LINEARLY RECURRENT CIRCLE MAP SUBSHIFTS AND AN APPLICATION TO SCHRÖDINGER OPERATORS

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Abstract. We discuss circle map sequences and subshifts generated by them. We give a characterization of those sequences among them which are linearly recurrent. As an application we deduce zero-measure spectrum for a class of discrete one-dimensional Schrödinger operators with potentials generated by circle maps.

1. Introduction and Results

1.1. **Introduction.** The concept of linear recurrence or linear repetitivity, LR in short, has been recently discussed and investigated by quite a number of researchers within various frameworks. For example, the articles [14, 16, 18] study the LR property from the point of view of combinatorics on words, whereas [13, 28, 34] discuss its implications within the theory of tilings.

In both cases one considers structures (e.g., an infinite word or a tiling of Euclidean space), or families of structures (e.g., a subshift or a family of tilings), and their local patterns (e.g., subwords or patches occurring in the given tiling) which are equivalence classes modulo translations. Fixing such a local pattern, one may look at the set of occurrences of the pattern in the structure and compare the distance between two "consecutive" occurrences with the size of the pattern. If the distance is bounded by a fixed linear function of the size, the structure is said to have the LR property. Although the concepts are the same in spirit, applied to words it is usually referred to as linear recurrence, whereas among tiling theorists this concept is usually called linear repetitivity. Since this article will be concerned with a class of words and subshifts, we will henceforth use the term linear recurrence.

The usefulness of the LR property has been independently realized by numerous people who had quite different applications in mind. LR has been shown to

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have consequences in mathematical disciplines as diverse as combinatorics [14, 18], ergodic theory [13, 28, 29], and spectral theory of Schrödinger operators [30].

For infinite words (i.e., sequences over a finite alphabet), linear recurrence has been studied for the most prominent examples. It is known that all primitive substitution sequences are linearly recurrent [14, 16, 34] and for Sturmian sequences [18] and, more generally, Arnoux-Rauzy sequences [32], the linearly recurrent cases have been characterized. Our main objective in this article is to perform a similar study for another prominent class of sequences, the so-called circle map sequences. Our approach is based on a recent paper of one of us [1]. Furthermore, there are two important points which motivated our study. Firstly, there is a standard way of proving that a given sequence/subshift is LR which is related to S-adicity and uses a result of Durand [18, 19]. To prove that a sequence (or subshift) is not LR, it suffices to show that the sequence has infinite index, that is, for every $N \in \mathbb{N}$, there is some word v such that v^N is a subword of the given sequence. This technique was applied in the Sturmian and Arnoux-Rauzy cases and in particular established that those sequences are LR if and only if they have finite index (i.e., there is some N_0 such that no v^{N_0} is a subword — also called power-free). This makes the proof of the LR characterization relatively simple in these cases and might even lead one to expect that such an equivalence holds in a much more general context. We will show here that this equivalence fails for circle map sequences, which is of interest in its own right since it provides the first example of a non-LR sequence having finite index. However, we are therefore faced with the problem of proving that certain sequences of finite index are not LR and hence have to develop a new proof strategy. To this end, we will rely on the concept of derived sequences which was introduced by Durand in [17]. Secondly, we want to exploit the consequences of the LR property in the context of Schrödinger operators that were established by Lenz in [30]. Namely, a Schrödinger operator has zero-measure spectrum if its potential is linearly recurrent. For Schrödinger operators with a circle map potential, a zero-measure spectrum had been expected (cf. [9]) but could not be proven with earlier approaches. The new approach suggested by Lenz is therefore shown to be applicable for an explicitly characterized subclass of circle map potentials. We will discuss this in more detail later in the paper.

The organization of the article is as follows. In the remainder of this section we will introduce circle map sequences, recall some key notions, and state our main result which provides a characterization of the circle map sequences/subshifts which are linearly recurrent. In Section 2 we will develop the general setup and in particular recall the connection between LR subshifts and primitive S-adic subshifts. The link between circle map sequences and interval exchange transformations, and particularly the results of [1], will be explained in Section 3. Section 4 contains the proof of our main result. The application of this theorem to Schrödinger operators is discussed in Section 5. Appendix A explains how to prove a finite index for some circle map sequences which are not LR. Finally, in Appendix B we discuss possible generalizations of the approach presented in this paper.

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1.2. **Circle maps.**

Definition 1. Let $(\alpha,\beta)\in(0,1)^2$. The **circle map corresponding to the parameters $(\boldsymbol{\alpha},\boldsymbol{\beta})$** is the symbolic sequence $U=(u_n)_{n\geq 0}$ defined over the binary alphabet $\{a,b\}$ by:

$$u_n=\begin{cases}1 & \text{if } \{n\alpha\}\in[0,\beta[,\\ 0 & \text{else.}\end{cases}$$

We will restrict our attention to circle maps where α is irrational and $\beta\notin\mathbb{Z}+\alpha\mathbb{Z}$. The case α rational is not interesting since the associated circle map is periodic. The case $\beta=\alpha$ gives a Sturmian sequence and, more generally, the case $\beta\in\mathbb{Z}+\alpha\mathbb{Z}$ corresponds to quasi-Sturmian sequences and will be not considered in this paper (see [7, 33]).

Definition 2. A circle map is called **nondegenerate** if its parameters satisfy:

- α is irrational,
- $\beta\notin\mathbb{Z}+\alpha\mathbb{Z}$.

Such a circle map is called **admissible** if in addition we have $\alpha<\beta$.

1.3. **The $\mathcal{D}$-expansion.** In a previous paper [1], one of us has investigated the links between nondegenerate circle maps and three-interval exchange transformations. An algorithm was introduced which can be regarded as a speed-up of the Rauzy induction for three-interval exchange transformations. This algorithm is also a generalization of the classical continued fraction algorithm. Let us introduce a map

$$\mathcal{D}:\ ]0,1[\times\mathbb{R}_+^*\longrightarrow]0,1[\times\mathbb{R}_+^*$$

given by

$$(x,y)\longmapsto\begin{cases}\left(\frac{\{\frac{x}{y-1}\}}{\frac{1}{y-1}-\lfloor\frac{x}{y-1}\rfloor},\frac{1}{\frac{1}{y-1}-\lfloor\frac{x}{y-1}\rfloor}\right) & \text{if}\quad y>1,\\[2ex] \left(\{\frac{x}{1-y}\},\frac{y}{1-y}-\lfloor\frac{x}{1-y}\rfloor\right) & \text{if}\quad y<1,\\[2ex] (0,1) & \text{if}\quad y=1.\end{cases}$$

Definition 3. Given an admissible circle map with parameters (α,β), the associated $\mathcal{D}$-expansion is given by the sequence $(a_n,i_n)_{n\in\mathbb{N}}$ which is defined as follows:

$$\begin{aligned}a_n &= \left\lfloor\left|\frac{x_n}{y_n-1}\right|\right\rfloor\\ i_n &= \begin{cases}1 & \text{if}\quad y_n<1\\ 0 & \text{if}\quad y_n>1\end{cases}\end{aligned}$$

where

$$(x_n,y_n)=\mathcal{D}^n(x_0,y_0)\ \text{ and }\ (x_0,y_0)=\left(\frac{1-\left\lfloor\frac{1-\beta}{\alpha}\right\rfloor\alpha-\beta}{1-\left(\left\lfloor\frac{1-\beta}{\alpha}\right\rfloor+1\right)\alpha},\frac{\alpha}{1-\left(\left\lfloor\frac{1-\beta}{\alpha}\right\rfloor+1\right)\alpha}\right).$$

For a circle map corresponding to $(\alpha,\beta)\in[0,1[^2$ which is nondegenerate and not admissible (i.e., $\alpha>\beta$), its $\mathcal{D}$-expansion is given by the $\mathcal{D}$-expansion associated with the admissible circle map corresponding to $(1-\alpha,1-\beta)$.

Conversely, for any sequence $(a_n, i_n)_{n\in\mathbb{N}}$ with $(a_n)_{n\in\mathbb{N}}$ not ultimately vanishing and $(i_n)_{n\in\mathbb{N}}$ not ultimately constant, and any $k \in \mathbb{N}$, there is exactly one nondegenerate pair (α, β) such that $\lfloor \frac{1-\beta}{\alpha} \rfloor = k$ and the corresponding circle map sequence has $\mathcal{D}$-expansion $(a_n, i_n)_{n\in\mathbb{N}}$.

We also want to mention the recent paper [21] of Ferenczi, Holton, and Zamboni which introduces a generalized continued fraction algorithm for three-interval exchange transformations which is based on a different induction process.

1.4. **Results.** Our main result is Theorem 4 which gives a characterization of linearly recurrent nondegenerate circle map subshifts.

Theorem 4. *A nondegenerate circle map subshift is linearly recurrent if and only if its $\mathcal{D}$-expansion $(a_n, i_n)_{n\in\mathbb{N}}$ satisfies the following: there exists an integer M such that for every integer n,*

(i) $a_n \leq M$,
(ii) $i_n = i_{n+1} = \ldots = i_{n+t} \Rightarrow t \leq M$,
(iii) $a_n = a_{n+1} = \ldots = a_{n+t} = 0 \Rightarrow t \leq M$.

In the following, we will call this condition the $(*)$-condition.

In particular, the class of LR nondegenerate circle map subshifts contains, but is not equal to, the circle map subshifts corresponding to parameters (α, β), where α and β lie in the same quadratic field. This follows directly from the fact proved in [16] that a primitive substitutive subshift is linearly recurrent and

Theorem 5 (Adamczewski [1]). *For a subshift associated with a nondegenerate circle map corresponding to parameters (α, β), the following are equivalent:*

(i) *It is primitive substitutive, that is, it can be generated by the morphic image of a fixed point of a primitive substitution.*
(ii) *The associated $\mathcal{D}$-expansion is ultimately periodic.*
(iii) *α and β lie in the same quadratic field.*

In terms of interval exchange transformations, Theorem 4 is a full geometric generalization of the following theorem.

Theorem 6 (Durand [19]). *A Sturmian subshift associated with an irrational number α is linearly recurrent if and only if the coefficients of the continued fraction expansion of α are bounded.*

2. Definitions and Background

2.1. **Symbolic sequences and substitutions.** A finite and nonempty set $\mathcal{A}$ is called alphabet. The elements of $\mathcal{A}$ are called letters. A finite word on $\mathcal{A}$ is a finite sequence of letters and an infinite word on $\mathcal{A}$ is a sequence of letters indexed by $\mathbb{N}$. The length of a finite word ω, denoted by $|\omega|$, is the number of letters it is built from. The empty word, ε, is the unique word of length 0. We denote by $\mathcal{A}^*$ the set of finite words on $\mathcal{A}$ and by $\mathcal{A}^{\mathbb{N}}$ the set of sequences over $\mathcal{A}$.

Let $U = (u_k)_{k\in\mathbb{N}}$ be a symbolic sequence defined over the alphabet $\mathcal{A}$. A factor of U is a finite word of the form $u_i u_{i+1} \ldots u_j$, $0 \leq i \leq j$. If ω is a factor of U and a a letter, then $|w|_a$ is the number of occurrences of the letter a in the word ω.

We denote by $\mathcal{L}(U)$ the set of all the factors of the sequence U, $\mathcal{L}(U)$ is called the language of U. A sequence in which all the factors have an infinite number of occurrences is called *recurrent*. When these occurrences have bounded gaps, the sequence is called *uniformly recurrent*. A sequence U is called K-power free if $u^K \in \mathcal{L}(U)$ implies $u = \varepsilon$. A sequence U is called power free if there exists an integer K such that U is K-power free.

Endowed with concatenation, the set $\mathcal{A}^*$ is a free monoid with unit element ε. A map from $\mathcal{A}$ to $\mathcal{A}^* \setminus \{\varepsilon\}$, called *substitution* on $\mathcal{A}$, can be extended by concatenation to an endomorphism of the free monoid $\mathcal{A}^*$ and then to a map from $\mathcal{A}^{\mathbb{N}}$ to itself. Given a substitution σ defined on $\mathcal{A}$, we call the matrix $M_\sigma = (|\sigma(j)|_i)_{(i,j)\in\mathcal{A}^2}$ the *incidence matrix* associated with σ. The composition of substitutions corresponds to the multiplication of incidence matrices. A substitution is called primitive if there exists a power of its incidence matrix for which all the entries are positive.

2.2. **Return words.** We present here the main definitions concerning the notion of return words introduced in [17]. Let U be a uniformly recurrent sequence over the alphabet $\mathcal{A}$ and let $u = u_1u_2\dots u_n$ be a nonempty prefix of U. A *return word* to u of U is a factor $u_{[i,j-1]}$ $(= u_iu_{i+1}\dots u_{j-1})$ of U such that i and j are two consecutive occurrences of u. The sequence U can be written in a unique way as a concatenation of return words to u. Let $\mathcal{R}_{U,u}$ be the set of return words to u in U. Then $U = \omega_0\omega_1\dots\omega_i\dots$, where $\omega_i \in \mathcal{R}_{U,u}$. The fact that U is uniformly recurrent implies that $\mathcal{R}_{U,u}$ is a finite set. We can therefore consider a bijective map $\Lambda_{U,u}$ from $\mathcal{R}_{U,u}$ to the finite set $\{1, 2, \dots, \mathrm{Card}(\mathcal{R}_{U,u})\} = \mathcal{A}_{U,u}$, where, for definiteness, the return words are ordered according to their first occurrence (i.e., $\Lambda_{U,u}^{-1}(1)$ is the first return word ω_0, $\Lambda_{U,u}^{-1}(2)$ is the first ω_i which is different from ω_0, and so on). The *derived sequence* of U on u is the sequence with values in the alphabet $\mathcal{A}_u$ given by

$$\mathcal{D}_u(U) = \Lambda_{U,u}(\omega_0)\Lambda_{U,u}(\omega_1)\dots\Lambda_{U,u}(\omega_i)\dots.$$

To such a sequence we can associate a morphism $\Theta_{U,u}$ from $\mathcal{A}_{U,u}$ to $\mathcal{A}^*$ defined by:

$$\Theta_{U,u}(i) = \omega_i.$$

We obtain $\Theta_{U,u}(\mathcal{D}_u(U)) = U$. The morphism $\Theta_{U,u}$ is called the return morphism to u of U. When $\mathcal{A}_{U,u} = \mathcal{A}$, we will call it *return substitution* to u of U. When it does not create confusion, we will suppress the "U" in the symbols $\mathcal{R}_{U,u}$, $\Theta_{U,u}$, and $\mathcal{A}_{U,u}$.

Proposition 7 (Durand [17])**.** *Let u be a nonempty prefix of U. Then the following holds.*

(i) *The set $\mathcal{R}_u$ is a code and the map Θ_u is one to one.*
(ii) *Let v be a nonempty prefix of $\mathcal{D}_u(U)$. Then there exists a nonempty prefix w of U such that $\mathcal{D}_v(\mathcal{D}_u(U)) = \mathcal{D}_w(U)$. Moreover, we have $\Theta_u \circ \Theta_v = \Theta_w$.*

A derived sequence of a derived sequence is hence a derived sequence.

Definition 8. Let U be a symbolic sequence defined over the alphabet $\mathcal{A}$ starting with the symbol $1 \in A$. We introduce the following notation: $\mathcal{D}^{(0)}(U) = U$ and, for $n \in \mathbb{N}$, $\mathcal{D}^{(n+1)}(U) = \mathcal{D}_1(\mathcal{D}^{(n)}(U))$; Θ_0 is the identity map and, for $n \in \mathbb{N}$, $\Theta_{n+1} = \Theta_n \circ \Theta_{\mathcal{D}^{(n)}(U),1}$.

Remark 9. According to Proposition 7, we obtain that $(\mathcal{D}^{(n)})_{n\in\mathbb{N}}$ is a sequence of derived sequences of U and $(\Theta_n)_{n\in\mathbb{N}}$ is a sequence of return morphisms of U.

2.3. **LR sequences.**

Definition 10. Let $\mathcal{A}$ be an alphabet, K a positive integer, and U a sequence over $\mathcal{A}$. The sequence U is called K-linearly recurrent (K-LR) if it is uniformly recurrent and for all $\omega \in \mathcal{R}_u$, we have $|\omega| \leq K|u|$. A sequence is called linearly recurrent (LR) if it is K-LR for some K.

Proposition 11 (DHS [16]). *Let U be an aperiodic K-LR sequence over an alphabet $\mathcal{A}$. Then:*

1. *For every n, each factor of U of length n has at least one occurrence in each factor of U of length $(K+1)n$.*
2. *U is $(K+1)$-power free.*
3. *For every nonempty prefix u of U and for all $\omega \in \mathcal{R}_u$, we have $\frac{1}{K}|u| < |\omega|$.*

2.4. **Subshifts and LR subshifts.** Let $\mathcal{A}$ be an alphabet. The topology of $\mathcal{A}^{\mathbb{N}}$ is given by the product of the discrete topologies on $\mathcal{A}$. We denote by T the standard *shift transformation* which associates to each symbolic sequence $U = (u_k)_{k\geq 0}$ the sequence $T(U) = (u_k)_{k\geq 1}$. To a sequence U in $\mathcal{A}^{\mathbb{N}}$ we associate the dynamical system $(\overline{\mathcal{O}(U)}, T)$, where $\overline{\mathcal{O}(U)}$ is the closure of the orbit of U under the shift. This dynamical system is called the *subshift* associated with U. A dynamical system is *minimal* if it has no nontrivial invariant closed set. For a subshift associated with a sequence U, minimality of the subshift is equivalent to uniform recurrence of U.

Definition 12. A subshift is called primitive substitutive if it contains a primitive substitutive sequence (i.e., a sequence which is the morphic image of a fixed point of a primitive substitution). A minimal subshift associated with a sequence U is called linearly recurrent (LR) if and only if U is LR.

2.5. **S-adic sequences and S-adic subshifts.** Let $\mathcal{A}$ be an alphabet, a a letter of $\mathcal{A}$, and S a finite set of substitutions from $\mathcal{A}$ to $\mathcal{A}^*$. We will say that a sequence $U \in \mathcal{A}^{\mathbb{N}}$ is an *S-adic* sequence (generated by $(\sigma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ and a) if there exists a sequence $(\sigma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ such that $U = \lim_{n\to\infty} \sigma_0\sigma_1 \ldots \sigma_n(aa\ldots)$. Let U be such a sequence. If there exists an integer s_0 such that for all $b \in \mathcal{A}$ and all $c \in \mathcal{A}$, the letter b has an occurrence in $\sigma_{r+1}\sigma_{r+2}\ldots\sigma_{r+s_0}(c)$, then U is called a *primitive S-adic* sequence (with constant s_0).

The subshift associated with an S-adic sequence (resp., a primitive S-adic sequence) is called an S-adic subshift (resp., a primitive S-adic subshift). These notions were introduced by S. Ferenczi in [20] and by F. Durand in [18].

It was claimed in [18] that a subshift is LR if and only if it is primitive S-adic. In [19], Durand provides a counterexample and exhibits a primitive S-adic subshift which is not LR. However, LR does imply primitive S-adic and with an additional condition we can obtain a partial converse given in Proposition 14 below.

Definition 13. Let $\mathcal{A}$ be an alphabet and σ a substitution on $\mathcal{A}$. The substitution σ is called (b,c)-proper if for any letter i in $\mathcal{A}$, $\sigma(i)$ begins with b and ends with c.

An S-adic sequence is called *proper* if there exist two letters b and c in $\mathcal{A}$ such that any substitution in S is a (b,c)-proper substitution. A subshift which contains a proper and primitive S-adic sequence is called a *proper primitive S-adic* subshift.

Proposition 14 (Durand [19]). *A subshift (X,T) is LR if and only if it is a proper primitive S-adic subshift.*

2.6. **Interval exchange transformations.** Interval exchange transformations are classical examples of dynamical systems.

Definition 15. Let $s \in \mathbb{N}$, $s \geq 2$. Let σ be a permutation of the set $\{1, 2, \dots, s\}$ and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ be a vector in $\mathbb{R}^s$ with strictly positive entries. Let

$$I = [0, |\lambda|[, \text{ where } |\lambda| = \sum_{i=1}^{s} \lambda_i \text{ and for } 1 \leq i \leq s,\ I_i = \left[\sum_{j<i} \lambda_j, \sum_{j \leq i} \lambda_j \right[.$$

The interval exchange transformation associated with (λ, σ) is the map E from I into itself, defined as the piecewise isometry which arises from ordering the intervals I_i with respect to σ. More precisely, if $x \in I_i$,

$$E(x) = x + a_i, \text{ where } a_i = \sum_{k<\sigma^{-1}(i)} \lambda_{\sigma_k} - \sum_{k<i} \lambda_k.$$

We can introduce a natural coding of the orbit of a point under the action of an interval exchange transformation by assigning to each element of this orbit the number of the interval which contains it.

Remark 16. Let us consider an interval exchange transformation E, and U the natural coding of the orbit of the point 0 under E. The natural coding of the orbit of the point 0 under the action of the induced map of E on its first interval is the derived sequence on the letter 1 of U. Moreover, the associated induced substitution corresponds to the return substitution to 1 of U. In the case of the Rauzy induction, one does not induce on the first interval but on an interval which is larger. However, the induction on the first interval can be decomposed into several steps of the Rauzy induction.

We refer the reader to [31] for information on the useful notion of Rauzy induction for interval exchange transformations.

3. A Geometric Interpretation

In this section, we investigate the geometric link between Theorems 4 and 6.

The symmetric Rauzy induction for two-interval exchange transformations is introduced in [2]. From the study of this induction process, the authors of [2] obtain an S-adic expresion for Sturmian subshifts. Let τ_1 and τ_2 be substitutions on $\{0, 1\}$ defined as follows:

$$\begin{array}{lll} \tau_1(0) = 01 & \text{and} & \tau_2(0) = 0 \\ \tau_1(1) = 1 & & \tau_2(1) = 10. \end{array}$$

Proposition 17. *Let $\alpha \in (0, 1)$ be an irrational number. The Sturmian subshift associated with α is generated by the sequence*

$$\lim_{n \to \infty} \tau_2^{i_1} \tau_1^{i_2} \tau_2^{i_3} \tau_1^{i_4} \dots \tau_2^{i_{2n-1}} \tau_1^{i_{2n}}(0),$$

where $[0; i_1 + 1, i_2, i_3, i_4, \dots]$ is the continued fraction expansion of α.

The decomposition of the two-interval exchange transformation associated to α under the symmetric Rauzy induction is symbolized in Figure 1. The fact that α is irrational implies that this two-interval exchange transformation satisfies the well-known I.D.O.C. condition introduced in [27]. It also implies that an orbit under the

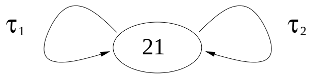

FIGURE 1. The symmetric Rauzy induction graph for two-interval exchange transformations.

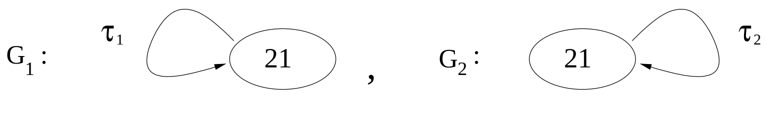

FIGURE 2. The primitivity subgraphs for two-interval exchange transformations.

symmetric Rauzy induction does not ultimately remain in one of the primitivity subgraphs G_1 or G_2 represented in Figure 2.

Moreover, an I.D.O.C. two-interval exchange is LR if and only if its orbit under the symmetric Rauzy induction can stay in any of the primitivity subgraphs G_1 and G_2 only for a bounded number of consecutive induction steps. This last remark provides a geometric interpretation of Theorem 6.

We present now an analogous study in the case of nondegenerate circle map subshifts. Let us introduce the following four substitutions, defined over the alphabet $\{1,2,3\}$, given by:

$$\begin{array}{ccc} & \sigma_1 & \\ 1 & \longmapsto & 13 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 3 \end{array} \qquad \begin{array}{ccc} & \sigma_2 & \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 23 \end{array} \qquad \begin{array}{ccc} & \sigma_3 & \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 23 \\ 3 & \longmapsto & 3 \end{array} \qquad \begin{array}{ccc} & \sigma_4 & \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 13 \\ 3 & \longmapsto & 2 \end{array}$$

For each integer k, we also consider the following morphism:

$$\begin{array}{rrcl} \Phi_k : & \{1,2,3\}^* & \longrightarrow & \{1,0\}^* \\ & 1 & \longmapsto & 1, \\ & 2 & \longmapsto & 10^{k+1}, \\ & 3 & \longmapsto & 10^k. \end{array}$$

If $(U_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$, the sequence $(\overline{U_n})_{n\in\mathbb{N}}$ is defined by

$$\overline{U_n} = \begin{cases} 1 & \text{if } U_n = 0, \\ 0 & \text{if } U_n = 1. \end{cases}$$

Having fixed the above notation, we can give the following S-adic expression for nondegenerate circle map subshifts.

Theorem 18 (Adamczewski [1]). *Let us consider nondegenerate parameters (α,β) $\in (0,1)$ and let $(a_n, i_n)_{n\in\mathbb{N}}$ be the $\mathcal{D}$-expansion associated with (α,β). The circle map subshift associated with parameters (α,β) is generated by the sequence*

$$\lim_{n\to\infty} \Phi_{\lfloor \frac{1-\beta}{\alpha} \rfloor} \left(\prod_{j=0}^{n} \left(\left(\sigma_1 \sigma_2^{a_j} \sigma_3\right)^{i_j} \circ \left(\sigma_4 \sigma_1^{a_j} \sigma_4\right)^{1-i_j} \right) (1) \right)$$

if $\alpha < \beta$ *and by*

$$\lim_{n\to\infty} 1T\left(\overline{\Phi_{\lfloor \frac{\beta}{1-\alpha}\rfloor}\left(\prod_{j=0}^{n}\left(\left(\sigma_1\sigma_2^{a_j}\sigma_3\right)^{i_j}\circ\left(\sigma_4\sigma_1^{a_j}\sigma_4\right)^{1-i_j}\right)(1)\right)}\right)$$

if $\alpha > \beta$.

The proof is based on a study of an induction process for three-interval exchange transformations close to that of Rauzy. We also obtain an analog to Proposition 17 in the case of nondegenerate circle map subshifts. Figure 3 is the analog of Figure 1 and Figure 4 is the analog of Figure 2. To a nondegenerate circle map we can associate an I.D.O.C. three-interval exchange transformation. The orbit of such an interval exchange transformation under the Rauzy induction does not ultimately remain in one of the primitivity subgraphs G_1, G_2, or G_3 represented in Figure 4.

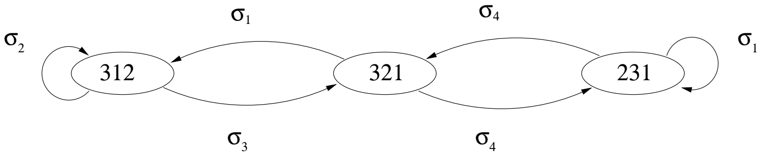

FIGURE 3. The Rauzy induction graph for three-interval exchange transformations.

Moreover, an I.D.O.C. three-interval exchange transformation is LR if and only if its orbit under the Rauzy induction can stay in any of the primitivity subgraphs G_1, G_2, and G_3 only for a bounded number of consecutive induction steps. This last remark provides a geometric interpretation of the $(*)$-condition in Theorem 4 and will be proved in Section 4.

A similar study could clearly be carried out in the general case of an I.D.O.C. interval exchange transformation. However, the results quickly become hard to read since the complexity of the equivalent to the $(*)$-condition increases rapidly (cf. Appendix B).

In this section we have exhibited some similarities between the Sturmian and the circle map cases. On the other hand, some aspects of the two cases do not have mutual counterparts. The strategy used to prove Theorem 6 is the following:

- Exhibit a primitive S-adic expression for Sturmian subshifts generated by an irrational α when the coefficients of the continued fraction expansion of α are bounded and use this to establish linear recurrence in this case.
- Show that otherwise a Sturmian sequence contains arbitrarily high powers.

We thus obtain that a Sturmian sequence is LR if and only if it is power free. This equivalence is also true for some generalizations of Sturmian sequences, namely Arnoux-Rauzy sequences (see [32]). However, such an equivalence does not hold for circle maps. We can therefore not mimic the strategy used in the Sturmian case. For example, the circle map sequences with $\mathcal{D}$-expansion $(a_n, i_n)_{n\in\mathbb{N}}$, where $(i_n)_{n\in\mathbb{N}}$ is the periodic sequence $(10)^\omega$, $a_n = 1$ if n is a power of 2 and 0 otherwise, are both non-LR and power free (see Appendix A).

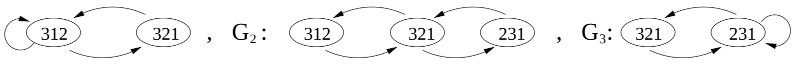

FIGURE 4. The primitivity subgraphs for three-interval exchange transformations.

4. Proof of Theorem 4

The proof of Theorem 4 is based on Theorem 18 and Proposition 14 which states that a proper primitive S-adic subshift is LR. Our strategy to prove this theorem is the following:

- We exhibit a proper primitive S-adic expression for three-interval exchanges associated with circle maps whose $\mathcal{D}$-expansion satisfies the $(*)$-condition (Proposition 20).
- We prove the existence of a uniform upper bound of the gaps between successive occurrences of letters in the different derived sequences of an LR-sequence (Lemma 24).
- Finally, we show that such a uniform bound does not exist for a circle map whose $\mathcal{D}$-expansion does not satisfy the $(*)$-condition (Proposition 23).

For $i \in \{1,2,3,4\}$, let A_i denote the incidence matrix of the substitution σ_i which has been defined in the previous section. For every integer k, we write

$$\mathcal{F}_k = (\sigma_1\sigma_2^k\sigma_3) \text{ and } \mathcal{G}_k = (\sigma_4\sigma_1^k\sigma_4), \tag{1}$$

and for the associated incidence matrices, we write

$$\mathcal{B}_k = (A_1A_2^kA_3) \text{ and } \mathcal{C}_k = (A_4A_1^kA_4). \tag{2}$$

Definition 19. Let $(C,D) \in \mathcal{M}_3(\mathbb{R})^2$, $C = (c_{i,j})$, $D = (d_{i,j})$. We say that $C \geq D$ if $c_{i,j} \geq d_{i,j}$, $\forall (i,j) \in \{1,2,3\}^2$. Similarly, we say that $C > D$ holds if $c_{i,j} > d_{i,j}$, $\forall (i,j) \in \{1,2,3\}^2$.

Proposition 20. *A nondegenerate circle map whose $\mathcal{D}$-expansion satisfies the $(*)$-condition is the image by a morphism of a proper primitive S-adic sequence.*

Lemma 21. *If C is a nonnegative matrix in $\mathcal{M}_3(\mathbb{Z})$, then for every integer k, the following four inequalities hold:*

$$\mathcal{B}_kC \geq C,\ C\mathcal{B}_k \geq C,\ \mathcal{C}_kC \geq C,\ \text{and } C\mathcal{C}_k \geq C.$$

Proof. This follows directly from $\mathcal{B}_k = I_3 + A'_k$ with $A'_k \geq 0$ and $\mathcal{C}_k = I_3 + B'_k$ with $B'_k \geq 0$. □

Lemma 22. *Let $(a_n,i_n)_{n\in\mathbb{N}}$ be a $\mathcal{D}$-expansion satisfying the $(*)$-condition with an integer M_0 and let $S = \left\{\prod_{j=kM_0}^{(k+1)M_0} \mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j},\ k \in \mathbb{N}\right\}$. Then S is a finite set of substitutions and each of its element is $(1,3)$-proper.*

Proof. The set S is clearly finite because the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded by M_0. In view of (1), we obtain for every integer k

$$\begin{array}{ccc} & \mathcal{F}_k & \\ 1 & \longmapsto & 13 \\ 2 & \longmapsto & 2^{k+1}3 \\ 3 & \longmapsto & 2^k3 \end{array} \qquad \begin{array}{ccc} & \mathcal{G}_k & \\ 1 & \longmapsto & 12^k \\ 2 & \longmapsto & 12^{k+1} \\ 3 & \longmapsto & 13 \end{array}$$

Let k be an integer and $i \in \{1,2,3\}$. Then $\mathcal{F}_k(i)$ ends with 3 and $\mathcal{F}_k(1)$ begins with 1. Moreover $\mathcal{G}_k(i)$ begins with 1 and $\mathcal{G}_k(1)$ ends with 3. It follows thus that each composition of substitutions of types $\mathcal{F}_k$ and $\mathcal{G}_k$ in which the two types both appear is $(1,3)$-proper. Part (ii) of the $(*)$-condition allows us to conclude. $\square$

Proof of Proposition 20. Let us consider a circle map U whose $\mathcal{D}$-expansion satisfies the $(*)$-condition with some integer M_0. Theorem 18 provides us with an S-adic expression for this circle map. Our goal is now to prove that we can extract a proper primitive S-adic expression for U from this representation.

We can suppose that U is admissible in order to simplify the notation. We have

$$U = \lim_{n\to\infty} \Phi_{\lfloor \frac{1-\beta}{\alpha} \rfloor} \left(\prod_{j=0}^{n} \left(\left(\sigma_1 \circ \sigma_2^{a_j} \circ \sigma_3\right)^{i_j} \circ \left(\sigma_4 \circ \sigma_1^{a_j} \circ \sigma_4\right)^{1-i_j} \right) (1) \right).$$

Let

$$V = \lim_{n\to\infty} \left(\prod_{j=0}^{n} \left(\left(\sigma_1 \circ \sigma_2^{a_j} \circ \sigma_3\right)^{i_j} \circ \left(\sigma_4 \circ \sigma_1^{a_j} \circ \sigma_4\right)^{1-i_j} \right) (1) \right). \tag{3}$$

Thus,

$$U = \Phi_{\lfloor \frac{1-\beta}{\alpha} \rfloor}(V) \tag{4}$$

and

$$V = \lim_{n\to\infty} \left(\prod_{j=0}^{n} \mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j} \right)(1).$$

We have then

$$V = \lim_{n\to\infty} \left(\prod_{k=0}^{n} \left(\prod_{j=kM_0}^{(K+1)M_0} \mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j} \right) \right)(1). \tag{5}$$

Let

$$S = \left\{ \prod_{j=kM_0}^{(k+1)M_0} \mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j},\ k \in \mathbb{N} \right\}.$$

Then Lemma 22 implies that (5) gives us a proper S-adic representation of V.

We have now to prove that this representation is primitive or more precisely that there exists an integer s_0 such that for every integer r and all $b \in \{1,2,3\}$ and $c \in \{1,2,3\}$, the letter b has an occurrence in

$$\left(\prod_{k=r}^{r+s_0} \left(\prod_{j=rM_0}^{(r+1)M_0} \mathcal{F}_{a_j}^{i_j} \circ \mathcal{G}_{a_j}^{1-i_j} \right) \right)(c).$$

Or similarly, we have to show that the corresponding product of matrices

$$\left(\prod_{k=r}^{r+s_0} \left(\prod_{j=rM_0}^{(r+1)M_0} \mathcal{B}_{a_j}^{i_j} \circ \mathcal{C}_{a_j}^{1-i_j} \right) \right)$$

is positive, where the matrices $\mathcal{B}_k$ and $\mathcal{C}_k$ are defined in (2). Let us consider the matrix

$$\mathcal{M}_r = \left(\prod_{j=rM_0}^{(r+1)M_0} \mathcal{B}_{a_j}^{i_j} \circ \mathcal{C}_{a_j}^{1-i_j} \right).$$

By the fact that the $\mathcal{D}$-expansion associated with U satisfies the $(*)$-condition with the integer M_0, we get

$$\begin{array}{lll} \exists j_1 \in \{1,2,\ldots,l\} & \text{such that} & i_{j_1} = 0, \\ \exists j_2 \in \{1,2,\ldots,l\} & \text{such that} & i_{j_2} = 1, \\ \exists j_3 \in \{1,2,\ldots,l\} & \text{such that} & a_{j_3} \geq 1. \end{array}$$

The previous remark and Lemma 21 show that at least one of the following inequalities holds:

$$\begin{array}{l} \mathcal{M}_r \geq \mathcal{B}_0\mathcal{C}_1, \\ \mathcal{M}_r \geq \mathcal{B}_1\mathcal{C}_0, \\ \mathcal{M}_r \geq \mathcal{C}_1\mathcal{B}_0, \\ \mathcal{M}_r \geq \mathcal{C}_0\mathcal{B}_1. \end{array}$$

Now we just have to remark that each element of $\{\mathcal{B}_0\mathcal{C}_1, \mathcal{B}_1\mathcal{C}_0, \mathcal{C}_1\mathcal{B}_0, \mathcal{C}_0\mathcal{B}_1\}^2$ is positive. Therefore, we obtain primitive S-adicity of our representation with constant $s_0 = 2$. We therefore obtain that U is the image under the morphism $\Phi_{\lfloor \frac{1-\beta}{\alpha} \rfloor}$ of the proper primitive S-adic sequence V, concluding the proof. □

Proposition 23. *A nondegenerate circle map subshift whose $\mathcal{D}$-expansion does not satisfy the $(*)$-condition is not linearly recurrent.*

Since we will work with the derived sequences of a given circle map sequence in our proof of Proposition 23, we start off by discussing LR properties of derived sequences of an LR sequence.

Lemma 24. *Let U be a K-linearly recurrent sequence defined over an alphabet $\mathcal{A}$ and let ω be a nonempty prefix of U. Then every word of length at least $K^2(K+1)$ in $D_\omega(U)$ contains all the elements of $\mathcal{A}_\omega$.*

Proof. Let ω be a factor of U and $i \in \mathcal{A}_\omega = \{1,2,\ldots,d\}$. Then there exists a unique word ω_i such that $\Theta_\omega(i) = \omega_i$. By definition we have

$$\forall j \in \mathcal{A}_\omega, \quad |\omega_j| \leq K|\omega|.$$

This inequality implies that ω_i appears in each word of length at least $(K+1)(K|\omega|)$, in view of Proposition 11. Moreover, again by Proposition 11, we have

$$\forall j \in \mathcal{A}_\omega, \quad \frac{1}{K}|\omega| \leq |\omega_j| \leq K|\omega|.$$

The set $\mathcal{R}_\omega$ is a code. We thus obtain that the letter i occurs in each word of length at least $K^2(K+1)$ in $D_\omega(U)$. □

Lemma 25. *Let U be a K-linearly recurrent sequence. Then, for every integer n, we have*

$$\forall i \in \mathcal{A}_n, \quad |\Theta_n(i)| \leq K^2(K+1),$$

where the maps Θ_n are introduced in Definition 8.

Proof. Let i be an element of $\mathcal{A}_n$ and $\Theta_n(i) = \omega_i$. By definition of the return words and the sequence $\mathcal{D}^{(n)}$, the letter 1 has just one occurrence in ω_i and 1 is the first letter of ω_i. Then, 1 does not appear in the maximal proper suffix of ω_i. Lemma 24 implies that the length of this suffix is at most $K^2(K+1)-1$. □

Lemma 26. *Let U be a K-linearly recurrent sequence defined over an alphabet $\mathcal{A}$ and let ω be a nonempty prefix of U. Then the sequence $D_\omega(U)$ is K^3-linearly recurrent.*

Proof. This statement and its proof are very similar in spirit to the previous two lemmas. Let x be a factor of $D_\omega(U)$. Consider any occurrence of x in $D_\omega(U)$ and the length of the corresponding return word to x in $D_\omega(U)$ (i.e., the length of the gap between this occurrence of x and the next, plus the length of x). We use again that $\mathcal{R}_\omega$ is a code. Namely, to this occurrence of x in $D_\omega(U)$ corresponds a word of length at most $K \cdot |\omega| \cdot |x|$ in U whose return words have length at most $K^2 \cdot |\omega| \cdot |x|$. Choose the one that corresponds to this particular occurrence and go back via $\Lambda_{U,\omega}$ to factors of $D_\omega(U)$. We conclude that the length of the return word to x in question is bounded by $K^3 \cdot |x|$. In the previous steps, we have made repeated use of Proposition 11. This shows that $D_\omega(U)$ is K^3-linearly recurrent since x and its occurrence were arbitrary. □

Lemma 27. *Let r be a positive integer. Then for every $(i_1, i_2, \dots, i_r) \in \{1,2,3,4\}^r$, we have*

$$|\sigma_{i_1} \circ \sigma_{i_2} \circ \dots \circ \sigma_{i_r}(123)| \geq \left|\sigma_{i_1} \circ \sigma_{i_2} \circ \dots \circ \sigma_{i_{r-1}}(123)\right| + 1.$$

Proof. We just have to remark that for each $k \in \{1,2,3,4\}$, there exists a letter $b \in \{1,2,3\}$ such that $|\sigma_k(b)| \geq 2$ and that 1, 2, and 3 occur in $\sigma_k(123)$. □

Lemma 28. *Let r be a positive integer and $(i_1, i_2, \dots, i_{3r+1}) \in \{1,2,3,4\}^{3r+1}$. Then there exists at least one letter $b \in \{1,2,3\}$ such that*

$$\left|\sigma_{i_1} \circ \sigma_{i_2} \circ \dots \circ \sigma_{i_{3r+1}}(b)\right| > r.$$

Proof. According to Lemma 27, it follows by induction that

$$\left|\sigma_{i_1} \circ \sigma_{i_2} \circ \dots \circ \sigma_{i_{3r+1}}(123)\right| \geq 3r+1.$$

The assertion follows immediately. □

Lemma 29. *Let n be an integer, $(m_0, m_1, \dots, m_n) \in \mathbb{N}^n$, and $(l_0, l_1, \dots, l_n) \in \{0,1\}^n$. Then, for each $b \in \{1,2,3\}$, we have*

(i) $\left|\prod_{j=0}^n \left(\sigma_1 \sigma_2^{m_j} \sigma_3\right)(b)\right|_1 \leq 1,$

(ii) $\left|\prod_{j=0}^n \left((\sigma_1\sigma_3)^{l_j} \circ (\sigma_4\sigma_4)^{1-l_j}\right)(b)\right|_2 \leq 1,$

(iii) $\left|\prod_{j=0}^n \left(\sigma_4 \sigma_1^{m_j} \sigma_4\right)(b)\right|_3 \leq 1.$

Here, $|w|_i$ denotes the number of occurrences of the symbol i in the word w.

Proof. (i) The incidence matrix associated with the substitution $\prod_{j=0}^n \left(\sigma_1 \sigma_2^{m_j} \sigma_3\right)$ is $\prod_{j=0}^n \mathcal{B}_{m_j}$, where the matrices $\mathcal{B}_{m_j}$ are defined in (2). For each integer k, the matrix $\mathcal{B}_k$ is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix},$$

and so the matrix $\prod_{j=0}^{n} \mathcal{B}_{m_j}$ is of course of the same form. Then, the definition of the incidence matrix allows us to conclude.

(ii) The incidence matrix associated with $\prod_{j=0}^{n} \left((\sigma_1\sigma_3)^{l_j} \circ (\sigma_4\sigma_4)^{1-l_j} \right)$ is equal to $\prod_{j=0}^{n} \left(\mathcal{B}_0^{l_j} \mathcal{C}_0^{1-l_j} \right)$. The matrices $\mathcal{B}_0$ and $\mathcal{C}_0$ are of the form

$$\begin{pmatrix} \times & \times & \times \\ 0 & 1 & 0 \\ \times & \times & \times \end{pmatrix},$$

and so the matrix $\prod_{j=0}^{n} \left(\mathcal{B}_0^{l_j} \mathcal{C}_0^{1-l_j} \right)$ is of the same form.

(iii) The incidence matrix associated with the substitution $\prod_{j=0}^{n} \left(\sigma_4 \sigma_1^{m_j} \sigma_4 \right)$ is equal to $\prod_{j=0}^{n} \mathcal{C}_{m_j}$, where the matrices $\mathcal{C}_{m_j}$ are defined in (2). For each integer k, the matrix $\mathcal{C}_k$ is of the form

$$\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & 0 & 1 \end{pmatrix},$$

and so the matrix $\prod_{j=0}^{n} \mathcal{C}_{m_j}$ is of the same form, concluding the proof. □

Proof of Proposition 23. Let U be a circle map whose $\mathcal{D}$-expansion $(a_n, i_n)_{n\in\mathbb{N}}$ does not satisfy the $(*)$-condition. Let V be as in (3) so that we have (4). Let us assume for the moment that $1-\beta > \alpha$ so that V is the derived sequence corresponding to the prefix 1 of U. We will comment later on the case $1-\beta < \alpha$.

Now assume there exists an integer K such that U is K-LR. We consider four cases.

(i) Let us suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ is unbounded. Then a direct consequence of the fact that $\sigma_2^{a_n}(3) = 2^{a_n}3$, $\sigma_1^{a_n}(1) = 13^{a_n}$, and that powers propagate by substitution is that U cannot be $(K+1)$-power free. Proposition 11 thus yields a contradiction.

(ii) Let us suppose that the sequence $(i_n)_{n\in\mathbb{N}}$ contains arbitrarily long blocks of 1's. In particular, there exists an integer n_0 such that

$$i_{n_0} = i_{n_0+1} = \ldots = i_{n_0+12K^2(K+1)} = 1. \tag{6}$$

We recall that there exists an increasing sequence of integers $(k_N)_{N\in\mathbb{N}}$ such that

$$\Theta_N = \prod_{j=k_{N-1}+1}^{k_N} \left(\left(\sigma_1 \sigma_2^{a_j} \sigma_3\right)^{i_j} \circ \left(\sigma_4 \sigma_1^{a_j} \sigma_4\right)^{1-i_j} \right),$$

where Θ_N is introduced in Definition 8. This follows from Remark 16 and the fact that, as was already observed in [1], certain steps of our induction process correspond to induction on the first interval of three-interval exchange transformations associated with U. According to Lemmas 25 and 28, the fact that U is K-LR implies that for each integer N,

$$k_{N+1} - k_N < 3K^2(K+1) + 1. \tag{7}$$

Now, let us consider two particular elements of the sequence $(k_N)_{N\in\mathbb{N}}$:

$$k_{N_1} = \min \left\{ k_N, n_0 \le k_N \le n_0 + 12K^2(K+1) \right\}$$

and

$$k_{N_2} = \max\left\{k_N, n_0 \leq k_N \leq n_0 + 12K^2(K+1)\right\}.$$

By the inequality (7), we obtain that k_{N_1} and k_{N_2} are well-defined and

$$k_{N_2} - k_{N_1} \geq 6K^2(K+1) + 1. \tag{8}$$

Let us introduce the substitution $\Theta = \Theta_{N_1+1}\Theta_{N_1+2}\dots\Theta_{N_2}$. Then,

$$\Theta = \prod_{j=k_{N_1}+1}^{k_{N_2}} \left(\left(\sigma_1\sigma_2^{a_j}\sigma_3\right)^{i_j} \circ \left(\sigma_4\sigma_1^{a_j}\sigma_4\right)^{1-i_j}\right).$$

More precisely, using condition (6), we have

$$\Theta = \prod_{j=k_{N_1}+1}^{k_{N_2}} \left(\sigma_1\sigma_2^{a_j}\sigma_3\right). \tag{9}$$

Proposition 7 implies that Θ is a return substitution for U since it is a composition of return substitutions. Thus there exists a nonempty prefix ω of U such that $\Theta = \Theta_{U,\omega}$. According to the inequality (8) and Lemma 28, we obtain that there exists a letter b in the alphabet $\{1,2,3\}$ such that

$$|\Theta(b)| \geq \frac{k_{N_2} - k_{N_1}}{3} > 2K^2(K+1),$$

and it follows from the equality (9) and Lemma 29 that

$$|\Theta(b)|_1 \leq 1.$$

But $\Theta(b)$ is necessarily a factor of $\mathcal{D}_\omega(U)$. Hence there exists a factor of $\Theta(b)$ of length greater or equal than $K^2(K+1)$ in which the letter 1 does not occur. We obtain finally that there exists a factor of $\mathcal{D}_\omega(U)$ of length greater than or equal to $K^2(K+1)$ in which the letter 1 does not occur. This last remark is in contradiction with the K-LR property of U in view of Lemma 24.

(iii) Let us suppose that the sequence $(i_n)_{n\in\mathbb{N}}$ contains arbitrarily long blocks of 0's. Then, we just have to mimic the above arguments in order to find a return substitution Θ' for U and a letter b in $\{1,2,3\}$ such that

$$|\Theta'(b)|_3 \leq 1 \text{ and } |\Theta'(b)| > 2K^2(K+1).$$

We thus obtain a nonempty prefix ω' of U such that $\mathcal{D}_{\omega'}(U)$ contains a factor of length greater than or equal to $K^2(K+1)$ in which the letter 3 does not occur.

(iv) Let us suppose that the sequence $(a_n)_{n\in\mathbb{N}}$ contains arbitrarily long blocks of 0. Then, analogous reasoning gives a return substitution Θ'' for U and a letter b in $\{1,2,3\}$ such that:

$$|\Theta''(b)|_2 \leq 1 \text{ and } |\Theta''(b)| > 2K^2(K+1).$$

We find a nonempty prefix ω'' of U such that $\mathcal{D}_{\omega''}(U)$ contains a factor of length greater than or equal to $K^2(K+1)$ in which the letter 2 does not occur.

Thus we arrive at a contradiction in each case. Recall that we assumed $1-\beta > \alpha$ at the beginning of the proof. Let us now discuss the case where $1-\beta < \alpha$. In this case V in (3) is not the derived sequence corresponding to the prefix 1 of U, that is, $\mathcal{D}_1(U) \neq V$. In fact, V takes three values, while 1 has only two return words, 1 and 10. However, for sufficiently large n, it is relatively easy to see that $\mathcal{D}^{(n)}(U)$ is one of the sequences obtained in the induction process of [1] (leading to the representation (4)) and hence there is a morphism Ψ such that $V = \Psi(\mathcal{D}^{(n)}(U))$. If we now again assume that U is LR, then so is $\mathcal{D}^{(n)}(U)$, by Lemma 26, and hence we get that V is LR. Now we can derive a contradiction following the steps given above. □

Proof of Theorem 4. In view of Proposition 14, Theorem 4 follows directly from Propositions 20 and 23. □

5. Application of Theorem 4 to Schrödinger Operators

In this section we apply our main result, Theorem 4, to discrete one-dimensional Schrödinger operators with potentials given by circle maps. This is in part motivated by previous results on their Sturmian counterparts and a recent result of Lenz which relates aspects of their spectral theory to LR properties.

A discrete one-dimensional Schrödinger operator acts in the Hilbert space $\ell^2(\mathbb{Z})$. If $\phi \in \ell^2(\mathbb{Z})$, then $H\phi$ is given by

$$(H\phi)(n) = \phi(n+1) + \phi(n-1) + V(n)\phi(n),$$

where $V : \mathbb{Z} \to \mathbb{R}$. The map V is called the potential. Given such an operator H, the spectrum $\sigma(H)$ of H is defined by

$$\sigma(H) = \mathbb{R} \setminus \{E \in \mathbb{R} : (H - E \cdot I)^{-1} \text{ exists as a bounded linear operator}\}.$$

Here, I denotes the identity operator. From a physical point of view, the unitary group e^{-itH} generated by H describes the time evolution of a wavepacket in a medium given by V and the spectrum of H corresponds to the set of allowed energies.

If A is an alphabet, $T : \mathbb{Z} \to \mathbb{Z}$ is the standard shift, $\Omega \subseteq A^{\mathbb{Z}}$ is T-invariant (i.e., $T\Omega = \Omega$) and closed (discrete topology on A and product topology on $A^{\mathbb{Z}}$), then Ω is called a (two-sided) subshift. Given such a subshift and a function $f : A \to \mathbb{R}$, we define, for $\omega \in \Omega$, a potential $V = V_\omega$ by

$$V_\omega(n) = f(\omega_n)$$

and an operator H_ω (as above, with this particular potential). It is a standard result that if Ω is minimal, then the spectrum of H_ω does not depend on ω, that is, there is a set $\Sigma \subseteq \mathbb{R}$ such that $\sigma(H_\omega) = \Sigma$ for every $\omega \in \Omega$.

A special case of a recent result of Lenz is given in the following theorem.

Theorem 30 (Lenz [30]). *If Ω is a linearly recurrent subshift and Ω and f are such that the resulting potentials V_ω are aperiodic, then Σ has Lebesgue measure zero.*

This result not only shows that the set of allowed energies is small in Lebesgue sense, it also has consequences for the dynamics governed by e^{-itH_ω}, $\omega \in \Omega$. Note in particular that the result is essentially independent of the function f. Moreover, it suffices that at least one V_ω is aperiodic. This implies that all V_ω are aperiodic.

Our goal is to apply this theorem to circle map subshifts. A circle map generates a two-sided subshift as follows. If $u \in \{0,1\}^{\mathbb{N}}$ is a circle map corresponding to parameters (α, β), the associated subshift is given by

$$\Omega = \Omega_{\alpha,\beta} = \{\omega \in \{0,1\}^{\mathbb{Z}} : \text{ every factor of } \omega \text{ is a factor of } u\}.$$

If we restrict the sequences in Ω to the right half-line, we get exactly the one-sided subshift that was introduced and discussed above. By recurrence, the languages associated with the one-sided and two-sided subshifts are the same. In particular, LR-properties are the same for both subshifts.

Combining our Theorem 4 and the theorem of Lenz, we obtain the following result.

Theorem 31. *Suppose that u is a nondegenerate circle map corresponding to parameters (α, β) whose $\mathcal{D}$-expansion $(a_n, i_n)_{n\in\mathbb{N}}$ satisfies the $(*)$-condition. Consider the associated subshift $\Omega = \Omega_{\alpha,\beta}$ and, for a nonconstant function $f : \{0,1\} \to \mathbb{R}$, the operators $(H_\omega)_{\omega\in\Omega}$. Then we have that for every $\omega \in \Omega$, the spectrum of H_ω has Lebesgue measure zero.*

Let us discuss this result and related results for Sturmian potentials. Schrödinger operators with potentials generated by a circle map have been studied extensively by physicists and mathematicians, in particular since the discovery of quasicrystals in 1984 by Shechtman et al. because they are common models for one-dimensional quasicrystals and hence describe the motion of electrons in such media; see, for example, [3, 4, 6, 10, 11, 12, 15, 22, 23, 24, 25, 26, 35, 36]. For such models and related ones, it was possible to exhibit several properties that seem to hold universally — zero-measure spectrum being one of them. Circle maps with $\alpha = \beta$ irrational are called Sturmian. For such sequences, one can generate Schrödinger operators in the exact same way and for them, zero-measure spectrum was proven by Bellissard et al. in 1989 [3], building upon earlier results by Sütő [35, 36]. A few years later, the zero-measure spectrum property was proven for potentials generated by a large class of primitive substitutions by Bovier and Ghez [5]. Both results were obtained through an analysis of trace maps, dynamical systems induced by the S-adic structure of the potentials. However, it has not been possible to perform a similar analysis in the non-Sturmian circle map case. While, as was noted above, an S-adic structure exists for such models, the trace map is far too complex to be studied with reasonable technical effort. It was therefore expected that for circle map potentials, a different method would have to be used for a proof of zero-measure spectrum.

Such a different method was then introduced by Lenz in the article referred to above. On the one hand, he complements the results of Bovier and Ghez by proving zero-measure spectrum for all primitive substitutions (since they lead to LR subshifts) and he also notes that one should try to apply his result to circle map models, that is, one should identify the circle map subshifts which are linearly recurrent. This is exactly what we have achieved in the above theorem which gives the first zero-measure spectrum result for circle map subshifts that are neither Sturmian nor primitive substitutive.

However, Lenz's approach is essentially limited to a class slightly larger than LR (see his paper for a discussion of this issue). It definitely does not apply to all circle map subshifts. While one may expect that zero-measure spectrum holds for many more, even all, circle map subshifts, this cannot be proven at this point.

Appendix A

In this section we give a proof (and a little bit more) of the power freeness of the sequence we consider in the end of the Section 3. This proof was suggested by J. Cassaigne [8].

Let us introduce the following two substitutions, defined over $\{1,2,3\}$, given by:

$$f = \sigma_1\sigma_3\sigma_4\sigma_4 \qquad\qquad g = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_4$$

$$\begin{array}{lll|lll} 1 & \longmapsto & 13 & 1 & \longmapsto & 13 \\ 2 & \longmapsto & 1323 & 2 & \longmapsto & 13223 \\ 3 & \longmapsto & 133 & 3 & \longmapsto & 1323 \end{array}$$

where the substitutions σ_i are defined in Section 3. We denote by $\mathcal{F}$ the largest language defined over the alphabet $\{1,2,3\}$ which satisfies the following three conditions:

- $\forall \omega \in \{1,2,3\}^*$, $\omega^4 \in \mathcal{F} \Rightarrow \omega = \varepsilon$,
- $\forall \omega \in \{1,2,3\}^*$ and $\forall z \in \{1,2,3\}$, $(\omega z)^3\omega \notin \mathcal{F}$,
- $11 \notin \mathcal{F}$.

The language $\mathcal{F}$ is naturally obtained as the union of all the languages defined over the alphabet $\{1,2,3\}$ which satisfy these three conditions.

Lemma 32. *If $\omega \in \mathcal{F}$, then $f(\omega)$ and $g(\omega)$ are two elements of $\mathcal{F}$.*

Proof. Let ω be an element of $\mathcal{F}$. We consider three cases to prove that $f(\omega) \in \mathcal{F}$.

1. Assume there exists a nonempty word M such that M^4 is a factor of $f(\omega)$. Then, M could be decomposed in a unique way in $xf(v)y$, where $(x,v,y) \in \{\varepsilon, 3, 23, 33, 323\} \times \{1,2,3\}^* \times \{\varepsilon, 1, 13, 132\}$ and the length of v is maximal with the convention that if v ends with the letter 1, then $y \neq \varepsilon$. We consider two subcases.
   (a) Let us suppose that $v = \varepsilon$. Then $M = xy$ and thus

   $$\begin{aligned} M \in\ & \{3, 33, 3313, 32313\} \cup \{31, 313, 3132, 2313, 331, 3231\} \\ & \cup \{23, 33132, 323, 323132\} \cup \{23132\} \cup \{231\}. \end{aligned}$$

   But $M \notin \{3, 33, 3313, 32313\}$ because 33 is always followed by a 1 in $f(\omega)$. If $M \in \{31, 313, 3132, 2313, 331, 3231\}$, we obtain that there exists a letter $z \in \{1,2,3\}$ such that z^3 is a factor of ω. This gives a contradiction because $\omega \in \mathcal{F}$. The word M cannot belong to the set $\{23, 33132, 323, 323132\}$ because 23 is always followed by a 1 in $f(\omega)$. M cannot belong to $\{23132\}$ because the letter 2 is always followed by a 3 in $f(\omega)$. Finally, M cannot belong to $\{231\}$ because the letter 1 is never followed by a 2 in $f(\omega)$.

   (b) Let us suppose that $v \neq \varepsilon$. Then

   $$M^4 = xf(v)yxf(v)yxf(v)yxf(v)y$$

   and necessarily $yx = f(z)$ with $z \in \{\varepsilon, 1, 2, 3\}$. If $z = \varepsilon$, then $M^4 = f(v^4)$. The fact that v does not end with a 1 allows us to infer that v^4 is a factor of ω. We obtain a contradiction because $\omega \in \mathcal{F}$. If z is a letter, then $f((vz)^3v)$ is a factor of $f(\omega)$. The fact that v does not end with a 1 shows that $(vz)^3v$ is a factor of ω. We obtain a contradiction because $\omega \in \mathcal{F}$.

2. Let us suppose that there exist a word M and a letter z such that $(Mz)^3M$ is a factor of $f(\omega)$. Then, M can be decomposed in a unique way in $xf(v)y$, where $(x,v,y) \in \{\varepsilon, 3, 23, 33, 323\} \times \{1,2,3\}^* \times \{\varepsilon, 1, 13, 132\}$ and the length of v is maximal with the convention that if v ends with the letter 1, then $y \neq \varepsilon$. We obtain that

$$(Mz)^3M = xf(v)yzxf(v)yzxf(v)yzxf(v)y,$$

and necessarily $xzy = f(m)$ with $m \in \{1,2,3\}$ and $|m| \leq 2$ because $|xzy|_1 \leq 2$ and the letter 1 has exactly one occurrence in the image of each letter. Again we consider two subcases.

(a) Let us suppose that $|m| = 2$. Then there exist two letters a and b such that $yzx = f(ab)$. But $|y|_1 \leq 1$ and $|x|_1 = 0$ imply that $y = f(a)$ and $z = 1$. We get $(Mz)^3M = xf((vab)^3va)$. If $a \neq 1$, then $(vab)^3va$ is a factor of $f(\omega)$ and we obtain a contradiction because $\omega \in \mathcal{F}$. If $a = 1$, $xf((v1b)^3v)$ is a factor of $f(\omega)$. We recall that $zx = 1x = f(b)$. It follows that if $b = 2$ or $b = 3$, then $x = 323$ or $x = 33$ and thus x is always preceded by the letter 1 in $f(\omega)$. This implies that $1xf((v1b)^3v)$ is a factor of $f(\omega)$. But since $1xf((v1b)^3v) = f((bv1)^3bv)$ and v does not end with the letter 1, it follows that $(bv1)^3bv$ is a factor of ω. This is in contradiction with $\omega \in \mathcal{F}$. Finally, if $b = 1$, then $f((v11)^3v)$ is a factor of $f(\omega)$. The fact that v does not end with the letter 1 gives that $(v11)^3v$ is a factor of ω and thus 11 is a factor of ω. We get a contradiction since $11 \notin \mathcal{F}$.

(b) Let us suppose that $|m| = 1$, then $(Mz)^3M = xf((vm)^3v)y$. In particular, $f((vm)^3v)$ is a factor of $f(\omega)$. But since v does not end with the letter 1, $(vm)^3v$ is a factor of ω. We obtain a contradiction because m is a letter and $\omega \in \mathcal{F}$.

3. Let us suppose that 11 is a factor of $f(\omega)$. This yields a contradiction immediately because the letter 1 is always followed by a 3 in $f(\omega)$ by definition of f.

The proof for g is exactly the same. $\square$

Proposition 33. *A circle map whose $\mathcal{D}$-expansion $(a_n, i_n)_{n\in\mathbb{N}}$ satisfies*

- $(i_n)_{n\in\mathbb{N}} = (10)^\omega$,
- $i_n = 0$ *implies* $a_n = 0$, *and*
- $i_n = 1$ *implies* $a_n \in \{0,1\}$

is power free.

Proof. Let U be such a circle map and V be the natural coding of the three-interval exchange transformation associated with U. Theorem 18 says that there exists a sequence of integers $(b_n)_{n\in\mathbb{N}}$ such that

$$U = \lim_{n\to\infty} \Phi_{\lfloor\frac{1-\beta}{\alpha}\rfloor}\left(f^{b_0}g^{b_1}f^{b_2}g^{b_3}\dots f^{b_{2n}}(1)\right)$$

and thus

$$V = \lim_{n\to\infty} f^{b_0}g^{b_1}f^{b_2}g^{b_3}\dots f^{b_{2n}}(1).$$

With the previous notation, $1 \in \mathcal{F}$. Then Lemma 32 implies that

$$f^{b_0}g^{b_1}f^{b_2}g^{b_3}\dots f^{b_{2n}}(1) \in \mathcal{F}$$

for every integer n. We thus obtain $\mathcal{L}(V) \subset \mathcal{F}$. This implies that V is 4-power free. Then, in view of the definition of the morphisms Φ_k, U is clearly power free if $\lfloor \frac{1-\beta}{\alpha} \rfloor > 0$ (i.e., $1-\beta > \alpha$). In the case where $1-\beta < \alpha$, we can use an argument similar to the one used in the proof of Proposition 23. It is relatively easy to see that if a sequence is not power free, then all of its derived sequences are not power free, either. We have already noticed at the end of the proof of Proposition 23 that for sufficiently large n, there is a morphism Ψ such that $V = \Psi(\mathcal{D}^{(n)}(U))$. Now, if we assume that U is not power free, then $\mathcal{D}^{(n)}(U)$ is not power free and hence V is not power free because morphisms propagate powers. We therefore obtain a contradiction to the 4-power freeness of V obtained above. □

In particular, we obtain the power freeness of the sequences mentioned in Section 3. These sequences are of course not LR in view of Theorem 4 and hence they are both power free and not LR. To the best of our knowledge, these are the first examples of sequences with these two properties.

We end this appendix with the following conjecture concerning the power freeness of circle maps.

Conjecture. A nondegenerate circle map is power free if and only if its $\mathcal{D}$-expansion $(a_n, i_n)_{n\in\mathbb{N}}$ satisfies the following: there exists an integer M such that for every integer n, we have

- $a_n \leq M$,
- $i_n = i_{n+1} = \ldots = i_{n+M} \Rightarrow \exists k,\ n \leq k \leq n+M$ such that $a_k \neq 0$.

Appendix B

We present here what would be the analog of the geometric considerations of Section 3 in the case of I.D.O.C. four-interval exchange transformations which lie in the Rauzy class of (4321). The notion of Rauzy class for an interval exchange transformation was introduced in [31].

Let us introduce the following substitutions, defined on the alphabet $\{1,2,3,4\}$, given by

$$\begin{array}{ccc} \sigma_1 & \sigma_2 & \sigma_3 \\ \begin{array}{lcl} 1 & \longmapsto & 1 \\ 2 & \longmapsto & 14 \\ 3 & \longmapsto & 2 \\ 4 & \longmapsto & 3 \end{array} & \begin{array}{lcl} 1 & \longmapsto & 14 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 3 \\ 4 & \longmapsto & 4 \end{array} & \begin{array}{lcl} 1 & \longmapsto & 1 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 3 \\ 4 & \longmapsto & 34 \end{array} \\ \sigma_4 & \sigma_5 & \sigma_6 \\ \begin{array}{lcl} 1 & \longmapsto & 1 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 34 \\ 4 & \longmapsto & 4 \end{array} & \begin{array}{lcl} 1 & \longmapsto & 1 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 24 \\ 4 & \longmapsto & 3 \end{array} & \begin{array}{lcl} 1 & \longmapsto & 1 \\ 2 & \longmapsto & 24 \\ 3 & \longmapsto & 3 \\ 4 & \longmapsto & 4 \end{array} \end{array}$$

The Rauzy induction graph for the Rauzy class of (4321) is given in Figure 5. The orbit of an I.D.O.C. four-interval exchange transformation in the Rauzy class of (4321) under the Rauzy induction cannot be ultimately confined to one of its primitivity subgraphs G_1, G_2, G_3 or G_4 represented in Figures 6, 7, 8, and 9, respectively. Moreover, an I.D.O.C. four-interval exchange in the Rauzy class of (4321) is LR if and only if its orbit under the Rauzy induction can stay in any of the primitivity subgraphs G_1, G_2, G_3, and G_4 only for a bounded number of consecutive induction steps.

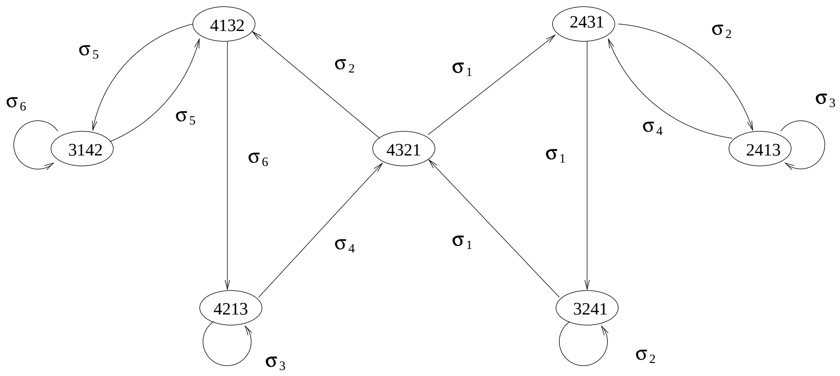

FIGURE 5. The Rauzy induction graph for the Rauzy class of (4321).

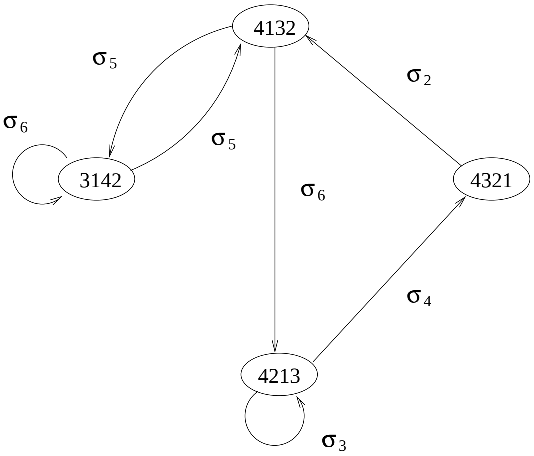

FIGURE 6. The primitivity subgraph G_1 for the Rauzy class of (4321).

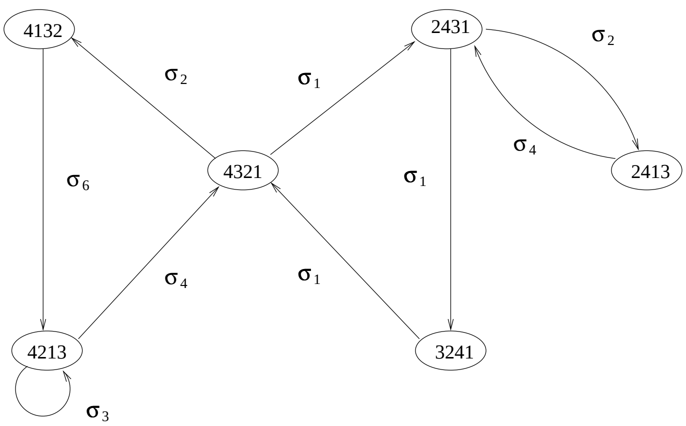

FIGURE 7. The primitivity subgraph G_2 for the Rauzy class of (4321).

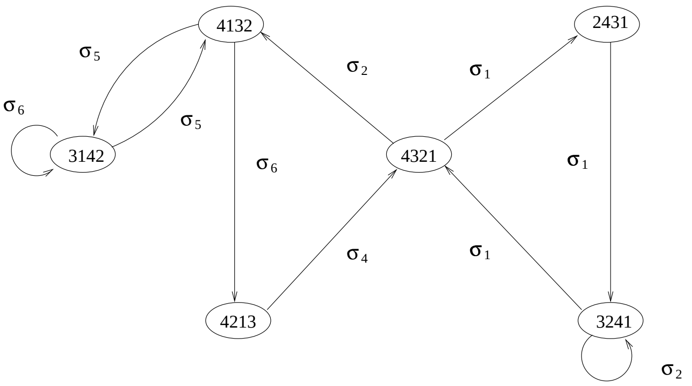

FIGURE 8. The primitivity subgraph G_3 for the Rauzy class of (4321).

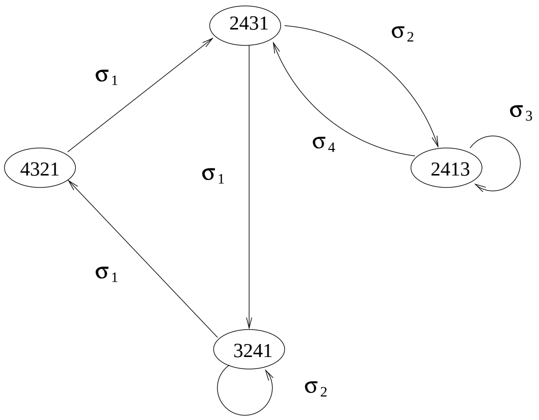

FIGURE 9. The primitivity subgraph G_4 for the Rauzy class of (4321).

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