Dirichlet Forms and Symmetric Markovian Semigroups on CCR Algebras with respect to Quasi-free States

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Abstract

Employing the construction method of Dirichlet forms on standard forms of von Neumann algebras developed in [Pa], we construct Dirichlet forms and associated symmetric Markovian semigroups on CCR algebras with respect to quasi-free states. More precisely, let $\mathcal{A}(\mathfrak{h}_0)$ be the CCR algebra over a complex separable pre-Hilbert space $\mathfrak{h}_0$ and let ω be a quasi-free state on $\mathcal{A}(\mathfrak{h}_0)$. For any normalized admissible function f and complete orthonormal system (CONS) $\{g_n\} \subset \mathfrak{h}_0$, we construct a Dirichlet form and corresponding symmetric Markovian semigroup on the natural standard form associated to the GNS representation of $(\mathcal{A}(\mathfrak{h}_0), \omega)$. It turns out that the form is independent of admissible function f and CONS $\{g_n\}$ chosen. By analyzing the spectrum of the generator (Dirichlet operator) of the semigroup, we show that the semigroup is ergodic and tends to the equilibrium exponentially fast.

1 Introduction

The purpose of this paper is to construct Dirichlet forms and associated symmetric Markovian semigroups on CCR algebras with respect to gauge invariant quasi-free states, and then investigate detailed properties such as ergodicity of the semigroups. Let $\mathcal{A}(\mathfrak{h}_0)$ be the CCR algebra over a complex separable pre-Hilbert space $\mathfrak{h}_0$ and let ω be a gauge invariant quasi-free state on $\mathcal{A}(\mathfrak{h}_0)$ [BR]. For any normalized admissible function f and complete orthonormal system (CONS) $\{g_n\} \subset \mathfrak{h}_0$, we use the general construction method of Dirichlet forms developed in [Pa] to construct a Dirichlet form and corresponding symmetric Markovian semigroup on the natural standard form associated to the GNS representation of the pair $(\mathcal{A}(\mathfrak{h}_0), \omega)$. We show that the Dirichlet form we constructed is independent of the admissible function f and the CONS $\{g_n\}$ chosen. By establishing a (chaos) decomposition of the quasi-free Hilbert space (see Section 5) and investigating the spectrum of the generator (Dirichlet operator) of the semigroup, we prove that the semigroup is ergodic and tends to the equilibrium exponentially fast.

The study of noncommutative Dirichlet forms was pioneered by Albeverio and Høegh-Krohn [AH-K], Sauvageot [Sau1, Sau2, Sua3] and extensively developed by Davies and Lindsay [DL], and Guido, Isola and Scarlatti [GIS]. All these authors considered Markovianity of

forms and semigroups only with respect to a tracial state ϕ_0. Recently, Goldstein and Lindsay [GL1, GL2], and Cipriani [Cip] have extended the abstract theory to a faithful normal state ϕ_0 in the context of standard forms.

The need to construct Markovian semigroups on von Neumann algebras, which are symmetric with respect to a non-tracial state, is clear for various applications to open systems [Dav], quantum statistical mechanics [BR], and quantum probability theory [Acc, AFL, Part]. Although on an abstract level we have quite well-developed theory as mentioned above, the progress in concrete application is very slow. We would like to mention a few recent works in this direction. The completely positive Hamiltonian semigroup for quantum spin chains in the ground state representation has been considered in [FNW, Mat, Na]. In [MZ1] and [MZ2] Majewski and Zegarlinski used the generalized conditional expectation to construct generators of spin-flip type dynamics for quantum spin systems. In [Pa], one of authors gave a general construction method of Dirichlet forms on standard forms of von Neumann algebras and applied the method to construct translation invariant Markovian semigroups for quantum spin systems. In [CFL], quantum Ornstein-Uhlenbeck semigroups were constructed by means of noncommutative Dirichlet forms. Extending the methods of [Pa] and this paper we construct symmetric Markovian semigroups on CAR algebras with respect to quasi-free states [BKP].

Let us describe the content of this paper briefly. Let $\mathfrak{h}_0$ be a complex separable pre-Hilbert space and $\mathfrak{h}$ the completion of $\mathfrak{h}_0$. Denote by $\mathcal{A}(\mathfrak{h}_0)$ the C^*-algebra over $\mathfrak{h}_0$ generated by the Weyl operators $W(g)$, $g \in \mathfrak{h}_0$. Let A be a self-adjoint operator on $\mathfrak{h}$ satisfying

$$0 < A \leq \alpha \mathbf{1} < \mathbf{1} \tag{1.1}$$

for some $\alpha \in (0,1)$. The gauge invariant quasi-free state ω on $\mathcal{A}(\mathfrak{h}_0)$ is given by

$$\omega(W(f)) = \exp\{-(f, \frac{1}{4}(\mathbf{1}+A)(\mathbf{1}-A)^{-1}f)\}, \quad f \in \mathfrak{h}_0.$$

We assume that A^{-1} exists as a (unbounded) self-adjoint operator and that any vector in $\mathfrak{h}_0$ is an analytic vector for $A^{-1/2}$ (Assumption 3.1).

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation of the pair $(\mathcal{A}(\mathfrak{h}_0), \omega)$ and $\mathcal{M} = \pi_\omega(\mathcal{A}(\mathfrak{h}_0))''$. We suppress ω and π_ω from the notation, i.e., $\mathcal{H} = \mathcal{H}_\omega$, $W(f) = \pi_\omega(W(f))$, etc. We also write $\xi_0 = \Omega_\omega$. Let $\sigma_t : \mathcal{M} \to \mathcal{M}$ be the one parameter group of automorphisms defined by

$$\sigma_t(W(f)) = W(A^{it}f), \quad f \in \mathfrak{h}_0, \ t \in \mathbb{R}.$$

Then ω satisfies σ-KMS conditions [BR]. We use Δ and J to denote the modular operator and the modular conjugation respectively. Then σ_t becomes the modular group ; $\sigma_t(B) = \Delta^{it}B\Delta^{-it}$, $B \in \mathcal{M}$. Let $\mathcal{M}'$ be the commutant of $\mathcal{M}$. The map $j : \mathcal{M} \to \mathcal{M}'$ is the antilinear $*$-isomorphism defined by $j(B) = JBJ$, $B \in \mathcal{M}$. The natural positive cone $\mathcal{P}$ associated with $(\mathcal{M}, \xi_0)$ is the closure of the set $\{Bj(B)\xi_0 : B \in \mathcal{M}\}$. The form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is the standard form associated with $(\mathcal{M}, \xi_0)$.

For any $g \in \mathfrak{h}_0$, let $\Phi(g)$ be the infinitesimal generator of the unitary group $W(tg)$, $t \in \mathbb{R}$, and

$$\begin{aligned} a(g) &= 2^{-1/2}(\Phi(g) + i\Phi(ig)), \\ a^*(g) &= 2^{-1/2}(\Phi(g) - i\Phi(ig)). \end{aligned}$$

For any $f, g \in \mathfrak{h}_0$, $a(f)$ and $a^*(g)$ are densely defined, closed and $a(f)^* = a^*(f)$, and satisfy the canonical commutation relations (CCRs) [BR]. Notice that $a^\#(g)$ and $j(a^\#(g))$, $g \in \mathfrak{h}_0$, are affiliated with $\mathcal{M}$ and $\mathcal{M}'$ respectively, where $a^\#(g)$ stands for either $a(g)$ or $a^*(g)$.

For any $B \in \mathcal{M}$ and $n \in \mathbb{N}$, we write

$$B_n = \left(\frac{n}{\pi}\right)^{\frac{1}{2}} \int \sigma_t(B) e^{-nt^2} dt.$$

Let $\mathcal{W}$ be the algebra generated by $W(f)$, $f \in \mathfrak{h}_0$. We use $\mathcal{W}_0$ and $\mathcal{M}_0$ to denote the algebras generated by $W_n(f)$, $f \in \mathfrak{h}_0$, $n \in \mathbb{N}$ and by B_n, $B \in \mathcal{M}$, $n \in \mathbb{N}$ respectively. We also use $\mathcal{H}_{fin}$ to denote the subspace of $\mathcal{H}$ spanned by the vectors of the form $(\prod_{j=1}^n \Phi(g_j))\xi_0$, $g_j \in \mathfrak{h}_0$, $j = 1, 2, \cdots, n$. Obviously, $\mathcal{W}\xi_0$, $\mathcal{W}_0\xi_0$, $\mathcal{M}_0\xi_0$ and $\mathcal{H}_{fin}$ are dense in $\mathcal{H}$.

For any normalized admissible function f (Definition 2.1) and CONS $\{g_n\} \subset \mathfrak{h}_0$ for $\mathfrak{h}$, we define a sesquilinear form $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{C}$ by

$$\begin{aligned} D(\mathcal{E}) &= \mathcal{W}_0\xi_0 \,(\text{ or } \mathcal{W}\xi_0,\, \mathcal{M}_0\xi_0), \\ \mathcal{E}(\eta, \xi) &= \sum_{n=1}^{\infty} \mathcal{E}^{(n)}(\eta, \xi), \quad \eta, \xi \in D(\mathcal{E}), \end{aligned} \tag{1.2}$$

where for each $n \in \mathbb{N}$, $\eta, \xi \in D(\mathcal{E}^{(n)}) = \mathcal{W}_0\xi_0$ (or $\mathcal{W}\xi_0$, $\mathcal{M}_0\xi_0$)

$$\begin{aligned} \mathcal{E}^{(n)}(\eta, \xi) &= \int \langle \big(\sigma_{t-i/4}(a(g_n)) - j\,(\sigma_{t-i/4}(a^*(g_n)))\big)\eta, \\ &\qquad \big(\sigma_{t-i/4}(a(g_n)) - j\,(\sigma_{t-i/4}(a^*(g_n)))\big)\xi\rangle f(t)\, dt \\ &\quad + \int \langle \big(\sigma_{t-i/4}(a^*(g_n)) - j\,(\sigma_{t-i/4}(a(g_n)))\big)\eta, \\ &\qquad \big(\sigma_{t-i/4}(a^*(g_n)) - j\,(\sigma_{t-i/4}(a(g_n)))\big)\xi\rangle f(t)\, dt. \end{aligned} \tag{1.3}$$

See Section 3 for the details.

It turns out that the forms $(\mathcal{E}, \mathcal{W}\xi_0)$ and $(\mathcal{E}, \mathcal{W}_0\xi_0)$ are closable and independent of the admissible function f and the CONS $\{g_n\}$ chosen (Proposition 3.1). For each $n \in \mathbb{N}$, the form $(\mathcal{E}^{(n)}, \mathcal{M}_0\xi_0)$ is closable and its closure $(\overline{\mathcal{E}}^{(n)}, D(\overline{\mathcal{E}}^{(n)}))$ is a Dirichlet form (Proposition 3.2). Let $\overline{\mathcal{E}}$ be the form defined by

$$\overline{\mathcal{E}} = \sum_{n=1}^{\infty} \overline{\mathcal{E}}^{(n)}.$$

Then $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is a densely defined Dirichlet form (Theorem 3.1). As a Corollary of Theorem 3.2, the form $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is also independent of f and $\{g_n\}$ we have chosen.

Let H be the infinitesimal generator of the Markovian semigroup $\{T_t\}_{t \geq 0}$ associated to $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$. In order to analyze the spectrum $\sigma(H)$ of H, we introduce a (chaos) decomposition of $\mathcal{H}$. Let B be the operator given by

$$B := A^{-1/2} - A^{1/2}. \tag{1.4}$$

For any $g \in \mathfrak{h}_0$, let $D_1(g)$ and $D_2(g)$ be the operators on $\mathcal{H}$ defined by

$$\begin{aligned} D_1(g) &:= \sigma_{-i/4}(a(B^{-1/2}g)) - j(\sigma_{-i/4}(a^*(B^{-1/2}g))), \\ D_2(g) &:= \sigma_{-i/4}(a^*(B^{-1/2}g)) - j(\sigma_{-i/4}(a(B^{-1/2}g))). \end{aligned}$$

Then $D_i(f)\xi_0 = 0$, $i = 1, 2$, $f \in \mathfrak{h}_0$ (Lemma 5.1) and the following CCRs hold (Proposition 5.1) : for any $f, g \in \mathfrak{h}_0$

$$\begin{aligned} [D_1(f), D_1(g)^*] &= (f, g)\mathbf{1}, \\ [D_2(f), D_2(g)^*] &= (g, f)\mathbf{1}, \\ [D_i(f), D_i(g)] &= 0, \quad i = 1, 2, \\ [D_1(f)^\#, D_2(g)^\#] &= 0, \end{aligned}$$

where $D_i(f)^\#$ is either $D_i(f)$ or $D_i(f)^*$, $i = 1, 2$. Thus $D_i(f)$ and $D_i(g)^*$ can be considered as annihilation and creation operators respectively and ξ_0 is the vacuum vector. Thus $\mathcal{H}$ has the following decomposition (Theorem 5.1):

$$\mathcal{H} = \bigoplus_{m,n=0}^{\infty} \mathcal{H}^{(m,n)},$$

where for each $m, n \in \mathbb{N} \cup \{0\}$, $\mathcal{H}^{(m,n)}$ is the closure of the subspace spanned by vectors of the form

$$(\prod_{j=1}^{m} D_1(g_j)^*)(\prod_{l=1}^{n} D_2(h_l)^*)\xi_0, \quad g_j,\, h_l \in \mathfrak{h}_0.$$

It follows that $\mathcal{H}_{fin} \subset D(H)$ and for any CONS $\{g_n\} \subset \mathfrak{h}_0$

$$H = \sum_{n=1}^{\infty} \{D_1(B^{1/2}g_n)^* D_1(B^{1/2}g_n) + D_2(B^{1/2}g_n)^* D_2(B^{1/2}g_n)\}$$

as a bilinear form on $\mathcal{H}_{fin} \times \mathcal{H}_{fin}$ (Lemma 5.4 (b)). Let $\mathcal{F}$ be the symmetric Fock space over $\mathfrak{h}$ [BR]. Then there is an antiunitary operator $V : \mathcal{F} \to \mathcal{F}$ and a unitary operator $U : \mathcal{H} \to \mathcal{F} \otimes V\mathcal{F}$ (Proposition 5.2) such that

$$UHU^{-1} = d\Gamma(B) \otimes \mathbf{1} + \mathbf{1} \otimes d\Gamma(B),$$

where $d\Gamma(B)$ is the second quantization of B [BR]. Thus H is essentially self-adjoint on $\mathcal{H}_{fin}$ and independent of the admissible function f and CONS $\{g_n\} \subset \mathfrak{h}_0$ chosen. Moreover the zero is a simple eigenvalue of H with eigenvector ξ_0 and $(0, \alpha^{-1/2} - \alpha^{1/2}) \cap \sigma(H) = \emptyset$ (Theorem 3.2).

We should mention that the main results in this paper can be generalized. For an instance, if one replace g_n by $B^\lambda g_n$, $n \in \mathbb{N}$, $\lambda \in [-\frac{1}{2}, \infty)$, in the definition of $\mathcal{E}(\eta, \xi)$ in (1.2) and (1.3), the results in Section 3 still hold with an appropriate modification on the spectral gap. See Remark 3.1.

We organize this paper as follows; In Section 2 we first introduce some terminologies in the theory of noncommutative Dirichlet forms [Cip] and then review the general construction

method of [Pa]. We extend the method of [Pa] slightly and produce its proof. In section 3, we give an explicit expression of a Dirichlet form for given normalized admissible function f and CONS $\{g_n\} \subset \mathfrak{h}_0$ for $\mathfrak{h}$, and then state main results in this paper. Section 4 is devoted to the proofs of Proposition 3.1, Proposition 3.2 and Theorem 3.1. In Section 5, we introduce a chaos decomposition of $\mathcal{H}$ and then prove the ergodicity of the semigroup and the existence of a spectral gap. In Appendix, we give the proofs of technical lemmas (Lemma 4.3 and Lemma 5.4).

2 Review on Construction of Dirichlet Forms on Standard Forms of von Neumann Algebras

In this section, we first introduce necessary terminologies in the theory of noncommutative Dirichlet forms in the sense of Cipriani [Cip], and then describe the general construction method of Dirichlet forms on the standard form of von Neumann algebras developed in [Pa]. We extend the result of [Pa] slightly and give its proof.

Let $\mathcal{M}$ be a σ-finite von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$. A self-dual cone $\mathcal{P}$ in $\mathcal{H}$ is a subset satisfying the property

$$\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \geq 0, \quad \forall \eta \in \mathcal{P}\} = \mathcal{P}.$$

$\mathcal{P}$ is then a closed convex cone and $\mathcal{H}$ is the complexification of the real subspace $\mathcal{H}^J := \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \in \mathbb{R}, \ \forall \eta \in \mathcal{P}\}$, which elements are called *J-real* : $\mathcal{H} = \mathcal{H}^J + i\mathcal{H}^J$. Such a $\mathcal{P}$ gives rise to a structure of ordered Hilbert space on $\mathcal{H}^J$ (denoted by $\leq$) and to an anti-unitary involution J on $\mathcal{H}$, which preserves $\mathcal{H}^J$: $J(\xi + i\eta) := \xi - i\eta$ for all $\xi, \eta \in \mathcal{H}^J$. Any J-real element $\xi \in \mathcal{H}^J$ can be decomposed uniquely as a difference, $\xi = \xi_+ - \xi_-$, of two orthogonal, positive elements, called the positive and the negative part of ξ : $\xi_+, \xi_- \in \mathcal{P}, \quad \langle \xi_+, \xi_- \rangle = 0$.

A *standard form* $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ of the von Neumann algebra $\mathcal{M}$ acting faithfully on the Hilbert space $\mathcal{H}$ consists of self-dual, closed, convex cone $\mathcal{P}$ in $\mathcal{H}$ and the anti-unitary involution J satisfying the properties:

(a) $J\mathcal{M}J = \mathcal{M}'$

(b) $JxJ = x^*, \quad \forall x \in \mathcal{M} \cap \mathcal{M}'$

(c) $J\xi = \xi, \quad \forall \xi \in \mathcal{P}$

(d) $xJxJ(\mathcal{P}) \subset \mathcal{P}, \quad \forall x \in \mathcal{M},$

where $\mathcal{M}'$ is the commutant of $\mathcal{M}$.

A bounded operator A on $\mathcal{H}$ is called *J-real* if $AJ = JA$ and *positive preserving* if $A\mathcal{P} \subset \mathcal{P}$. The semigroup $\{T_t\}_{t \geq 0}$ is said to be *J-real* if T_t is J-real for any $t \geq 0$ and it is called *positive preserving* if T_t is positive preserving for any $t \geq 0$.

Let us fix a cyclic and separating vector ξ_0 in $\mathcal{P}$. A bounded operator $A : \mathcal{H} \to \mathcal{H}$ is called *sub-Markovian* (with respect to ξ_0) if $0 \leq \xi \leq \xi_0$ implies $0 \leq A\xi \leq \xi_0$. A is called *Markovian* if it is sub-Markovian and also $A\xi_0 = \xi_0$. A semigroup $\{T_t\}_{t \geq 0}$ is said to be *sub-Markovian*

(with respect to ξ_0) if T_t is sub-Markovian for every $t \geq 0$. The semigroup $\{T_t\}_{t\geq 0}$ is called *Markovian* if T_t is Markovian for every $t \geq 0$.

Next, we consider a sesquilinear form on some linear manifold of $\mathcal{H}$: $\mathcal{E}(\cdot,\cdot) : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{C}$. We also consider the associated quadratic form: $\mathcal{E}[\cdot] : D(\mathcal{E}) \to \mathbb{C}$, $\mathcal{E}[\xi] := \mathcal{E}(\xi,\xi)$. A real valued quadratic form $\mathcal{E}[\cdot]$ is said to be *semi-bounded* if $\inf\{\mathcal{E}[\xi] : \xi \in D(\mathcal{E}),\ ||\xi|| = 1\} = -b > -\infty$. A quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is said to be *J-real* if $JD(\mathcal{E}) \subset D(\mathcal{E})$ and $\mathcal{E}[J\xi] = \overline{\mathcal{E}[\xi]}$ for any $\xi \in D(\mathcal{E})$. For a given semi bounded quadratic form $\mathcal{E}$, one consider the inner product given by $\langle \xi, \eta\rangle_\lambda := \mathcal{E}(\xi,\eta) + \lambda\langle \xi, \eta\rangle$, for $\lambda > b$. The form $\mathcal{E}$ is *closed* if $D(\mathcal{E})$ is a Hilbert space for some of the above norms. The form $\mathcal{E}$ is called *closable* if it admits a closed extension.

Associated to a semi bounded closed form $\mathcal{E}$, there are a self-adjoint operator $(H, D(H))$ and a strongly continuous, symmetric semigroup $\{T_t\}_{t\geq 0}$. Each of the above objects determines uniquely the others according to well known relations (see Section 3.1 of [BR] and Section 1.3 of [Fuk]).

From now on we will consider only J-real, real-valued, semi bounded, densely defined quadratic forms. It is easy to check that these forms satisfy the relation : $\mathcal{E}[\xi + i\eta] = \mathcal{E}[\xi] + \mathcal{E}[\eta]$ for all $\xi + i\eta \in D(\mathcal{E})^J + iD(\mathcal{E})^J = D(\mathcal{E})$ where $D(\mathcal{E})^J := D(\mathcal{E}) \cap H^J$.

Let us denoted by $\mathrm{Proj}\,(\xi, \mathcal{Q})$ the projection of the vector $\xi \in \mathcal{H}^J$ onto the closed, convex cone $\mathcal{Q} \subset \mathcal{H}^J$. For $\xi, \eta \in \mathcal{H}^J$, define

$$\begin{aligned} \xi \vee \eta &:= \mathrm{Proj}\,(\xi, \eta + \mathcal{P}) \\ \xi \wedge \eta &:= \mathrm{Proj}\,(\xi, \eta - \mathcal{P}). \end{aligned}$$

A J-real, real-valued, densely defined quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is called *Markovian* with respect to $\xi_0 \in \mathcal{P}$ if

$$\xi \in D(\mathcal{E})^J \text{ implies } \xi \wedge \xi_0 \in D(\mathcal{E}) \text{ and } \mathcal{E}[\xi \wedge \xi_0] \leq \mathcal{E}[\xi].$$

A closed Markovian form is called a *Dirichlet form*. Let $\{T_t\}_{t\geq 0}$ be the semigroup associated to a semi bounded closed form $(\mathcal{E}, D(\mathcal{E}))$. Then one of the main results in [Cip] is that $\{T_t\}_{t\geq 0}$ is sub-Markovian if and only if $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form (Theorem 4.11 of [Cip]).

Next, we describe a general construction method of Dirichlet forms on the natural standard forms of von Neumann algebras associated with the Tomita-Takesaki theory [BR, Tak]. The method has been developed in [Pa]. However we generalize the result of [Pa] slightly and produce the proof.

Let $\mathcal{M}$ be a σ-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and let $\xi_0 \in \mathcal{H}$ be a cyclic and separating vector for $\mathcal{M}$. We use Δ and J to denote respectively, the modular operator and the modular conjugation associated with the pair $(\mathcal{M}, \xi_0)$[Ara, BR]. The associated modular automorphism group is denoted by $\sigma_t : \sigma_t(A) = \Delta^{it} A \Delta^{-it}, A \in \mathcal{M}$. Finally, $j : \mathcal{M} \to \mathcal{M}'$ is the anti linear $*$-isomorphism defined by $j(A) = JAJ$, $A \in \mathcal{M}$. The natural positive cone $\mathcal{P}$ associated with $(\mathcal{M}, \xi_0)$ is the closure of the set

$$\{Aj(A)\xi_0 : A \in \mathcal{M}\}.$$

By a general result the cone $\mathcal{P}$ can be obtained by the closure of the set

$$\{\Delta^{1/4} A A^* \xi_0 : A \in \mathcal{M}\}.$$

The natural cone $\mathcal{P}$ is self-dual [Ara, BR]. For the detailed properties of $\mathcal{P}$, we refer to Section 2.5.4 of [BR]. The form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is the standard form associated with the pair $(\mathcal{M}, \xi_0)$.

In order to express Dirichlet forms, let us introduce the notion of admissible functions:

Definition 2.1 *An analytic function $f : D \to \mathbb{C}$ on a domain D containing the strip $Im\ z \in [-1/4, 1/4]$ is said to be admissible if the following properties hold:*

(a) $f(t) \geq 0$ for $\forall t \in \mathbb{R}$,

(b) $f(t + i/4) + f(t - i/4) \geq 0$ for $\forall t \in \mathbb{R}$,

(c) there exist $M > 0$ and $p > 1$ such that the bound

$$|f(t + is)| \leq M(1 + |t|)^{-p}$$

holds uniformly in $s \in [-1/4, 1/4]$.

Let us give an example of an admissible function. Using the residue integration method it is easy to check that

$$\int_{-\infty}^{\infty} (\cosh k)^{-1} e^{ikt}\, dk = 2\pi (e^{\frac{\pi}{2}t} + e^{-\frac{\pi}{2}t})^{-1}.$$

See also the expression in p. 94 of [BR]. Using the above, one may check that the function given by

$$f(t) = \frac{2}{\sqrt{2\pi}} \int (e^{k/4} + e^{-k/4})^{-1} e^{-\frac{1}{2}k^2} e^{-ikt}\, dk \tag{2.1}$$

is an admissible function.

We are ready to give a construction of Dirichlet forms on the standard form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ associated with the pair $(\mathcal{M}, \xi_0)$. Denote by $\mathcal{M}_{an}$ the dense subset of $\mathcal{M}$ consisting of every σ_t-analytic element with a domain containing the strip $I_{1/2} := \{z : |Im\ z| \leq \frac{1}{2}\}$ [BR]. By Prop. 2.5.21 of [BR], any $A \in \mathcal{M}_{an}$ is strongly analytic. In the following, the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ is conjugate linear in the first and linear in the second variable. For given admissible function f and $x \in \mathcal{M}_{an}$, define a sesquilinear form $\mathcal{E} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$\begin{aligned}
\mathcal{E}(\eta, \xi) &= \int \langle \big(\sigma_{t-i/4}(x) - j\,(\sigma_{t-i/4}(x^*))\big)\eta, \\
&\qquad \big(\sigma_{t-i/4}(x) - j\,(\sigma_{t-i/4}(x^*))\big)\xi \rangle f(t)\, dt \\
&\quad + \int \langle \big(\sigma_{t-i/4}(x^*) - j\,(\sigma_{t-i/4}(x))\big)\eta, \\
&\qquad \big(\sigma_{t-i/4}(x^*) - j\,(\sigma_{t-i/4}(x))\big)\xi \rangle f(t)\, dt \\
&\equiv \mathcal{E}^{(1)}(\eta, \xi) + \mathcal{E}^{(2)}(\eta, \xi).
\end{aligned} \tag{2.2}$$

Then the associated quadratic form is given by

$$\begin{aligned}\mathcal{E}[\xi] &= \int || \, [\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*))]\xi||^2 f(t)\, dt \\ &\quad + \int || \, [\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x))]\xi||^2 f(t)\, dt \\ &\equiv \mathcal{E}^{(1)}[\xi] + \mathcal{E}^{(2)}[\xi].\end{aligned} \tag{2.3}$$

In [Pa], we have considered the case $x = x^* \in \mathcal{M}_{an}$. The following is the result corresponding to Theorem 3.1 of [Pa].

Theorem 2.1 *For a given admissible function f and $x \in \mathcal{M}_{an}$, let $(\mathcal{E}, \mathcal{H})$ be defined as in (2.2). Let H be the self-adjoint operator associated with $(\mathcal{E}, \mathcal{H})$. Assume that there exists a constant $M > 0$ such that the bound*

$$\sup_{s \in [-1/4, 1/4]} || \, \sigma_{t+is}(x) \, || \leq M$$

holds uniformly in $t \in \mathbb{R}$. Then the following properties hold:

(a) $H\xi_0 = 0$,

(b) $\mathcal{E}$ is J-real,

(c) $\mathcal{E}(\xi_+, \xi_-) \leq 0 \quad \forall \xi \in \mathcal{H}^J$.

Furthermore the form $(\mathcal{E}, \mathcal{H})$ is a Dirichlet form.

We will produce the proof of Theorem 2.1 at the end of this section. The following is a consequence of Theorem 2.1:

Theorem 2.2 *Let $\{T_t\}_{t \geq 0}$ be the semigroup generated by the form $(\mathcal{E}, \mathcal{H})$ in Theorem 2.1. Then $\{T_t\}_{t \geq 0}$ is a J-real, strongly continuous, symmetric Markovian semigroup.*

Proof. It follows from Theorem 2.1 (a) that $T_t(\xi_0) = \xi_0$ for any $t \geq 0$. Thus the theorem follows from Theorem 4.11 of [Cip]. □

We now produce the proof of Theorem 2.1.

Proof of Theorem 2.1. (a) Notice that $JA\xi_0 = \Delta^{1/2} A^* \xi_0$ for any $A \in \mathcal{M}$. Thus we have that

$$\begin{aligned}\left(\sigma_{t-i/4}(x) - j\left(\sigma_{t-i/4}(x^*)\right)\right)\xi_0 &= \Delta^{1/4}\sigma_t(x)\xi_0 - J\Delta^{1/4}\sigma_t(x)^*\Delta^{-1/4}\xi_0 \\ &= \Delta^{1/4}\sigma_t(x)\xi_0 - \Delta^{1/4}\sigma_t(x)\xi_0 \\ &= 0.\end{aligned}$$

Replacing x by x^* in the above, we get

$$\Big(\sigma_{t-i/4}(x^*) - j\left(\sigma_{t-i/4}(x)\right)\Big)\,\xi_0 = 0.$$

Thus (a) follows from (2.2) and the above facts.

(b) A direct estimate shows that

$$\begin{aligned}
&|| \Big(\sigma_{t-i/4}(x) - j\left(\sigma_{t-i/4}(x^*)\right)\Big)\, J\xi \,||^2 \\
&\quad = || \,-J\left(\sigma_{t-i/4}(x^*) - j\left(\sigma_{t-i/4}(x)\right)\right)\xi \,||^2 \\
&\quad = || \Big(\sigma_{t-i/4}(x^*) - j\left(\sigma_{t-i/4}(x)\right)\Big)\,\xi \,||^2,
\end{aligned}$$

which implies $\mathcal{E}^{(1)}[J\xi] = \mathcal{E}^{(2)}[\xi]$. The method used in the above also implies that $\mathcal{E}^{(2)}[J\xi] = \mathcal{E}^{(1)}[\xi]$.

(c) By the expression of $\mathcal{E}(\eta,\xi)$ in (2.2), $\mathcal{E}(\xi_+,\xi_-)$ can be written as

$$\begin{aligned}
\mathcal{E}(\xi_+,\xi_-) &= \mathcal{E}^{(1)}(\xi_+,\xi_-) + \mathcal{E}^{(2)}(\xi_+,\xi_-) \\
&= (\mathrm{I}^{(1)} + \mathrm{II}^{(1)}) + (\mathrm{I}^{(2)} + \mathrm{II}^{(2)})
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
\mathrm{I}^{(1)} &= \int \Big(\langle \sigma_{t-i/4}(x)\xi_+, \sigma_{t-i/4}(x)\xi_-\rangle \\
&\qquad + \langle \sigma_{t-i/4}(x^*)\xi_-, \sigma_{t-i/4}(x^*)\xi_+\rangle\Big) f(t)\,dt, \qquad (2.5)\\
\mathrm{II}^{(1)} &= -\int \Big(\langle \sigma_{t-i/4}(x)\xi_+, j\left(\sigma_{t-i/4}(x^*)\right)\xi_-\rangle \\
&\qquad + \langle j\left(\sigma_{t-i/4}(x^*)\right)\xi_+, \sigma_{t-i/4}(x)\xi_-\rangle\Big) f(t)\,dt
\end{aligned}$$

and $\mathrm{I}^{(2)}$ and $\mathrm{II}^{(2)}$ are obtained from $\mathrm{I}^{(1)}$ and $\mathrm{II}^{(1)}$ respectively replacing x by x^* in the above.

As a consequence of Theorem 4(7) of [Ara], $\mathcal{M}\xi_+ \perp \mathcal{M}\xi_-$, which implies $\mathrm{I}^{(1)} = 0$ and $\mathrm{I}^{(2)} = 0$. See also the proof of Proposition 5.3 (ii) of [Cip]. Next, we first consider $\mathrm{II}^{(1)}$. It can be checked that $\sigma_{t-is}(x)^* = \sigma_{t+is}(x^*)$, for any $x \in \mathcal{M}_{an}$ and $s \in \mathbb{R}$, and so

$$\begin{aligned}
&\langle \sigma_{t-i/4}(x)\xi_+ \,,\, j\left(\sigma_{t-i/4}(x^*)\right)\xi_-\rangle \\
&\qquad = \langle\, \xi_+ \,,\, \sigma_{t+i/4}(x^*) j(\sigma_{t+i/4}(x)^*)\xi_-\rangle
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&\langle j\left(\sigma_{t-i/4}(x^*)\right)\xi_+ \,,\, \sigma_{t-i/4}(x)\xi_-\rangle \\
&\qquad = \langle \xi_+ \,,\, \sigma_{t-i/4}(x) j(\sigma_{t-i/4}(x^*)^*)\xi_-\rangle.
\end{aligned} \tag{2.7}$$

It follows from (2.5)-(2.7) that

$$\begin{aligned} \mathrm{II}^{(1)} &= -\int \langle\, \xi_+ \,,\, \sigma_{t+i/4}(x^*) j(\sigma_{t+i/4}(x)^*)\xi_- \rangle\, f(t)\, dt \\ &\quad -\int \langle \xi_+ \,,\, \sigma_{t-i/4}(x) j(\sigma_{t-i/4}(x^*)^*)\xi_- \,\rangle f(t)\, dt. \end{aligned}$$

Notice that the map $A \mapsto j(A^*)$ from $\mathcal{M}$ to $\mathcal{M}'$ is linear. It can be shown that for any $x \in \mathcal{M}_{an}$ and $\xi \in \mathcal{H}$, the map

$$z \mapsto j(\sigma_z(x)^*)\xi$$

is analytic on a domain containing the strip $I_{1/2}$. In fact, the analyticity follows from the facts that $\langle \eta \,,\, j(\sigma_z(x)^*)\xi \rangle = \langle \sigma_z(x)^* J\xi \,,\, J\eta \,\rangle = \langle J\xi, \sigma_z(x) J\eta \,\rangle$ for any $\eta, \xi \in \mathcal{H}$, and that weak analyticity implies strong analyticity (see Theorem VI.4 of [RS]).

Using the Cauchy integral theorem, the assumption in the theorem, the property (c) in the Definition 2.1 and $\sigma_t(x)^* = \sigma_t(x^*)$, we obtain that

$$\begin{aligned} \mathrm{II}^{(1)} = \quad & -\int \langle \xi_+, \sigma_t(x)^* j(\sigma_t(x)^*)\xi_- \rangle f(t - i/4)\, dt \\ & -\int \langle \xi_+, \sigma_t(x) j(\sigma_t(x))\xi_- \rangle f(t + i/4)\, dt. \end{aligned}$$

Replacing x and x^* in the above, we obtain the expression of $\mathrm{II}^{(2)}$. Thus we get

$$\begin{aligned} \mathrm{II} &= -\int \langle \xi_+, [\sigma_t(x) j(\sigma_t(x)) + \sigma_t(x)^* j(\sigma_t(x)^*)]\xi_- \rangle \\ &\quad \cdot (f(t - i/4) + f(t + i/4))\, dt. \end{aligned}$$

Since $\sigma_t(x) j(\sigma_t(x))\xi_- \in \mathcal{P}$, $\langle \xi_+, [\sigma_t(x) j(\sigma_t(x)) + \sigma_t(x)^* j(\sigma_t(x)^*)]\xi_- \rangle \geq 0$ for $t \in \mathbb{R}$. By the property (b) in the Definition 2.1, we conclude that $\mathrm{II} \leq 0$. This proved the part (c) of the theorem.

Clearly $\mathcal{E}[\cdot] \geq 0$. Note that $\mathcal{E}(\xi_0, \xi) = 0$, $\forall \xi \in \mathcal{H}$. By Theorem 2.1 (b)-(c), Proposition 4.5 (b) of [Cip] and Proposition 4.10(ii) of [Cip], $(\mathcal{E}, \mathcal{H})$ is a Dirichlet form. $\square$

3 Dirichlet Forms on CCR Algebras with respect to Quasi-free States : Preliminaries and Main Results

Let $\mathfrak{h}_0$ be a complex separable pre-Hilbert space. Denote by $\mathfrak{h}$ the completion of $\mathfrak{h}_0$. The inner product $(\cdot, \cdot)$ on $\mathfrak{h}$ is conjugate linear in the first and linear in the second variable. Let $\mathcal{A}(\mathfrak{h}_0)$ be the C^*-algebra generated by the Weyl operators $W(f)$, $f \in \mathfrak{h}_0$, satisfying

$$\begin{aligned} W(-f) &= W(f)^*, \\ W(f)W(g) &= e^{-\frac{i}{2}\mathrm{Im}(f,g)} W(f+g), \quad \forall f, g \in \mathfrak{h}_0. \end{aligned} \tag{3.1}$$

For the abstract properties of $\mathcal{A}(\mathfrak{h}_0)$, see Theorem 5.2.8 of [BR].

Next, we describe quasi-free states on $\mathcal{A}(\mathfrak{h}_0)$. Let A be a bounded and non-negative operator on $\mathfrak{h}$. Recall that $\varphi \in \mathfrak{h}$ is an analytic vector for an operator B on $\mathfrak{h}$ if $\varphi \in D(B^n)$, $n \in \mathbb{N}$, and if

$$\sum_{n=0}^{\infty} \frac{\|B^n \varphi\|}{n!} t^n < \infty$$

for some $t > 0$. In the rest of this paper, we assume that A satisfies the following properties.

Assumption 3.1 (a) There exists $\alpha \in (0,1)$ such that

$$0 < A \leq \alpha \mathbf{1} < \mathbf{1}.$$

(b) The inverse A^{-1} of A exists as a (unbounded) self-adjoint and positive operator on $\mathfrak{h}$.

(c) For any $z \in \mathbb{C}$, A^z leaves $\mathfrak{h}_0$ invariant, i.e., $A^z \mathfrak{h}_0 \subset \mathfrak{h}_0$. Moreover, $z \mapsto A^z \varphi$ is entire analytic for any $\varphi \in \mathfrak{h}_0$.

(d) Any $\varphi \in \mathfrak{h}_0$ is an analytic vector for $A^{-1/2}$.

We remark that a dense submanifold $\mathfrak{h}_0$ of $\mathfrak{h}$ satisfying the assumption exists by the spectral theorem.

Example 3.1. (Ideal Bose gases) Let $\mathfrak{h}$ be $L^2(\mathbb{R}^d, dx)$ and Δ the Laplacian operator on $L^2(\mathbb{R}^d, dx)$. Let A be given by

$$A = \exp\{-\beta(-\frac{1}{2}\Delta + \mu \mathbf{1})\},$$

where $\beta > 0$ and $\mu > 0$. Then $\alpha = \exp(-\beta\mu)$. For $f \in L^2(\mathbb{R}^d, dx)$, denote by $\hat{f}$ the Fourier transform of f. Choose $\mathfrak{h}_0 = \{f \in L^2(\mathbb{R}^d, dx) : \hat{f} \in C_c(\mathbb{R}^d)\}$. Clearly Assumption 3.1 satisfied.

For given $A \in \mathcal{L}(\mathfrak{h})$ satisfying Assumption 3.1, the gauge invariant quasi-free state ω on $\mathcal{A}(\mathfrak{h}_0)$ is defined by

$$\omega(W(f)) = \exp\{-(f, \frac{1}{4}(\mathbf{1} + A)(\mathbf{1} - A)^{-1} f)\}, \quad f \in \mathfrak{h}_0. \tag{3.2}$$

Let$(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation [BR] of $(\mathcal{A}(\mathfrak{h}_0), \omega)$, and let $\mathcal{M} = \pi_\omega(\mathcal{A}(\mathfrak{h}_0))''$. Notice that the representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of the CCR algebra $\mathcal{A}(\mathfrak{h}_0)$ is regular [BR] in the sense that the unitary groups $t \in \mathbb{R} \longmapsto \pi_\omega(W(tf))$ are strongly continuous for all $f \in \mathfrak{h}_0$. We use the notation $\Phi_\omega(f)$ to denote the infinitesimal generator of the unitary group $\pi_\omega(W(tf))$:

$$\pi_\omega(W(tf)) = \exp(it\Phi_\omega(f)). \tag{3.3}$$

Notice that, since ω is an entire analytic state[BR] by (3.2) and Assumption 3.1, the cyclic vector Ω_ω is an entire analytic vector for all $\Phi_\omega(f)$, $f \in \mathfrak{h}_0$. The annihilation and creation operators defined for each $f \in \mathfrak{h}_0$ by

$$D(a_\omega(f)) = D(\Phi_\omega(f)) \cap D(\Phi_\omega(if)) = D(a_\omega^*(f))$$

and

$$\begin{aligned} a_\omega(f) &:= 2^{-1/2}(\Phi_\omega(f) + i\Phi_\omega(if)), \\ a_\omega^*(f) &:= 2^{-1/2}(\Phi_\omega(f) - i\Phi_\omega(if)) \end{aligned} \tag{3.4}$$

are densely defined, closed and $a_\omega(f)^* = a_\omega^*(f)$, and satisfy the canonical commutation relations(CCRs): for $\forall f, g \in \mathfrak{h}_0$

$$[a_\omega(f), a_\omega(g)] = 0,$$

$$[a_\omega(f), a_\omega^*(g)] = (f, g)\mathbf{1}. \tag{3.5}$$

See Section 5.2.3 of [BR] for the details.

In the rest of this paper, we suppress ω and π_ω from the notations. Thus $\mathcal{H} = \mathcal{H}_\omega$, $W(f) = \pi_\omega(W(f))$, $\Phi(f) = \Phi_\omega(f), a(f) = a_\omega(f)$ and $a^*(f) = a_\omega^*(f)$ for any $f \in \mathfrak{h}_0$. We also use the notation $\xi_0 = \Omega_\omega$. Let $\sigma_t : \mathcal{M} \to \mathcal{M}$ be the group of automorphisms on $\mathcal{M}$ defined by

$$\sigma_t(W(f)) = W(A^{it}f), \quad f \in \mathfrak{h}_0.$$

It can be checked that the state ω defined by (3.2) is a σ-KMS state [BR]. Thus σ_t, $t \in \mathbb{R}$, is the modular automorphism by Theorem 5.3.10 of [BR]. Let Δ and J be the modular operator and modular conjugation respectively. Then $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is the natural standard form associated to $(\mathcal{M}, \xi_0)$.

Next, we introduce several dense manifolds of $\mathcal{H}$. For any $B \in \mathcal{M}$, define

$$B_n = \left(\frac{n}{\pi}\right)^{\frac{1}{2}} \int \sigma_t(B) e^{-nt^2} dt, \quad n \in \mathbb{N}. \tag{3.6}$$

Then B_n is an entire analytic element for σ_t, $\|B_n\| \leq \|B\|$ for all $n \in \mathbb{N}$ and $B_n \to B$ strongly. See the proof of Proposition 2.5.22 of [BR]. Put

$$\begin{aligned} \mathcal{M}_0 &:= \text{ the algebra generated by } B_n, \quad B \in \mathcal{M}, n \in \mathbb{N} \\ \mathcal{W}_0 &:= \text{ the algebra generated by } W_n(f), \quad f \in \mathfrak{h}_0, n \in \mathbb{N}, \\ \mathcal{W} &:= \text{ the algebra generated by } W(f), \quad f \in \mathfrak{h}_0. \end{aligned} \tag{3.7}$$

Denote by $\mathcal{H}_{fin}$ the subset of finite linear combinations of the vectors of the following type:

$$\psi_n = \left[\prod_{j=1}^{n} \Phi(f_j)\right] \xi_0, \quad \text{for } f_j \in \mathfrak{h}_0,\ j = 1, \cdots, n,\ n \in \mathbb{N}.$$

Clearly $\mathcal{M}_0\xi_0$, $\mathcal{W}_0\xi_0$, $\mathcal{W}\xi_0$ and $\mathcal{H}_{fin}$ are dense in $\mathcal{H}$ (See Lemma 4.1).

We denote by $a^\#(f)$ either $a(f)$ or $a^*(f)$, for any $f \in \mathfrak{h}_0$. Notice that $a^\#(f)$ and $j(a^\#(f))$, $f \in \mathfrak{h}_0$, are affiliated with $\mathcal{M}$ and $\mathcal{M}'$ respectively. For any $f \in \mathfrak{h}_0$ and $z \in \mathbb{C}$, we write

$$\begin{aligned} \sigma_z(a(f)) &:= a(A^{i\bar{z}}f), \\ \sigma_z(a^*(f)) &:= a^*(A^{i\bar{z}}f). \end{aligned} \tag{3.8}$$

In fact, one may be able to show that for any $f \in \mathfrak{h}_0$ and $\xi \in \mathcal{M}_0\xi_0$ the function $t \mapsto \sigma_t(a^\#(f))\xi$ has an analytic extension on $\mathbb{C}$, which is denoted by $\sigma_z(a^\#(f))\xi$, and that $\sigma_z(a^\#(f))$ is equal to that in the right hand side of (3.8) on $\mathcal{M}_0\xi_0$. See the proof of Lemma 4.3.

We are ready to describe Dirichlet forms. An admissible function f is said to be *normalized* if $\int f(t)dt = 1$. For given normalized admissible function f and a complete orthonormal system(CONS) $\{g_n\}_{n=1}^{\infty} \subset \mathfrak{h}_0$ of $\mathfrak{h}$, define a sesquilinear form $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{C}$ as follows:

$$D(\mathcal{E}) = \mathcal{W}_0\xi_0 \text{ (or } \mathcal{W}\xi_0,\ \mathcal{M}_0\xi_0),$$

$$\mathcal{E}(\eta,\xi) = \sum_{n=1}^{\infty} \mathcal{E}^{(n)}(\eta,\xi), \quad \eta,\xi \in D(\mathcal{E}), \tag{3.9}$$

where for each $n \in \mathbb{N}$, $\eta,\xi \in D(\mathcal{E}^{(n)}) = \mathcal{W}_0\xi_0$ (or $\mathcal{W}\xi_0,\ \mathcal{M}_0\xi_0$)

$$\begin{aligned}\mathcal{E}^{(n)}(\eta,\xi) &= \int \langle \big(\sigma_{t-i/4}(a(g_n)) - j\,(\sigma_{t-i/4}(a^*(g_n)))\big)\eta, \\ &\qquad \big(\sigma_{t-i/4}(a(g_n)) - j\,(\sigma_{t-i/4}(a^*(g_n)))\big)\xi\ \rangle f(t)\,dt \\ &\quad + \int \langle \big(\sigma_{t-i/4}(a^*(g_n)) - j\,(\sigma_{t-i/4}(a(g_n)))\big)\eta, \\ &\qquad \big(\sigma_{t-i/4}(a^*(g_n)) - j\,(\sigma_{t-i/4}(a(g_n)))\big)\xi\rangle f(t)\,dt.\end{aligned} \tag{3.10}$$

We also define the associated quadratic forms by

$$\begin{aligned}\mathcal{E}^{(n)}[\xi] &= \mathcal{E}^{(n)}(\xi,\xi), \quad \xi \in D(\mathcal{E}^{(n)}),\ n \in \mathbb{N} \\ \mathcal{E}[\xi] &= \sum_{n=1}^{\infty} \mathcal{E}^{(n)}[\xi], \quad \xi \in D(\mathcal{E}).\end{aligned}$$

We remark that the expression $\mathcal{E}^{(n)}(\eta,\xi)$ in (3.10) can be obtained from $\mathcal{E}(\eta,\xi)$ in (2.2) by replacing x by $a(g_n)$.

We state main results. It turns out that the form defined in (3.9) and (3.10) is independent of the normalized admissible function f and the CONS $\{g_n\} \subset \mathfrak{h}_0$ we have chosen:

Proposition 3.1 *Let $D(\mathcal{E})$ be either $\mathcal{W}_0\xi_0$ or else $\mathcal{W}\xi_0$ and let $(\mathcal{E}, D(\mathcal{E}))$ be defined as in (3.9) and (3.10). The form $(\mathcal{E}, D(\mathcal{E}))$ is closable. Moreover,$(\mathcal{E}, D(\mathcal{E}))$ is independent of the normalized admissible function f and the CONS $\{g_n\} \subset \mathfrak{h}_0$ we have chosen.*

Proposition 3.2 *For each $n \in \mathbb{N}$, $(\mathcal{E}^{(n)}, \mathcal{M}_0\xi_0)$ is closable. The closure $(\overline{\mathcal{E}}^{(n)}, D(\overline{\mathcal{E}}^{(n)}))$ is a Dirichlet form.*

Theorem 3.1 *Let us consider the following form :*

$$\begin{aligned}D(\overline{\mathcal{E}}) &= \{\xi \in \bigcap_{n=1}^{\infty} D(\overline{\mathcal{E}}^{(n)}) : \sum_{n=1}^{\infty} \overline{\mathcal{E}}^{(n)}[\xi] < \infty\}, \\ \overline{\mathcal{E}}[\xi] &= \sum_{n=1}^{\infty} \overline{\mathcal{E}}^{(n)}[\xi],\end{aligned}$$

where for each $n \in \mathbb{N}$, $(\overline{\mathcal{E}}^{(n)}, D(\overline{\mathcal{E}}^{(n)}))$ is the Dirichlet form obtained in Proposition 3.2. Then $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is a densely defined Dirichlet form.

Theorem 3.2 *Let $\{T_t\}$ be the symmetric Markovian semigroup associated to the Dirichlet form $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$, and let H be the Dirichlet operator, i.e., $T_t = e^{-tH}, t \geq 0$. Then the following results hold:*

(a) H is essentially self-adjoint on $\mathcal{H}_{fin}$.

(b) H is independent of the normalized admissible function f and the CONS $\{g_n\} \subset \mathfrak{h}_0$ we have chosen.

(c) The zero is a simple eigenvalue of H with eigenvector ξ_0. Moreover $(0, \alpha^{-1/2} - \alpha^{1/2}) \cap \sigma(H) = \emptyset$. Thus $\{T_t\}$ is ergodic.

By the spectral theorem, Theorem 3.2 (c) implies that for any $\xi \in \mathcal{H}$ and $t \geq 0$,

$$\|T_t\xi - \langle \xi_0, \xi \rangle \xi_0\|_{\mathcal{H}} \leq e^{-mt} \|\xi - \langle \xi_0, \xi \rangle \xi_0\|_{\mathcal{H}} \tag{3.11}$$

where $m = \alpha^{-1/2} - \alpha^{1/2}$. Thus $\{T_t\}_{t \geq 0}$ converges to the equilibrium exponentially fast. Before closing this section, we would like to make the following remark:

Remark 3.1 *The main results in this paper can be generalized in several ways. For an instance, let B be the operator on $\mathfrak{h}$ defined in (1.4). If one replace g_n by $B^{\lambda} g_n$, $n \in \mathbb{N}$, for some $\lambda \in [-\frac{1}{2}, \infty)$ in the definition of $\mathcal{E}(\eta, \xi)$ in (3.9) and (3.10), and modify Assumption 3.1 (d) appropriately, then all of the results in this section still hold with a modified spectral gap in Theorem 3.2 (c), i.e. $(0, (\alpha^{-1/2} - \alpha^{1/2})^{1+2\lambda}) \cap \sigma(H) = \emptyset$.*

4 Markovianity of Forms : Proofs of Proposition 3.1 - Theorem 3.1

In this section, we produce the proofs of Proposition 3.1, Proposition 3.2 and Theorem 3.1. The most difficult part is the proof of Proposition 3.2 which states the Dirichlet property of each component $\mathcal{E}^{(n)}$, $n \in \mathbb{N}$, of $\mathcal{E}$ in (3.10). In Theorem 2.1, we established the Dirichlet property for any σ_t-analytic element $x \in \mathcal{M}$. However $a(g_n)$ in the definition of $\mathcal{E}^{(n)}$ in (3.10) is an unbounded operator affiliated with $\mathcal{M}$ for any $g_n \in \mathfrak{h}_0$. Thus we have to employ several limiting processes which make the paper lengthy.

Recall the definitions of $\mathcal{W}$, $\mathcal{W}_0$, and $\mathcal{M}_0$ in (3.7) and $\mathcal{H}_{fin}$ in the below of (3.7). In the rest of this paper, we denote by $a^{\#}(f)$ either $a(f)$ or $a^*(f)$, $f \in \mathfrak{h}_0$. We first state elementary facts.

Lemma 4.1 *(a) $\mathcal{W}\xi_0$, $\mathcal{W}_0\xi_0$, $\mathcal{M}_0\xi_0$ and $\mathcal{H}_{fin}$ are dense in $\mathcal{H}$.*

(b) The inclusions

$$\mathcal{W}\xi_0 \subset D(a^{\#}(f)), \quad \mathcal{W}_0\xi_0 \subset D(a^{\#}(f)) \text{ and } \quad \mathcal{H}_{fin} \subset D(a^{\#}(f))$$

hold for any $f \in \mathfrak{h}_0$.

(c) The inclusion

$$\mathcal{M}_0\xi_0 \subset D(a^{\#}(f))$$

holds for any $f \in \mathfrak{h}_0$. Moreover the relation

$$a^{\#}(f)B\xi_0 = j(\sigma_{-i/2}(B^*))a^{\#}(f)\xi_0$$

holds for any $f \in \mathfrak{h}_0$ and $B \in \mathcal{M}_0$.

Proof. (a) Since $\mathcal{W}$ is norm-dense in $\pi_\omega(\mathcal{A}(\mathfrak{h}_0))$, $\mathcal{W}\xi_0$ is dense in $\mathcal{H}$. Let $f \in \mathfrak{h}_0$ be given. Using (3.6) and (3.2) it is easy to show that the sequence $\{W_n(f)\xi_0\}$ converges to $W(f)\xi_0$. This implies that $\mathcal{W}_0\xi_0$ is dense in $\mathcal{H}$. Since $\mathcal{W}_0\xi_0 \subset \mathcal{M}_0\xi_0$, $\mathcal{M}_0\xi_0$ is dense in $\mathcal{H}$. It follows from (3.2) and Assumption 3.1 that ω is an entire analytic state [BR] and so the cyclic vector ξ_0 is an entire analytic vector for $\Phi(f)$, $f \in \mathfrak{h}_0$. See also Section 5.2.3 of [BR]. Thus the sequence of the vectors

$$\sum_{k=0}^{n} \frac{i^k}{k!}\Phi(f)^k\xi_0$$

converge to $W(f)\xi_0$, which implies that $\mathcal{H}_{fin}$ is dense in $\mathcal{H}$.

(b) For any $f \in \mathfrak{h}_0$ and $n \in \mathbb{N}$, we write that

$$\Phi(f,n) = -in\Big(W(\frac{1}{n}f) - 1\Big).$$

For given $g \in \mathfrak{h}_0$, consider the sequence $\{\Phi(f,n)W(g)\xi_0\}$. Using (3.1) and (3.2), it can be checked that the sequence is a Cauchy sequence. Thus $W(g)\xi_0 \in D(\Phi(f))$ for any $f, g \in \mathfrak{h}_0$, and so $\mathcal{W}\xi_0 \subset D(\Phi(f))$, $f \in \mathfrak{h}_0$. By (3.4), this implies that $\mathcal{W}\xi_0 \subset D(a^{\#}(f))$, $f \in \mathfrak{h}_0$. The method similar to that used in the above implies that $\mathcal{W}_0\xi_0 \subset D(a^{\#}(f))$, $f \in \mathfrak{h}_0$. Since ξ_0 is an entire analytic vector for $\Phi(f)$, $f \in \mathfrak{h}_0$, $\mathcal{H}_{fin} \subset D(a^{\#}(f))$, $f \in \mathfrak{h}_0$.

(c) Notice that for any $B \in \mathcal{M}_0$, $j(\sigma_{-i/2}(B^*))\xi_0 = B\xi_0$. Thus we have that

$$\begin{aligned}\Phi(f,n)B\xi_0 &= \Phi(f,n)j(\sigma_{-i/2}(B^*))\xi_0 \\ &= j(\sigma_{-i/2}(B^*))\Phi(f,n)\xi_0.\end{aligned}$$

Since the sequence $\{\Phi(f,n)\xi_0\}$ converges to $\Phi(f)\xi_0$, it follows that $B\xi_0 \in D(\Phi(f))$ and the relation

$$\Phi(f)B\xi_0 = j(\sigma_{-i/2}(B^*))\Phi(f)\xi_0$$

hold for any $f \in \mathfrak{h}_0$ and $B \in \mathcal{M}_0$. By (3.4), this proved the part (c). □

We next state a well-known formula ((4.2)) on quasi-free expectations which we will use repeatedly in the sequel.

Lemma 4.2 *Let ω be the quasi-free state given in (3.2). The equalities*

$$\omega(\Phi(f)\Phi(g)) = \frac{1}{2}\Big((f,(\mathbf{1}-A)^{-1}g) + (g,A(\mathbf{1}-A)^{-1}f)\Big) \tag{4.1}$$

and

$$\omega\left(\Phi(f)\left(\prod_{j=1}^{m}\Phi(g_j)\right)\left(\prod_{l=1}^{n}W(h_l)\right)\right) \tag{4.2}$$
$$=\sum_{k=1}^{m}\omega(\Phi(f)\Phi(g_k))\omega\left(\Big(\prod_{j=1}^{k-1}\Phi(g_j)\Big)\Big(\prod_{j=k+1}^{m}\Phi(g_j)\Big)\Big(\prod_{l=1}^{n}W(h_l)\Big)\right)$$
$$+i\sum_{k=1}^{n}\omega(\Phi(f)\Phi(h_k))\omega\left(\Big(\prod_{j=1}^{m}\Phi(g_j)\Big)\Big(\prod_{l=1}^{n}W(h_l)\Big)\right)$$

hold for any $f, g, g_j, h_l \in \mathfrak{h}_0$, $j = 1, \cdots, m$, $l = 1, \cdots, n$.

Proof. The Weyl relations in (3.1) and the quasi-free expectation in (3.2) yield

$$\omega(W(f)W(g)) = \exp\{-\frac{i}{2}\mathrm{Im}(f,g) - \frac{1}{4}(f+g, D(f+g))\}$$

for any $f, g \in \mathfrak{h}_0$, where $D = (\mathbf{1}+A)(\mathbf{1}-A)^{-1}$. Replacing f and g by tf and sg respectively in the above, and differentiating both sides with respect to $t \in \mathbb{R}$ and $s \in \mathbb{R}$ at the zero, one obtains (4.1).

Again, replacing f by tf in the above relation and differentiating with respect to $t \in \mathbb{R}$ at the zero, one can deduce that

$$\omega(\Phi(f)W(g)) = i\omega(\Phi(f)\Phi(g))\omega(W(g)).$$

Setting $g = g_1 + \cdots + g_p$ in the above and using the relation in (3.1), one obtains

$$\omega(\Phi(f)(\prod_{j=1}^{p}W(g_j))) = i(\sum_{k=1}^{p}\omega(\Phi(f)\Phi(g_k)))\omega(\prod_{j=1}^{p}W(g_j)).$$

Choose $p = m + n$. Replacing g_j by $t_j g_j$, $j = 1, \cdots, m$ and g_{m+l} by h_l, $l = 1, \cdots, n$, in the above, and differentiating with respect to t_j, $j = 1, \cdots, m$, at the zero, we obtain (4.2). □

We remark that $f \mapsto a(f)$ is conjugate linear and $f \mapsto a^*(f)$ is linear. See (3.4), (3.1) and (3.2). Recall the definitions of $\sigma_z(a^{\#}(f))$, $f \in \mathfrak{h}_0$, $z \in \mathbb{C}$ in (3.8), i.e.,

$$\begin{aligned}\sigma_z(a(f)) &:= a(A^{i\overline{z}}f),\\ \sigma_z(a^*(f)) &:= a^*(A^{iz}f).\end{aligned}$$

We also write that

$$\sigma_z(\Phi(f)) := \frac{1}{\sqrt{2}}\{a(A^{i\overline{z}}f) + a^*(A^{iz}f)\} \tag{4.3}$$

for any $z \in \mathbb{C}$ and $f \in \mathfrak{h}_0$. Using CCRs in (3.5), one can check that for any $f, g \in \mathfrak{h}_0$ the relations

$$\begin{aligned} [a(f), W(g)] &= \frac{i}{\sqrt{2}}(f,g)W(g), \\ [a^*(f), W(g)] &= -\frac{i}{\sqrt{2}}(g,f)W(g), \end{aligned} \tag{4.4}$$

hold on $\mathcal{W}\xi_0$ (also on $\mathcal{W}_0\xi_0$). See also the proof of Proposition 5.2.4.(1) of [BR].

For any $f \in \mathfrak{h}_0$ and $m, n \in \mathbb{N}$, we write that

$$\begin{aligned} \Phi(f,n) &:= -in\{\exp(\frac{i}{n}\Phi(f)) - 1\}, \\ \Phi_m(f) &:= \sqrt{\frac{m}{\pi}} \int \sigma_t(\Phi(f)) e^{-mt^2}\, dt, \\ \Phi_m(f,n) &:= \sqrt{\frac{m}{\pi}} \int \sigma_t(\Phi(f,n)) e^{-mt^2}\, dt. \end{aligned} \tag{4.5}$$

Notice that for any $f \in \mathfrak{h}_0$ and $m, n \in \mathbb{N}$, $\Phi_m(f,n)$ is an entire analytic element for σ_t and $\sigma_z(\Phi_m(f,n))$ is given by

$$\sigma_z(\Phi_m(f,n)) = \sqrt{\frac{m}{\pi}} \int \sigma_t(\Phi(f,n)) e^{-m(t-z)^2}\, dt \tag{4.6}$$

for any $z \in \mathbb{C}$. See the proof of Proposition 2.5.22 of [BR].

We establish technical lemmas which will be used later.

Lemma 4.3 *(a) For any $f \in \mathfrak{h}_0$, $m, n \in \mathbb{N}$ and $z = t + is \in \mathbb{C}$, there exist constants $M_1(f,s)$ and $M_2(f,s)$ depending only on $f \in \mathfrak{h}_0$ and $s = Im\ z$ such that the bounds*

$$\begin{aligned} \|\sigma_z(\Phi_m(f,n))\xi_0\| &\leq M_1(f,s), \\ \|\sigma_z(\Phi(f))\xi_0\| &\leq M_2(f,s) \end{aligned}$$

hold (uniformly on $m, n \in \mathbb{N}$).

(b) For any $f \in \mathfrak{h}_0$, $n \in \mathbb{N}$ and $z \in \mathbb{C}$, the equality

$$\sigma_z(\Phi(f))\xi_0 = \lim_{n\to\infty} \sigma_z(\Phi_n(f,n))\xi_0$$

holds.

The proof of the above lemma will be given in Appendix.

We define that for any $f \in \mathfrak{h}_0$ and $m, n \in \mathbb{N}$

$$\begin{aligned} a_m(f,n) &:= \frac{1}{\sqrt{2}}\{\Phi_m(f,n) + i\Phi_m(if,n)\}, \\ a_m^*(f,n) &:= \frac{1}{\sqrt{2}}\{-\Phi_m(-f,n) + i\Phi_m(-if,n)\}. \end{aligned} \tag{4.7}$$

We denote by $a_m^\#(f,n)$ either $a_m(f,n)$ or $a_m^*(f,n)$. The following is a consequence of Lemma 4.3.

Corollary 4.1 *(a) For any $f \in \mathfrak{h}_0$, $m, n \in \mathbb{N}$ and $z = t + is \in \mathbb{C}$, there exist constants $M_3(f,s)$ and $M_4(f,s)$ depending only on $f \in \mathfrak{h}_0$ and $s = Im\ z$ such that the bounds*

$$\begin{aligned} \|\sigma_z(a_m^{\#}(f,n))\xi_0\| &\leq M_3(f,s), \\ \|\sigma_z(a^{\#}(f))\xi_0\| &\leq M_4(f,s) \end{aligned}$$

hold (uniformly on $m, n \in \mathbb{N}$).

(b) For any $f \in \mathfrak{h}_0$, $n \in \mathbb{N}$ and $z \in \mathbb{C}$, the equality

$$\sigma_z(a^{\#}(f))\xi_0 = \lim_{n\to\infty} \sigma_z(a_n^{\#}(f,n))\xi_0$$

holds.

Proof. The corollary follows from Lemma 4.3 and the definition of $a_m^{\#}(f,n)$ in (4.7). □

Lemma 4.4 *(a) For any $f \in \mathfrak{h}_0$, $m, n \in \mathbb{N}$, $z = t + is \in \mathbb{C}$ and $B \in \mathcal{M}_0$, there exist constants $M_5(B,f,s)$ and $M_6(B,f,s)$ such that the bounds*

$$\begin{aligned} \|\sigma_z(a_m^{\#}(f,n))B\xi_0\| &\leq M_5(B,f,s), \\ \|\sigma_z(a^{\#}(f))B\xi_0\| &\leq M_6(B,f,s) \end{aligned}$$

hold (uniformly on $m, n \in \mathbb{N}$).

(b) The equality

$$\sigma_z(a^{\#}(f))B\xi_0 = \lim_{n\to\infty} \sigma_z(a_n^{\#}(f,n))B\xi_0$$

holds for any $f \in \mathfrak{h}_0$, $z \in \mathbb{C}$ and $B \in \mathcal{M}_0$.

Proof. (a) As in the proof of Lemma 4.1 (c), one has that for any $m, n \in \mathbb{N}$, $B \in \mathcal{M}_0$ and $z \in \mathbb{C}$,

$$\sigma_z(a_m^{\#}(f,n))B\xi_0 = j(\sigma_{-i/2}(B^*))\sigma_z(a_m^{\#}(f,n))\xi_0. \tag{4.8}$$

Thus the bounds follow from Corollary 4.1 (a), Lemma 4.1 (c) and the above relations.

(b) This follows from Corollary 4.1 (b), (4.8) and Lemma 4.1 (c). □

In order to show the closability of the form $(\mathcal{E}^{(n)}, \mathcal{M}_0\xi_0)$, $n \in \mathbb{N}$, in Proposition 3.2, we will use the proposition listed below. The proposition is probably well known to the experts.

Proposition 4.1 *Let (Ω, μ) be a probability space, $\mathcal{H}$ a separable Hilbert space and D a dense subset in $\mathcal{H}$. For any $t \in \Omega$, let $A(t)$ be an operator defined on D satisfying the following properties:*

(a) $(A(t), D)$ is closable for each $t \in \Omega$,

(b) for any $\xi \in D$, $\Omega \ni t \mapsto \|A(t)\xi\|^2$ is an integrable function on Ω.

Then the quadratic form defined by

$$\begin{aligned} D(\mathcal{E}) &= D, \\ \mathcal{E}[\xi] &= \int_\Omega \|A(t)\xi\|^2 d\mu(t), \quad \xi \in D, \end{aligned}$$

is closable.

Proof. Let $\{\xi_n\}$ be a sequence in D such that $\xi_n \to 0$ as $n \to \infty$ and $\mathcal{E}[\xi_n - \xi_m] \to 0$ as $n, m \to \infty$. One has to show that $\mathcal{E}[\xi_n] \to 0$ as $n \to \infty$. Since $\{\xi_n\}$ is a $\mathcal{E}$-Cauchy sequence, one can choose a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\mathcal{E}[\xi_{n_{k+1}} - \xi_{n_k}] < \frac{1}{2^{2k}}$ for any $k \in \mathbb{N}$, which implies that

$$\sum_{k=1}^{\infty} \mathcal{E}[\xi_{n_{k+1}} - \xi_{n_k}]^{1/2} < \infty.$$

It follows from the Schwarz inequality and the above bound that

$$\sum_{k=1}^{\infty} \int_{\Omega} \|A(t)(\xi_{n_{k+1}} - \xi_{n_k})\| \, d\mu(t) < \infty.$$

The monotone convergence theorem and the above bound imply that

$$\sum_{k=1}^{\infty} \|A(t)(\xi_{n_{k+1}} - \xi_{n_k})\| < \infty, \quad \mu - a.e..$$

This implies that $\{A(t)\xi_{n_k}\}$ is a Cauchy sequence $\mu - a.e.$. Since $A(t)$ is closable on D for any $t \in \Omega$, we conclude that

$$A(t)\xi_{n_k} \to 0 \quad as\, k \to \infty, \mu - a.e..$$

Using Fatou's lemma and the above result, we obtain that

$$\begin{aligned} \mathcal{E}[\xi_n] &= \int \underline{\lim}_{k\to\infty} \|A(t)(\xi_n - \xi_{n_k})\|^2 \, d\mu(t) \\ &\leq \underline{\lim}_{k\to\infty} \mathcal{E}[\xi_n - \xi_{n_k}], \end{aligned}$$

which implies that $\mathcal{E}[\xi_n] \to 0$ as $n \to \infty$. This proved the proposition. □

We write that

$$\delta(a^{\#}(f)) := a^{\#}(f) - j(\sigma_{-i/2}((a^{\#}(f))^*)) \tag{4.9}$$

for $f \in \mathfrak{h}_0$. By Lemma 4.1, $\delta(a^{\#}(f))$ is well defined on $\mathcal{W}\xi_0$ and also on $\mathcal{M}_0\xi_0$ for any $f \in \mathfrak{h}_0$. Notice that for any $B \in \mathcal{M}_0$, $j(\sigma_{-i/2}(B^*))\xi_0 = B\xi_0$. Thus it follows from Lemma 4.1 (b) and Corollary 4.1 (b) that for any $f, h \in \mathfrak{h}_0$

$$\delta(a^{\#}(f))W(h)\xi_0 = [a^{\#}(f), W(h)]\xi_0. \tag{4.10}$$

Since $(\sigma_{-i/4}(a^{\#}(f)))^* = \sigma_{i/4}((a^{\#}(f))^*)$ by (3.8), we have that

$$\delta(\sigma_{-i/4}(a^{\#}(f))) = \sigma_{-i/4}(a^{\#}(f)) - j(\sigma_{-i/4}((a^{\#}(f))^*)) \tag{4.11}$$

for any $f \in \mathfrak{h}_0$.

We are ready to prove Proposition 3.1, Proposition 3.2 and Theorem 3.1. We first produce the proof of Proposition 3.1.

Proof of Proposition3.1. Let us first consider $(\mathcal{E}, \mathcal{W}\xi_0)$. Recall the definition of $\mathcal{E}(\eta, \xi)$ in (3.9)-(3.10). Let f be a normalized admissible function and let $\{g_n\} \subset \mathfrak{h}_0$ be a CONS for $\mathfrak{h}$. We first note that by (4.10) and (4.4)

$$\begin{aligned}
\delta(\sigma_{t-\frac{i}{4}}(a(g_n)))W(h)\xi_0 &= \delta(a(A^{it-\frac{1}{4}}g_n))W(h)\xi_0 \\
&= \frac{i}{\sqrt{2}}(A^{it-\frac{1}{4}}g_n, h)W(h)\xi_0, \\
\delta(\sigma_{t-\frac{i}{4}}(a^*(g_n)))W(h)\xi_0 &= -\frac{i}{\sqrt{2}}(h, A^{it+\frac{1}{4}}g_n)W(h)\xi_0.
\end{aligned}$$

It follows from (3.9), the above relations, the dominated convergence theorem and the Parserval relations that for g, $h \in \mathfrak{h}_0$

$$\begin{aligned}
&\mathcal{E}(W(g)\xi_0, W(h)\xi_0) \\
= &\Big\{\frac{1}{2}\sum_{n=1}^{\infty}\int (A^{-it-\frac{1}{4}}g, g_n)(g_n, A^{-it-\frac{1}{4}}h)f(t)\,dt\Big\}\langle W(g)\xi_0, W(h)\xi_0\rangle \\
&+\Big\{\frac{1}{2}\sum_{n=1}^{\infty}\int (A^{-it+\frac{1}{4}}h, g_n)(g_n, A^{-it+\frac{1}{4}}g)f(t)\,dt\Big\}\langle W(g)\xi_0, W(h)\xi_0\rangle \\
= &\frac{1}{2}\{(g, A^{-\frac{1}{2}}h) + (h, A^{\frac{1}{2}}g)\}\langle W(g)\xi_0, W(h)\xi_0\rangle. \qquad (4.12)
\end{aligned}$$

Here we have used the fact that by the Schwarz inequality and the Bessel inequality

$$\left|\sum_{n=1}^{m}(A^{-it}h_1, g_n)(g_n, A^{-it}h_2)\right| \le \|h_1\|\|h_2\|$$

for any $m \in \mathbb{N}$, $t \in \mathbb{R}$ and h_1, $h_2 \in \mathfrak{h}_0$. Thus $\mathcal{E}$ is well defined on $\mathcal{W}\xi_0$ and independent of the normalized admissible function f and the CONS $\{g_n\}$ chosen.

In order to show that the closability of $(\mathcal{E}, \mathcal{W}\xi_0)$, we introduce the operator $(H, \mathcal{W}\xi_0)$ defined by

$$\begin{aligned}
D(H) &= \mathcal{W}\xi_0, \\
HW(h)\xi_0 &= \frac{i}{\sqrt{2}}(\delta(a(A^{-\frac{1}{2}}h)))^*W(h)\xi_0 \qquad (4.13) \\
&- \frac{i}{\sqrt{2}}(\delta(a^*(A^{\frac{1}{2}}h)))^*W(h)\xi_0.
\end{aligned}$$

By (4.9) and Lemma 4.1 (b), H is well-defined on $\mathcal{W}\xi_0$. It follows from (4.10) and (4.4) that for any $g, h \in \mathfrak{h}_0$

$$\begin{aligned}
&\langle W(g)\xi_0, HW(h)\xi_0\rangle \\
&= \frac{i}{\sqrt{2}}\langle\delta(a(A^{-\frac{1}{2}}h))W(g)\xi_0, W(h)\xi_0\rangle \\
&-\frac{i}{\sqrt{2}}\langle\delta(a^*(A^{\frac{1}{2}}h))W(g)\xi_0, W(h)\xi_0\rangle \\
&= \frac{1}{2}\{(g, A^{-\frac{1}{2}}h) + (h, A^{\frac{1}{2}}g)\}\langle W(g)\xi_0, W(h)\xi_0\rangle. \qquad (4.14)
\end{aligned}$$

By the method used in the above, we get that for any $g, h \in \mathfrak{h}_0$, $\langle HW(g)\xi_0, W(h)\xi_0\rangle$ is equal to the right hand side of (4.14). Thus H is symmetric on $\mathcal{W}\xi_0$. It follows from (4.12) and (4.14) that

$$\langle \eta, H\xi\rangle = \mathcal{E}(\eta, \xi)$$

for $\eta, \xi \in \mathcal{W}\xi_0$. Since $(H, \mathcal{W}\xi_0)$ is a positive symmetric operator, the form $(\mathcal{E}, \mathcal{W}\xi_0)$ is closable (Theorem X.23 of [RS]).

Next, we consider $(\mathcal{E}, \mathcal{W}_0\xi_0)$. Employing the method similar to that used to derive (4.12), one can check that $(\mathcal{E}, \mathcal{W}_0\xi_0)$ is independent of admissible function f and the CONS $\{g_n\} \subset \mathfrak{h}_0$ chosen. To prove the closability, one may introduce the Dirichlet operator $(H, \mathcal{W}_0\xi_0)$ similar to that in (4.13) and then use the argument in the below of (4.13). We leave the details to the reader. $\square$

Remark 4.1 *For any CONS $\{g_n\} \subset \mathfrak{h}_0$, the relation*

$$\begin{aligned} \mathcal{E}(\eta, \xi) &= \sum_{n=1}^{\infty} \langle \delta(\sigma_{-i/4}(a(g_n)))\eta, \delta(\sigma_{-i/4}(a(g_n)))\xi\rangle \\ &\quad + \sum_{n=1}^{\infty} \langle \delta(\sigma_{-i/4}(a^*(g_n)))\eta, \delta(\sigma_{-i/4}(a^*(g_n)))\xi\rangle \end{aligned} \tag{4.15}$$

holds for any $\eta, \xi \in \mathcal{W}\xi_0$. In fact, the relation follows from the method used to derive (4.12).

In order to show Proposition 3.2, we introduce the following forms: For given normalized admissible function f and $g \in \mathfrak{h}_0$, let $(\widetilde{\mathcal{E}}, \mathcal{M}_0\xi_0)$ be a sesquilinear form defined by

$$\begin{aligned} D(\widetilde{\mathcal{E}}) &= \mathcal{M}_0\xi_0, \\ \widetilde{\mathcal{E}}(\eta, \xi) &= \int \langle \delta(\sigma_{t-\frac{i}{4}}(a(g)))\eta, \delta(\sigma_{t-\frac{i}{4}}(a(g)))\xi\rangle f(t)dt \\ &\quad + \int \langle \delta(\sigma_{t-\frac{i}{4}}(a^*(g)))\eta, \delta(\sigma_{t-\frac{i}{4}}(a^*(g)))\xi\rangle f(t)dt \end{aligned} \tag{4.16}$$

where for any $h \in \mathfrak{h}_0$, $\delta(a^{\#}(h))$ has been defined in (4.9). For any $n \in \mathbb{N}$, let $(\widetilde{\mathcal{E}}_n, \mathcal{H})$ be the form given by

$$\begin{aligned} \widetilde{\mathcal{E}}_n(\eta, \xi) &= \int \langle \delta(\sigma_{t-\frac{i}{4}}(a_n(g,n)))\eta, \delta(\sigma_{t-\frac{i}{4}}(a_n(g,n)))\xi\rangle f(t)dt \\ &\quad + \int \langle \delta(\sigma_{t-\frac{i}{4}}(a_n^*(g,n)))\eta, \delta(\sigma_{t-\frac{i}{4}}(a_n^*(g,n)))\xi\rangle f(t)dt \end{aligned} \tag{4.17}$$

where for $n \in \mathbb{N}$ and $g \in \mathfrak{h}_0$, $a_n^{\#}(g,n)$ has been defined in (4.7). Since $a_n^{\#}(g,n) \in \mathcal{M}_0$, it follows from Theorem 2.1 that $(\widetilde{\mathcal{E}}_n, \mathcal{H})$ is a Dirichlet form for each $n \in \mathbb{N}$.

Lemma 4.5 *For any η, $\xi \in \mathcal{M}_0\xi_0$,*

$$\widetilde{\mathcal{E}}(\eta, \xi) = \lim_{n\to\infty} \widetilde{\mathcal{E}}_n(\eta, \xi).$$

Proof. The lemma follows from Lemma 4.4 and the dominated convergence theorem. □

Proposition 4.2 *The form $(\widetilde{\mathcal{E}}, \mathcal{M}_0\xi_0)$ defined in (4.16) is closable. Denote by $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ the closure of $(\widetilde{\mathcal{E}}, \mathcal{M}_0\xi_0)$ and by $\widetilde{H}$ the positive self -adjoint operator associated to $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$. Then the following properties hold :*

(a) $\xi_0 \in D(\widetilde{H})$ and $\widetilde{H}\xi_0 = 0$,

(b) $\widetilde{\mathcal{E}}$ is J-real,

(c) $\widetilde{\mathcal{E}}(\xi_+, \xi_-) \leq 0$ for any $\xi \in D(\widetilde{\mathcal{E}})^J$.

Furthermore the form $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ is a Dirichlet form.

Proof of Proposition 3.2. By setting $g = g_n$, $n \in \mathbb{N}$, the proposition follows from Proposition 4.2. □

Proof of Proposition 4.2. The closability of $(\widetilde{\mathcal{E}}, \mathcal{M}_0\xi_0)$ follows from Lemma 4.1 (c), Lemma 4.4 and Proposition 4.1.

(a) It follows from (4.16) that for $m, n \in \mathbb{N}$ and $h_1, h_2 \in \mathfrak{h}_0$

$$\widetilde{\mathcal{E}}(W_m(h_1)\xi_0, W_n(h_2)\xi_0) = \langle W_m(h_1)\xi_0, \widetilde{H}W_n(h_2)\xi_0\rangle,$$

where

$$\begin{aligned}\widetilde{H} &= \int \big(\delta(\sigma_{t-i/4}(a(g)))\big)^*\delta(\sigma_{t-i/4}(a(g)))\, f(t)\, dt \\ &\quad + \int \big(\delta(\sigma_{t-i/4}(a^*(g)))\big)^*\delta(\sigma_{t-i/4}(a^*(g)))\, f(t)\, dt.\end{aligned}$$

Using (4.10) and Lemma 4.4 (a), it is not hard to show that $\widetilde{H}W_n(h_2)\xi_0$ is defined as a vector in $\mathcal{H}$. Since $\mathcal{W}_0\xi_0 \subset \mathcal{M}_0\xi_0$, we conclude that $\mathcal{W}_0\xi_0 \subset D(\widetilde{H})$. By (4.10), we get that

$$\delta(\sigma_{t-\frac{i}{4}}(a^{\#}(g)))\xi_0 = 0$$

for any $t \in \mathbb{R}$ and $g \in \mathfrak{h}_0$, and so $\widetilde{H}\xi_0 = 0$.

(b) This follows from the method in the proof of the property (b) in Theorem 2.1 and the fact that $\mathcal{M}_0\xi_0$ is a form core.

(c) The proof of the property (c) is the hardest part in the proof. We have to employ several limiting processes. We assert that

$$\xi \in \mathcal{M}_0\xi_0 \cap \mathcal{H}^J \Rightarrow \xi_+, \xi_- \in D(\tilde{\mathcal{E}}) \quad \textit{and} \quad \tilde{\mathcal{E}}[\xi_\pm] = \lim_{m\to\infty} \tilde{\mathcal{E}}_m[\xi_\pm]. \tag{4.18}$$

Let us prove our assertion. Let s_+ and s_- be the projections onto the closure of $\mathcal{M}'\xi_+$ and $\mathcal{M}'\xi_-$, where s_+, $s_- \in \mathcal{M}$. See [Ara]. For $\xi = A\xi_0$, $A \in \mathcal{M}_0$, we write that

$$\xi_{n,\pm} = (s_\pm A)_n\xi_0$$

where

$$(s_\pm A)_n = \sqrt{\frac{n}{\pi}} \int \sigma_t(s_\pm A) e^{-nt^2}\, dt.$$

Notice that $\|(s_\pm A)_n\| \leq \|A\|$, $n \in \mathbb{N}$, and that for any $\eta \in \mathcal{H}$, $(s_\pm A)_n \eta \to s_\pm A\eta$ as $n \to \infty$. See, i.e., the proof of Proposition 2.5.22 of [BR]. Since $\Delta^{it}\mathcal{P} \subset \mathcal{P}$ for $t \in \mathbb{R}$, $\xi_{n,\pm} = (s_\pm A)_n \xi_0 = j((s_\pm A)_n)\xi_0$, and so

$$\sigma_z(a^\#(g))\xi_{n,\pm} = j((s_\pm A)_n)\sigma_z(a^\#(g))\xi_0$$

for any $z \in \mathbb{C}$. Since $\sigma_z(a^\#(g))$ is a closed operator for any $g \in \mathfrak{h}_0$, $z \in \mathbb{C}$, we take n to infinity to conclude that $\xi_\pm \in D(\sigma_z(a^\#(g)))$, and

$$\sigma_z(a^\#(g))\xi_\pm = j(s_\pm A)\sigma_z(a^\#(g))\xi_0,$$

and so

$$\begin{aligned} & \delta(\sigma_z(a^\#(g)))\xi_\pm \\ = \;& j(s_\pm A)\sigma_z(a^\#(g))\xi_0 - (s_\pm A) j(\sigma_{-\frac{i}{2}}(\sigma_z(a^\#(g)))^*)\xi_0. \end{aligned} \tag{4.19}$$

Thus, it follows from Corollary 4.1 (a) and the dominated convergence theorem that $\widetilde{\mathcal{E}}[\xi_{m,\pm} - \xi_{n,\pm}] \to 0$ as $m, n \to \infty$. Since $(\widetilde{\mathcal{E}}, D(\widetilde{\mathcal{E}}))$ is closed, $\xi_\pm \in D(\widetilde{\mathcal{E}})$. Next we will prove that $\widetilde{\mathcal{E}}_n[\xi_\pm]$ converges to $\widetilde{\mathcal{E}}[\xi_\pm]$ as n tends to infinity. Using (4.19), the analogous relation for $\sigma_z(a_n^\#(g,n))$, and Corollary 4.1 (a), we conclude that for $\xi = A\xi_0$, there exists a constant $M > 0$ such that the bounds

$$\|\delta(\sigma_{t-\frac{i}{4}}(a_n^\#(g,n)))\xi_\pm\| + \|\delta(\sigma_{t-\frac{i}{4}}(a^\#(g)))\xi_\pm\| \leq M$$

hold uniformly in $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Using the above bounds, (4.19) and Corollary 4.1, we get that

$$\begin{aligned} & |\widetilde{\mathcal{E}}_n[\xi_\pm] - \widetilde{\mathcal{E}}[\xi_\pm]| \\ & \leq 2M \int \|\{\delta(\sigma_{t-\frac{i}{4}}(a_n^\#(g,n))) - \delta(\sigma_{t-\frac{i}{4}}(a^\#(g)))\}\xi_\pm\| f(t)dt \\ & \leq M' \int \|\sigma_{t-\frac{i}{4}}(a_n^\#(g,n))\xi_0 - \sigma_{t-\frac{i}{4}}(a^\#(g))\xi_0\| f(t)dt \\ & \to 0 \;\text{ as }\; n \to \infty. \end{aligned}$$

Here we have used Corollary 4.1 (b) and the dominated convergence theorem to obtain the last conclusion in the above. This completes the proof of our assertion.

We turn to the proof of the property (c). A direct computation shows that

$$\|\xi_\pm - \eta_\pm\| \leq \|\xi - \eta\|, \quad \forall \xi, \eta \in \mathcal{H}^J. \tag{4.20}$$

See also the proof of Proposition 1.2 of [DL]. Since $|\xi| = \xi_+ + \xi_-$, the assertion (4.18) implies that

$$\xi \in \mathcal{M}_0\xi_0 \cap \mathcal{H}^J \Rightarrow |\xi| \in D(\widetilde{\mathcal{E}}) \quad \text{and} \quad \widetilde{\mathcal{E}}[|\xi|] = \lim_{n\to\infty} \widetilde{\mathcal{E}}_n[|\xi|]. \tag{4.21}$$

Let $\xi \in D(\widetilde{\mathcal{E}}) \cap \mathcal{H}^J$ be given. Choose $\{\xi_k\}$ in $\mathcal{M}_0\xi_0 \cap \mathcal{H}^J$ such that $\xi_k \to \xi$ and $\widetilde{\mathcal{E}}[\xi_k] \to \widetilde{\mathcal{E}}[\xi]$ as $k \to \infty$. By (4.20), $|\xi_k| \to |\xi|$ as $k \to \infty$. Notice that $\widetilde{\mathcal{E}}[|\xi|] \leq \widetilde{\mathcal{E}}[\xi]$ is equivalent to $\widetilde{\mathcal{E}}(\xi_+, \xi_-) \leq 0$ and that each $\widetilde{\mathcal{E}}_m$, $m \in \mathbb{N}$, satisfies the property (c) by Theorem 2.1. By the lower semi-continuity of $\widetilde{\mathcal{E}}$ and (4.21), we obtain that

$$\begin{aligned}
\widetilde{\mathcal{E}}[|\xi|] &\leq \underline{\lim}_{k\to\infty} \widetilde{\mathcal{E}}[|\xi_k|] \\
&= \underline{\lim}_{k\to\infty} (\lim_{n\to\infty} \widetilde{\mathcal{E}}_n[|\xi_k|]) \\
&\leq \underline{\lim}_{k\to\infty} (\lim_{n\to\infty} \widetilde{\mathcal{E}}_n[\xi_k]) \\
&= \underline{\lim}_{k\to\infty} \widetilde{\mathcal{E}}[\xi_k] \\
&= \widetilde{\mathcal{E}}[\xi].
\end{aligned}$$

Thus $|\xi| \in D(\widetilde{\mathcal{E}})$ and $\widetilde{\mathcal{E}}[|\xi|] \leq \widetilde{\mathcal{E}}[\xi]$. This completes the proof of the property (c).

Since $\widetilde{\mathcal{E}}(\xi, \xi_0) = 0$ for any $\xi \in D(\widetilde{\mathcal{E}})$, the properties (b) and (c) imply that $\widetilde{\mathcal{E}}$ is a Dirichlet form. □

Proof of Theorem 3.1. By Proposition 3.1, $\mathcal{W}_0\xi_0 \subset D(\overline{\mathcal{E}})$. Since each component $\overline{\mathcal{E}}_n$ is a Dirichlet form, it follows from Theorem 5.2 of [Cip] that $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is a Dirichlet form. □

5 Decomposition of Quasi-free Hilbert Space : Ergodicity

For given quasi-free state ω, we will decompose the Hilbert space $\mathcal{H} = \mathcal{H}_\omega$ into direct sum of $\mathcal{H}^{(m,n)}, m, n \in \mathbb{N} \cup \{0\}$, where $\mathcal{H}^{(m,n)}$ is the Hilbert space of m quasi-particles and n anti quasi-particles. We then use the result to show that the symmetric Markovian semigroup is ergodic.

Recall the definition of $\delta(a^{\#}(g)), g \in \mathfrak{h}_0$ in (4.9). Denote by B the operator given by

$$B = A^{-\frac{1}{2}} - A^{\frac{1}{2}}. \tag{5.1}$$

It follows from (4.9) that for $g \in \mathfrak{h}_0$

$$\begin{aligned}
\delta(a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)) &= a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g) - j(\sigma_{-i/2}(a^*(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g))) \\
&= a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g) - j(a^*(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)),
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
\delta(a^*(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)) &= a^*(B^{-\frac{1}{2}}A^{\frac{1}{4}}g) - j(\sigma_{-i/2}(a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g))) \\
&= a^*(B^{-\frac{1}{2}}A^{\frac{1}{4}}g) - j(a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)).
\end{aligned}$$

The above operators are well defined on $\mathcal{M}_0\xi_0$ and also on $\mathcal{H}_{fin}$. Since $(j(a(g)))^* = j(a^*(g))$, etc, a computation shows that

$$\begin{aligned}
(\delta(a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)))^* &= a^*(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g) - j(a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)) \\
&= a^*(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g) - j(\sigma_{-\frac{i}{2}}(a(B^{-\frac{1}{2}}A^{\frac{3}{4}}g))) \\
&= a^*(B^{\frac{1}{2}}A^{\frac{1}{4}}g) + \delta(a^*(B^{-\frac{1}{2}}A^{\frac{3}{4}}g)).
\end{aligned} \tag{5.3}$$

Here we have used the fact that $B^{-\frac{1}{2}}(A^{-\frac{1}{4}} - A^{\frac{3}{4}}) = B^{\frac{1}{2}}A^{\frac{1}{4}}$. Using the method similar to that used in the above, we get that

$$(\delta(a^*(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)))^* = -a(B^{\frac{1}{2}}A^{-\frac{1}{4}}g) + \delta((a(B^{-\frac{1}{2}}A^{-\frac{3}{4}}g)). \tag{5.4}$$

From notational brevity, we write that for $g \in \mathfrak{h}_0$

$$\begin{aligned} D_1(g) &:= \delta(a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)), \\ D_2(g) &:= \delta(a^*(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)). \end{aligned} \tag{5.5}$$

Then it follows from (5.3) and (5.4) that

$$\begin{aligned} D_1(g)^* &= a^*(B^{\frac{1}{2}}A^{\frac{1}{4}}g) + D_2(A^{\frac{1}{2}}g), \\ D_2(g)^* &= -a(B^{\frac{1}{2}}A^{-\frac{1}{4}}g) + D_1(A^{-\frac{1}{2}}g). \end{aligned} \tag{5.6}$$

We first collect some properties of $D_i(g)$ for $g \in \mathfrak{h}_0$ and $i = 1, 2$.

Lemma 5.1 *$D_i(g)\xi_0 = 0$ for any $g \in \mathfrak{h}_0$ and $i = 1, 2$.*

Proof. This follows from (5.5) and (4.10). □

Lemma 5.2 *As operators defined on $\mathcal{W}\xi_0$ and $\mathcal{W}_0\xi_0$, the following relations hold for any g, $h \in \mathfrak{h}_0$:*

(a) $[D_i(g), D_j(h)] = 0,\ i = 1, 2,\ j = 1, 2$

(b) $[D_1(g), a(h)] = 0,$

(c) $[D_2(g), a^*(h)] = 0,$

(d) $[D_1(g), a^*(B^{\frac{1}{2}}A^{\frac{1}{4}}h)] = (g, h)\mathbf{1},$

(e) $[D_2(g), a(B^{\frac{1}{2}}A^{-\frac{1}{4}}h)] = -(h, g)\mathbf{1}.$

Proof. We first remark that each operator in the commutators in the lemma is defined on $\mathcal{W}\xi_0$ and also $\mathcal{W}_0\xi_0$. (a) This follows from (4.10), (4.4) and the definition of $D_i(g)$, $i = 1, 2$, in (5.5). (b) and (c) follow from (5.5) and (4.9). (d) and (e) follow from (5.5), (4.9) and the CCRs in (3.5). □

Proposition 5.1 *As operators defined on $\mathcal{W}\xi_0$, $\mathcal{W}\xi_0$ and also on $\mathcal{H}_{fin}$, the following canonical commutation relations (CCRs) hold for any g, $h \in \mathfrak{h}_0$:*

(a) $[D_1(g), D_1(h)^*] = (g, h)\mathbf{1},$
$[D_1(g), D_1(h)] = 0, \quad [D_1(g)^*, D_2(h)^*] = 0,$

(b) $[D_2(g), D_2(h)^*] = (h, g)\mathbf{1}$,
$[D_2(g), D_2(h)] = 0, \quad [D_2(g)^*, D_2(h)^*] = 0$

(c) $[D_1(g), D_2(h)] = 0, \quad [D_1(g), D_2(h)^*] = 0,$
$[D_1(g)^*, D_2(h)] = 0, \quad [D_1(g)^*, D_2(h)^*] = 0.$

Proof. The commutation relations on $\mathcal{W}\xi_0$and $\mathcal{W}_0\xi_0$ in the proposition follow from Lemma 5.2 and (5.6). Thus we need to extend the relations to $\mathcal{H}_{fin}$. Recall the definition of $\Phi(f, n)$, $f \in \mathfrak{h}_0, n \in \mathbb{N}$, in (4.5). Using Lemma 4.2 and (3.2), it is not hard to check that for any $g, h, f_j, \in \mathfrak{h}_0$, $n_j \in \mathbb{N}$, $j = 1, ..., m$,

$$\Phi(g)\Phi(h)(\prod_{j=1}^{m} \Phi(f_j, n_j))\xi_0 \to \Phi(g)\Phi(h)(\prod_{j=1}^{m} \Phi(f_j))\xi_0 \tag{5.7}$$

as $n_j \to \infty$, $j = 1, ..., m$. This implies that the relations in the proposition extend to $\mathcal{H}_{fin}$. □

We are ready to decompose the Hilbert space $\mathcal{H} = \mathcal{H}_\omega$, called quasi-free Hilbert space. According to Lemma 5.1 and the CCRs in Proposition 5.1, $D_i(g)$ and $D_i(h)^*$, $g, h \in \mathfrak{h}_0$, $i = 1, 2$, can be thought as annihilation and creation operators respectively. We remark that $h \mapsto D_1(h)^*$ is linear, but $g \to D_2(g)^*$ is conjugate linear. With an abuse of terminology, we call $D_1(h)^*$ and $D_2(h)^*$ the creation operators for quasi-particles and anti quasi-particles respectively for $h \in \mathfrak{h}_0$. The following is the decomposition of $\mathcal{H}$:

Theorem 5.1 *The following decomposition holds:*

$$\mathcal{H} = \bigoplus_{m,n=0}^{\infty} \mathcal{H}^{(m,n)}$$

where for each $m, n \in \mathbb{N} \cup \{0\}$, $\mathcal{H}^{(m,n)}$ is the closure of the subspace spanned by the vectors of the form

$$(\prod_{j=1}^{m} D_1(g_j)^*)(\prod_{l=1}^{n} D_2(h_l)^*)\xi_0,\ g_j,\ h_l \in \mathfrak{h}_0.$$

In the case in which $m = 0$ $(n = 0)$, we replace the operator in the first(second) parenthesis in the above by the identity.

Proof. It follows from (5.6) that any $\Phi(g)$, $g \in \mathfrak{h}_0$, can be written as sum of four $D_i(h)^\#, h \in \mathfrak{h}_0$, $i = 1, 2$. Thus any $(\prod_{l=1}^{m} \Phi(g_l))\xi_0$, $g_l \in \mathfrak{h}_0$, $l = 1, \cdots, m$, can be expressed as a finite linear combination of the vectors of the form

$$(\prod_{j=1}^{p} D_1(g'_j)^\#)(\prod_{l=1}^{q} D_2(h'_l)^\#)\xi_0,\ g'_j,\ h'_l \in \mathfrak{h}_0,$$

where $D_i^{\#}(g)$ is either $D_i(g)$ or $D_i^*(g)$, $i = 1, 2$. As a consequence of Lemma 5.1 and the CCRs in Proposition 5.1, the above vector can be expressed as a finite linear combination of the vector of the form

$$(\prod_{j=1}^{m'} D_1(g_j')^*)(\prod_{l=1}^{n'} D_2(h_l')^*)\xi_0,\ g_j',\ h_l' \in \mathfrak{h}_0,\ m', n' \in \mathbb{N} \cup \{0\}.$$

Since $\mathcal{H}_{fin}$ is dense in $\mathcal{H}$, we conclude that the set of finite linear combinations of the vectors of the above form is dense in $\mathcal{H}$. Thus the decomposition follows from Lemma 5.1 and the CCRs in Proposition 5.1. □

Recall that $\mathfrak{h}_0$ is a dense subspace of a complex Hilbert space $\mathfrak{h}$. Let $\mathcal{F} = \mathcal{F}(\mathfrak{h})$ be the symmetric Fock space over $\mathfrak{h}$, and $a(g)$ and $a^*(g)$, $g \in \mathfrak{h}_0$, the annihilation and creation operator respectively. Denote by Ω the vacuum vector in $\mathcal{F}$. Let $C : \mathfrak{h} \to \mathfrak{h}$ be an anti-unitary operator. If $\mathfrak{h}$ is a L^2-space, one may consider that C is the complex conjugation. Denote by $\Gamma(C)$ the second quantization of C. See Section 5.2.1 of [BR]. Let $\mathcal{F}_1$, Ω_1, $a_1(g)$ and $a_1^*(g)$, $g \in \mathfrak{h}_0$ be the identical copies of $\mathcal{F}$, Ω, $a(g)$ and $a^*(g)$, $g \in \mathfrak{h}_0$ respectively. Notice that $\Gamma(C)a^{\#}(g)\Gamma(C)^{-1} = a^{\#}(Cg)$. We write that $\mathcal{F}_2 = \Gamma(C)\mathcal{F}(= \mathcal{F})$, $\Omega_2 = \Omega$, $a_2(g) = a(Cg)$, and $a_2^*(g) = a^*(Cg)$, $g \in \mathfrak{h}_0$. Then the following commutation relations hold : for g, $h \in \mathfrak{h}_0$,

$$\begin{aligned} &[a_2(g), a_2^*(h)] = (h, g)\mathbf{1}, \\ &[a_2(g), a_2(h)] = 0. \end{aligned} \tag{5.8}$$

One may compare the above relations to those in Proposition 5.1 (b).

Proposition 5.2 *Let U be the operator defined by*

$$U : \mathcal{H} \to \mathcal{F}_1 \otimes \mathcal{F}_2$$

$$(\prod_{j=1}^{m} D_1(g_j)^*)(\prod_{l=1}^{n} D_2(h_l)^*)\xi_0 \mapsto (\prod_{j=1}^{m} a_1^*(g_j))\Omega_1 \otimes (\prod_{l=1}^{n} a_2^*(h_l))\Omega_2$$

for g_j, $h_l \in \mathfrak{h}_0$, $j = 1, ..., m, l = 1, ..., n$. Then U is unitary.

Proof. Since $D_1^{\#}(g)$ and $a_1^{\#}(g)$, and $D_2^{\#}(g)$ and $a_2^{\#}(g)$ for $g \in \mathfrak{h}_0$ satisfy the same commutation relations respectively by Proposition 5.1 and (5.8), the unitarity of U follows from the fact that $a(g)\Omega = 0$ for any $g \in \mathfrak{h}_0$. □

We next turn to the spectral analysis of $\overline{H}$, where $\overline{H}$ is the generator of the symmetric Markovian semigroup $\{T_t\}_{t\geq 0}$ associated to the Dirichlet form $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$.

Let us first describe the basic idea of the proof Theorem 3.2. Recall the definitions of $D_1(g)$ and $D_2(g)$ in (5.5). Let $\{f_n\} \subset \mathfrak{h}_0$ be a CONS for $\mathfrak{h}$. By Remark 4.1 and (5.5), we have that

for any $g,\, h \in \mathfrak{h}_0$

$$\begin{aligned}
& \overline{\mathcal{E}}(W(g)\xi_0, W(h)\xi_0) \\
= & \sum_{n=1}^{\infty} \langle D_1(B^{\frac{1}{2}} f_n) W(g)\xi_0, D_1(B^{\frac{1}{2}} f_n) W(h)\xi_0 \rangle \\
& + \sum_{n=1}^{\infty} \langle D_2(B^{\frac{1}{2}} f_n) W(g)\xi_0, D_2(B^{\frac{1}{2}} f_n) W(h)\xi_0 \rangle \\
= & \langle W(g)\xi_0, HW(h)\xi_0 \rangle,
\end{aligned}$$

where

$$H = \sum_{n=1}^{\infty} \{ D_1^*(B^{\frac{1}{2}} f_n) D_1(B^{\frac{1}{2}} f_n) + D_2^*(B^{\frac{1}{2}} f_n) D_2(B^{\frac{1}{2}} f_n) \} \tag{5.9}$$

as a bilinear form on $\mathcal{W}\xi_0 \times \mathcal{W}\xi_0$. By Proposition 5.2, one can see that H is unitary equivalent to sum of two second quantizations of B. If one can show that $\mathcal{H}_{fin} \subset D(\overline{H})$, $\overline{H} = H$ on $\mathcal{H}_{fin}$, and that $\mathcal{H}_{fin}$ is a core for $\overline{H}$, then one expect that spectrum of $\overline{H}$ can be analyzed completely.

We first establish technical lemmas. Recall that $\mathcal{W}_0\xi_0 \subset D(\overline{\mathcal{E}})$.

Lemma 5.3 $\mathcal{W}\xi_0 \subset D(\overline{\mathcal{E}})$

Proof. Since $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is closed and $\mathcal{W}_0\xi_0 \subset D(\mathcal{E})$, it is sufficient to show that for any $g \in \mathfrak{h}_0$

$$\overline{\mathcal{E}}[W_n(g)\xi_0 - W(g)\xi_0] \to 0 \quad \text{as} \quad n \to \infty.$$

Notice that by Remark 4.1

$$\begin{aligned}
\overline{\mathcal{E}}[(W_n(g) - W(g))\xi_0] &= \sum_{m=1}^{\infty} \int \|\delta(\sigma_{\tau - i/4}(a(g_m)))(W_n(g) - W(g))\xi_0\|^2 f(\tau) d\tau \\
&\quad + \sum_{m=1}^{\infty} \int \|\delta(\sigma_{\tau - i/4}(a^*(g_m)))(W_n(g) - W(g))\xi_0\|^2 f(\tau) d\tau \\
&\equiv A_n^{(1)} + A_n^{(2)}.
\end{aligned}$$

Employing the method similar to that used to derive (4.12), it is easy to show that

$$A_n^{(1)} = \frac{n}{2\pi} \int \int K(t_1, t_2, g) e^{-nt_1^2} e^{-nt_2^2} dt_1 dt_2$$

where

$$\begin{aligned}
K(t_1, t_2, g) &= (A^{it_1} g, (A^{it_2} - \mathbf{1}) A^{-1/2} g) \langle W(A^{it_1} g)\xi_0, W(A^{it_2} g)\xi_0 \rangle \\
&\quad + (A^{it_1} g, A^{-1/2} g) \langle W(A^{it_1} g)\xi_0, (W(A^{it_2} g) - W(g))\xi_0 \rangle \\
&\quad + (g, A^{-1/2} A^{it_2} g) \langle W(g)\xi_0, (W(g) - W(A^{it_2} g))\xi_0 \rangle \\
&\quad + (g, A^{-1/2}(\mathbf{1} - A^{it_2}) g) \langle W(g)\xi_0, W(g)\xi_0 \rangle.
\end{aligned}$$

Notice that $K(t_1, t_2, g)$ is bounded uniformly with respect to $(t_1, t_2) \in \mathbb{R}^2$. Changing the variables ($t_1' = n^{1/2}t_1$, $t_2' = n^{1/2}t_2$) and using the dominated convergence theorem, we conclude that

$$A_n^{(1)} \to 0 \quad \text{as} \quad n \to \infty.$$

By the similar calculation, we get that

$$A_n^{(2)} \to 0 \quad \text{as} \quad n \to \infty.$$

This proved the lemma completely. □

Lemma 5.4 *(a)* $\mathcal{H}_{fin} \subset D(\overline{\mathcal{E}})$.

(b)Let $\{g_n\} \subset \mathfrak{h}_0$ *be a CONS for* $\mathfrak{h}$. *Then for* $\xi \in \mathcal{H}_{fin}$ *the equality*

$$\overline{\mathcal{E}}[\xi] = \sum_{n=1}^{\infty} \{\|D_1(B^{\frac{1}{2}}g_n)\xi\|^2 + \|D_2(B^{\frac{1}{2}}g_n)\xi\|^2\} \tag{5.10}$$

holds.

The proof of the above lemma will be given in Appendix.

The following is one of main results in this section:

Theorem 5.2 *(a) Let* $H : \mathcal{H}_{fin} \to \mathcal{H}$ *be the operator defined by*

$$\begin{aligned} & H \prod_{p=1}^{m} D_1(g_p)^* \prod_{q=1}^{n} D_2(h_q)^* \xi_0 \qquad (5.11) \\ = & \sum_{k=1}^{m} \prod_{p=1}^{k-1} D_1(g_p)^* D_1(Bg_k)^* \prod_{p=k+1}^{m} D_1(g_p)^* \prod_{q=1}^{n} D_2(h_q)^* \xi_0 \\ & + \sum_{k=1}^{n} \prod_{p=1}^{m} D_1(g_p)^* \prod_{q=1}^{k-1} D_2(h_q)^* D_2(Bh_k)^* \prod_{q=k+1}^{n} D_2(h_q)^* \xi_0 \end{aligned}$$

for any $m, n \in \mathbb{N} \cup \{0\}$ *and* $g_p, h_q \in \mathfrak{h}_0$, $p = 1, ..., m, q = 1, ..., n$. *Then the relation*

$$\overline{\mathcal{E}}(\eta, \xi) = \langle \eta, H\xi \rangle$$

holds for any $\eta, \xi \in \mathcal{H}_{fin}$.

(b) H *is essential self-adjoint and the self-adjoint extension denoted by* H *again is equal to the Dirichlet operator* $\overline{H}$.

Proof. (a) Let $(\mathcal{E}_1, \mathcal{H}_{fin})$ be the form given by

$$\mathcal{E}_1[\xi] = \sum_{k=1}^{\infty} \|D_1(B^{\frac{1}{2}}f_k)\xi\|^2,$$

where $\{f_k\}$, $f_k \in \mathfrak{h}_0$ is a CONS for $\mathfrak{h}$. We write $H = H_1 + H_2$, where the image under H_1(resp.H_2) is defined by the first(resp. second) vector in the right hand side of (5.11). The CCRs in Proposition 5.1 (a) and Lemma 5.1 imply that

$$\begin{aligned}&\mathcal{E}_1[(\prod_{j=1}^{m} D_1(g_j)^*)\xi_0]\\ &= \sum_{k=1}^{\infty}\sum_{p=1}^{m}\sum_{q=1}^{m}(B^{\frac{1}{2}}g_p, f_k)(f_k, B^{\frac{1}{2}}g_q)G(g_1, ..., g_m; p, q)\end{aligned}$$

where

$$\begin{aligned}&G(g_1, ..., g_m; p, q)\\ &:= \Big\langle \prod_{\tau=1}^{p-1} D_1(g_\tau)^* \prod_{\tau=p+1}^{m} D_1(g_\tau)^*\xi_0\,,\, \prod_{\tau=1}^{q-1} D_1(g_\tau)^* \prod_{\tau=q+1}^{m} D_1(g_\tau)^*\xi_0\Big\rangle.\end{aligned}$$

Using the Parserval relations and the fact that

$$\begin{aligned}&\sum_{p=1}^{m}(g_p, Bg_q)G(g_1, ..., g_m; p, q)\\ &= \Big\langle D_1(Bg_q)\prod_{k=1}^{m} D_1(g_k)^*\xi_0\,,\, \prod_{k=1}^{q-1} D_1(g_q)^* \prod_{k=q+1}^{m} D_1(g_k)^*\xi_0\Big\rangle,\end{aligned}$$

we have that

$$\begin{aligned}&\mathcal{E}_1\left[\prod_{j=1}^{m} D_1(g_j)^*)\xi_0\right]\\ &= \sum_{q=1}^{m}\Big\langle \prod_{j=1}^{m} D_1(g_j)^*\xi_0, \prod_{j=1}^{q-1} D_1(g_j)^* D_1(Bg_q)^* \prod_{j=q+1}^{m} D_1(g_j)^*\xi_0\Big\rangle\\ &= \Big\langle \prod_{j=1}^{m} D_1(g_j)^*\xi_0\,,\, H_1\prod_{j=1}^{m} D_1(g_j)^*\xi_0\Big\rangle.\end{aligned}$$

Notice that $H_2 = H - H_1$ commutes with $D_1(g)^*$ for any $g \in \mathfrak{h}_0$ by (5.11). Thus by the polarization identity, we proved that

$$\mathcal{E}_1(\eta, \xi) = \langle \eta, H_1\xi\rangle$$

for any η, $\xi \in \mathcal{H}_{fin}$. The method similar to that used in the above implies that

$$\mathcal{E}_2(\eta, \xi) = \langle \eta, H_2\xi\rangle$$

for any $\eta, \xi \in \mathcal{H}_{fin}$.This proved the part (a) of the theorem.

(b) By Proposition 5.2, we have that

$$UHU^{-1} = d\Gamma_1(B) \otimes \mathbf{1} + \mathbf{1} \otimes d\Gamma_2(B), \tag{5.12}$$

where each i, $i = 1, 2$, $d\Gamma_i(B)$ is the second quantization of B on $\mathcal{F}_i$. We remark that $d\Gamma_2(B)$ is anti-unitary equivalent to $d\Gamma_1(B)$. By Assumption 3.1, any $g \in \mathfrak{h}_0$ is an analytic vector for B, and so it is easy to check that $(\prod_{j=1}^m a^*(g_j))\Omega$ is an analytic vector for $d\Gamma(B)$ for any $g_j \in \mathfrak{h}_0$, $j = 1, ..., m$. Thus it follows that any $\xi \in \mathcal{H}_{fin}$ is an analytic vector for H. Since $H\mathcal{H}_{fin} \subset \mathcal{H}_{fin}$ by (5.11) and since $H = \overline{H}$ on $\mathcal{H}_{fin}$ by the part (a) of the theorem, it follows from Corollary 2 of Theorem X. 39 in [RS] that H and $\overline{H}$ are essentially self-adjoint on $\mathcal{H}_{fin}$, and so $H = \overline{H}$. □

Finally we are able to produce the proof of Theorem 3.2.

Proof of Theorem 3.2. (a) and (b) follow from Theorem 5.2.

To show (c), recall $0 < A \leq \alpha \mathbf{1}$, $0 < \alpha < 1$. Since $B = A^{-\frac{1}{2}} - A^{\frac{1}{2}}$,

$$\inf \sigma(B) \geq \alpha^{-1/2} - \alpha^{1/2}.$$

It follows from the above lower bound and (5.12) that zero is a simple eigenvalue with eigenvector ξ_0 and

$$\inf(\sigma(H) - \{0\}) \geq \alpha^{-1/2} - \alpha^{1/2}.$$

Thus the Markovian semigroup $\{T_t\}_{t \geq 0}$ is ergodic. This completes the proof of Theorem 3.2. □

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A Appendix : Proofs of Lemma 4.3 and Lemma 5.4

Proof of Lemma 4.3. (a) Let us prove the first bound. Recall the definition of $\Phi(f, n)$, $f \in \mathfrak{h}_0$, $n \in \mathbb{N}$, in (4.5). Notice that

$$\Phi(f, n) = \int_0^1 W(\frac{s}{n} f)\Phi(f) ds. \tag{A.1}$$

It follows from (4.6) and (A.1) that

$$\begin{aligned} &\|\sigma_z(\Phi_m(f, n))\xi_0\|^2 \\ &\quad = (\frac{m}{\pi}) \int \int \left(\int_0^1 \int_0^1 F(t_1, t_2, s_1, s_2; f, n)\, ds_1 ds_2 \right) \\ &\qquad \cdot e^{-m(t_1 - \overline{z})^2} e^{-m(t_2 - z)^2}\, dt_1 dt_2, \end{aligned}$$

where

$$F(t_1,t_2,s_1,s_2;f,n)$$
$$:=\omega\Big(\sigma_{t_1}(\Phi(f))\sigma_{t_1}(W(-\frac{s_1}{n}f))\sigma_{t_2}(W(\frac{s_2}{n}f))\sigma_{t_2}(\Phi(f))\Big).$$

We use (4.4) and the fact that $\sigma_t(\Phi(f))=\Phi(A^{it}f)$, etc, to obtain that

$$\begin{aligned}&F(t_1,t_2,s_1,s_2;f,n)\\&=\omega\Big(\Phi(A^{it_1}f)\Phi(A^{it_2}f)W(-\frac{s_1}{n}A^{it_1}f)W(\frac{s_2}{n}A^{it_2}f)\Big)\\&\quad+\frac{s_1}{n}\mathrm{Im}\,(A^{it_1}f,A^{it_2}f)\omega\Big(\Phi(A^{it_1}f)W(-\frac{s_1}{n}A^{it_1}f)W(\frac{s_2}{n}A^{it_2}f)\Big).\end{aligned}$$

Iterating (4.2) and then using (4.1), (3.1) and (3.2), the above function can be calculated explicitely. More precisely, let us use (4.2) iteratively in the above to obtain that

$$F(t_1,t_2,s_1,s_2;f,n)=\sum_{k=1}^{7}F^{(k)}(t_1,t_2,s_1,s_2;f,n),$$

where

$$\begin{aligned}&F^{(1)}(t_1,t_2,s_1,s_2;f,n)\\&\quad=\omega\Big(\Phi(A^{it_1}f)\Phi(A^{it_2}f)\Big)\omega\Big(W(-\frac{s_1}{n}A^{it_1}f)W(\frac{s_2}{n}A^{it_2}f)\Big),\end{aligned}$$

and appropriate expressions for $F^{(k)}(t_1,t_2,s_1,s_2;f,n)$, $k=2,\cdots,7$. It follows from (4.1), (3.1) and (3.2) that

$$\begin{aligned}&F^{(1)}(t_1,t_2,s_1,s_2;f,n)\\&\quad=\frac{1}{2}\left((f,(\mathbf{1}-A)^{-1}A^{-it_1}A^{it_2}f)+(f,A(\mathbf{1}-A)^{-1}A^{it_1}A^{-it_2}f)\right)\\&\qquad\cdot\exp\{\frac{i}{2}\mathrm{Im}\,(f,\frac{s_1}{n}\frac{s_2}{n}A^{-it_1}A^{it_2}f)\}\\&\qquad\cdot\exp\{-\frac{1}{4}((\frac{s_2}{n}A^{it_2}-\frac{s_1}{n}A^{it_1})f,D(\frac{s_2}{n}A^{it_2}-\frac{s_1}{n}A^{it_1})f)\},\end{aligned}$$

where $D=(\mathbf{1}+A)(\mathbf{1}-A)^{-1}$. By Assumption 3.1, $F^{(1)}$ has an analytic extension on $\mathbb{C}\times\mathbb{C}$ in t_1 and t_2 variables. From the above expression, it is easy to see that there exists a constant $C_1(f,\mathrm{Im}\,z)$ independent of n such that

$$\sup_{s_1,s_2\in[0,1]}|F^{(1)}(t_1+\overline{z},t_2+z,s_1,s_2;f,n)|\leq C_1(f,\mathrm{Im}\,z)$$

for any $n\in\mathbb{N}$. Using the Cauchy integral theorem and the above bound, we conclude that the contribution of $F^{(1)}$ in F is bounded by $C_1(f,\mathrm{Im}\,z)$. Now, it is obvious that the function $F(t_1,t_2,s_1,s_2;f,n)$ has an analytic extension on $\mathbb{C}\times\mathbb{C}$ in t_1 and t_2 variables and there exists a

constant $M_1(f, \text{Im } z)$ depending only on $f \in \mathfrak{h}_0$ and Im z such that the bound similar to that in the above holds. We use the Cauchy integral theorem to conclude that

$$\begin{aligned}
&\|\sigma_z(\Phi_m(f,n))\xi_0\|^2 \\
&\quad = (\frac{m}{\pi}) \int\int \Big(\int_0^1 \int_0^1 F(t_1+\overline{z}, t_2+z, s_1, s_2; f, n)\, ds_1 ds_2 \Big) \\
&\qquad \cdot e^{-mt_1^2} e^{-mt_2^2}\, dt_1 dt_2 \\
&\quad \le M_1(f, \text{Im } z)^2.
\end{aligned}$$

We leave the details to the reader.

We next consider the second bound. It follows from (4.1) that

$$\begin{aligned}
\omega(a^*(f)a(g)) &= (g, A(\mathbf{1}-A)^{-1}f), \\
\omega(a(g)a^*(f)) &= (g, (\mathbf{1}-A)^{-1}f).
\end{aligned} \tag{A.2}$$

A direct computation yields that

$$\begin{aligned}
&\|\sigma_z(\Phi(f))\xi_0\|^2 \\
&\quad = \frac{1}{2}\Big(\omega(a^*(A^{i\overline{z}}f)a(A^{i\overline{z}}f)) + \omega(a(A^{iz}f)a^*(A^{iz}f))\Big).
\end{aligned}$$

Thus the bound follows from (A.2) and the above result.

(b) Notice that

$$\begin{aligned}
&\|\sigma_z(\Phi_n(f,n))\xi_0 - \sigma_z(\Phi(f))\xi_0\| \\
&\quad \le \|\sigma_z(\Phi_n(f,n))\xi_0 - \sigma_z(\Phi_n(f))\xi_0\| + \|\sigma_z(\Phi_n(f))\xi_0 - \sigma_z(\Phi(f))\xi_0\| \\
&\quad \equiv A_n^{(1)} + A_n^{(2)}.
\end{aligned} \tag{A.3}$$

It follows from (4.5) and (A.1) that

$$\begin{aligned}
(A_n^{(1)})^2 &= (\frac{n}{\pi}) \int\int \Big(\int_0^1 \int_0^1 G(t_1, t_2, s_1, s_2; f, n)\, ds_1 ds_2 \Big) \\
&\qquad \cdot e^{-n(t_1-\overline{z})^2} e^{-n(t_2-z)^2}\, dt_1 dt_2,
\end{aligned}$$

where

$$\begin{aligned}
&G(t_1, t_2, s_1, s_2; f, n) \\
&\quad := \omega(\Phi(A^{it_1}f)(W(-\frac{s_1}{n}A^{it_1}f) - \mathbf{1})(W(\frac{s_2}{n}A^{it_2}f) - \mathbf{1})\Phi(A^{it_2}f)).
\end{aligned} \tag{A.4}$$

Notice that for $s, t \in \mathbb{R}$

$$W(\frac{s}{n}A^{it}f) - \mathbf{1} = i\frac{s}{n}\Phi(A^{it}f) \int_0^1 W(\frac{s\tau}{n}A^{it}f) d\tau. \tag{A.5}$$

We substitute (A.5) into (A.4). Iterating (4.2), and then using (4.1) and (3.2), the function given in (A.4) can be calculated explicitely. Using Assumption 3.1, it can be proved that

$G(t_1, t_2, s_1, s_2; f, n)$ has an analytic extension on $\mathbb{C} \times \mathbb{C}$ in the t_1 and t_2 variables. By using the method similar to that used in the proof of the part (a) of the lemma, it is not hard to check that there exists constant $C_2(f, \text{Im } z)$ such that

$$\begin{aligned} &\sup_{s_1, s_2 \in [0,1]} |G(t_1 + \overline{z}, t_2 + z, s_1, s_2; f, n)| \\ &\qquad \leq \frac{1}{n^2} C_2(f, \text{Im}\, z). \end{aligned}$$

It follows from the Cauchy integral theorem and the above bound that

$$(A_n^{(1)})^2 \to 0 \quad \text{as } n \to \infty.$$

We leave again the details to the reader.

Next, consider $A_n^{(2)}$ defined in (A.3). Notice that

$$(A_n^{(2)})^2 = (\frac{n}{\pi}) \int \int \widetilde{G}(t_1, t_2; f) e^{-n(t_1 - \overline{z})^2} e^{-n(t_2 - z)^2} \, dt_1 dt_2,$$

where

$$\widetilde{G}(t_1, t_2; f) = \omega\Big((\Phi(A^{it_1} f) - \sigma_{\overline{z}}(\Phi(f)))(\Phi(A^{it_2} f) - \sigma_z(\Phi(f))) \Big).$$

We use (A.2) to obtain that

$$\begin{aligned} \widetilde{G}(t_1, t_2; f) &= \frac{1}{2} \omega(\{a(A^{it_1} f) - a(A^{iz} f)\}\{a^*(A^{it_2} f) - a^*(A^{iz} f)\}) \\ &\quad + \frac{1}{2} \omega(\{a^*(A^{it_1} f) - a^*(A^{i\overline{z}} f)\}\{a(A^{it_2} f) - a(A^{i\overline{z}} f)\}) \\ &= \frac{1}{2} ((A^{it_1} - A^{iz}) f, (\mathbf{1} - A)^{-1} (A^{it_2} - A^{iz}) f) \\ &\quad + \frac{1}{2} ((A^{it_2} - A^{i\overline{z}}) f, A(\mathbf{1} - A)^{-1} (A^{it_1} - A^{i\overline{z}}) f). \end{aligned}$$

Thus $\widetilde{G}(t_1, t_2; f)$ has an analytic extension on $\mathbb{C} \times \mathbb{C}$ in the t_1 and t_2 variables and

$$\begin{aligned} &\widetilde{G}(t_1 + \overline{z}, t_2 + z; f) \qquad (A.6) \\ &\quad = \frac{1}{2} (f, (\mathbf{1} - A)^{-1} (A^{-it_1} - \mathbf{1})(A^{it_2} - \mathbf{1}) A^{-i\overline{z}} A^{iz} f) \\ &\qquad + \frac{1}{2} (f, A(\mathbf{1} - A)^{-1} (A^{it_1} - \mathbf{1})(A^{-it_2} - \mathbf{1}) A^{-iz} A^{i\overline{z}} f). \end{aligned}$$

We use the Cauchy integral theorem to get

$$(A_n^{(2)})^2 = (\frac{n}{\pi}) \int \int \widetilde{G}(t_1 + \overline{z}, t_2 + z; f) e^{-nt_1^2} e^{-nt_2^2} \, dt_1 dt_2.$$

By (A.6), $\widetilde{G}(t_1 + \overline{z}, t_2 + z; f)$ is bounded uniformly with respect to $(t_1, t_2) \in \mathbb{R}^2$. Changing the variables ($t_1' = n^{1/2} t_1$ and $t_2' = n^{1/2} t_2$) and using the dominated convergence theorem, we conclude that

$$A_n^{(2)} \to 0 \quad \text{as } n \to \infty.$$

This proved the part (b) of the lemma completely. $\square$

Proof of Lemma 5.4. (a) Recall the definition of $\Phi(f,n)$ in (4.5) for any $f \in \mathfrak{h}_0$ and $n \in \mathbb{N}$. For given(fixed) $m \in \mathbb{N} \cup \{0\}$ and $h_l \in \mathfrak{h}_0$, $l = 1, ..., m$ we will use the following abbreviated notations:

$$\begin{aligned} &\Phi_l(n) := \Phi(h_l, n),\ l = 1, ..., m, \\ &\Phi_l := \Phi(h_l),\ l = 1, ..., m, \\ \xi(n) \ &:= (\prod_{l=1}^{m} \Phi_l(n))\xi_0, \text{ and } \ \xi := (\prod_{l=1}^{m} \Phi_l)\xi_0. \end{aligned} \tag{A.7}$$

Notice that $\xi(n) \in \mathcal{W}\xi_0$, $n \in \mathbb{N}$ and $\xi \in \mathcal{H}_{fin}$. It is not hard to show that $\xi(n) \to \xi$ as $n \to \infty$. See the method used in the below. We will show that $\overline{\mathcal{E}}[\xi(n) - \xi] \to 0$ as $n \to \infty$. Since $\overline{\mathcal{E}}$ is closed, this implies that $\mathcal{H}_{fin} \subset D(\overline{\mathcal{E}})$ and

$$\overline{\mathcal{E}}[\xi] = \lim_{n\to\infty} \overline{\mathcal{E}}[\xi(n)]. \tag{A.8}$$

By Remark 4.1 and (5.5), the relation (5.10) holds for $\xi \in \mathcal{W}\xi_0$. Let ξ and $\xi(n)$, $n \in \mathbb{N}$ be defined as in (A.7). Notice that

$$\xi(n) - \xi = \sum_{p=1}^{m} (\prod_{l=1}^{p-1} \Phi_l(n))(\Phi_p(n) - \Phi_p)(\prod_{l=p+1}^{m} \Phi_l)\xi_0.$$

We use the Schwarz inequality twice to obtain that

$$\begin{aligned} &\|D_1(B^{\frac{1}{2}} g_k)(\xi(n) - \xi)\|^2 \\ &\leq m \sum_{p=1}^{m} \|D_1(B^{\frac{1}{2}} g_k)(\prod_{l=1}^{p-1} \Phi_l(n))(\Phi_p(n) - \Phi_p)(\prod_{l=p+1}^{m} \Phi_l)\xi_0\|^2. \end{aligned} \tag{A.9}$$

Recall that $D_1(B^{\frac{1}{2}} g_k) = a(A^{-\frac{1}{4}} g_k) - j(\sigma_{-\frac{i}{2}}(a(A^{-\frac{1}{4}} g_k)))$. It follows from the CCRs and Lemma 5.1 that

$$\begin{aligned} &D_1(B^{\frac{1}{2}} g_k)(\prod_{l=1}^{p-1} \Phi_l(n))(\Phi_p(n) - \Phi_p)(\prod_{l=p+1}^{m} \Phi_l)\xi_0 \\ &= \frac{1}{\sqrt{2}} \sum_{q=1}^{m} (g_k, A^{-\frac{1}{4}} h_q) \Psi^{(m)}(p, q; n) \end{aligned} \tag{A.10}$$

where

$$\begin{aligned}
\Psi^{(m)}(p,q;n) &:= (\prod_{l=1}^{q-1}\Phi_l(n))W(\frac{1}{n}h_q)(\prod_{l=q+1}^{p-1}\Phi_l(n)) \\
&\quad \cdot(\Phi_p(n)-\Phi_p)(\prod_{l=p+1}^{m}\Phi_l)\xi_0, \quad 1\le q\le p-1, \\
\Psi^{(m)}(p,q;n) &:= (\prod_{l=1}^{p-1}\Phi_l(n))(W(\frac{1}{n}h_p)-\mathbf{1})(\prod_{l=p+1}^{m}\Phi_l)\xi_0, \quad q=p, \\
\Psi^{(m)}(p,q;n) &:= (\prod_{l=1}^{p-1}\Phi_l(n))(\Phi_p(n)-\Phi_p)(\prod_{l=p+1}^{q-1}\Phi_l(n)) \\
&\quad \cdot(\prod_{l=q+1}^{m}\Phi_l)\xi_0, \quad p+1\le q\le m.
\end{aligned}$$

We use the Schwarz inequality twice again to (A.10) and substitute the result into (A.9) to conclude that

$$\|D_1(B^{\frac{1}{2}}g_k)(\xi(n)-\xi)\|^2 \le \frac{1}{2}m^2\sum_{p=1}^{m}\sum_{q=1}^{m}|(g_k,A^{-\frac{1}{4}}h_q)|^2\|\Psi^{(m)}(p,q;n)\|^2.$$

Using the Parserval relation, we obtain that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\|D_1(B^{\frac{1}{2}}g_k)(\xi(n)-\xi)\|^2 \qquad (A.11)\\
&\quad\le \frac{1}{2}m^2\sum_{p=1}^{m}\sum_{q=1}^{m}\|A^{-\frac{1}{4}}h_q\|^2\|\Psi^{(m)}(p,q;n)\|^2.
\end{aligned}$$

Iterating (4.2), $\|\Psi^{(m)}(p,q;n)\|^2$ can be calculated explicitly for any p,q and n. One notes that each $\Psi^{(m)}(p,q;n)$ contains either $(\Phi_p(n)-\Phi_p)$ or else $(W(\frac{1}{n}h_p)-\mathbf{1})$, which implies that

$$\|\Psi^{(m)}(p,q;n)\|^2\to 0 \ \text{ as } \ n\to\infty$$

for any p, q. Thus it follows from (A.11) that

$$\sum_{k=1}^{\infty}\|D_1(B^{\frac{1}{2}}g_k)(\xi(n)-\xi)\|^2\to 0 \ \ as \ \ n\to\infty.$$

The method similar to that used in the above implies that

$$\sum_{k=1}^{\infty}\|D_2(B^{\frac{1}{2}}g_k)(\xi(n)-\xi)\|^2\to 0 \quad \text{as } \ n\to\infty.$$

This proved the part (a) of this lemma.

(b) This follows from (5.9) and (A.8). $\square$

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