WEYL-TITCHMARSH M-FUNCTION ASYMPTOTICS FOR MATRIX-VALUED SCHRÖDINGER OPERATORS

STEVE CLARK AND FRITZ GESZTESY

Abstract. We explicitly determine the high-energy asymptotics for Weyl-Titchmarsh matrices associated with general matrix-valued Schrödinger operators on a half-line.

This is a revised and updated version of a previously archived file.

1. Introduction

The high-energy asymptotics $z \to \infty$ of scalar-valued Weyl-Titchmarsh functions $m_+(z, x_0)$ associated with general half-line Schrödinger differential expressions of the type $-\frac{d^2}{dx^2} + q(x)$, with real-valued coefficients $q^{(n)} \in L^1([x_0, c])$ for some $n \in \mathbb{N}_0 (= \mathbb{N} \cup \{0\})$ and all $c > x_0$, received enormous attention over the past three decades as can be inferred, for instance from [6], [7], [8], [9], [11], [25], [31], [32], [33], [46]–[49], [50], [51], [61], [62], [86] and the literature therein. Hence it may perhaps come as a surprise that the corresponding matrix extension of this problem, considering general matrix-valued differential expressions of the type $-I_m \frac{d^2}{dx^2} + Q(x)$, with I_m the identity matrix in $\mathbb{C}^m$, $m \in \mathbb{N}$, and $Q(x)$ a self-adjoint matrix satisfying $Q^{(n)} \in L^1([x_0, c])^{m \times m}$ for some $n \in \mathbb{N}_0$ and all $c > x_0$, received hardly any attention at all. In fact, we are only aware of a total of two papers devoted to Weyl-Titchmarsh M-function asymptotics for general half-line matrix Schrödinger operators: one by Clark [20] in the case $m = 2$, $n = 0$, and a second one, an apparently unpublished manuscript by Atkinson [7] in the case $m \in \mathbb{N}$, $n = 0$. Both authors are solely concerned with determining the leading term in the asymptotic high-energy expansion of the Weyl-Titchmarsh matrix $M_+(z, x_0)$ as $z \to \infty$ without investigating higher-order expansion coefficients. (It should be noted that this observation discounts papers in the special scattering theoretic case concerned with short-range coefficients $Q^{(n)} \in L^1([x_0, \infty); (1 + |x|)dx)^{m \times m}$, where straightforward iterations of Volterra-type integral equations yield the asymptotic high-energy expansion of $M_+(z, x_0)$ to any order, cf. Lemma 4.1.) A quick look at Atkinson's preprint reveals the nontrivial nature of determining even the leading-order term of the asymptotic expansion of $M_+(z, x_0)$ as $z \to \infty$,

$$M_+(z, x_0) \underset{z \to \infty}{=} i z^{1/2} I_m + o(|z|^{1/2}) \tag{1.1}$$

for z in sectors in the open upper half-plane due to a variety of variable transformations, which finally reduce the problem to the study of a managable Riccati system of differential equations.

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Our principal motivation in studying this problem stems from our general interest in operator-valued Herglotz functions (cf. [12], [37], [38], [39], [40], [41], [44], [85]) and its possible applications in the areas of inverse spectral theory and completely integrable systems. More precisely, using higher-order expansions of the type (1.1), one can prove trace formulas for $Q(x)$ and certain higher-order differential polynomials in $Q(x)$, following an approach pioneered in [42] (see also [35], [36]). These trace formulas, in turn, then can be used to prove various results in inverse spectral theory for matrix-valued Schrödinger operators $H = -I_m \frac{d^2}{dx^2} + Q$ in $L^2(\mathbb{R})^m$. For instance, using the principal result of this paper, Theorem 4.6, and its straightforward application to the asymptotic high-energy expansion of the diagonal Green's matrix $G(z,x,x) = (H-z)^{-1}(x,x)$ of H, the following matrix-valued extension of a classical uniqueness result of Borg [10] has been obtained in [22].

Theorem 1.1 ([22]). *Suppose H is reflectionless (e.g., Q is periodic and H has uniform spectral multiplicity $2m$) and spectrum $[E_0,\infty)$ for some $E_0 \in \mathbb{R}$. Then*

$$Q(x) = E_0 I_m \text{ for a.e. } x \in \mathbb{R}. \tag{1.2}$$

For related results see, for instance, [2], [13], [14], [15], [17], [18], [19], [23], [26], [59], [67], [68], [69], [77], [82]. Incidentally, the higher-order differential polynomials in $Q(x)$ just alluded to represent the Korteweg-deVries (KdV) invariants (i.e., densities associated with KdV conservation laws) and hence open the link to infinite-dimensional completely integrable systems (cf. [3], [16], [27], [28], [29], [30], [34], [45], [70], [71], [72], [75], [83], [84] and the references therein).

In Section 2 we briefly recall some of the basic notions of Weyl-Titchmarsh theory for singular matrix Schrödinger operators on a half-line as developed in detail by Hinton and Shaw [54]–[58] (see also [5], [50], [60], [63], [64], [65], [66], [76], [78], [79], [80], [83], [89]). In fact, most of these references deal with more general singular Hamiltonian systems and hence we here specialize this material to the matrix-valued Schrödinger operator case at hand. The subsequent section is devoted to a proof of the leading-order asymptotic high-energy expansion (1.1) of $M_+(z,x_0)$, originally due to Atkinson. Since the result is a fundamental ingredient for our principal Section 4, and to the best of our knowledge, his manuscript [7] remained unpublished, we provide a detailed description of his approach in Section 3. Finally, Section 4 develops a systematic higher-order high-energy asymptotic expansion of $M_+(z,x_0)$, combining Atkinson's result in Section 3 with matrix-valued extensions of some methods based on an associated Riccati equation. More precisely, following a technique in [43] in the scalar-valued context, we show how to derive the general high-energy asymptotic expansion of $M_+(z,x_0)$ as $z \to \infty$ by combining Atkinson's leading-order term in (1.1) and the corresponding asymptotic expansion of $M_+(z,x_0)$ in the special case where Q has compact support.

Analogous results for Dirac-type operators are in preparation [21].

2. Weyl-Titchmarsh Matrices

In this section we briefly recall the Weyl–Titchmarsh theory for matrix-valued Schrödinger operators on half-lines (x_0,∞) for some $x_0 \in \mathbb{R}$.

Throughout this note all $m \times m$ matrices $M \in \mathbb{C}^{m\times m}$ will be considered over the field of complex numbers $\mathbb{C}$. Moreover, I_p denotes the identity matrix in $\mathbb{C}^{p\times p}$ for $p \in \mathbb{N}$, M^* the adjoint (i.e., complex conjugate transpose), M^t the transpose of the matrix M, and $\mathrm{AC}([a,b])$ denotes the set of absolutely continuous functions on $[a,b]$.

The basic assumption of this paper will be the following.

Hypothesis 2.1. *Let $m \in \mathbb{N}$ and suppose $Q = Q^* \in L^1([x_0,c])^{m\times m}$ for all $c > x_0$.*

Given Hypothesis 2.1 we consider the matrix-valued Schrödinger operator H_+ in $L^2((x_0,\infty))^m$,

$$H_+ = -I_m \frac{d^2}{dx^2} + Q, \tag{2.1}$$

$$\begin{aligned}\mathrm{dom}(H_+) = \{g \in L^2((x_0,\infty))^m \mid g, g' \in AC([x_0,c])^m \text{ for all } c > x_0; \lim_{x\downarrow x_0} g(x) = 0,\\ \text{s.-a.b.c. at } \infty; (-I_m g'' + Qg) \in L^2((x_0,\infty))^m\},\end{aligned}$$

where the abbreviation s.-a.b.c. denotes a fixed self-adjoint boundary condition at ∞ throughout this paper if the differential expression $-I_m\frac{d^2}{dx^2} + Q(x)$ is not in the limit point case at ∞ (cf. the paragraph following Definition 2.7) and is disregarded in the case where $-I_m\frac{d^2}{dx^2} + Q(x)$ is in the limit point case at ∞. Naturally associated with the operator H_+ is the matrix Schrödinger equation

$$-\psi''(z,x) + Q(x)\psi(z,x) = z\psi(z,x), \quad z \in \mathbb{C},\ x \geq x_0, \tag{2.2a}$$

where z represents a spectral parameter and $\psi(z,x)$ is assumed to satisfy

$$\psi'(z,\cdot) \in AC([x_0,c])^{m\times r} \text{ for all } c > 0, \tag{2.2b}$$

with the value of r depending upon the context involved. Equation (2.2a) may also be expressed as the system

$$J\Psi(z,x)' = (zA(x) + B(x))\Psi(z,x), \quad z \in \mathbb{C},\ x \geq x_0, \tag{2.3a}$$

where

$$\Psi(z,x) = \begin{bmatrix}\psi(z,x)\\ \psi'(z,x)\end{bmatrix}, \quad J = \begin{bmatrix}0 & -I_m\\ I_m & 0\end{bmatrix}, \tag{2.3b}$$

$$A(x) = \begin{bmatrix}I_m & 0\\ 0 & 0\end{bmatrix}, \quad B(x) = \begin{bmatrix}-Q(x) & 0\\ 0 & I_m\end{bmatrix}, \tag{2.3c}$$

and where

$$\Psi(z,\cdot) \in AC([x_0,c])^{2m\times r} \text{ for all } c > 0; \tag{2.3d}$$

again, with the value of r depending upon the context involved.

Next we turn to Weyl-Titchmarsh theory associated with (2.1) and briefly recall some of the results developed by Hinton and Shaw in a series of papers devoted to spectral theory of (singular) Hamiltonian systems [54]–[58] (see also [5], [50], [60], [63], [64], [65], [66], [76], [78], [79], [80], [83], [89]). While they discuss much more general systems, we here confine ourselves to the special case of matrix-valued Schrödinger operators governed by Hypothesis 2.1.

Let $\Psi(z,x,x_0)$ be a normalized fundamental matrix of solutions of (2.3) at some $x_0 \in \mathbb{R}$; that is, $\Psi(z,x,x_0)$ is of the type

$$\begin{aligned}\Psi(z,x,x_0) &= [\psi_{j,k}(z,x,x_0)]_{j,k=1}^2\\ &= \begin{bmatrix}\Theta(z,x,x_0) & \Phi(z,x,x_0)\end{bmatrix} = \begin{bmatrix}\theta(z,x,x_0) & \phi(z,x,x_0)\\ \theta'(z,x,x_0) & \phi'(z,x,x_0)\end{bmatrix},\end{aligned} \tag{2.4a}$$

$$\Psi(z,x_0,x_0) = I_{2m}, \tag{2.4b}$$

where $\Theta(z,x,x_0)$, $\Phi(z,x,x_0) \in \mathbb{C}^{2m\times m}$, and where $\theta(z,x,x_0)$ and $\phi(z,x,x_0)$ represent a fundamental system of $m\times m$ matrix-valued solutions of (2.2), entire with respect to $z\in\mathbb{C}$, and normalized according to (2.4b); that is,

$$\theta(z,x_0,x_0) = \phi'(z,x_0,x_0) = I_m, \quad \theta'(z,x_0,x_0) = \phi(z,x_0,x_0) = 0. \tag{2.4c}$$

Let $\beta = [\beta_1\ \beta_2]$; with β_1, $\beta_2 \in \mathbb{C}^{m\times m}$. We assume

$$\ker(\beta_1^*)\cap\ker(\beta_2^*) = \{0\}, \text{ or equivalently that } \operatorname{rank}(\beta) = m, \tag{2.5a}$$

and that either

$$\pm(1/2i)\beta J\beta^* = \pm\operatorname{Im}(\beta_2\beta_1^*) > 0, \quad \text{or} \quad \operatorname{Im}(\beta_2\beta_1^*) = 0. \tag{2.5b}$$

Here we denote, as usual, $\operatorname{Im}(M) = (M-M^*)/(2i)$ and $\operatorname{Re}(M) = (M+M^*)/2$.

We now prove the following result concerning $\Phi(z,x,x_0)$.

Lemma 2.2. *Let $\Phi(z,x,x_0)$ be defined in (2.4), and let $\beta\in\mathbb{C}^{m\times 2m}$ satisfy (2.5). If $\beta\Phi(z,c,x_0) = \beta_1\phi(z,c,x_0)+\beta_2\phi'(z,c,x_0)$ is singular for $c > x_0$ and $\operatorname{Im}(\beta_2\beta_1^*) \gtrless 0$, then $\operatorname{Im}(z) \lesseqgtr 0$. If $\beta\Phi(z,c,x_0)$ is singular for $c>x_0$ and $\operatorname{Im}(\beta_2\beta_1^*) = 0$, then $\operatorname{Im}(z) = 0$.*

Proof. Let $\Psi(z,x)$ satisfy (2.3). Then,

$$\Psi(z,c)^*J\Psi(z,c) - \Psi(z,x_0)^*J\Psi(z,x_0) = 2i\operatorname{Im}(z)\int_{x_0}^c ds\,\Psi(z,s)^*A(s)\Psi(z,s). \tag{2.6a}$$

Let $\Psi(z,x) = \Phi(z,x,x_0)$ and note that

$$\Psi(z,x_0)^*J\Psi(z,x_0) = [0\ I_m]J[0\ I_m]^t = 0. \tag{2.6b}$$

Moreover, note that

$$\Psi(z,c)^*J\Psi(z,c) = \phi'(z,c,x_0)^*\phi(z,c,x_0) - \phi(z,c,x_0)^*\phi'(z,c,x_0). \tag{2.6c}$$

Now suppose that $\beta\Phi(z,c,x_0)v = 0$ for $v\in\mathbb{C}^m$, $v\neq 0$. If $\operatorname{Im}(\beta_2\beta_1^*)\gtrless 0$, then $\phi'(z,c,x_0)v = -\beta_2^{-1}\beta_1\phi(z,c,x_0)v$. Thus by (2.6) we obtain,

$$\begin{aligned} &v^*\phi(z,c,x_0)^*\beta_2^{-1}[\operatorname{Im}(\beta_2\beta_1^*)]\beta_2^{*\,-1}\phi(z,c,x_0)v \\ &= -\operatorname{Im}(z)\int_{x_0}^c ds\ v^*\phi(z,s,x_0)^*\phi(z,s,x_0)v, \end{aligned} \tag{2.7}$$

by which we obtain the first part of the lemma. However, if $0 = \operatorname{Im}(\beta_2\beta_1^*) = (1/2i)\beta J\beta^*$ then by (2.5a) there is a $w\in\mathbb{C}^m$ such that

$$\Phi(z,c,x_0)v = J\beta^*w. \tag{2.8}$$

For this case, by (2.6) we obtain

$$0 = \operatorname{Im}(z)\int_{x_0}^c ds\ v^*\phi(z,s,x_0)^*\phi(z,s,x_0)v, \tag{2.9}$$

which implies that $\operatorname{Im}(z) = 0$. ☐

One can also prove the following result.

Lemma 2.3. *Let $\Theta(z,x,x_0)$ and $\Phi(z,x,x_0)$ be defined in (2.4), and let $\beta\in\mathbb{C}^{m\times 2m}$ satisfy (2.5). For $c>x_0$, $\beta\Phi(z,c,x_0)$ is singular if and only if z is an eigenvalue for the regular boundary value problem given by (2.3) together with the boundary conditions*

$$[I_m\ 0]\Psi(z,x_0) = 0, \quad \beta\Psi(z,c) = 0. \tag{2.10}$$

Lemmas 2.2 and 2.3 show that with appropriate values of $z \in \mathbb{C}$ one may define a certain $m \times m$ matrix.

Definition 2.4. Let $\Theta(z,x)$ and $\Phi(z,x)$ be defined in (2.4), and let $\beta \in \mathbb{C}^{m\times 2m}$ satisfy (2.5). Then, for $c > x_0$ and for $\beta\Phi(z,c,x_0)$ nonsingular, let

$$M(z,c,x_0,\beta) = -(\beta\Phi(z,c,x_0))^{-1}(\beta\Theta(z,c,x_0)). \tag{2.11}$$

$M(z,c,x_0,\beta)$ is said to be the *Weyl-Titchmarsh M-function* for the regular boundary value problem described in Lemma 2.3. Also, let $D(z,c,x_0)$ denote the set of all $M(z,c,x_0,\beta)$ defined in (2.11) together with the assumption (2.5b).

About $M(z,c,x_0,\beta)$, we prove the following result.

Lemma 2.5. *Let $c > x_0$. If $\mathrm{Im}(\beta_2\beta_1^*) > 0$, then $M(z,c,x_0,\beta)$ is well-defined if $\mathrm{Im}(z) \geq 0$; if $\mathrm{Im}(\beta_2\beta_1^*) < 0$, then $M(z,c,x_0,\beta)$ is well-defined for $\mathrm{Im}(z) \leq 0$; and if $\mathrm{Im}(\beta_2\beta_1^*) = 0$ then $M(z,c,x_0,\beta)$ is well-defined for $z \in \mathbb{C}\backslash\mathbb{R}$. Moreover, if $\mathrm{Im}(\beta_2\beta_1^*) \gtreqless 0$ and $\mathrm{Im}(z) \gtrless 0$, then*

$$\mathrm{Im}(\pm M(z,c,x_0,\beta)) > 0. \tag{2.12}$$

Proof. Except for the conditions given in (2.12), the claims made in the statement of this lemma are an immediate consequence of Lemma 2.2.

To verify the conditions given in (2.12), we begin by letting

$$U(z,x,x_0) = \begin{bmatrix}\Theta(z,x,x_0) & \Phi(z,x,x_0)\end{bmatrix}\begin{bmatrix}I_m \\ M\end{bmatrix}, \tag{2.13}$$

where $\Theta(z,x,x_0)$ and $\Phi(z,x,x_0)$ are defined in (2.4), and where $M \in \mathbb{C}^{m\times m}$. Now, let $\Psi(z,x) = U(z,x,x_0)$ in (2.6a) and note that

$$\Psi(z,x_0)^* J\Psi(z,x_0) = 2i\mathrm{Im}(M). \tag{2.14}$$

Moreover, if $M = M(z,c,x_0,\beta)$, then also note that

$$\beta\Psi(z,c) = \beta U(z,c,x_0) = 0. \tag{2.15}$$

If $(1/2i)\beta J\beta^* = \mathrm{Im}(\beta_2\beta_1^*) = 0$, then by (2.5a) and (2.15) there is a matrix $C \in \mathbb{C}^{m\times m}$ such that $\Psi(z,c) = U(z,c,x_0) = J\beta^* C$. Thus,

$$\Psi(z,c)^* J\Psi(z,c) = C^*\beta J\beta^* C = 0, \tag{2.16}$$

and (2.12) follows immediately from (2.14), (2.16) and (2.6a) when $\mathrm{Im}(z) \neq 0$. On the other hand, if $\mathrm{Im}(\beta_2\beta_1^*) \gtrless 0$, then (2.15) implies that $u'(z,c) = -\beta_2^{-1}\beta_1 u(z,c)$. As a result,

$$\Psi(z,c)^* J\Psi(z,c) = -2iu(z,c)^*\beta_2^{-1}\mathrm{Im}(\beta_2\beta_1^*)\beta_2^{*\,-1}u(z,c), \tag{2.17}$$

and again (2.12) follows from (2.14), (2.17), and (2.6a) when $\mathrm{Im}(z) \neq 0$. $\square$

Remark 2.6. By Lemma 2.5, $M(z,c,x_0,\beta)$ is a Herglotz function of rank m. Moreover, $D(z,c,x_0)$ is contained in the standard *Weyl disk* of matrices associated with (2.2) or (2.3) for fixed values of z, c, and x_0 (cf. [21]). These disks nest with respect to increasing values of c; that is, $D(z,c_2,x_0) \subseteq D(z,c_1,x_0)$ whenever $x_0 < c_1 \leq c_2$. This leads to the existence of a closed limiting set $D_+(z,x_0)$, which we call a limiting disk and which corresponds to the standard limiting Weyl disk (cf. [21], [54], [56], [57], [76]).

Definition 2.7. Let $D_+(z, x_0)$ denote the limit, as $c \to \infty$, of the nested collection of sets $\overline{D(z, c, x_0)}$ given in Definition 2.4, and let $M_+(z, x_0)$ denote a point of the limit disk $D_+(z, x_0)$ associated with (2.2).

We note that $D_+(z, x_0)$ consists of a unique matrix if and only if $-I_m d^2/dx^2 + Q(x)$ is in the limit point case at $+\infty$ and that $D_+(z, x_0)$ consists of a closed, convex, and bounded set with nonempty interior if and only if $-I_m d^2/dx^2 + Q(x)$ is in the limit circle case at $+\infty$. In both of these cases, we note that elements of $\partial D_+(z, x_0)$ represent *half-line Weyl-Titchmarsh matrices* where each such element is associated with the construction of self-adjoint operator H_+ on the half-line. However, for those operators that fall between the limit point and limit circle cases, Hinton and Schneider have noted that not every element of $\partial D_+(z, x_0)$ is a half-line Weyl-Titchmarsh matrix and have characterized those elements of the boundary that are (cf. [52], [53]).

For convenience of the reader we quickly summarize the principal results on half-line Weyl-Titchmarsh matrices in the following theorem.

Theorem 2.8 ([4], [12], [44], [54], [55], [58], [64]). *Assume Hypothesis* 2.1, $x_0 \in \mathbb{R}$, $z \in \mathbb{C}\backslash\mathbb{R}$, *and let* $M_+(z, x_0)$ *be a half-line Weyl-Titchmarsh matrix. Then*
(i) $M_+(z, x_0)$ *is a matrix-valued Herglotz function of maximal rank. In particular,*

$$\mathrm{Im}(M_+(z, x_0)) > 0, \quad z \in \mathbb{C}_+, \tag{2.18}$$

$$M_+(\bar{z}, x_0) = M_+(z, x_0)^*, \tag{2.19}$$

$$\mathrm{rank}(M_+(z, x_0)) = m, \tag{2.20}$$

$$\lim_{\varepsilon \downarrow 0} M_+(\lambda + i\varepsilon, x_0) \text{ exists for a.e. } \lambda \in \mathbb{R}. \tag{2.21}$$

Isolated poles of $M_+(z, x_0)$ *and* $-M_+(z, x_0)^{-1}$ *are at most of first order, are real, and have a nonpositive residue.*
(ii) $M_+(z, x_0)$ *admits the representations*

$$M_+(z, x_0) = F_+(x_0) + \int_{\mathbb{R}} d\Omega_+(\lambda, x_0) \big((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}\big) \tag{2.22}$$

$$= \exp\left(C_+(x_0) + \int_{\mathbb{R}} d\lambda\, \Xi_+(\lambda, x_0)\big((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}\big)\right), \tag{2.23}$$

where

$$F_+(x_0) = F_+(x_0)^*, \quad \int_{\mathbb{R}} \frac{\|d\Omega_+(\lambda, x_0)\|}{1 + \lambda^2} < \infty, \tag{2.24}$$

$$C_+(x_0) = C_+(x_0)^*, \quad 0 \le \Xi_+(\,\cdot\,, x_0) \le I_m \text{ a.e.} \tag{2.25}$$

Moreover,

$$\Omega_+((\lambda, \mu], x_0) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda + \delta}^{\mu + \delta} d\nu\, \mathrm{Im}(M_+(\nu + i\varepsilon, x_0)), \tag{2.26}$$

$$\Xi_+(\lambda, x_0) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \mathrm{Im}(\ln(M_+(\lambda + i\varepsilon, x_0)) \text{ for a.e. } \lambda \in \mathbb{R}. \tag{2.27}$$

(iii) With $\phi(z, x, x_0)$ *and* $\theta(z, x, x_0)$ *given in* (2.4)*, define the* $m \times m$ *matrices*

$$u_+(z, x, x_0) = \theta(z, x, x_0) + \phi(z, x, x_0) M_+(z, x_0), \quad x, x_0 \in \mathbb{R},\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{2.28}$$

Then

$$u_+(z, \cdot, x_0) \in L^2((x_0, \infty))^{m \times m}, \tag{2.29}$$

$u_+(z,x,x_0)$ is invertible, $u_+(z,x,x_0)$ satisfies the boundary condition of H_+ at infinity (if any), and

$$\operatorname{Im}(M_+(z,x_0)) = \operatorname{Im}(z)\int_{x_0}^{\infty} dx\, u_+(z,x,x_0)^* u_+(z,x,x_0). \tag{2.30}$$

3. Atkinson's Leading-Order Argument

The purpose of this section is to prove the following result due to Atkinson.

Theorem 3.1. (Atkinson [7].) *Assume Hypothesis 2.1 and denote by $C_\varepsilon \subset \mathbb{C}_+$ the open sector with vertex at zero, symmetry axis along the positive imaginary axis, and opening angle ε, with $0<\varepsilon<\pi/2$. Let $M_+(z,x_0)$ be either the unique limit point or a point of the limit disk $D_+(z,x_0)$ associated with (2.2). Then*

$$M_+(z,x_0) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} iz^{1/2}I_m + o(|z|^{1/2}). \tag{3.1}$$

More precisely, Atkinson proved the analog of (3.1) in [7] for more general Sturm-Liouville-type situations rather than matrix-valued Schrödinger operators. To the best of our knowledge, Atkinson's manuscript [7] remains unpublished. Since the result is crucial for our principal Section 4, we will specialize his arguments to the simplified situation at hand and provide a complete account of his approach (at times going beyond some of the arguments sketched in [7]).

One of our tools in studying $M_+(z,x_0)$ will be to relate it to a matrix Riccati-type equation with solution $M_+(z,x)$ defined as follows. We first note that

$$M_+(z,x_0) = \tilde u_+'(z,x_0)\tilde u_+(z,x_0)^{-1}, \quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{3.2}$$

where $\tilde u_+(z,\cdot\,)\in L^2([x_0,\infty))^{m\times m}$ is a nonnormalized solution of (2.2) satisfying

$$\tilde u_+(z,x) = u_+(z,x,x_0)C_+, \tag{3.3}$$

with u_+ defined in (2.28), and with $C_+\in\mathbb{C}^{m\times m}$ a nonsingular matrix. Varying the reference point x_0 we may define

$$M_+(z,x) = \tilde u_+'(z,x)\tilde u_+(z,x)^{-1}, \quad z\in\mathbb{C}\backslash\mathbb{R},\ x\ge x_0. \tag{3.4}$$

Then $M_+(z,x)$ is independent of the chosen normalization C_+ and is well-known (see, e.g., [22], [35], [60]) to satisfy (3.5) below.

Lemma 3.2. *Assume Hypothesis 2.1, suppose $z\in\mathbb{C}\backslash\mathbb{R}$, $x\in[x_0,\infty)$, and define $M_+(z,x)$ as in (3.4). Then $M_+(z,x)$ satisfies the standard Riccati-type equation,*

$$M_+'(z,x) + M_+(z,x)^2 = Q(x) - zI_m. \tag{3.5}$$

Proof. Differentiating (3.4) with respect to x and inserting $\tilde u_+''(z,x) + (Q(x)-z)\tilde u_+(z,x) = 0$ immediately yields (3.5). □

Utilizing

$$\Psi(z,x,x_1) = \Psi(z,x,x_0)\Psi(z,x_0,x_1) \text{ and } \Psi(z,x_0,x_1)\Psi(z,x_1,x_0) = I_{2m}, \tag{3.6}$$

(cf. (2.4)) one computes for all $x_1\in\mathbb{R}$

$$\begin{aligned}&\theta(z,x,x_1) + \phi(z,x,x_1)M_+(z,x_1)\\ &= u_+(z,x,x_0)[\theta(z,x_0,x_1) + \phi(z,x_0,x_1)M_+(z,x_1)], \quad z\in\mathbb{C}\backslash\mathbb{R}.\end{aligned} \tag{3.7}$$

Since $u_+(z,\cdot,x_0) \in L^2([R,\infty))^{m\times m}$ for all $R \in \mathbb{R}$, the left-hand side of (3.7) is in $L^2([R,\infty))^{m\times m}$. Thus, for $M_+(z,x_1)$ defined according to (3.4), one concludes that

$$M_+(z,x_1) \in D_+(z,x_1) \text{ for all } x_1 \in \mathbb{R} \tag{3.8}$$

since $M_+(z,x_0) \in D_+(z,x_0)$ by hypothesis. This justifies our choice of notation $M_+(z,x)$ when compared with the notation employed in Definition 2.7.

Atkinson's proof of Theorem 3.1, which we follow very closely, is first concerned with the relationship between $D(z,c,x_0)$, described in Definition 2.4, and $D^{\mathcal{R}}(z,c,x_0)$, a *Riccati disk*, which we now define.

Definition 3.3. With $Q(x)$ given in Hypothesis 2.1 and $z \in \mathbb{C}_+$, $D^{\mathcal{R}}(z,c,x_0)$ is defined to be the set of all $M(z,x_0) \in \mathbb{C}^{m\times m}$ where $\operatorname{Im}(M(z,x_0)) > 0$, and such that the solution of

$$\begin{aligned}\vartheta'(z,x) = \frac{1}{2}\begin{bmatrix} I_m + \vartheta(z,x) & I_m - \vartheta(z,x)\end{bmatrix} \begin{bmatrix} -i|z|^{-1/2}(zI_m - Q(x)) & 0 \\ 0 & i|z|^{1/2}I_m\end{bmatrix} \times \\ \times \begin{bmatrix} I_m + \vartheta(z,x) \\ I_m - \vartheta(z,x)\end{bmatrix},\end{aligned} \tag{3.9a}$$

$$\begin{aligned}\vartheta(z,x_0) &= (I_m + i|z|^{-1/2}M(z,x_0))(I_m - i|z|^{-1/2}M(z,x_0))^{-1} \\ &= (i|z|^{1/2}I_m - M(z,x_0))(i|z|^{1/2}I_m + M(z,x_0))^{-1},\end{aligned} \tag{3.9b}$$

satisfies

$$\vartheta(z,x)^*\vartheta(z,x) \le I_m \text{ for all } x \in [x_0,c]. \tag{3.10}$$

Note that there is a correspondence between elements of $D^{\mathcal{R}}(z,c,x_0)$ and certain solutions of (3.9a). If $\vartheta(z,x)$ satisfies (3.9a) and (3.10) then note that a matrix $M(z,x_0) \in D^{\mathcal{R}}(z,c,x_0)$ is defined by the initial data $\vartheta(z,x_0)$ if $\vartheta(z,x_0)^*\vartheta(z,x_0) < I_m$. Note also that (3.9a) can be written as

$$\begin{aligned}\vartheta' =& \frac{1}{2}\begin{bmatrix} I_m + \vartheta & I_m - \vartheta\end{bmatrix}\begin{bmatrix} I_m & 0 \\ 0 & i|z|^{1/2}I_m\end{bmatrix}(zA+B)\begin{bmatrix} -i|z|^{-1/2}I_m & 0 \\ 0 & I_m\end{bmatrix} \times \\ &\times \begin{bmatrix} I_m + \vartheta \\ I_m - \vartheta\end{bmatrix}.\end{aligned} \tag{3.11}$$

Remark 3.4. We emphasize that (3.9a) formally results from the Cayley-type transformation,

$$\vartheta(z,x) = (i|z|^{1/2}I_m - M(z,x))(i|z|^{1/2}I_m + M(z,x))^{-1}, \tag{3.12}$$

where $M(z,x)$ satisfies the Riccati-type equation (3.5). In the scalar context this corresponds to a conformal mapping of the complex upper half-plane to the unit disk. Moreover, we note that Atkinson proves the asymptotic result of Theorem 3.1 for elements of $D^{\mathcal{R}}(z,c,x_0)$. Recall that $D_+(z,x_0) \subseteq D(z,c,x_0)$ (cf. Remark 2.6). Theorem 3.1 then follows upon showing that $D(z,c,x_0) \subseteq D^{\mathcal{R}}(z,c,x_0)$ for $c > x_0$. This containment is shown in Theorem 3.6.

Lemma 3.5. *If $u(z,x)$ is defined by* (2.13) *with $M \in D(z,c,x_0)$, then $u(z,x) - i|z|^{-1/2}u'(z,x)$ is invertible for all $z \in \mathbb{C}_+$, and all $x \in [x_0,c]$.*

Proof. With $M \in D(z,c,x_0)$ and with $U(z,c)$ defined in (2.13), $U(z,c)$ satisfies (2.15). If $\mathrm{Im}(\beta_2\beta_1^*) > 0$, then $u'(z,c) = -\beta_2^{-1}\beta_1 u(z,c)$. $U(z,c)$ has rank m, hence $u(z,c)$ is nonsingular. As a result,

$$\mathrm{Im}(u(z,c)^*u'(z,c)) = u(z,c)^*\beta_2^{-1}\mathrm{Im}(\beta_2\beta_1^*)\beta_2^{*-1}u(z,c) > 0. \tag{3.13}$$

However, if $\mathrm{Im}(\beta_2\beta_1^*) = 0$, then

$$0 = \beta_1\beta_2^* - \beta_2\beta_1^* = \begin{bmatrix}\beta_1 & \beta_2\end{bmatrix}\begin{bmatrix}\beta_2^* \\ -\beta_1^*\end{bmatrix}. \tag{3.14}$$

By (2.5a), $\mathrm{rank}[\beta_1\ \beta_2] = \mathrm{rank}[\beta_2\ \ -\beta_1]^* = m$; thus, by (2.15) and (3.14) there is a $w \in \mathbb{C}^{m\times m}$ such that $U(z,c) = [\beta_2\ \ -\beta_1]^*w$. From this, we obtain

$$\mathrm{Im}(u(z,c)^*u'(z,c)) = w^*\mathrm{Im}(\beta_2\beta_1^*)w = 0. \tag{3.15}$$

Hence for β which satisfy (2.5),

$$\mathrm{Im}(u(z,c)^*u'(z,c)) \geq 0. \tag{3.16}$$

For all $x \in [x_0,c]$ and $z \in \mathbb{C}_+$, note that

$$(\mathrm{Im}(u(z,x)^*u'(z,x)))' = -\mathrm{Im}(z)u(z,x)^*u(z,x) \leq 0. \tag{3.17}$$

Together, (3.16) and (3.17) imply that

$$\mathrm{Im}(u(z,x)^*u'(z,x)) \geq 0 \text{ for all } x \in [x_0,c],\ z \in \mathbb{C}_+. \tag{3.18}$$

If for some $\xi \in [x_0,c]$ there is an $f \in \mathbb{C}^m$, $f \neq 0$, such that $u(z,\xi)f = i|z|^{-1/2}u'(z,\xi)f$, then

$$-i|z|^{1/2}f^*u(z,\xi)^*u(z,\xi)f = f^*u(z,\xi)^*u'(z,\xi)f. \tag{3.19}$$

Together, (3.18) and (3.19) imply that $f^*u(z,\xi)^*u(z,\xi)f \leq 0$, and hence that $u(z,\xi)f = u'(z,\xi)f = 0$. Given the uniqueness of solutions for system (2.3), we conclude that $0 = u(z,x_0)f = f$; thereby producing a contradiction which completes the proof. ☐

Theorem 3.6. *$D(z,c,x_0) \subseteq D^{\mathcal{R}}(z,c,x_0)$ for all $c > x_0$ and $z \in \mathbb{C}_+$.*

Proof. With $z \in \mathbb{C}_+$ and $M = M(z,c,x_0,\beta) \in D(z,c,x_0)$, we see that $\mathrm{Im}(M) > 0$ by (2.12). With $u(z,x)$ defined in (2.13), then by Lemma 3.5, we can define

$$\vartheta(z,x) = (u(z,x) + i|z|^{-1/2}u'(z,x))(u(z,x) - i|z|^{-1/2}u'(z,x))^{-1} \tag{3.20}$$

for $x \in [x_0,\ c]$. As defined, $\vartheta(z,x)$ satisfies (3.9b). Then for $x \in [x_0\ c]$, (3.18) implies that

$$\begin{aligned}&2i|z|^{-1/2}\{u(z,x)^*u'(z,x) - u^{*\prime}(z,x)u(z,x)\}\\ &= -4|z|^{-1/2}\mathrm{Im}(u^*(z,x)u'(z,x)) \leq 0.\end{aligned} \tag{3.21}$$

This is equivalent to

$$\begin{aligned}&(u(z,x)^* - i|z|^{-1/2}u'(z,x)^*)(u(z,x) + i|z|^{-1/2}u'(z,x))\\ &\leq (u(z,x)^* + i|z|^{-1/2}u'(z,x)^*)(u(z,x) - i|z|^{-1/2}u'(z,x)).\end{aligned} \tag{3.22}$$

Given the invertibility of $u(z,x) - i|z|^{-1/2}u'(z,x)$ shown in Lemma 3.5, we infer that $\vartheta(z,x)$ satisfies (3.10). The proof of this theorem will be completed upon showing that $\vartheta(z,x)$ satisfies (3.9a).

First, observe that

$$\vartheta(z,x)(u(z,x)-i|z|^{-1/2}u'(z,x)) = u(z,x)+i|z|^{-1/2}u'(z,x), \tag{3.23}$$

and hence that

$$I_m+\vartheta = 2u(u-i|z|^{-1/2}u')^{-1}, \ \ I_m-\vartheta = -2i|z|^{-1/2}u'(u-i|z|^{-1/2}u')^{-1}. \tag{3.24}$$

Upon differentiating (3.23) and using the fact that $u(z,x)$ satisfies (2.2), we obtain

$$\vartheta'(u-i|z|^{-1/2}u') = (I_m-\vartheta)u' - i|z|^{-1/2}(I_m+\vartheta)(zI_m-Q)u. \tag{3.25}$$

By (3.24), it follows that

$$\vartheta' = -\frac{1}{2i|z|^{-1/2}}(I_m-\vartheta)^2 - \frac{i|z|^{-1/2}}{2}(I_m+\vartheta)(zI_m-Q)(I_m+\vartheta), \tag{3.26}$$

which is precisely (3.9a). □

The following relation holds for the sets in Theorem 3.6, that is,

$$\overline{D(z,c,x_0)} = D^{\mathcal{R}}(z,c,x_0) \text{ for all } c>x_0 \text{ and } z\in\mathbb{C}_+. \tag{3.27}$$

This will be discussed in detail in [21].

An associated system for (3.9) is obtained by introducing a change of independent variable: With $z\in\mathbb{C}_\varepsilon$, $c\in[x_0,\infty)$, and $x\in[x_0,c]$, let

$$t \ = \ (x-x_0)|z|^{1/2}. \tag{3.28}$$

Introducing

$$\varphi(z,t)=\vartheta(z,x), \tag{3.29}$$

with x and t related as in (3.28), (3.9) becomes

$$\begin{aligned}\varphi'(z,t) =& \frac{1}{2}\begin{bmatrix} I_m+\varphi(z,t) & I_m-\varphi(z,t)\end{bmatrix}\begin{bmatrix} -i|z|^{-1}(zI_m-\widehat{Q}(t)) & 0 \\ 0 & iI_m\end{bmatrix}\times \\ &\times\begin{bmatrix} I_m+\varphi(z,t) \\ I_m-\varphi(z,t)\end{bmatrix},\end{aligned} \tag{3.30a}$$

$$\varphi(z,0) = (i|z|^{1/2}I_m - M(z,x_0))(i|z|^{1/2}I_m+M(z,x_0))^{-1}, \tag{3.30b}$$

and (3.10) becomes

$$\varphi(z,t)^*\varphi(z,t)\le I_m \text{ for all } t\in[0,(c-x_0)|z|^{1/2}]. \tag{3.31}$$

Note that in (3.30a)

$$\widehat{Q}(t) = Q(x_0+t|z|^{-1/2}). \tag{3.32}$$

In (3.30) and (3.31), we have a set of conditions equivalent to (3.9) and (3.10) for $D^{\mathcal{R}}(z,c,x_0)$ given in Defintion 3.3.

Now consider a sequence, $z_n\in\mathbb{C}_\varepsilon$, such that $|z_n|\to\infty$ as $n\to\infty$ and such that

$$0<\varepsilon<\delta_n=\arg(z_n)<\pi-\varepsilon. \tag{3.33}$$

By choosing an appropriate subsequence, we may assume that

$$\delta_n\to\delta\in[\varepsilon,\pi-\varepsilon]. \tag{3.34}$$

Let $\varphi(z_n,t)$ denote a corresponding sequence of functions that satisfy (3.30a) and (3.31), with initial data, $\varphi(z_n,0)$, defined by (3.30b) for a sequence of points $M(z_n,x_0)$, $n\in\mathbb{N}$, where each $M(z_n,x_0)$ is chosen to lie in the disk $D^{\mathcal{R}}(z_n,c,x_0)$.

Note that as $z_n \to \infty$, the intervals described in (3.31) eventually cover all compact subintervals of $\mathbb{R}_+$. Given the uniform boundedness of $\varphi_n(t) = \varphi(z_n, t)$ described in (3.31), we assume, upon passing to an appropriate subsequence still denoted by $\varphi_n(0)$, that

$$\varphi_n(0) = \varphi(z_n, 0) \to \varphi_0(\delta), \text{ as } n \to \infty, \tag{3.35}$$

and as a consequence, that

$$\varphi_0(\delta)^* \varphi_0(\delta) \leq I_m. \tag{3.36}$$

With $\varphi_0(\delta)$ defined in (3.35) as $|z_n| \to \infty$, we consider a limiting system associated with (3.30):

$$\begin{aligned}\eta'(\delta, t) =& \frac{1}{2} \begin{bmatrix} I_m + \eta(\delta, t) & I_m - \eta(\delta, t) \end{bmatrix} \begin{bmatrix} -ie^{i\delta} I_m & 0 \\ 0 & iI_m \end{bmatrix} \times \\ & \times \begin{bmatrix} I_m + \eta(\delta, t) \\ I_m - \eta(\delta, t) \end{bmatrix}, \quad t \geq 0, \end{aligned} \tag{3.37a}$$

$$\eta(\delta, 0) = \varphi_0(\delta). \tag{3.37b}$$

Theorem 3.7. *The solution $\eta(\delta, t)$ of (3.37) satisfies*

$$\eta(\delta, t)^* \eta(\delta, t) \leq I_m \tag{3.38}$$

for $0 \leq t < \infty$. Moreover, the solutions $\varphi_n(t) = \varphi(z_n, t)$ of (3.30) converge to $\eta(\delta, t)$ uniformly on $[0, T]$ for every $T > 0$, as $n \to \infty$.

Proof. Let $T \in \mathbb{R}_+$ be the greatest value such that (3.38) holds for $t \in [0,\ T]$. We show that (3.38) must hold for some $[0,\ T']$ with $T' > T$, thus proving $T = \infty$.

The solution of (3.37), $\eta(\delta, t)$, presumed to be defined on $[0,\ T]$, can be continued onto some $[0,\ T']$ with $T' > T$; $\eta(\delta, t)$ then satisfies

$$\eta(\delta, t)^* \eta(\delta, t) \leq k^2 I_m \tag{3.39}$$

for $0 \leq t \leq T'$ and for some $k \geq 1$.

For brevity, let $\varphi_n' = G_n(\varphi_n, t)$ denote (3.30a) with $z = z_n$, and let $\eta' = H(\eta, t)$ denote (3.37a) in the following. Integrating (3.37) and (3.30), we obtain

$$\begin{aligned}\varphi_n(t) - \eta(\delta, t) = \varphi_n(0) - \varphi_0(\delta) + \int_0^t ds \{G_n(\eta, s) - H(\eta, s)\} \\ + \int_0^t ds \{G_n(\varphi_n, s) - G_n(\eta, s)\}. \end{aligned} \tag{3.40}$$

Note that

$$\begin{aligned} G_n(\eta, s) - H(\eta, s) = & \frac{1}{2} i (e^{i\delta} - e^{i\delta_n})(I_m + \eta(\delta, s))^2 \\ & + \frac{1}{2} i |z_n|^{-1} (I_m + \eta(\delta, s)) \widehat{Q}(s) (I_m + \eta(\delta, s)), \end{aligned} \tag{3.41}$$

and by (3.39), as $n \to \infty$ and for $t \in [0, T']$, that

$$\frac{1}{2} i (e^{i\delta} - e^{i\delta_n}) \int_0^t ds\ (I_m + \eta(\delta, s))^2 = O(\delta - \delta_n), \tag{3.42}$$

and by (3.28), (3.32), and (3.39), that

$$\frac{1}{2} i |z_n|^{-1} \int_0^t ds\ (I_m + \eta(\delta, s)) \widehat{Q}(s) (I_m + \eta(\delta, s))$$

$$= |z_n|^{-1/2} O\left(\int_{x_0}^{x_0+t|z_n|^{-1/2}} dx\, ||Q(x)||\right) \tag{3.43}$$

$$= o(|z_n|^{-1/2}) \text{ as } n \to \infty. \tag{3.44}$$

Together with (3.35), we observe that as $n \to \infty$ (3.40) is equivalent to

$$\varphi_n(t) - \eta(\delta, t) = o(1) + \int_0^t ds (G_n(\varphi_n, s) - G_n(\eta, s)) \tag{3.45}$$

uniformly so, for $0 \le t \le T'$. Since $||\varphi_n|| \le 1$ and $||\eta|| \le k$ for $t \in [0,\ T']$,

$$||G_n(\varphi_n, s) - G_n(\eta, s)|| \le \frac{1}{2}(3+k)||\varphi_n(s) - \eta(\delta, s)||(2 + |z_n|^{-1}\ ||\widehat{Q}(s)||). \tag{3.46}$$

Thus (3.40) yields

$$\begin{aligned} &||\varphi_n(t) - \eta(\delta, t)|| \\ &\le o(1) + \int_0^t ds\ \frac{1}{2}(3+k)||\varphi_n(s) - \eta(\delta, s)||(2 + |z_n|^{-1}\ ||\widehat{Q}(s)||). \end{aligned} \tag{3.47}$$

In light of (3.44), an application of Gronwall's inequality to (3.47) yields

$$\varphi_n(t) - \eta(\delta, t) \to 0 \text{ as } n \to \infty \tag{3.48}$$

uniformly for $0 \le t \le T'$. For n sufficiently large and $t \in [0,\ T']$, $\varphi_n(t)$ satisfies (3.31) with $z = z_n$, and $\eta(\delta, t)$ satisfies (3.39), hence $\eta(\delta, t)$ satisfies (3.38) for $t \in [0,\ T']$, where $T' > T$. □

The following computation identifies $\varphi_0(\delta)$ and proves that $\eta(\delta, t)$ is constant with respect to $t \ge 0$.

Corollary 3.8.

$$\eta(\delta, t) = \eta(\delta, 0) = \varphi_0(\delta) = \frac{1 - \exp(i\delta/2)}{1 + \exp(i\delta/2)} I_m, \quad t \ge 0. \tag{3.49}$$

Proof. Utilizing the standard connection between the explicit exponential solutions of the second-order Schrödinger equation (2.2a) and the Riccati equation (3.5) (in the special case $Q(x) = 0$, cf. (3.2) and Lemma 3.2), performing a conformal map of the type (3.12), and the variable transformations (3.28) and (3.29) then yields the following solution for (3.30a),

$$\begin{aligned} \varphi(z, t) = &\big(-(i|z|^{1/2} - iz^{1/2})(M_0(z) + iz^{1/2} I_m) + \exp(-2it\exp(i\delta/2)) \times \\ &\times (i|z|^{1/2} + iz^{1/2})(M_0(z) - iz^{1/2} I_m)\big) \times \\ &\times \big(-(i|z|^{1/2} + iz^{1/2})(M_0(z) + iz^{1/2} I_m) + \exp(-2it\exp(i\delta/2)) \times \\ &\times (i|z|^{1/2} - iz^{1/2})(M_0(z) - iz^{1/2} I_m)\big)^{-1}, \end{aligned} \tag{3.50}$$

associated with the general initial condition

$$\varphi(z, 0) = \big(i|z|^{1/2} I_m - M_0(z)\big)\big(i|z|^{1/2} I_m + M_0(z)\big)^{-1} \tag{3.51}$$

for some $M_0(z) \in \mathbb{C}^{m \times m}$ with $\operatorname{Im}(M_0(z)) > 0$, $z \in \mathbb{C}_+$. Since by hypothesis $0 < \delta < \pi$, the exponential terms in (3.50) enforce

$$||\varphi(z, t)|| > 1 \text{ as } t \uparrow \infty \tag{3.52}$$

unless

$$M_0(z) = iz^{1/2} I_m, \tag{3.53}$$

implying (3.49). □

Given these facts we proceed to the

Conclusion of the proof of Theorem 3.1.

With $M(z_n, x_0) \in D^{\mathcal{R}}(z, c, x_0)$, and for δ as in (3.34) let

$$C(\delta) = \frac{1 - \exp(i\delta/2)}{1 + \exp(i\delta/2)}. \tag{3.54}$$

From (3.30b), (3.35),(3.48), and (3.49), we infer that

$$(I_m + i|z_n|^{-1/2} M(z_n, x_0))(I_m - i|z_n|^{-1/2} M(z_n, x_0))^{-1} \to C(\delta) I_m \text{ as } n \to \infty. \tag{3.55}$$

As a consequence,

$$\begin{aligned} i|z_n|^{-1/2} M(z_n, x_0) &= \frac{C(\delta) - 1 + o(1)}{C(\delta) + 1 + o(1)} I_m = \frac{C(\delta) - 1}{C(\delta) + 1}(1 + o(1)) I_m \\ &= -e^{i\delta/2}(1 + o(1)) I_m. \end{aligned} \tag{3.56}$$

Thus, by (3.54) and (3.56) we conclude that

$$M(z_n, x_0) = i z_n^{1/2}(1 + o(1)) I_m, \tag{3.57}$$

and hence by Theorem 3.6, that (3.1) holds. □

In (3.1) an asymptotic expansion is given that is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε. However, we observe that the proof just completed shows more: Allowing the reference point x_0 to vary, the asymptotic expansion given in (3.1) is also uniform in x_0 for x_0 in a compact subset of $\mathbb{R}$.

Theorem 3.9. *Assume Hypothesis* 2.1, *let* $z \in \mathbb{C}_+$, $x \in \mathbb{R}$, *and denote by* $C_\varepsilon \subset \mathbb{C}_+$ *the open sector with vertex at zero, symmetry axis along the positive imaginary axis, and opening angle* ε, *with* $0 < \varepsilon < \pi/2$. *Let* $M_+(z, x)$, $x \geq x_0$, *be as in Definition* 2.7. *Then*

$$M_+(z, x) \underset{\substack{|z| \to \infty \\ z \in C_\varepsilon}}{=} i z^{1/2} I_m + o(|z|^{1/2}) \tag{3.58}$$

uniformly with respect to $\arg(z)$ *for* $|z| \to \infty$ *in* C_ε *and uniformly in* x *as long as* x *varies in compact subsets of* $[x_0, \infty)$.

Proof. Though stated for elements of the Weyl disk $D_+(z, x_0)$, the proof of Theorem 3.1 shows that the asymptotic expansion given in (3.1) holds, uniformly with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε, for all elements of the Riccati disk $D^{\mathcal{R}}(z, c, x_0)$ as noted in (3.57).

Note that the system (3.37) is independent of the reference point x_0. Recall that δ, defined in (3.34), is determined by our apriori choice of the sequence z_n subject only to z_n being in C_ε (c.f. (3.33)). Next note that $\varphi_0(\delta)$, which is defined as a limit in (3.35), which is described explicitly in Corollary 3.8, and which gives solutions of (3.37) which satisfy (3.38) for $0 \leq t < \infty$, is also independent of the reference point x_0. Thus, had we chosen $x_0' \neq x_0$ as our reference point at the start, the asymptotic analysis begun in Theorem 3.7 and continued in (3.54)–(3.57) would remain the same after the variable change in (3.28) except for the integral expression present in (3.43) in which x_0 would be replaced by x_0'. However, given the local integrability assumption on $Q(x)$ present in Hypothesis 2.1, we see that the integral expression in (3.43) is uniformly continuous for x_0 in a compact subset of $\mathbb{R}$. Thus (3.44) and consequently (3.48) are uniform for t, and for x_0, in compact subsets of $\mathbb{R}$.

As a consequence, we see that (3.57) holds for elements of the Riccati disk $D^{\mathcal{R}}(z, c, x_0)$, that this asymptotic expansion is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε and that it is uniform in x_0 as long as x_0 varies in compact subsets of $\mathbb{R}$. The asymptotic expansion described in (3.58) then follows by Theorem 3.6 $\square$

We will see in the next section that the remainder term $o(|z|^{1/2})$ in (3.58) can be improved to $o(1)$.

4. Higher-Order Asymptotic Expansions

In this section we shall prove our principal result, the asymptotic high-energy expansion of $M_+(z, x)$ to arbitrarily high orders in sectors of the type $C_\varepsilon \subset \mathbb{C}_+$ as defined in Theorem 3.1.

Throughout this section we choose $\mathrm{Im}(z^{1/2}) > 0$ for $z \in \mathbb{C}_+$. We also recall the following notion: $x \in [a, b)$ (resp., $x \in (a, b]$) is called a right (resp., left) Lebesgue point of an element $q \in L^1((a, b))$, $a < b$, if $\int_0^\varepsilon dx' \, |q(x+x') - q(x)| = o(\varepsilon)$ (resp., $\int_0^\varepsilon dx' \, |q(x - x') - q(x)| = o(\varepsilon)$) as $\varepsilon \downarrow 0$. Similarly, $x \in (a, b)$ is called a Lebesgue point of $q \in L^1((a, b))$ if $\int_{-\varepsilon}^\varepsilon dx' \, |q(x + x') - q(x)| = o(\varepsilon)$ as $\varepsilon \downarrow 0$. The set of all such points is then denoted by the right (resp., left) Lebesgue set of q on $[a, b]$ in the former case and simply the Lebesgue set of q on $[a, b]$ in the latter case. The analogous notions are applied to $m \times m$ matrices $Q \in L^1((a, b))^{m\times m}$ by simultaneously considering all m^2 entries of Q. The right (resp., left) Lebesgue set of Q on $[a, b]$ is then simply the intersection of the right (resp., left) Lebesgue sets of $Q_{j,k}$ for all $1 \le j, k \le m$, and similarly for the Lebesgue set of Q, etc.

We start with the simple case where Q has compact support and hence the underlying half-line matrix-valued Schrödinger operator is in the limit point case at $+\infty$.

Lemma 4.1. *Fix $x_0 \in \mathbb{R}$ and let $x \ge x_0$. In addition to Hypothesis 2.1 suppose that $\widetilde{Q}$ has compact support in $[x_0, \infty)$, that $\widetilde{Q}^{(N-1)} \in L^1([x_0, \infty))^{m\times m}$ for some $N \in \mathbb{N}$, and that x is a right Lebesgue point of $\widetilde{Q}^{(N-1)}$. Denote by $\widetilde{M}_+(z, x)$ the Weyl-Titchmarsh matrix associated with $\widetilde{Q}$. Then, as $|z| \to \infty$ in C_ε, $\widetilde{M}_+(z, x)$ has an asymptotic expansion of the form* ($\mathrm{Im}(z^{1/2}) > 0$, $z \in \mathbb{C}_+$)

$$\widetilde{M}_+(z, x) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} iI_m z^{1/2} + \sum_{k=1}^{N} \tilde{m}_{+,k}(x) z^{-k/2} + o(|z|^{-N/2}), \quad N \in \mathbb{N}. \tag{4.1}$$

The expansion (4.1) *is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε and uniform in x as long as x varies in compact subintervals of $[x_0, \infty)$ intersected with the right Lebesgue set of $\widetilde{Q}^{(N-1)}$. The expansion coefficients $\tilde{m}_{+,k}(x)$ can be recursively computed from*

$$\tilde{m}_{+,1}(x) = \frac{1}{2i}\widetilde{Q}(x), \quad \tilde{m}_{+,2}(x) = \frac{1}{4}\widetilde{Q}'(x),$$
$$\tilde{m}_{+,k+1}(x) = \frac{i}{2}\left(\tilde{m}'_{+,k}(x) + \sum_{\ell=1}^{k-1} \tilde{m}_{+,\ell}(x)\tilde{m}_{+,k-\ell}(x)\right), \quad k \ge 2. \tag{4.2}$$

If one merely assumes $\widetilde{Q} \in L^1([x_0,\infty))^{m\times m}$ with compact support in $[x_0,\infty)$, then (4.1) *holds with $N=0$ (interpreting $\sum_{k=1}^{0} \cdot = 0$), uniformly with respect to $\arg(z)$ for $|z|\to\infty$ in C_ε and uniformly in x as long as x varies in compact subsets of $[x_0,\infty)$.*

Proof. In the following let $z \in \mathbb{C}_+$, $\operatorname{Im}(z^{1/2}) > 0$, and $x \geq x_0$. The existence of an expansion of the type (4.1) is shown as follows. First one considers a matrix Volterra integral equation of the type (cf. [1, Ch. I], [24], [74], [84], [88])

$$\tilde{u}_+(z,x) = \exp(iz^{1/2}x)I_m - \int_x^\infty dx' \, \frac{\sin(z^{1/2}(x-x'))}{z^{1/2}} \widetilde{Q}(x')\tilde{u}_+(z,x'), \tag{4.3}$$

and observes that the solution $\tilde{u}_+(z,x)$ of (4.3) satisfies $\tilde{u}_+(z,\cdot) \in L^2([x_0,\infty))^{m\times m}$ in accordance with (3.3) and (3.4). Next, introducing

$$\tilde{v}_+(z,x) = \exp(-iz^{1/2}x)\tilde{u}_+(z,x), \tag{4.4}$$

one rewrites (4.3) in the form

$$\tilde{v}_+(z,x) = I_m - \frac{1}{2iz^{1/2}} \int_x^\infty dx' \, [1-\exp(-2iz^{1/2}(x-x'))]\widetilde{Q}(x')\tilde{v}_+(z,x'), \tag{4.5}$$

and thus infers,

$$\widetilde{M}_+(z,x) = \tilde{u}_+'(z,x)\tilde{u}_+(z,x)^{-1} = iz^{1/2}I_m + \tilde{v}_+'(z,x)\tilde{v}_+(z,x)^{-1}. \tag{4.6}$$

Iterating (4.5) then yields

$$\|\tilde{v}_+(z,x)\| \leq C, \quad z \in \mathbb{C}_+, \ \operatorname{Im}(z^{1/2}) > 0, \ x \geq x_0 \tag{4.7}$$

for some $C>0$ depending on $\widetilde{Q}$. Finally, we need one more ingredient, proven in [81, Lemma 3] using appropriate maximal functions. Let $q \in L^1([x_0,\infty))$, $\operatorname{supp}(q) \subseteq [x_0,R]$, for some $R>0$, and suppose $x \in [x_0,R]$ is a right Lebesgue point of q. Then

$$\int_x^R dx' \, \exp(2iz^{1/2}(x'-x))q(x') \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} -\frac{q(x)}{2iz^{1/2}} + o(|z|^{-1/2}). \tag{4.8}$$

An alternative proof of (4.8) follows from [87, Theorem I.13], which implies

$$\lim_{\substack{|z|\to\infty \\ z\in C_\varepsilon}} z^{-1/2} \int_x^R dx' \, \exp(2iz^{1/2}(x'-x))|q(x')-q(x)| = 0 \tag{4.9}$$

for any right Lebesgue point x of q. Given these facts, one iterates (4.5) and its x-derivative, integrates by parts, applies (4.8) to $q = \|Q_{j,k}\|$ for all $1 \leq j,k \leq m$, and estimates $\|\tilde{v}_+(z,x')\|$ by (4.7). Inserting the expansions for $\tilde{v}_+'(z,x)$ and $\tilde{v}_+(z,x)^{-1}$into (4.6) (using a geometric series expansion for $\tilde{v}_+(z,x)^{-1}$) then yields the existence of an expansion of the type (4.1). The actual expansion coefficients and the associated recursion relation (4.2) then follow upon inserting expansion (4.1) into the Riccati-type equation (3.5). The assertion following (4.2) is an immediate consequence of (4.5) and its derivative with respect to x, (4.6), and the Riemann-Lebesgue lemma. □

The corresponding scalar case $m=1$ (for a sufficiently smooth coefficient q) was treated in this manner in [36]. The original version of our preprint assumed $\widetilde{Q}^{(N)} \in L^1([x_0,\infty))^{m\times m}$ in Lemma 4.1. Prompted by a recent preprint by A. Rybkin [81], who used an m-function-type approach to trace formulas for scalar

Schrödinger operators and succeeded in removing any smoothness hypotheses on q, we reconsidered our original approach and extended the asymptotic expansion (4.1) to the case $\widetilde{Q}^{(N-1)} \in L^1(\mathbb{R})$ and x a right Lebesgue point of $\widetilde{Q}^{(N-1)}$.

Next we recall an elementary result on finite-dimensional evolution equations essentially taken from [73].

Lemma 4.2. ([73].) *Let $A \in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$. Then any $m\times m$ matrix-valued solution $X(x)$ of*

$$X'(x) = A(x)X(x) + X(x)A(x) \text{ for a.e. } x \in \mathbb{R}, \tag{4.10}$$

is of the type

$$X(x) = Y(x)CZ(x), \tag{4.11}$$

where C is a constant $m\times m$ matrix and $Y(x)$ is a fundamental system of solutions of

$$\Psi'(x) = A(x)\Psi(x) \tag{4.12}$$

and $Z(x)$ is a fundamental system of solutions of

$$\Phi'(x) = \Phi(x)A(x). \tag{4.13}$$

Proof. Clearly (4.11) satisfies (4.10) since

$$X' = Y'CZ + YCZ' = AYCZ + YCZA = AX + XA. \tag{4.14}$$

Conversely, let X be a solution of (4.10) and Y a fundamental matrix of solutions of (4.12). Define

$$K(x) = Y(x)^{-1}X(x), \text{ that is, } X(x) = Y(x)K(x). \tag{4.15}$$

Then

$$X' = Y'K + YK' = AYK + YK' = AX + YK' = AX + XA \tag{4.16}$$

implies

$$YK' = XA, \quad K' = Y^{-1}XA = KA. \tag{4.17}$$

Thus, there exists a constant $m\times m$ matrix C (possibly singular), such that

$$K(x) = CZ(x), \tag{4.18}$$

with Z a fundamental matrix of solutions of (4.13). Hence, $X = YCZ$. ☐

The next result provides the proper extension of Proposition 2.1 in [43] to the matrix-valued case.

Lemma 4.3. *Suppose $Q_1, Q_2 \in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$ with $Q_1(x) = Q_2(x)$ for a.e. $x \in [x_0, x_1]$, $x_0 < x_1$. Denote by $M_{j,+}(z,x)$ the Weyl-Titchmarsh matrices corresponding to Q_j, $j = 1, 2$. Then*

$$\begin{aligned}(M'_{1,+}&(z,x) - M'_{2,+}(z,x)) \\ &= -(1/2)(M_{1,+}(z,x) + M_{2,+}(z,x))(M_{1,+}(z,x) - M_{2,+}(z,x)) \\ &\quad - (1/2)(M_{1,+}(z,x) - M_{2,+}(z,x))(M_{1,+}(z,x) + M_{2,+}(z,x))\end{aligned} \tag{4.19}$$

for a.e. $x \in [x_0, x_1]$.

Proof. This is obvious from (3.5). ☐

Lemma 4.4. *Suppose $Q_1, Q_2 \in L^1_{\mathrm{loc}}(\mathbb{R})^{m\times m}$ and denote by $M_{j,+}(z,x)$ the corresponding Weyl-Titchmarsh matrices associated with Q_j, $j=1,2$. Define*

$$A_+(z,x) = -(1/2)(M_{1,+}(z,x) + M_{2,+}(z,x)) \tag{4.20}$$

for $x \in [x_0, x_1]$, $x_0 < x_1$ and assume $Y_+(z,x)$ and $Z_+(z,x)$ to be fundamental matrix solutions of

$$\Psi'(z,x) = A_+(z,x)\Psi(z,x) \text{ and } \Phi'(z,x) = \Phi(z,x)A_+(z,x) \tag{4.21}$$

respectively, with

$$Y_+(z,x_1) = Z_+(z,x_1) = I_m. \tag{4.22}$$

Then, as $|z|\to\infty$, $z \in C_\varepsilon$,

$$\|Y_+(z,x_0)\|, \|Z_+(z,x_0)\| \leq \exp(-(x_1-x_0)\mathrm{Im}(z^{1/2})(1+o(1))). \tag{4.23}$$

Proof. Define $B_+(z,x)$ by

$$B_+(z,x) = A_+(z,x) + iz^{1/2} I_m, \tag{4.24}$$

then

$$\int_{x_0}^{x_1} dx\, \|B_+(z,x)\| \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} o(|z|^{1/2}) \tag{4.25}$$

due to the uniform nature of the asymptotic expansion (3.58) for x varying in compact intervals. Next, introduce

$$E_+(z,x,x_1) = \exp(i(x_1-x)z^{1/2})I_m, \quad x \leq x_1, \tag{4.26}$$

then

$$Y_+(z,x) = E_+(z,x,x_1) - \int_x^{x_1} dx'\, E_+(z,x,x')B_+(z,x')Y_+(z,x'), \tag{4.27}$$

$$Z_+(z,x) = E_+(z,x,x_1) - \int_x^{x_1} dx'\, Z_+(z,x')B_+(z,x')E_+(z,x,x'). \tag{4.28}$$

Using

$$\|E_+(z,x_0,x_1)\| \leq \exp(-(x_1-x_0)\mathrm{Im}(z^{1/2})), \tag{4.29}$$

a standard Volterra-type iteration argument in (4.27), (4.28) then yields

$$\|Y_+(z,x_0)\|, \|Z_+(z,x_0)\| \leq \exp\left(-(x_1-x_0)\mathrm{Im}(z^{1/2}) + \int_{x_0}^{x_1} dx\, \|B_+(z,x)\|\right), \tag{4.30}$$

and hence (4.23). □

Theorem 4.5. *Suppose $Q_1, Q_2 \in L^1_{\mathrm{loc}}(\mathbb{R})^{m\times m}$ with $Q_1(x) = Q_2(x)$ for a.e. $x \in [x_0, x_1]$, $x_0 < x_1$ and let $M_{j,+}(z,x)$ be the corresponding Weyl-Titchmarsh matrices associated with Q_j, $j=1,2$. Then, as $|z|\to\infty$ in C_ε,*

$$\|M_{1,+}(z,x_0) - M_{2,+}(z,x_0)\| \leq C(1+|z|^{1/2})\exp(-2(x_1-x_0)\mathrm{Im}(z^{1/2})(1+o(1))) \tag{4.31}$$

for some constant $C > 0$.

Proof. Define for $x \in [x_0, x_1]$, $z \in \mathbb{C}\backslash\mathbb{R}$
$$X_+(z,x) = M_{1,+}(z,x) - M_{2,+}(z,x), \tag{4.32}$$
$$A_+(z,x) = -(1/2)(M_{1,+}(z,x) + M_{2,+}(z,x)). \tag{4.33}$$
By Lemma 4.3,
$$X_+' = A_+X_+ + X_+A_+ \tag{4.34}$$
and hence by Lemma 4.2,
$$X_+(z,x) = Y_+(z,x)X_+(z,x_1)Z_+(z,x), \tag{4.35}$$
where $Y_+(z,x)$ and $Z_+(z,x)$ are fundamental matrices of
$$\Psi'(z,x) = A_+(z,x)\Psi(z,x) \text{ and } \Phi'(z,x) = \Phi(z,x)A_+(z,x), \tag{4.36}$$
respectively, with
$$Y_+(z,x_1) = I_m, \quad Z_+(z,x_1) = I_m. \tag{4.37}$$
By Lemma 4.4,
$$\|Y_+(z,x_0)\|, \|Z_+(z,x_0)\| \leq \exp(-(x_1 - x_0)\mathrm{Im}(z^{1/2})(1 + o(1))) \tag{4.38}$$
as $|z| \to \infty$, $z \in C_\varepsilon$. Thus, as $|z| \to \infty$, $z \in C_\varepsilon$,
$$\begin{aligned}\|X_+(z,x_0)\| &\leq \|X_+(z,x_1)\| \, \|Y_+(z,x_0)\| \, \|Z_+(z,x_0)\| \\ &\leq C(1 + |z|^{1/2})\exp(-2(x_1 - x_0)\mathrm{Im}(z^{1/2})(1 + o(1)))\end{aligned} \tag{4.39}$$
for some constant $C > 0$ by (3.58), (4.35), and (4.38). $\square$

The result (4.31) can be slightly improved as will be discussed in Remark 4.7.

Given these preparations we can now drop the compact support assumption on Q in Lemma 4.1 and hence arrive at the principal result of this paper.

Theorem 4.6. *Fix $x_0 \in \mathbb{R}$. In addition to Hypothesis* 2.1 *suppose that for some $N \in \mathbb{N}$, $Q^{(N-1)} \in L^1([x_0, c])^{m \times m}$ for all $c > x_0$, and that x_0 is a right Lebesgue point of $Q^{(N-1)}$. Let $M_+(z, x_0)$ be either the unique limit point or a point of the limit disk $D_+(z, x_0)$ associated with* (2.2). *Then, as $|z| \to \infty$ in C_ε, $M_+(z, x_0)$ has an asymptotic expansion of the form* ($\mathrm{Im}(z^{1/2}) > 0$, $z \in \mathbb{C}_+$)
$$M_+(z,x_0) \underset{\substack{|z|\to\infty \\ z \in C_\varepsilon}}{=} iI_m z^{1/2} + \sum_{k=1}^{N} m_{+,k}(x_0) z^{-k/2} + o(|z|^{-N/2}), \quad N \in \mathbb{N}. \tag{4.40}$$
The expansion (4.40) *is uniform with respect to* $\arg(z)$ *for $|z| \to \infty$ in C_ε. The expansion coefficients $m_{+,k}(x_0)$ can be recursively computed from* (4.2) (*replacing $\tilde{m}_{+,k}(x)$ by $m_{+,k}(x)$*).
If one merely assumes Hypothesis 2.1, *then* (4.40) *holds with* $N = 0$ (*interpreting* $\sum_{k=1}^{0} \cdot = 0$), *uniformly with respect to* $\arg(z)$ *for $|z| \to \infty$ in C_ε.*

Proof. Define
$$\widetilde{Q}(x) = \begin{cases} Q(x) & \text{for } x \in [x_0, x_1], \quad x_0 < x_1 \\ 0 & \text{otherwise} \end{cases}$$
and apply Theorem 4.5 with $Q_1 = Q$, $Q_2 = \widetilde{Q}$. Then (in obvious notation)
$$\|M_+(z,x_0) - \widetilde{M}_+(z,x_0)\| \leq C(1 + |z|^{1/2})\exp(-2(x_1 - x_0)\mathrm{Im}(z^{1/2})(1 + o(1))) \tag{4.41}$$

as $|z| \to \infty$, $z \in C_\varepsilon$, and hence the asymptotic expansion (4.1) for $\widetilde{M}_+(z,x_0)$ in Lemma 4.1 coincides with that of $M_+(z,x_0)$ (this also applies to the case $N = 0$). ☐

Remark 4.7. A comparison of (3.58) and (4.40) with $N = 0$ shows that one can improve the remainder term $o(|z|^{1/2})$ in (3.58) to $o(1)$. In particular, this also yields an improvement of our rough estimate (4.39) to the effect that the term $(1+|z|^{1/2})$ can be replaced by 1 in (4.39) and hence in (4.31).

In analogy to Theorem 3.9, the asymptotic expansion (4.40) extends to one for $M_+(z,x)$ valid uniformly with respect to x as long as x varies in compact subintervals of $[x_0,\infty)$ intersected with the right Lebesgue set of $Q^{(N-1)}$. Since this result represents a fundamental input for deriving trace formulas in [22], we provide the facts in the next theorem.

Theorem 4.8. *Fix $x_0 \in \mathbb{R}$ and let $x \geq x_0$. In addition to Hypothesis* 2.1 *suppose that for some $N \in \mathbb{N}$, $Q^{(N-1)} \in L^1([x_0,c))^{m\times m}$ for all $c > x_0$, and that x is a right Lebesgue point of $Q^{(N-1)}$. Let $M_+(z,x)$, $x \geq x_0$, be either the unique limit point or a point of the limit disk $D_+(z,x)$ associated with* (2.2)*. Then, as $|z| \to \infty$ in C_ε, $M_+(z,x)$ has an asymptotic expansion of the form* ($\mathrm{Im}(z^{1/2}) > 0$, $z \in \mathbb{C}_+$)

$$M_+(z,x) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} iI_m z^{1/2} + \sum_{k=1}^{N} m_{+,k}(x) z^{-k/2} + o(|z|^{-N/2}), \quad N \in \mathbb{N}. \tag{4.42}$$

The expansion (4.42) *is uniform with respect to* $\arg(z)$ *for $|z| \to \infty$ in C_ε and uniform in x as long as x varies in compact subsets of $[x_0,\infty)$ intersected with the right Lebesgue set of $Q^{(N-1)}$. The expansion coefficients $m_{+,k}(x)$ can be recursively computed from* (4.2) (*replacing $\tilde m_{+,k}(x)$ by $m_{+,k}(x)$*)*.*
If one merely assumes Hypothesis 2.1*, then* (4.42) *holds with $N = 0$* (*interpreting $\sum_{k=1}^{0} \cdot = 0$*)*, uniformly with respect to* $\arg(z)$ *for $|z| \to \infty$ in C_ε and uniformly in x as long as x varies in compact subsets of $[x_0,\infty)$.*

Remark 4.9. Our asymptotic results in Theorem 4.8 are not necessarily confined to $M_+(z,x_0)$-matrices associated with half-line Schrödinger operators H_+ on $[x_0,\infty)$. In fact, introducing in addition the analogous Weyl-Titchmarsh matrix $M_-(z,x_0)$ associated with a half-line Schrödinger operator H_- on $(-\infty,x_0]$, and noticing that the diagonal Green's matrix $G(z,x,x) = (H-z)^{-1}(x,x)$ of a matrix-valued Schrödinger operator $H = -I_m \frac{d^2}{dx^2} + Q$ in $L^2(\mathbb{R})^m$ is given by

$$G(z,x,x) = (M_-(z,x) - M_+(z,x))^{-1}, \tag{4.43}$$

Theorem 4.6 then yields an analogous asymptotic expansion for $G(z,x,x)$ in C_ε of the form

$$G(z,x,x) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \begin{cases} (i/2)I_m z^{-1/2} + o(|z|^{-1}) & \text{for } N = 0, \\ (i/2)\sum_{k=0}^{N} G_k(x) z^{-k-1/2} + o(|z|^{-N-1/2}) & \text{for } N \in \mathbb{N}, \end{cases} \tag{4.44}$$

where

$$G_0(x) = I_m, \quad G_1(x) = \frac{1}{2}Q(x), \text{ etc.} \tag{4.45}$$

The expansion (4.44) is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε and uniform in $x \in \mathbb{R}$ as long as x varies in compact intervals interesected with the

Lebesgue set of $Q^{(N-1)}$ (if $N \in \mathbb{N}$). We refer to [22] for further details and applications of this fact in connection with trace formulas and Borg-type uniqueness results for $Q(x)$, $x \in \mathbb{R}$.

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Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65409, USA
E-mail address: sclark@umr.edu
URL: http://www.umr.edu/~clark

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: fritz@math.missouri.edu
URL: http://www.math.missouri.edu/people/fgesztesy.html