta^{n-1}, \tag{6.51}$$

which, since $\mu_k^2 \geq ck^{2\gamma}$ by hypothesis (H1), implies that $\mu^2 + \sum_{m=0}^{n} a_m \equiv \tilde{\mu}_{n+1}^2$ is positive definite for $\varepsilon = \varepsilon(c, \eta)$ small enough. Next, it follows from a_n being hermitian that $\sigma(\tilde{\mu}_{n+1}) \subset \mathbb{R}^+$. Furthermore, using (5.11) and the fact that $\mathcal{J}_{\mathbf{k}}^n$ is by definition an invariant subspace for $\tilde{\mu}_n$, one infers from $\mu_k \geq ck^\gamma$, the asymptotic (5.13) for $\tilde{\mu}_n$, and the estimate (6.51), that

$$|a_n \tilde{\mu}_n^{-1} \mathcal{P}_{\mathbf{k}}^n|_{\mathcal{L}(\mathcal{J}_{\mathbf{k}}^n)} \leq 3c^{-1}k^{-\xi}\varepsilon\eta^{n-1}.$$

Therefore, denoting by $\mathcal{P}_k$ the projector onto the k^{th} component of $\hat{\mathcal{R}}^\infty = \bigoplus_{k\geq 1} \mathbb{C}^{d_k}$, one obtains

$$\tilde{\mu}_{n+1}\mathcal{P}_k = \left[\tilde{\mu}_n^2 + a_n\right]^{\frac{1}{2}}\mathcal{P}_k = \tilde{\mu}_n\mathcal{P}_k + \mathcal{O}(k^{-\xi}\varepsilon\eta^{n-1}), \tag{6.52}$$

which, since $\mu\mathcal{P}_k = \mu_k \mathbb{1}_{d_k}$, implies by recursion that

$$\tilde{\mu}_{n+1}\mathcal{P}_k = \mu_k \mathbb{1}_{d_k} + \mathcal{O}(\varepsilon k^{-\xi}).$$

Hence, the asymptotic (5.13) holds, where for each $k \geq 1$ the sequence of clusters $\mathcal{C}_{k,i}^{n+1}$, $i = 1, \ldots, M_k^{n+1}$, forms a partition of the component $\sigma(\tilde{\mu}_{n+1}\mathcal{P}_k)$ satisfying $d(\mathcal{C}_{k,i}^{n+1}, \mathcal{C}_{k,j}^{n+1}) > \eta^{n+1}$ for $i \neq j$. This partition is unique if M_k^{n+1} is required to be maximal. Furthermore, it follows from (1.13) and (1.14) in (H1) that for $\varepsilon = \varepsilon(c)$ small enough, the components $\sigma(\tilde{\mu}_{n+1}\mathcal{P}_k)$ are well separated. Therefore, the sets $S_{\mathbf{k}}^{n+1}$, $\mathbf{k} \in \mathcal{I}^{n+1}$, defined according to (3.25) are pairwise disjoint. Next, (6.52) and the gap condition (5.12) with $n+1$ replaced by n imply that for $\varepsilon = \varepsilon(c, \eta)$ small enough, every cluster $\mathcal{C}_{k,i}^{n+1}$ is composed of perturbed eigenvalues belonging to a unique $\mathcal{C}_{k,j_i^{n+1}}^n$. The distance between these two clusters is at most of order $\mathcal{O}(k^{-\xi}\varepsilon\eta^{n-1})$, so that (3.31) follows for $n+1$ by induction. In order to iterate (3.32), we note that by definition, $\mathcal{J}_{\mathbf{k}}^{n+1}$ is the eigenspace of $\tilde{\mu}_{n+1}$ associated with $\mathcal{C}_{\mathbf{k}}^{n+1}$, $\mathbf{k} \in \mathcal{I}^{n+1}$, and that every $\mathcal{J}_{\mathbf{k}'}^n$, $\mathbf{k}' \in \mathcal{I}^n$, is also an invariant subspace for $\tilde{\mu}_{n+1}$ by (5.11). Therefore, each $\mathcal{J}_{k,i}^{n+1}$ is contained in a unique $\mathcal{J}_{k,j_i^{n+1}}^n$, namely, the eigenspace associated with $\mathcal{C}_{k,j_i^{n+1}}^n$. Finally, we check that (3.33) iterates. This is a simple consequence of (3.32) and $S_{k,i}^{n+1} \subset S_{k,j_i^{n+1}}^n$, the latter following from (6.52) for ε small enough.

7. Measure Estimate

In this section, we prove Proposition 5.2, namely, that $\Omega^*(K) = \bigcap_{n\geq 1} \Omega_n(K)$ satisfies

$$\lim_{K\to 0} \text{meas}(\Omega \setminus \Omega^*(K)) = 0, \tag{7.1}$$

for all bounded $\Omega \subset \mathbb{R}^d$. The strategy is standard and consists in studying the complementary sets of $\Omega_n(K)$. For $n \geq 1$, $b > 0$, and $q \in \mathbb{Z}^d$, let us define

$$\Sigma_{q,b}^n \equiv \Big(\bigcup_{\mathbf{k}\in\mathcal{I}^n} \Sigma_{q,b}^{n;\mathbf{k}}\Big) \cup \Big(\bigcup_{\mathbf{k},\mathbf{k}'\in\mathcal{I}^n} \Sigma_{q,b}^{n;\mathbf{k},\mathbf{k}'}\Big),$$

where

$$\Sigma_{q,b}^{n;\mathbf{k}} = \{\omega \in \mathbb{R}^d \mid d(|\omega\cdot q|, \mathcal{C}_{\mathbf{k}}^n) \leq b\},$$
$$\Sigma_{q,b}^{n;\mathbf{k},\mathbf{k}'} = \{\omega \in \mathbb{R}^d \mid d(|\omega\cdot q|, |\mathcal{C}_{\mathbf{k}}^n \pm \mathcal{C}_{\mathbf{k}'}^n|) \leq b\}.$$

Next, with

$$\mathcal{Z}_n \equiv \{q \in \mathbb{Z}^d \mid K^{\frac{1}{\nu}}\eta^{-\frac{n-1}{\nu}} \leq |q| < K^{\frac{1}{\nu}}\eta^{-\frac{n}{\nu}}\},$$

and

$$\Sigma^*(K) \equiv \bigcup_{n\geq 1} \bigcup_{q\in\mathcal{Z}_n} \Sigma_{q,2K|q|^{-\nu}}^n,$$

one shows first, that for all bounded $\Omega \subset \mathbb{R}^d$,

$$\text{meas}\big(\Omega \cap \Sigma^*(K)\big) \leq C_\Omega\, K^{\frac{\xi}{\xi+1}}, \tag{7.2}$$

for some constant C_Ω depending on Ω only, and, second, that

$$\big[\Sigma^*(K)\big]^c \subseteq \Omega^*(K). \tag{7.3}$$

Obviously, (7.1) follows from (7.2) and (7.3). Below, C_Ω will denote a generic constant that may change from place to place but depends on Ω only.

Let us start with the bound (7.2). One has

$$\text{meas}\big(\Omega \cap \Sigma^*(K)\big) \leq \sum_{n\geq 1}\sum_{q\in\mathcal{Z}_n} \big(T_{q,2K|q|^{-\nu}}^n + \hat{T}_{q,2K|q|^{-\nu}}^n\big), \tag{7.4}$$

where

$$T_{q,b}^n = \sum_{\mathbf{k}\in\mathcal{I}^n} \text{meas}\big(\Omega \cap \Sigma_{q,b}^{n;\mathbf{k}}\big),\ \hat{T}_{q,b}^n = \text{meas}\Big(\Omega \cap \bigcup_{\mathbf{k},\mathbf{k}'\in\mathcal{I}^n} \Sigma_{q,b}^{n;\mathbf{k},\mathbf{k}'}\Big). \tag{7.5}$$

To treat the terms on the right hand side of (7.4) involving the quantities $T_{q,b}^n$, we first use (3.30) to estimate,

$$\text{meas}\big(\Omega \cap \Sigma_{q,b}^{n;\mathbf{k}}\big) \leq C_\Omega(b + \bar{d}\eta^n).$$

Next, we note that the asymptotic behavior of the clusters $\mathcal{C}^n_{\mathbf{k}}$, cf. (1.12) and (5.13), implies that $\Omega \cap \Sigma^{n;\mathbf{k}}_{q,b}$ is empty if $\mathbf{k} = (k,\cdot)$ satisfies $k \geq C_\Omega |q|$ for some constant C_Ω. Hence, since the number of indices $\mathbf{k}$ of the form $(k,\cdot)$ is uniformly bounded in k, the number of terms which are non-zero in the sum defining $T^n_{q,b}$ is proportional to $|q|$, and one obtains the estimate $T^n_{q,b} \leq C_\Omega |q|(b + \bar{d}\eta^n)$. Finally, the fact that $q \in \mathcal{Z}_n$ satisfies $\eta^n \leq K|q|^{-\nu}$ leads to

$$\sum_{n\geq 1} \sum_{q\in\mathcal{Z}_n} T^n_{q,2K|q|^{-\nu}} \leq C_\Omega \big(2K + \bar{d}K\big) \sum_{q\in\mathbb{Z}^d} |q|^{1-\nu} \leq C_\Omega\, K, \tag{7.6}$$

for $\nu = \nu(d)$ large enough. To treat the remaining terms in (7.4), we first note that, as above,

$$\operatorname{meas}\big(\Omega \cap \Sigma^{n;\mathbf{k},\mathbf{k}'}_{q,b}\big) \leq C_\Omega (b + 2\bar{d}\eta^n). \tag{7.7}$$

Next, one distinguishes the cases $\gamma = 1$ and $\gamma > 1$. If $\gamma > 1$, then for $k' > k$ the inequality $k'^\gamma - k^\gamma > k'^{\gamma-1}$ and the asymptotic (1.13) imply that $\Omega \cap \Sigma^{n;\mathbf{k},\mathbf{k}'}_{q,b}$ is empty for $\mathbf{k} = (k,\cdot)$ and $\mathbf{k}' = (k',\cdot)$ such that $k' \geq C_\Omega |q|^{1/(\gamma-1)} \equiv k_q$. Furthermore, it follows from (5.13) that for $k_b = b^{-\frac{1}{\xi+1}}$,

$$\operatorname{meas}\Big(\bigcup_{\substack{k>Ck_b\\ i,j}} \Sigma^{n;(k,i),(k,j)}_{q,b}\Big) \leq C k_b^{-\xi}.$$

Therefore, one obtains with (7.7)

$$\begin{aligned}\hat{T}^n_{q,b} &\leq C b^{\frac{\xi}{\xi+1}} + \sum_{k=1}^{Ck_b} \operatorname{meas}(\Omega \cap \Sigma^{n;(k,i),(k,j)}_{q,b}) + \sum_{\substack{k'=2\\ k<k'}}^{k_q} \operatorname{meas}(\Omega \cap \Sigma^{n;(k,i),(k',j)}_{q,b}) \\ &\leq C b^{\frac{\xi}{\xi+1}} + C_\Omega (b + 2\bar{d}\eta^n)\Big(b^{-\frac{1}{\xi+1}} + |q|^{\frac{2}{\gamma-1}}\Big).\end{aligned}$$

This finally leads to, using again that $\eta^n \leq K|q|^{-\nu}$ for $q \in \mathcal{Z}_n$,

$$\sum_{n\geq 1} \sum_{q\in\mathcal{Z}_n} \hat{T}^n_{q,2K|q|^{-\nu}} \leq C K^{\frac{\xi}{1+\xi}} \sum_{q\in\mathbb{Z}^d} |q|^{-\nu\frac{\xi}{1+\xi}} + C_\Omega K \sum_{q\in\mathbb{Z}^d} |q|^{\frac{2}{\gamma-1}-\nu} \leq C_\Omega K^{\frac{\xi}{1+\xi}}, \tag{7.8}$$

for $\xi > 0$ and $\nu = \nu(d,\xi)$ large enough. We now consider the case $\gamma = 1$. From (5.13) and the asymptotic behavior (1.14), it follows first that $\Omega \cap \Sigma^{n;\mathbf{k},\mathbf{k}'}_{q,b}$ is empty for $\mathbf{k} = (k,\cdot)$ and $\mathbf{k}' = (k',\cdot)$ with $k' - k = l \geq C|q|$, and second that for all $l \geq 0$

$$\operatorname{meas}\Big(\bigcup_{\substack{k>Ck_b\\ i,j}} \Sigma^{n;(k,i),(k+l,j)}_{q,b}\Big) \leq C k_b^{-\xi},$$

where $k_b = b^{-\frac{1}{\xi+1}}$. Therefore, (7.7) leads to

$$\hat{T}^n_{q,b} \leq C|q|b^{\frac{\xi}{\xi+1}} + C_\Omega b^{-\frac{1}{\xi+1}}|q|(b + 2\bar{d}\eta^n),$$

and one finally obtains for $\nu = \nu(d, \xi)$ large enough,

$$\sum_{n\geq 1}\sum_{q\in\mathcal{Z}_n} \hat{T}^n_{q,2K|q|^{-\nu}} \leq C_\Omega K^{\frac{\xi}{1+\xi}} \sum_{q\in\mathbb{Z}^d} |q|^{1-\nu\frac{\xi}{1+\xi}} \leq C_\Omega K^{\frac{\xi}{1+\xi}}. \tag{7.9}$$

Inserting (7.6) and (7.9) into (7.4) yields (7.2).

We now check that (7.3) holds. If $\omega \notin \Sigma^*(K)$, then the following is true for all $n \geq 1$, $q \in \mathcal{Z}_n$ and $\mathbf{k}, \mathbf{k}' \in \mathcal{I}^n$,

$$d(|\omega\cdot q|, \mathcal{C}^n_{\mathbf{k}}) > 2K|q|^{-\nu}, \tag{7.10}$$
$$d(|\omega\cdot q|, |\mathcal{C}^n_{\mathbf{k}} \pm \mathcal{C}^n_{\mathbf{k}'}|) > 2K|q|^{-\nu}. \tag{7.11}$$

Next, we verify that for such ω, this implies that bounds (7.10) and (7.11) hold for all $q \in \bigcup_{m=1}^n \mathcal{Z}_m$ provided one replaces the constant $2K$ on the right hand side by K. This in turn implies that $\omega \in \Omega_n(K)$ for all $n \geq 1$, so that $\omega \in \Omega^*(K)$. Let $m < n$ and fix some $\mathbf{k} \in \mathcal{I}^n$. Then, recalling (3.31), namely that there is at least one $\mathbf{k}' \in \mathcal{I}^m$ for which

$$\sup_{x\in I^n_{\mathbf{k}}} \inf_{y\in I^m_{\mathbf{k}'}} d(x,y) \leq \eta^{m+1},$$

and since, on the other hand, $\eta^m < K|q|^{-\nu}$ whenever $q \in \mathcal{Z}_m$, one infers from (7.10) with n replaced by m that for $q \in \mathcal{Z}_m$ and $\eta < 1$,

$$\begin{aligned} d(|\omega\cdot q|, \mathcal{C}^n_{\mathbf{k}}) &\geq \left| d(|\omega\cdot q|, \mathcal{C}^m_{\mathbf{k}'}) - \eta^{m+1} \right| \\ &\geq (2K - \eta K)|q|^{-\nu} \\ &> K|q|^{-\nu}. \end{aligned} \tag{7.12}$$

Since (7.12) holds for all $q \in \mathcal{Z}_m$, $1 \leq m \leq n$, one concludes that $d(|\omega\cdot q|, \mathcal{C}^n_{\mathbf{k}}) > K|q|^{-\nu}$ whenever $0 < |q| < K\eta^{-n/\nu}$. In a similar way, one derives an identical lower bound on $d(|\omega\cdot q|, |\mathcal{C}^n_{\mathbf{k}} \pm \mathcal{C}^n_{\mathbf{k}'}|)$, thus achieving the proof of (7.3) and (7.1).

8. Proof of Theorem 1.1

Defining $z_n \equiv F_n(0)$, we now show that z_n converges in h_s, as $n \to \infty$, to a function z whose Fourier transform is real analytic and provides a solution of equation (3.4). Using $F_n(0) = F_{n-1}(R_n(0))$, cf. (3.13), one computes that

$$z_n - z_{n-1} = \delta_1 F_{n-1}\big(R_n(0)\big).$$

According to (6.5), $R_n(0) = H_n \Gamma_n w_{n-1}(0) + u(0)$, so that (5.6), (5.17), (6.9), (6.10) and the identity $\Gamma_n = \Gamma_n \hat{P}_{n-1}$ lead to

$$||R_n(0)||_{h_s^{n-1}} \leq \eta^{-n} r^{2(n-1)}.$$

Therefore, since, $F_{n-1} \in \mathcal{A}_{n-1} = H^\infty(B_{n-1}, h_{s'})$, one can apply (4.12) to $\delta_1 F_{n-1}$ with $\gamma = \eta^{-n} r^{n-2}$ to obtain

$$||z_n - z_{n-1}||_s \leq C\eta^{-n} r^{n-2} |||F_{n-1}|||_{\mathcal{A}_{n-1}},$$

and the convergence of z_n in h_s follows from the uniform bound (6.16) by taking $r = r(\eta)$ small enough. Bound (6.16) also implies $||z_\beta|| \leq \varepsilon$ uniformly in the strip $|\operatorname{Im} \beta| < \alpha' = \alpha \prod_{n=2}^\infty (1 - n^{-2})$. This yields the pointwise estimate

$$|z(q)| \leq \varepsilon e^{-\alpha' |q|},$$

and, consequently, ensures the real analyticity of the Fourier transform of z.

In order to prove that the limit z solves equation (3.6), namely, $\mathcal{K}_0 z = w_0(z)$, we will show below that

$$\mathcal{K}_0 z_n = Q_n w_0(z_n) + A_{<n} P_n R_n(0). \tag{8.1}$$

where one has defined $A_{<m} \equiv \sum_{k=0}^{m-1} A_k$ for $m = 1 \ldots, n$. Since it follows from (6.12) and (5.9) that

$$||A_k R_n(0)||_{s'} \leq C\varepsilon \eta^{k-1} r^n,$$

the second term in the right hand side of (8.1) converges to zero in $h_{s'}$ as $n \to \infty$. Moreover, $\lim_{n\to\infty} Q_n = \mathbb{1}$ for each $\omega \in \Omega^*$, and since w_0 is analytic, one can take the $n \to \infty$ limit of equation (8.1) to conclude that z solves (3.6). It thus remains to check that identity (8.1) holds. We will use the following relations

$$z_n = \sum_{m=1}^{n} R_m(F_n^{m+1}(0)), \tag{8.2}$$

$$w_m(F_n^{m+1}(0)) = w_0(z_n) - \sum_{k=0}^{m-1} A_k F_n^{k+1}(0), \tag{8.3}$$

where F_n^m is defined by (6.14) for $m \le n$, whereas $F_n^{n+1} \equiv \mathbb{1}$. The first relation simply follows from $z_n = F_n^1(0)$ by using recursively the definitions (6.13) and (6.14). The second relation is obtained by using (3.19) and (3.16) to get

$$w_m(F_n^{m+1}(0)) = w_{m-1}\big(F_n^m(0)\big) - A_{m-1}F_n^m(0),$$

which one applies recursively. Next, it follows from (8.2) that

$$\mathcal{K}_0 z_n = \sum_{m=1}^{n} \mathcal{K}_0 R_m(F_n^{m+1}(0)), \tag{8.4}$$

whereas (8.3) and (3.19) imply, since R_m solves equation (3.14) with n replaced by m,

$$\begin{aligned} \mathcal{K}_m R_m(F_n^{m+1}(0)) &= Q_m P_{m-1} \tilde{w}_{m-1}(F_n^m(0)) \\ &= Q_m P_{m-1}\Big(w_0(z_n) - \sum_{k=0}^{m-1} A_k F_n^{k+1}(0)\Big), \end{aligned} \tag{8.5}$$

where, for $m = 1$, one denotes $P_0 \equiv \mathbb{1}$. Therefore, since $\sum_{m=1}^n Q_m P_{m-1} = Q_n$ and $\mathcal{K}_0 = \mathcal{K}_m + \sum_{k=0}^{m-1} P_k A_k$, (8.4) and (8.5) yield

$$\mathcal{K}_0 z_n = Q_n w_0(z_n) + \sum_{m=1}^{n} T_m, \tag{8.6}$$

where T_m is given by

$$T_m = \sum_{k=0}^{m-1}\Big(P_k A_k R_m(F_n^{m+1}(0)) - Q_m P_{m-1} A_k F_n^{k+1}(0)\Big). \tag{8.7}$$

If P_k and Q_k were true projectors, namely, if the scales were defined in terms of sharp cut-off functions, cf. (3.26), then a straightforward calculation would show that $T_m \equiv 0$. Nevertheless, although none of the quantities T_m are zero, we check that

$$T_1 = A_{<1}\hat{R}_1 \quad \text{and} \quad T_m = A_{<m}\hat{R}_m - A_{<m-1}\hat{R}_{m-1} \quad \text{for} \quad m = 2, \ldots, n, \tag{8.8}$$

where $\hat{R}_n = P_n R_n(0)$ and for $m = 1, \ldots, n-1$,

$$\hat{R}_m = P_m R_m(F_n^{m+1}(0)) - Q_m R_{m+1}(F_n^{m+2}(0)).$$

Thus, one is left with the small correction term $\sum_{m=1}^n T_m = A_{<n} P_n R_n(0)$ as claimed in (8.1). To show (8.8), we note that A_k commutes with P_m and Q_m for all k, m. Furthermore, one easily checks that $Q_m R_{m-1} = R_{m-1}$, $P_{m-1}R_{m+1} = R_{m+1}$, and $Q_m P_{m-1} R_l = 0$ if $l \ne m-1, m$, or $m+1$. Hence, using in addition

$$F_n^{k+1}(0) = \sum_{l=k+1}^{n} R_l(F_n^{l+1}(0)),$$

one obtains, decomposing the expression for T_m (8.7) as $T_m = T_m^{(1)} - T_m^{(2)}$,

$$T_m^{(1)} = A_{<m-1} R_m + A_{m-1} P_{m-1} R_m \,,$$
$$T_m^{(2)} = A_{<m-1} P_{m-1} R_{m-1} + A_{<m}\big(Q_m P_{m-1} R_m + Q_m R_{m+1}\big),$$

where $A_{<m-1} \equiv 0$ for $m = 1$ and the last term in $T_m^{(2)}$ is absent for $m = n$. Substracting $T_m^{(2)}$ from $T_m^{(1)}$ finally leads to (8.8) by using the identities $(1 - Q_m)P_{m-1} = P_m$ and $1 - Q_m P_{m-1} = Q_{m-1} + P_m$. This completes the proof that $z = \lim_{n\to\infty} z_n$ solves (3.6).

The resulting solution $Z = Z(\lambda, \Phi, J)$ of equation (3.4) depends analytically on λ for $|\lambda| < \lambda_0$ and vanishes for $\lambda = 0$. Its uniqueness follows from the fact that equation (3.4) completely determine the coefficients of the Taylor expansion of its solution in powers of λ. Furthermore, recall that Φ and J are parameters in w_n, the latter being analytic in $(\Phi, J) \in \mathcal{O}_{\mathcal{B}}$. Thus, Z is also analytic in $(\Phi, J) \in \mathcal{O}_{\mathcal{B}}$.

These properties can be used, together with Theorem 4.1, to conclude the proof of Theorem 1.1, namely, to check that equation (1.7) has a unique solution $\mathcal{T} = (\Phi, J, Z)$, up to translation (1.10), analytic in λ and vanishing for $\lambda = 0$. To this end, introducing the variable $Y = (\Phi, J)$, we denote by $Y_s(\lambda, Z)$ the solution of (3.3) and by $Z_s(\lambda, Y)$ the solution of (3.4). Then, the solution $\mathcal{T}(\lambda)$ of (1.7) is given by $\mathcal{T} = (Y_s(\lambda, \mathcal{Z}), \mathcal{Z})$ where $\mathcal{Z} = \mathcal{Z}(\lambda)$ solves the functional fixed point equation

$$\mathcal{Z} = Z_s(\lambda, Y_s(\lambda, \mathcal{Z})) \equiv \mathcal{F}(\mathcal{Z}, \lambda). \tag{8.9}$$

To solve (8.9) for $\mathcal{Z}(\lambda)$, we use the implicit function theorem. We first note that by Theorem 4.1, $Y_s(\lambda, Z)$ is well defined in $\mathcal{B}$ for $|\lambda| < \lambda_1$ and Z in a small enough neighborhood of the origin $\mathcal{O}_s \subset h_s$, with $Y_s(\lambda, Z)|_{\lambda=0} = 0$ for all $Z \in \mathcal{O}_s$. Hence, there is a $\lambda_2 > 0$ small enough such that $Y_s(\lambda, Z) \in \mathcal{O}_{\mathcal{B}}$ for $|\lambda| < \lambda_2$ and $Z \in \mathcal{O}_s$. It thus follows from the previous discussion that $\mathcal{F}$ is analytic in $|\lambda| < \lambda_2$ and $\mathcal{Z} \in \mathcal{O}_s$ with $\mathcal{F}(\lambda, \mathcal{Z}) \in h_s$ and $\mathcal{F}(\lambda, \mathcal{Z})|_{\lambda=0} = 0$ for all $\mathcal{Z} \in \mathcal{O}_s$. One infers, in particular, that the solution of (8.9) at $\lambda = 0$ is given by $\mathcal{Z}(\lambda)|_{\lambda=0} = 0$. Next, one computes

$$D_{\mathcal{Z}}\mathcal{F}(\mathcal{Z}, \lambda) = D_Y Z_s(\lambda, Y_s(\lambda, \mathcal{Z})) D_{\mathcal{Z}} Y_s(\lambda, \mathcal{Z}).$$

Since $Y_s(\lambda, Z)|_{\lambda=0} = 0$ for all $Z \in \mathcal{O}_s$ and $Z_s(\lambda, Y)|_{\lambda=0} = 0$ for all $Y \in \mathcal{O}_{\mathcal{B}}$, it follows that $D_{\mathcal{Z}} Y_s(\lambda, \mathcal{Z})|_{\lambda=0} = D_Y Z_s(\lambda, Y_s(\lambda, \mathcal{Z}))|_{\lambda=0} = 0$ for $\mathcal{Z} \in \mathcal{O}_s$, which, in turn, implies

$$D_{\mathcal{Z}}\mathcal{F}(\lambda, \mathcal{Z})|_{\substack{\lambda=0\\ \mathcal{Z}=0}} = 0.$$

Therefore, the existence for all $|\lambda| < \lambda_2$ of a unique $\mathcal{Z}(\lambda) \in \mathcal{O}_s$ solving (8.9) follows by the implicit function theorem.

Appendix

Proof of Lemma 5.3.

We first consider the estimate (5.17). Since Γ_n has a diagonal kernel, one has

$$||\Gamma_n||_{\sigma,\sigma+\gamma} = \sup_{q\in\mathbb{Z}^d} |\mathcal{K}_n^{-1}(q)Q_n(q)P_{n-1}(q)|_{\sigma,\sigma+\gamma} \ ,$$

and using (3.32) one easily computes that

$$\Gamma_n(q) = \sum_{\mathbf{k}\in\mathcal{I}^n} \mathcal{K}_n^{-1}(q)\hat{\chi}_{\mathbf{k}}^n(q)\mathcal{P}_{\mathbf{k}}^n, \tag{9.1}$$

whith

$$\hat{\chi}_{\mathbf{k}}^n(q) = \big(1-\chi_{\mathbf{k}}^n(\omega\cdot q)\big)\chi_{\mathbf{k}_{n-1}}^{n-1}(\omega\cdot q), \tag{9.2}$$

where $\mathbf{k}_{n-1}\in\mathcal{I}^{n-1}$ labels the unique subspace $\mathcal{J}_{\mathbf{k}_{n-1}}^{n-1}$ containing $\mathcal{J}_{\mathbf{k}}^n$. Note that $\hat{\chi}_{\mathbf{k}}^n(q)\neq 0$ only for q in the set

$$\hat{S}_{\mathbf{k}}^n = \{q\in\mathbb{Z}^d \mid d(|\omega\cdot q|,\mathcal{C}_{\mathbf{k}}^n)\geq \tfrac{1}{8}\eta^n \text{ and } d(|\omega\cdot q|,\mathcal{C}_{\mathbf{k}_{n-1}}^{n-1})\leq \tfrac{1}{4}\eta^{n-1}\}. \tag{9.3}$$

Although the sets $\hat{S}_{\mathbf{k}}^n$ are not pairwise disjoint, $\hat{S}_{\mathbf{k}}^n\cap\hat{S}_{\mathbf{k}'}^n\neq\emptyset$ only if $\mathbf{k}_{n-1}={\mathbf{k}'}_{n-1}$. Since for $\mathbf{k}=(k,i)$ this happens only if $\mathbf{k}'=(k,j)$, and since the original frequencies μ_k are by assumption finitely many times degenerate (uniformly in k), there are for all $\mathbf{k}\in\mathcal{I}^n$ no more than $\bar{d}$ $\mathbf{k}'$ such that ${\mathbf{k}'}_{n-1}=\mathbf{k}_{n-1}$. Therefore, one obtains, with $0\leq\hat{\chi}_{\mathbf{k}}^n\leq 1$,

$$||\Gamma_n||_{\sigma,\sigma+\gamma}\leq \bar{d}\sup_{(k,i)\in\mathcal{I}^n}\ \sup_{q\in\hat{S}_{(k,i)}^n} k^\gamma|\mathcal{K}_n^{-1}(q)\mathcal{P}_{k,i}^n|. \tag{9.4}$$

Let us now fix some $\mathbf{k}=(k,i)\in\mathcal{I}^n$ and $q\in\hat{S}_{\mathbf{k}}^n$ with $\omega\cdot q>0$. Thus, $P_m(q)=\mathcal{P}_{\mathbf{k}_m}^m$ for all $0\leq m\leq n-2$, whereas $P_{n-1}(q)=\chi_{\mathbf{k}_{n-1}}^{n-1}(q)\mathcal{P}_{\mathbf{k}_{n-1}}^{n-1}$. This in turn imply, since $\mathcal{K}_n=\mathcal{K}_0-\sum_{m=0}^{n-1}A_mP_m$ and $q\in\mathcal{Q}_\omega^+$, that $\mathcal{K}_n(q)$ can be rewritten as

$$\mathcal{K}_n(q)=\hat{\mathcal{K}}_n(q)+\hat{\mu}_n(q), \tag{9.5}$$

where

$$\hat{\mathcal{K}}_n(q)=|\omega\cdot q|^2-\mu^2-\sum_{m=0}^{n-1}a_m\mathcal{P}_{\mathbf{k}_m}^m, \tag{9.6}$$

$$\hat{\mu}_n(q)=\big(1-\chi_{\mathbf{k}_{n-1}}^{n-1}(\omega\cdot q)\big)a_{n-1}\mathcal{P}_{\mathbf{k}_{n-1}}^{n-1}. \tag{9.7}$$

Note that $\hat{\mathcal{K}}_n$ is defined in such a way that $(|\omega\cdot q|^2-\hat{\mathcal{K}}_n(q))\mathcal{P}_{\mathbf{k}}^n=\tilde{\mu}_n^2$, cf. (3.21). One thus infers from the definition of $\mathcal{J}_{\mathbf{k}}^n$ that $\mathcal{J}_{\mathbf{k}}^n$ is an invariant subspace of $\hat{\mathcal{K}}_n(q)$. Therefore, since the spectrum of $(|\omega\cdot q|-\tilde{\mu}_n)\mathcal{P}_{\mathbf{k}}^n$ is bounded away from zero by $\eta^n/8$, one concludes from the identity $|\omega\cdot q|^2-\tilde\mu_n^2=(|\omega\cdot q|-\tilde\mu_n)(|\omega\cdot q|+\tilde\mu_n)$ and the asymptotic behavior (1.12), (5.13), that

$$|\hat{\mathcal{K}}_n^{-1}(q)\mathcal{P}_{\mathbf{k}}^n|\le Ck^{-\gamma}\eta^{-n}. \tag{9.8}$$

On the other hand, denoting $f(q)=1-\chi_{\mathbf{k}_{n-1}}^{n-1}(\omega\cdot q)$, we compute that

$$\hat{\mathcal{K}}_n^{-1}(q)\hat\mu_n(q)=f(q)\big(|\omega\cdot q|^2-\tilde\mu_n^2\big)^{-1}\mathcal{P}_{\mathbf{k}_{n-1}}^{n-1}a_{n-1}.$$

Since $f(q)=0$ whenever $d(|\omega\cdot q|,\mathcal{C}_{\mathbf{k}'}^n)\le\eta^n$ for all $\mathbf{k}'$ such that $\mathcal{J}_{\mathbf{k}'}^n\subset\mathcal{J}_{\mathbf{k}_{n-1}}^{n-1}$, and since (5.9) (with n replaced by $n-1$) implies $|a_{n-1}|\le 3\varepsilon k^{\gamma-\xi}\eta^{n-2}$, one estimates for ε small enough that $\big|\hat{\mathcal{K}}_n^{-1}(q)\hat\mu_n(q)\big|\le k^{-\xi}/4\le 1/4$, which leads to

$$\big|\big(1+\hat{\mathcal{K}}_n^{-1}(q)\hat\mu_n(q)\big)^{-1}\big|\le 2. \tag{9.9}$$

Bound (5.17) finally follows from (9.4) by applying (9.8) and (9.9) to

$$\mathcal{K}_n^{-1}(q)\mathcal{P}_{\mathbf{k}}^n=\big(1+\hat{\mathcal{K}}_n^{-1}(q)\hat\mu_n(q)\big)^{-1}\hat{\mathcal{K}}_n^{-1}(q)\mathcal{P}_{\mathbf{k}}^n,$$

and by noting that for $q\in\hat S_{\mathbf{k}}^n\cap\mathcal{Q}_\omega^-$, the previous analysis must be carried out with $\overline{a_m}$ instead of a_m, $m=0,\dots,n-1$, and leads to identical bounds since a_m being hermitian implies $\sigma\big(\overline{\tilde\mu_n}\big)=\sigma(\tilde\mu_n)$.

To conclude the proof of Lemma 5.3, it remains to check bound (5.18). If $p\in\mathbb{Z}^d$ is such that $|\omega\cdot p|\ge\eta^{n+1}$, one can estimate

$$||\Delta_p\Gamma_n||_{\sigma,\sigma+\gamma}\le 2||\Gamma_n||_{\sigma,\sigma+\gamma}\le 2\eta^{-n-1}||\Gamma_n||_{\sigma,\sigma+\gamma}\,|\omega\cdot p|,$$

which, with (5.17), leads to (5.18) for some other constant C. Let us assume now that $|\omega\cdot p|<\eta^{n+1}$. One computes from (9.1) that

$$\begin{aligned}\Delta_p\Gamma_n(q)&=\sum_{\mathbf{k}\in\mathcal{I}^n}\Big(\mathcal{K}_n^{-1}(q+p)\hat\chi_{\mathbf{k}}^n(q+p)-\mathcal{K}_n^{-1}(q)\hat\chi_{\mathbf{k}}^n(q)\Big)\mathcal{P}_{\mathbf{k}}^n\\&=\sum_{\mathbf{k}\in\mathcal{I}^n}\Delta_p\mathcal{K}_n^{-1}(q)\mathcal{P}_{\mathbf{k}}^n t_p\hat\chi_{\mathbf{k}}^n(q)+\sum_{\mathbf{k}\in\mathcal{I}^n}\mathcal{K}_n^{-1}(q)\mathcal{P}_{\mathbf{k}}^n\Delta_p\hat\chi_{\mathbf{k}}^n(q).\end{aligned} \tag{9.10}$$

We now fix some $\mathbf{k}=(k,i)\in\mathcal{I}^n$ and start by considering the second sum on the right hand side of (9.10). Since p is such that $|\omega\cdot p|<\eta^{n+1}$, $\Delta_p\hat\chi_{\mathbf{k}}^n(q)$ is non zero only for q in a set $\tilde S_{\mathbf{k}}^n$ that satisfies, with respect to the cluster $\mathcal{C}_{\mathbf{k}}^n$, similar gap condition as $\hat S_{\mathbf{k}}^n$. Therefore, the bounds derived previously imply that $|\mathcal{K}_n^{-1}(q)\mathcal{P}_{\mathbf{k}}^n|\le Ck^{-\gamma}\eta^{-n}$ for $q\in\tilde S_{\mathbf{k}}^n$, and one concludes by noting that

$$|\Delta_p\hat\chi_{\mathbf{k}}^n(q)|\le C\eta^{-n}|\omega\cdot p|,$$

for all $q \in \mathbb{Z}^d$. We now consider the first sum on the right hand side of (9.10). Let us fix q satisfying $t_p\hat{\chi}^n_{\mathbf{k}}(q) \neq 0$. Using the same notation, we decompose $\mathcal{K}_n$ as in (9.5) and express

$$\Delta_p \mathcal{K}_n^{-1}(q) = \Big[\big(1 + t_p(\hat{\mathcal{K}}_n^{-1}\hat{\mu}_n)\big)^{-1}\Delta_p\hat{\mathcal{K}}_n^{-1}\big(1 + \hat{\mu}_n\hat{\mathcal{K}}_n^{-1}\big)^{-1}\Big](q).$$

Bound (9.9) implies that $\big|\big(1 + t_p(\hat{\mathcal{K}}_n^{-1}\hat{\mu}_n)(q)\big)^{-1}\big| \leq 2$. Since $|\omega \cdot p| < \eta^{n+1}$, it follows that $\big|\big(1 + \hat{\mathcal{K}}_n^{-1}\hat{\mu}_n(q)\big)^{-1}\big|$ satisfies a similar bound for q with $t_p\hat{\chi}^n_{\mathbf{k}}(q) \neq 0$. Therefore, using in addition (9.8), one obtains

$$\begin{aligned}
\big|\big(\Delta_p\mathcal{K}_n^{-1}\big)(q)\mathcal{P}^n_{\mathbf{k}}\big| &\leq 2\big|\Delta_p\hat{\mathcal{K}}_n^{-1}(q)\mathcal{P}^n_{\mathbf{k}}\big| \\
&\leq 2\big|t_p\hat{\mathcal{K}}_n^{-1}(q)\mathcal{P}^n_{\mathbf{k}}\big|\,\big||\omega\cdot q|^2 - |\omega\cdot(q+p)|^2\big|\,\big|\hat{\mathcal{K}}_n^{-1}(q)\mathcal{P}^n_{\mathbf{k}}\big| \\
&\leq Ck^{-\gamma}\eta^{-2n}|\omega\cdot p|,
\end{aligned}$$

where $|\omega \cdot p| < \eta^{n+1}$ has been used again to conclude that $|\hat{\mathcal{K}}_n^{-1}(q)\mathcal{P}^n_{\mathbf{k}}| \leq Ck^{-\gamma}\eta^{-n}$ is also verified. This concludes the proof of bound (5.18) and Lemma 5.3.

■

Proof of Bound (6.44).

Bound (6.44) is a simple consequence of Lemma 5.3, Lemma 6.1, and the a priori bound

$$||\hat{P}_m\Delta_p Dw_m(z)||^{(m)}_{s,s'} \leq \varepsilon 4^m|\omega\cdot p|, \tag{9.11}$$

valid for all $m = 1, \ldots, n$, $z \in B_m$ and p satisfying $|\omega \cdot p| < \frac{1}{16}\eta^{m-1}$. Indeed, bounds (5.17), (6.27) and (6.28) lead to

$$||H_m(\tilde{z})||^{(m-1)}_s \leq 2 \quad \text{and} \quad ||\hat{P}_{m-1}T_m(z)||^{(m-2,m-1)}_{s,s'} \leq \varepsilon^2 r^{\frac{m}{2}}. \tag{9.12}$$

Using in addition (5.18) and the a priori bound (9.11), one estimates that for $r = r(\eta)$ small enough,

$$||\hat{P}_m\Delta_p T_m(z)||^{(m-1,m)}_{s,s'} \leq \varepsilon^2 r^{\frac{m}{4}}|\omega\cdot p|. \tag{9.13}$$

Hence, (6.44) follows from (9.12), (9.13) and (6.46) by taking $r = r(\eta)$ small enough and noting that if p satisfies $|\omega \cdot p| < \eta^{m-1}/16$, then the following estimate holds for any operator $B \in \mathcal{L}(h^m_s, h_{s'})$,

$$||\hat{P}_m t_p B||^{(m)}_{s,s'} \leq ||\hat{P}_{m-1}B||^{(m-1)}_{s,s'}. \tag{9.14}$$

It thus remains to check the a priori estimate (9.11). In the sequel, we use the shorter notation $\pi_n = Dw_n$. Using $\Delta_p(ab) = \Delta_p ab + t_p a\Delta_p b$, one computes from the recursive relation (6.24) that for all $m = 1, \ldots, n$,

$$\Delta_p\pi_m(z) = t_p\tilde{H}_m(z)\Delta_p\tilde{\pi}_{m-1}(\tilde{z})H_m(\tilde{z}) + t_p\pi_m(z)\Delta_p\Gamma_m\pi_m(z), \tag{9.15}$$

where $H_m(\tilde{z})$ is given by (6.22), and

$$\tilde{H}_m(z) = 1 + \pi_m(z)\Gamma_m. \tag{9.16}$$

To treat the first term on the right hand side of (9.15), one first estimates, as previously, that for ε small enough, $||H_m(\tilde{z})||_s^{(m-1)} \leq 2$ and, using (5.17) and the a priori bound (6.25),

$$||\hat{P}_{m-1}\tilde{H}_m(z)||_{s'}^{(m-1)} \leq 2. \tag{9.17}$$

Next, remarking that $\hat{P}_m t_p \tilde{H}_m = \hat{P}_m t_p \tilde{H}_m t_p \hat{P}_{m-1}$, one computes

$$\begin{aligned} \hat{P}_m t_p \tilde{H}_m \Delta_p \tilde{\pi}_{m-1}(\tilde{z}) &= \hat{P}_m t_p \Big(\tilde{H}_m \hat{P}_{m-1} t_{-p} \Delta_p \tilde{\pi}_{m-1}(\tilde{z}) \Big), \\ &= -\hat{P}_m t_p \Big(\tilde{H}_m \hat{P}_{m-1} \Delta_{-p} \tilde{\pi}_{m-1}(\tilde{z}) \Big), \end{aligned} \tag{9.18}$$

which, with (9.14) and (9.17), leads to

$$||\hat{P}_m t_p \tilde{H}_m(z) \Delta_p \tilde{\pi}_{m-1}(\tilde{z}) H_m(\tilde{z})||_{s,s'}^{(m)} \leq 4||\hat{P}_{m-1}\Delta_{-p}\tilde{\pi}_{m-1}(\tilde{z})||_{s,s'}^{(m-1)}. \tag{9.19}$$

In order to treat the second term on the right hand side of (9.15), we first note that $\Delta_p\Gamma_m = \hat{P}_{m-2}\Delta_p\Gamma_m\hat{P}_{m-2}$. Hence, using (5.18) and (9.14), one estimates that for $\varepsilon = \varepsilon(\eta)$ small enough,

$$||\hat{P}_m t_p \pi_m(z) \Delta_p \Gamma_m \pi_m(z)||_{s,s'}^{(m)} \leq \varepsilon|\omega \cdot p|. \tag{9.20}$$

Finally, collecting (9.19) and (9.20), one obtains, with the relation $\Delta_p\tilde{\pi}_{m-1} = \Delta_p\pi_{m-1}$,

$$||\hat{P}_m \Delta_p \pi_m(z)||_{s,s'}^{(m)} \leq 4||\hat{P}_{m-1}\Delta_{-p}\pi_{m-1}(\tilde{z})||_{s,s'}^{(m-1)} + \varepsilon|\omega \cdot p|.$$

Since $\Delta_p\pi_0 = 0$ for all $p \in \mathbb{Z}^d$, applying the previous inequality recursively leads to

$$||\hat{P}_m \Delta_p \pi_m(z)||_{s,s'}^{(m)} \leq \varepsilon \sum_{k=0}^{m-1} 4^k |\omega \cdot p|,$$

which finally yields (9.11).

■

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