THE PROBLEM OF BLOW-UP IN NONLINEAR PARABOLIC EQUATIONS

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1. Introduction

A great number of processes of the applied sciences can be modeled by means of evolution equations involving differential operators, or systems of such equations. In order to obtain a well-defined solution the equations are supplemented with suitable additional conditions (usually, initial and boundary conditions). The standard differential theories involve linear operators and for them an extensive theory has been developed. Nonlinear partial differential equations and systems exhibit a number of properties which are absent from the linear theories; these nonlinear properties are often related to important features of the real world phenomena which the mathematical model is supposed to describe; at the same time these new properties are closely connected with essential new difficulties of the mathematical treatment. The study of nonlinear processes has been a continuous source of new problems and it has motivated the introduction of new methods in the areas of mathematical analysis, partial differential equations and other disciplines, becoming a most active area of mathematical research in the last part of this century. The success of the new methods of modern analysis has enabled mathematicians to give rigorous answers to important questions of the nonlinear world.

One of the most remarkable properties that distinguish nonlinear evolution problems from the linear ones is the possibility of eventual occurrence of **singularities** starting from perfectly smooth data, more specifically, from classes of data for which a theory of existence, uniqueness and continuous dependence can be established for small time intervals, so-called *well-posedness in the small*. While singularities can arise in linear problems, this happens through the singularities contained in the coefficients or data or the problem (*fixed singularities*). On the contrary, in nonlinear systems they may arise from the nonlinear mechanisms of the problem and their time and location are to be determined by the mathematical analysis (*moving singularities*).

Blow-up and ordinary differential equations. The simplest form of spontaneous singularities in nonlinear problems appears when the variable or variables tend to infinity when time approaches a certain finite limit $T > 0$. This is what we call a **blow-up**

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phenomenon. Blow-up happens in an elementary (but quite representative) form in the theory of ordinary differential equations (ODE's), and the simplest example is the initial-value problem for a real scalar variable $u = u(t)$:

$$u_t = u^2, \;\; t > 0; \quad u(0) = a. \tag{1.1}$$

For data $a > 0$ it is immediate that a unique solution exists in the time interval $0 < t < T = 1/a$. Since it is given by the formula $u(t) = 1/(T-t)$, one sees that it is a smooth function for $t < T$ and also that $u(t) \to \infty$ as $t \to T^-$ (limit from the left). We say that the solution **blows up** at $t = T$ and also that $u(t)$ has **blow-up** at that time. Blow-up is referred to in Latin languages as **explosion**, and in fact the mathematical problems involved aim in many cases at (partially) describing explosive phenomena. Starting from this example, the concept of blow-up can be widely generalized as the phenomenon whereby solutions cease to exist *globally in time* because of infinite growth of the variables describing the evolution process. Thus, a first step is given by ODE's of the form $u_t = u^p$, with $p > 1$ and, more generally,

$$u_t = f(u), \tag{1.2}$$

where f is positive and, say, continuous, under the condition

$$\int_1^\infty ds/f(s) < \infty. \tag{1.3}$$

This *Osgood's condition* in the ODE theory established in 1898 [145] (see [113], Part I.3, and also [171], p. 140) is *necessary and sufficient* for the occurrence of blow-up in finite time for any solution with positive initial data. More generally, we can think of systems $u_t = f(t,u)$ for a vector variable $u \in I\!R^n$. In this case we may have blow-up due to the same mechanism if f is super-linear with respect to u for $|u|$ large, and also blow-up due to the singular character of f with respect to t at certain given times. It is the generalization of the first form that will of concern in these notes. The study of ODE's supplies basic tools and intuitions for the whole theory of blow-up, and, more generally, the study of singularities.

Blow-up in reaction-diffusion equations: basic problems and a bit of history. The preceding questions gain in mathematical difficulty, and also in interest for the application to different sciences, when the problem has a spatial structure, so that the unknowns depend not only on time but also on a space variable, $u = u(x,t)$, with $x \in \Omega$, a domain in $I\!R^n$, while time stretches to an interval $0 \le t < T$. Prominent in the study of blow-up problems appear Reaction-Diffusion equations, cf. [158]. In the theories of thermal propagation and combustion it is natural to consider quasilinear equations of the form:

$$u_t = \nabla \cdot \mathcal{A}(u, \nabla u, x, t) + B(u, \nabla u, x, t) \tag{1.4}$$

with standard ellipticity conditions on the operator $\mathcal{A}$ and growth and regularity conditions on both $\mathcal{A}$ and B. We typically think of (1.4) as a nonlinear heat propagation model in a reactive medium, and then u is a temperature. It is clear that, as a model of an evolution process and compared with equation (1.2), this equation introduces the new feature of taking into account the *spatial structure* of the solutions. The *concept of blow-up* is now formulated in its simplest form in the following framework.

(i) We start from the well-posedness of the mathematical problem in a certain framework and for small times; thus, assuming nice regularity conditions on $\mathcal{A}$ and B, we will have an existence and uniqueness theory for, say, the Cauchy problem or one of the initial-boundary value problems, in a certain class of bounded and nonnegative data, so that the solutions evolve being bounded for some time $0 < t < T$.

(ii) We have also a regularity and continuation theory in this framework which says that bounded solutions have the necessary smoothness so that they can be continued locally in time (cf. a well-known result in [159]). For classical solutions of parabolic equations this theory is based on Schauder estimates. Corresponding estimates exist for weak solutions of divergence-form equations in Sobolev spaces or for fully nonlinear equations, as described in Prof. Cabré's lectures in this volume.

Blow-up occurs then if the solution becomes infinite at some (or many) points as t approaches a certain finite time T. Namely, there exists a time $T < \infty$, called the *blow-up time*, such that the solution is well defined for all $0 < t < T$, while

$$\sup_{x\in\Omega} |u(x,t)| \to \infty \quad \text{as} \quad t \to T^-. \tag{1.5}$$

Let us review some **historical facts** with special attention to the early developments by the Russian school. The subject of blow-up was posed in the 1940's and 50's in the context of *Semenov's chain reaction theory*, *adiabatic explosion* and *combustion theory*, see [97] or [175]. A strong influence was also due to blow-up singularities in *gas dynamics*, the *intense explosion (focusing) problem* with second kind self-similar solutions considered by Bechert, Guderley and Sedov in the 1940's [15], p. 127. First analysis of the most striking effect of space *localization* of blow-up boundary regime (*S-regime of blow-up* generated by a blow-up standing wave) in quasilinear diffusion equations was performed by Samarskii and Sobol' in 1963 [155]. An essential increase of attention to the blow-up research in gas dynamics, laser fusion and combustion in the 70's was initiated by the numerical results [141] (announced by E. Teller in 1972) on the possibility of the laser blow-up-like compression of deuterium-tritium (DT) drop to super-high densities *without* shock waves. The problem of localization of blow-up solutions in reaction-diffusion equations was first proposed by Kurdyumov [122] in 1974, see Chapters 3 and 4 in [154] devoted to localization analysis. It was shown that, in a quasilinear model, the heat and burning localization arises in plasma with an electron heat conductivity and heat source due to amalgamation of DT nuclei [176]. Such a model, as well as different aspects of localization effects, are presented in [125]. An extensive list of references on localization of blow-up dissipative structures with the historical review can be found in the survey paper by Kurdyumov [123].

The **mathematical theory** has been actively investigated by researchers in the 60's mainly after general approaches to blow-up by Kaplan [117], Fujita [62], [63], Friedman [56] and some others; there is as yet no complete theory developed in the generality presented above, but detailed studies have been performed on a hierarchy of models of increasing complexity and there is nowadays a very extensive literature on the subject.

There are two classical scalar models. One of them is the *exponential reaction model*

$$u_t = \Delta u + \lambda e^u, \quad \lambda > 0, \tag{1.6}$$

which is important in combustion theory [175] under the name of *solid-fuel model* (*Frank-Kamenetsky equation*), and also in other areas, cf. [17], [97], [147], [154]. It is also of interest in differential geometry, [119], and other applications. The occurrence and type of blow-up depends on the parameter $\lambda > 0$, the initial data and the domain. The other classical blow-up equation is

$$u_t = \Delta u + u^p. \tag{1.7}$$

Both semilinear equations were studied in the pioneering works by Fujita. For exponents $p > 1$ we have the property of blow-up in (1.7); depending on the value of p it may happen not only for some but for all the solutions in a given class. There are a number of other popular models of evolution problems involving nonlinear parabolic equations, possibly degenerate. With the *porous medium* and *p-Laplace* operators,

$$u_t = \Delta u^m + u^p \ \ (m > 0) \quad \text{and} \quad u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u) + u^p \ \ (\sigma > -1), \tag{1.8}$$

first blow-up results were proved in [65] and [163] respectively. Local solvability and general regularity results for these and other degenerate quasilinear equations can be found in the survey paper [115]. All these models take the form

$$u_t = \mathbf{A}(u) + f(u), \tag{1.9}$$

where $\mathbf{A}$ is a second-order elliptic operator, maybe nonlinear and degenerate, representing a *diffusion*, and $f(u)$ is a superlinear function of u representing *reaction*.

There exist some good texts which display many of the results which are known and contain extended lists of references, like the books by Bebernes and Eberly [17], Samarskii et al. [154] and Vol'pert and Khudyaev [170]. However, this is a very active field where there are many new developments. In that sense there are also a few surveys on blow-up results in different periods, see e.g. [78], [132] (both include other nonlinear PDE's) and [12] (which treats semilinear heat equations and numerical methods; this whole issue of J. Comput. Appl. Math. is devoted to different blow-up aspects including numerical results on some open mathematical problems). It is our intention to complement the basic literature by discussing some of the most recent developments.

A more general framework. As we have said, it is very useful to insert the study of blow-up in a more general framework by considering it as a special type of **singularity** that develops for a certain evolution process. To be specific, we may have an evolution process described by a law and an initial condition:

$$u_t = \mathcal{A}(u) \quad \text{for } t > 0, \quad u(0) = u_0, \tag{1.10}$$

and we want to study the existence of a solution $u = u(t)$ as a curve living in a certain functional space, $u(t) \in X$. Frequent instances of X are $C(\Omega) \cap L^\infty(\Omega)$, or $L^p(\Omega)$, or the Sobolev space $H^1(\Omega)$. Typically, we are able to prove that for initial data $u_0 \in X$ the problem is well-posed in the small (i.e., the solution u with initial data u_0 is well-defined and lives in X for some time $0 < t < T = T(u_0) > 0$), and we face the problem that the solution leaves the space X as $t \to T^-$. As we see, the occurrence of a singularity becomes in this view contingent upon the type of ambient space and the concept of solution used. However, an *essential blow-up* will defy all standard choices of solution and functional framework. This will be apparent in the type of blow-up called *complete blow-up* which will be introduced in Section 2 and discussed in Sections 6-7.

Extinction. The general view of treating blow-up problems as singularities gives a unified approach in which to address very interesting related problems like **extinction**. A typical example of the latter is the semilinear heat equation with absorption

$$u_t = \Delta u - u^p, \quad p < 1, \tag{1.11}$$

where solutions with positive data are considered and the exponent p may be allowed to pass to the so-called singular range, $p < 0$, see the survey paper [131]. The difference with the blow-up problem lies in the fact that here the singularity that hinders the continuation of the solution past a given time is not a blow-up of the unknown u, but rather the blow-up of its derivative u_t and the absorption term $f(u)$. Blow-up of derivatives is therefore another reasonable way in which nonlinear evolution equations develop singularities which may or may not stop the evolution of their solutions.

Other examples of the more general framework will occur for instance when the reaction term depends on the spatial gradient and the latter blows up, even if the solution stays bounded, cf. [4], or in free boundary problems when the free boundary develops a cusp while the solution is regular in its domain, like in some Stefan or Hele-Shaw flows, see Section 12. We will delay the comment on further applications of the general framework for evolution equations to the concluding sections.

Let us point out that there is a great amount of work on blow-up for elliptic and other stationary equations which shares with the above presentation the idea that a singularity develops at a certain point (or at some points). Its scope and techniques fall out of the scope of these notes.

2. **The basic questions**

We concentrate next on the analysis of the main questions raised in the study of blow-up for reaction-diffusion equations. This list can be suitably adapted to other *singularity formation* problems. According for instance to [17] the basic list includes the questions of *when, where* and *how*. We propose here an expanded list of six items: (1) *Does blow-up occur?* (2) *When?* (3) *Where?* (4) *How?* (5) *What happens later?* (6) *How to compute it numerically.*

Let us describe what these questions mean in brief terms. For definiteness we are thinking of reaction-diffusion problems like those proposed in the last paragraphs, posed in a spatial domain $\Omega \subseteq \mathbb{R}^n$, but all the concepts can be easily adapted to a more abstract setting.

(1) The first question is: **Does blow-up occur?** The blow-up problem is properly formulated only when a suitable class of solutions is chosen. Usually, the existence and uniqueness of the solutions of the problem can be formulated in different functional settings, and blow-up is exactly the inability to continue the solutions in that framework up to or past a given time. By default we deal with classical solutions, but weak, viscosity or other kinds of generalized solutions can be more natural to a given problem. We may consider cases where blow-up happens in a functional framework and not in another one, for instance for classical solutions but not for weak L^1 solutions. This seemingly strange situation does happen in practice, cf. [140], and only reflects the rich structure of blow-up problems. The general question can be split into these two aspects:

(1.i) Which equations and problems do exhibit blow-up in finite time? The form of the equation (in terms of its coefficients or more generally its structural conditions) and the form of the data determine the answer. In case the explosion is created by the reaction term a condition of super-linearity like in the ODE analysis is necessary, but we can find examples where the problem with a space structure is different from the ODE problem. We can also have explosive phenomena due to the boundary conditions, see Section 10.

(1.ii) In case the previous question has a positive answer, we may ask which solutions do blow up in finite time? The possibilities for the last question are two-fold: blow-up occurs for all solutions in the given class or only for some solutions (which should be identified). If a particular solution does not blow up then it lives globally in time. A problem for which all solutions blow up is called a *Fujita problem*. The classical example is the semilinear heat equation $u_t = \Delta u + u^p$ posed in a bounded domain with zero Dirichlet conditions where all classical nonnegative solutions blow up in finite time when the exponent lies in the range $p \in (1, (n+2)/n)$. The upper bound is the so-called *Fujita exponent*.

(2) The second question is: **When?** Granted that blow-up occurs in finite time, can we estimate the blow-up time? Indeed, the property of blow-up can also happen in a less striking form in infinite time, when the solution exists in the given functional framework for all $0 < t < \infty$ but becomes unbounded as $t \to \infty$. Thus, we have the alternative: finite versus infinite-time blow-up. Indeed, a four-option table occurs for the solutions of reaction-diffusion systems:

(2.i) global solutions which remain uniformly bounded in time (i.e., no blow-up),

(2.ii) global solutions with blow-up at infinity, *infinite-time blow-up*,

(2.iii) solutions with *finite-time blow-up* (the standard blow-up case), and

(2.iv) *instantaneous blow-up*, i.e., the solution blows up at $t = 0$ in a sense to be specified.

The latter is a very striking nonlinear phenomenon, but we have shown that it occurs for such a simple equation as the exponential reaction equation $u_t = \Delta u + \lambda e^u$, cf. [147], [165], see Section 4 below.

(3) Next comes the question of **Where?** Firstly, for a solution $u = u(x,t)$ in $Q_T = \Omega \times (0,T)$, which blows up at a time $T > 0$, we define the **blow-up set** as

$$B(u_0) = \{x \in \Omega : \exists \{x_n, t_n\} \subset Q_T, t_n \to T^-, x_n \to x,\ u(x_n, t_n) \to \infty\}. \tag{2.1}$$

This is a closed set. Its points are the **blow-up points**. A smaller blow-up set is

$$B_1(u_0) = \{x \in \Omega : \exists \{t_n\} \subset (0,T),\ \{t_n\} \to T^-,\ u(x, t_n) \to \infty\}. \tag{2.2}$$

Typical alternatives when $\Omega = I\!R^n$ are: *single-point blow-up*, where $B(u_0)$ consists of a single point (or of a finite number of points), *regional blow-up*, where the measure of $B(u_0)$ is finite and positive, and *global blow-up*, where $B(u_0) = I\!R^n$. These notions are naturally adapted when Ω is not the whole Euclidean space. In the Russian literature of the 70-80s these types of blow-up are called LS-regime, S-regime and HS-regime of blow-up, respectively [154]. In the first two cases the blow-up solutions are called *localized*.

(4) Next question is: **How** does blow-up occur? There are two aspects:
(4.i) to calculate the **rate** at which u diverges as t approaches the blow-up time and x approaches a blow-up point, and
(4.ii) to calculate the **final-time blow-up profiles** as limits of $u(x,t)$ when $t \to T^-$ at the non-blowing points.

Usually the former information is replaced by some norm estimate of $u(\cdot,t)$ as t approaches blow-up, or better by an asymptotic expansion. In general, the question How? proceeds via a change of variables (*renormalization*) that rescales the evolution orbit to bounded size, followed by the study of the limits of these orbits, which are now not restricted to the non blowing points, but rather to the blow-up set. It usually leads to generic shape in the form of a *stable blow-up pattern*. The general classification of the singularity implies the further study of other unstable patterns. Typically, finite-time singularities, like blow-up, generate a countable discrete (not continuous!) spectra of structurally different patterns called *eigenfunctions of nonlinear media* [154], [123]. This is not the case with infinite time singularities.

An interesting question concerns the rate of blowup as t approaches T. For many equations like (1.7) the scale-invariance implies the existence of solutions which blow-up at a power rate, [99], [84]. *Self-similar blow-up* becomes then the usual form of blow-up and fixes the blow-up rates. However, it has been found that for large values of the exponent p in (1.7) the actual rate of blow-up can be faster than the self-similar power, a phenomenon called *fast blow-up*. It should be noticed that these patterns are unstable. Fast blow-up has been detected in a number of problems and its rates and profiles are difficult to obtain, cf. [6], [106]. There are problems where all solutions blow up fast.

(5) A question that has received until recently less attention, but is of great importance for the practical application of mathematical models involving blow-up, is: What happens after a finite-time blow-up singularity occurs? This is the problem of **continuation after blow-up**, referred to as **beyond blow-up**. A basic technical prerequisite is to find a suitable concept of continued solution. This is done in our research, see Sections 6-7, by means of monotone approximation of the reaction term and the data so that the approximate problems have global solutions. Passing then to the limit we may decide whether the solution becomes trivial (i.e., identically infinite) after T or not, [13]. This is the natural approach in the application to thermal propagation and combustion. With this method essentially three alternatives appear:

(5i) The solution cannot be continued. In the models we discuss below this happens because if continued it must be infinite everywhere in a natural sense. We call it *Complete Blow-up*.

(5ii) The solution can be continued in some region of space-time after T, but it is infinite in the complement, *Incomplete Blow-Up*.

(5iii) The solution becomes bounded again after T. This is a *Transient Blow-up*. We have found it in the form of *peaking blow-up*, where it becomes bounded immediately after T. In the models investigated so far, this is a very unstable phenomenon, a transition between more stable evolution patterns.

Alternative methods of continuation are not excluded and can be useful in suitable contexts, like *continuation in complex time*, cf. [137], who shows a way to continue the solution of $u_t = \Delta u + u^2$ past $t = T$ along a certain sector of times in the complex plane, avoiding the singularity. Unfortunately, this analytic continuation is not unique. Pointing out the interest in the study of continuation after blow-up is one of the main concerns of these notes.

(6) The final question refers to the **numerical methods** to detect the blow-up phenomenon and compute or approximate the blow-up solutions, times and profiles. The first problem is how to produce the *computational solution*. For instance, semi-discretization in space leads to an initial-value problem for a system of nonlinear ODEs. We can then use finite differences, collocation or finite elements to treat the spatial derivatives. Some of the relevant questions when we want the computational solution to display the properties of the blow-up phenomenon are the choice of spatial and temporal meshes, the choice of time integrator, the use of adaptive methods, and in a more theoretical direction the analysis of convergence. A good numerical approximation should be able to give an explanation of why a solution cannot be continued in complete blow-up. We refer to the survey paper [12] for more information.

3. **Outline of the paper. Some answers**

As it may be apparent to the reader, mathematical blow-up is a large field and much progress has been done recently or is being done at the moment. Therefore, we will concentrate on explaining some of the above problems in which we have been involved and on the answers given to them. Thus, Section 4 presents the different times of blow-up. We will devote particular attention to the phenomenon of instantaneous blow-up. The modes, rates and profiles of blow-up occupy Section 5. There we will be interested in explaining one of the simplest (but critical) models of regional blow-up, the case of linear diffusion with superlinear reaction of logarithmic type, together with an idea of the mathematical tools used in the rigorous blow-up analysis [88]. Section 6 deals with the problem of characterizing Continuation after Blow-Up. This problem has been solved in [90] in one space dimension for equations of the general form

$$u_t = \phi(u)_{xx} + f(u) \tag{3.1}$$

under quite general assumptions on ϕ and f. In particular, ϕ and f are positive for $u > 0$ and ϕ is increasing. It is to be noted that only possibilities (5.i) and (5.ii) of previous section occur. In several space dimensions (Section 7) the situation is much more complicated. Actually, the third possibility mentioned above can appear, as we show by constructing a family of unstable blow-up solutions in self-similar form, called *peaking solutions*, which have only one instant of blow-up, [94].

The same techniques allowed us to treat the problem of Continuation after Extinction for equations of the form

$$u_t = \phi(u)_{xx} - f(u), \tag{3.2}$$

which was also open. The results are explained in Section 8. In Sections 9-15 we shortly comment on some basic and quite recent results on blow-up for other nonlinear parabolic and PDEs of different types. We have tried to reflect a wide number of subjects and results, but we do not claim to have achieved in such a short space a balanced report of the state of the art in such a vast and continuously growing field and we apologize for any omissions.

4. **On the time of blow-up**

We present here evidence of the possibilities of global solutions with stabilization, global solutions with blow-up at infinity, solutions with finite time blow-up and instantaneous blow-up.

Finite time blow-up. It is treated in detail in a vast literature, cf. the books [17], [154], [175]. Many of the results refer in particular to the semilinear equations (1.7) and (1.6).

Quasilinear model. Critical Fujita exponent. In order to briefly review some typical results for a less standard equation, let us consider the quasilinear heat equation with source having power-like nonlinearities

$$u_t = \Delta u^m + u^p, \tag{4.1}$$

which for $m = 1$ is the most popular mathematical models for nonlinear reaction-diffusion phenomena, among them blow-up. We consider nonnegative solutions $u(x,t)$ defined in the whole n-dimensional space $x \in I\!R^n$ for some time interval $0 < t < T$. We take $p > 1$ because otherwise there can be no blow-up. The diffusion exponent includes the standard linear case $m = 1$, all porous medium cases, $m > 1$, and also values in the fast-diffusion range, $0 < m < 1$, though technical reasons recommend the condition $m > m_c = (n-2)_+/n$, a well-known critical value below which the very fast diffusion implies peculiar extinction effects which complicate the picture and fall out of our present interest (but implies unusual blowup patterns and final-time profiles like delta-function). The initial data $u_0(x)$ are assumed to be nonnegative and bounded. By standard theory we may also assume that they are smooth in the positivity domain (after delaying a bit the origin of time).

In view of the high intensity of the heat source, $f(u) = u^p$, for large $u > 1$, the Cauchy problem is known to admit a local-in-time solution which may blow-up in a finite time. The precise result depends on the relative values of m and p. Thus, it is known that if $1 < p \le m + 2/n$, then any solution $u \not\equiv 0$ blows up in finite time, namely, there exists a time $T = T(u_0) < \infty$, such that the solution is well defined, bounded and smooth for all $x \in I\!R^n$ and $0 < t < T$, while

$$\sup_{x\in I\!R^n} u(x,t) \to \infty \quad \text{as} \quad t \to T^-.$$

On the other hand, when $p > m + 2/n$ blow-up occurs if u_0 is large enough, but there also exist small solutions which live globally in time. The dividing value $p_c = m + (2/n)$ is the so-called *critical Fujita exponent*, calculated for $m = 1$ in [62] and for $m > 1$ in [77]. Blow-up of any $u \not\equiv 0$ in the *critical* case $p = p_c$ was proved in [102] ($n = 1, 2$) and [121], [11] ($n \geq 1$) for $m = 1$, and in [72] for $m > 1$. This type of alternative and the corresponding Fujita exponents have been subsequently found in many related problems, see [154] and the survey [132]. Finally, the possible modes of blow-up, or the continuation after blow-up, depend on m and p, as we will see in the next sections. For the equation with the p-Laplace operator in (1.8), the Fujita exponent is $p_c = 1 + \sigma + (\sigma + 2)/n$ [67], blow-up for $p = p_c$ was proved in [72].

Exponential equation. Stationary solutions. Instantaneous blow-up. Let us also review the exponential model (1.6) in a bounded domain $\Omega \subset I\!R^n$ with initial data u_0 and $u = 0$ on the smooth boundary $\partial\Omega$. In this equation small solutions will exist globally in time while solutions for large data will blow up. An important role in the separation between both classes of data is played by the *stationary solutions* of (1.6), i.e., solutions of the elliptic equation

$$\Delta u + \lambda e^u = 0 \quad \text{in } \Omega. \tag{4.2}$$

In order to give specific results we consider the Cauchy-Dirichlet problem, posed (for simplicity) in the ball $\Omega = B_R(0) = \{x \in I\!R^n : |x| < R\}$ as space domain, with initial and boundary data $u(x,0) = u_0(x), \quad x \in \Omega; \quad u(x,0) = 0 \quad \text{for} \quad x \in \partial\Omega = \{x : |x| = R\}$. The study of the *stationary equation* is due to Gel'fand-Barenblatt [97] and Joseph-Lundgren [114] in one of the most celebrated papers of this area (see also [170]), in which they describe the bifurcation diagram of equation (4.2) posed in a ball $B_R(0)$ with zero boundary data. By symmetry radial solutions are only considered. Let us recall some of the main results of this bifurcation analysis:

(i) There exists a value of the parameter $\lambda_*(n) > 0$ such that problem has at least a solution if $0 < \lambda \leq \lambda_* < \infty$ and no solution for $\lambda > \lambda_*$. We can defined a branch of uniquely defined *minimal solutions* $u_\lambda(x)$, for $0 < \lambda < \lambda_*$, which are smooth and stable.

(ii) If $1 \leq n \leq 2$ any weak solution $u(\cdot, t)$ of (1.6), say in $L^2_{\rm loc}([0,\infty) : W^{1,2}(B_R))$, is bounded.

(iii) If $n \geq 3$ there exists a unique unbounded radial stationary solution, $S(x) = -2\ln|x|$, corresponding to $\lambda_0 = 2(n-2)$. We call this solution the *singular stationary solution*.

(iv) $\lambda_0 = \lambda_*$ if and only if $n \geq 10$. In that case the singular solution is also the minimal stationary solution.

Consequently, since for $3 \leq n \leq 9$ we have $0 < \lambda_0 < \lambda_*$, the stationary problem with $\lambda_0 = 2(n-2)$ has a nontrivial minimal solution $\underline{u} > 0$ which is bounded and one unbounded solution, viz, the singular solution. Moreover, it is proved that there are infinitely many bounded, smooth solutions larger than $\underline{u}$; besides, whenever $u_1 \leq u_2$, and $u_1 \not\equiv u_2$, then necessarily $u_1 = \underline{u}$. The singular stationary solution is extremely important in understanding the global structure of the solutions $u(x,t)$. We list the following key results related to S assuming that $\lambda_0 = 2(n-2)$.

(v) If $3 \le n \le 9$, then the only stable stationary solution in the minimal one, $\underline{u}$, and for any initial data $u_0(x) \le S(x)$, $u_0 \not\equiv S$, $u(\cdot,t) \to \underline{u}(\cdot)$ uniformly (once $u(\cdot,t) \in L^\infty$ for $t > 0$, the classical stability theory applies).

(vi) There are effects for $n \ge 10$. In particular, in this case S becomes the unique and hence the minimal stationary solution which is stable from below with exponentially fast convergence in L^2 [147]. On the other hand, we then face a new stabilization problem, the convergence as $t \to \infty$ of an orbit of bounded solutions $\{u(\cdot,t),\ t > 0\}$ which are uniformly bounded in $L^2(\Omega)$ to a singular stationary solution $S \notin L^\infty(\Omega)$. The rate of convergence in $L^\infty(\Omega)$ is then obtained by a delicate matched asymptotic expansion, and the $L^\infty(\Omega)$ rate is not exponential, but linear in time, [40]. This is a case of L^∞ **blow-up in infinite time.** Infinite time blow-up cannot occur in dimensions 3-9 for radial solutions and in dimensions 1 and 2 in general, cf. [47, 52].

(vii) The **uniqueness** question becomes important for singular initial data, say, $u_0(x) = S(x)$. Is the solution of (1.6) unique? Of course, $u(x,t) \equiv S(x)$ for $t > 0$ is a solution. The answer is as follows: if $n \ge 10$ then S is the unique solution, while for $n \in [3,9]$ there exists another (minimal) solution $u(\cdot,t) \in L^\infty$ for any $t > 0$. The construction of a second self-similar solution is performed in [94], Sect. 16.1.

(viii) If we take a singular initial data above $S(x)$,

$$u_0(x) \ge S(x),\ \ u_0 \not\equiv S,$$

and assume that $n \ge 10$ then we obtain cases of **instantaneous blow-up**: $u(x,t) \equiv \infty$ for any arbitrarily small $t > 0$ [147]. This means the complete singularity at $t = 0$ and nonexistence of a non-trivial local in time solution of the problem. The basic argument is very simple. Suppose such a solution exists for a small time $0 < t < T$ and let v the solution of the initial-value problem with data $v_0(x) = 2S(x) - u_0(x) \le S$, so that it exists globally in time and moreover it lies below S. We can prove that v is bounded for all $t > 0$. Let now $\phi = u - S$, and $\psi = S - v$, which solve the equations

$$\phi_t - \Delta\phi = \frac{\lambda_0}{x^2}(e^\phi - 1), \quad \psi_t - \Delta\psi = \frac{\lambda_0}{x^2}(1 - e^{-\psi}).$$

Since the initial and boundary data coincide it is very easy to prove that $\phi \ge \psi$ as long as both solutions exist and the Maximum Principle holds, which can be justified for $n \ge 10$. Therefore, $u \ge 2S - v$, which means that u has a singularity at the origin at least twice that of S. Passing to the reaction term e^u we get $e^u \ge Cx^{-4}$ near $x = 0$ for $t > 0$. Further iteration produces lower estimates of the singularity of the form $CS(x)$ with large C so that e^u is no more integrable. In this moment we apply simple properties of the heat semigroup to conclude that u must be infinite everywhere.

In case $3 \le n \le 9$ the singular solution is not unique, the Maximum Principle is not necessarily true, we have to define a class of minimal of proper solutions (see Section 6 below) and the following curious alternative exists between the two extreme cases: global existence or instantaneous blow-up.

Theorem 4.1. *Let us pose for simplicity equation (1.6) in the whole space $I\!R^n$, $3 \le n \le 9$, with locally bounded data for $x \ne 0$. There exists a constant $c(n) > 0$ such that for* *every initial datum $u_0(x)$ such that $u_0(x) \le S(x) + c$ the minimal solution is global in time, while for $u_0(x) > S(x) + c$ it blows up at $t = 0$.*

We also have $c(n) \to 0$ as $n \to 10$ which is consistent with the fact that the *anomalous excess* constant $c > 0$ does not occur for $n \ge 10$. For proofs and further details see [165]. Extensive information about the exponential equation and related combustion models is found in [17].

5. Blow-up patterns: mode and rates of explosion. Regional blow-up

We have explained that for bell-shaped initial data the occurrence of blow-up for all solutions depends on the critical Fujita exponent. For these data it is also known (Chapter 4 in [154] and references therein) that for $p > m$ the blow-up set of an exploding solution reduces to a single point, so-called *single-point blow-up*. The first results on single point blow-up for $m = 1$, $p > 1$ were obtained in [172] for $n = 1$ and in [58] for $n \ge 1$ and radial solutions; for the quasilinear equation with $m > 1$, $p > m$ in [68]. For $1 < p < m$ the blow-up set is unbounded (in fact, $B(u_0) = I\!R^n$), the so-called *global blow-up* [77]. There is a third possibility, *regional blow-up* for $p = m$.

On self-similar blow-up for quasilinear equations. It is important that the asymptotic blow-up properties of the solutions to the Cauchy problem in $I\!R^n \times I\!R_+$ for the quasilinear heat equation (4.1) with $p > 1$, $m > 1$ and bounded initial data $u_0(x) \ge 0$ in $I\!R^n$, in all the three parameter ranges are governed by the nontrivial *self-similar solutions* (unlike the semilinear $m = 1$ and the fast diffusion case $m \in (m_c, 1)$, see below). See Chapter 4 in [154] and an extended list of references therein. The first formal and numerical results on the general classification of blow-up including exact and self-similar solutions for $m > 1$, $p > 1$ were obtained in 70s, see [156], [153] and references in comments to Chapter 4 in [154] and in [123].

There are still many open problems in dimensions $n > 1$. Monotone decreasing radial solutions are known exist for all the three types of blow-up, HS-, S- and LS-regime for all $1 < p < p_s = m(N + 2)/(n - 2)_+$ [154], p. 197, and for $p \ge p_s$ [94], and they are stable in the rescaled sense (known asymptotic techniques apply). In the S-regime of blow-up, $p = m$, self-similar separate patterns with compact connected supports are always symmetric [38]. In the LS-regime, $p > m$, finite sets of *nonsymmetric blow-up patterns* with a special "linearized" *architecture* were first constructed numerically in [124], see references to Chapter 4 [154], [122] and recent numerical results in [41]. The problem of spectra of nonsymmetric blow-up patterns remains open.

For the quasilinear heat equations with p-Laplacian diffusion operator

$$u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u) + u^p, \;\; p > \sigma + 1,$$

countable sets of self-similar symmetric patterns of single point blow-up (*nonlinear spectra*) were proven to exist, see [154], p. 314, where global and regional blow-up are described. Countable spectra of blow-up patterns for $u_t = (|u_x|^\sigma u_x)_x + e^u$ were obtained in [31] where *transition phenomena* between nonlinear and semilinear (see below) spectra of blow-up patterns was described.

Approximate self-similarity for the semilinear equations. We next briefly describe blow-up patterns for the classical semilinear equations (1.7) and (1.6). For (1.7) the stable generic pattern of the symmetric blow-up in 1D was derived in [108] for $p = 3$ and in [84] for any $p > 1$ by using a formal method of asymptotic expansion. It has the form

$$u(x,t) \sim u_*(x,t) \equiv [(p-1)(T-t)]^{-1/(p-1)}\theta(\xi), \quad \xi = |x|/[(T-t)|\ln(T-t)|]^{1/2}$$

(u_* is an *approximate* similarity solution of $u_t = u^p$ *without* diffusion), where $\theta \geq 0$ solves the semilinear hyperbolic equation (a *Hamilton-Jacobi* equation, HJ for short)

$$-\xi/2\,\theta' - \theta/(p-1) + \theta^p = 0, \;\; \theta(\infty) = 0,$$

and is uniquely given by

$$\theta(\xi) = (p - 1 + c_*\xi^2)^{-1/(p-1)}, \quad c_* = (p-1)^2/4p.$$

This blow-up structure implies that on smaller self-similar compact subsets in the variable $\eta = |x|/(T-t)^{1/2}$ without the logarithmic correction, the rescaled solution converges to the flat blow-up profile $[(p-1)(T-t)]^{-1/(p-1)}$, a result which first proved in [84] for $n = 1$ and in [99] for $n \geq 1$, $p \leq p_s = (n+2)/(n-2)_+$.

A formal centre/stable manifold analysis describing the semilinear countable spectrum of blow-up patterns was presented in [169] for $n = 1$, the spatial structure of all the patterns is governed by the *Hermite* polynomials (similar to zeros of semilinear parabolic equations [103]). The proofs of some of these results were first performed in [16] [54], [104], see detailed references to Chapter 4 in [154]. A classification of possible blow-up patterns for (1.7), $n \geq 1$, is presented in [167]. Actual construction of a countable spectrum of semilinear patterns was performed in [2] using the topological method [26]. Other unstable patterns in the supercritical range $p > p_u$ ($n \geq 11$, see Section 7 below) were considered in [106]. A general characterization of blow-up sets is given in [166]. For the exponential equation (1.6), a detailed formal expansion of a similar generic blow-up pattern was done in [39] and was proved first in [26], see also [169] and [104].

Study of a limit weakly nonlinear model with linear diffusion. We now describe a family of delicate blow-up patterns which are essentially of non-self-similar structure. We observe that the modes of blow-up under the standard assumption of linear diffusion, $m = 1$, in the previous model reduce to single-point blow-up for any $p > 1$. Therefore, the occurrence of the other two modes needs a different type of reaction term. Such a semilinear heat equation

$$u_t = u_{xx} + (1+u)\ln^\beta(1+u), \quad \beta > 1, \tag{5.1}$$

was introduced in 1979 [76], is a particular case of more general quasilinear models with common properties of convergence to Hamilton-Jacobi (HJ) equations studied in [79], where the asymptotic of global in time solutions with $\beta \leq 1$ were established. The weak nonlinearity in (5.1) seems to be the natural choice after the powers. This equation was derived on the basis that the flat blow-up $u_t \sim u \ln^2 u$ with $\beta = 2$ would create the behaviour $u(t) = e^{1/(T-t)}$ which is a *localized boundary blow-up S-regime* with the *fundamental length* of total penetration $L_S = 2$ for the *heat equation* $u_t = u_{xx}$, $x >$

0, $t \in (0,T)$; $u(0,t) = e^{(T-t)^{-1}}$, $t \in (0,T)$ (the *final-time profile* $u(x,T^-)$ is bounded for $x > L_S = 2$), see Chapter 3 in [154].

The mathematical theory of blow-up for (5.1) was created in [88] and [92]. It was shown that blow up is global for $\beta < 2$, single-point if $\beta > 2$ and it is regional precisely if $\beta = 2$. Let us explain the main lines of the regional blow-up result as described in [88]. The main result is as follows.

Theorem 5.1. *Consider the solution of equation (5.1) with $\beta = 2$, posed for $x \in I\!R$ and $t > 0$ with initial data $u_0(x)$ which are assumed to be bell-shaped, symmetric around $x = 0$ and sufficiently concentrated around the maximum. Then blow-up occurs in finite time and the blow-up set is exactly the interval $[-\pi, \pi]$. Moreover, the solutions diverge as $t \to T^-$ with the rate $O(e^{1/(T-t)})$, which is universal; the asymptotic profile is also universal and given by the following asymptotic law: as $t \to T^-$*

$$(T-t)\ln(1+u(x,t)) \to \cos^2(x/2) \tag{5.2}$$

uniformly in $[-\pi, \pi]$; the limit is zero otherwise.

It may seem strange to see that the asymptotic profile is given by a very precise function, $\cos^2(x/2) = (1 + \cos(x))/2$ with a single hump around the origin. This is a case of blow-up *pattern formation*, a very important phenomenon in nonlinear processes, *theory of dissipative structures* in nonlinear media, especially in reaction-diffusion ones. It is also very interesting to check that such a geometrical form can be rationally derived from the equation after some simple but smart calculations. Finally, such a process can be rigorously justified.

Let us see the formal derivation. The ln term in the equation is best eliminated if a power scaling lies somewhere in the problem. This can be actually done by means of the change of variables $v = \ln(1 + u)$. Then equation (5.1) with $\beta = 2$ transforms into

$$v_t = v_{xx} + (v_x)^2 + v^2, \tag{5.3}$$

which has only *quadratic nonlinearities*, a very reassuring fact. We must investigate this equation for blow-up. This is where the hand-waving begins. If we forget the terms with x derivatives (the spatial structure) (5.3) predicts a blow-up similar to $v_t = v^2$, i.e., behaving like $v = O((T-t)^{-1})$. Such a guess leads to the second change of variables $\theta(x,t) = (T-t)v(x,t)$, which must describe an evolution with finite θ. Next, we apply the usual trick of stretching the blow-up time to infinity by means of the slow-time variable $\tau = -\ln(T-t) \to \infty$ as $t \to T^-$. With these two changes (5.3) becomes

$$\theta_\tau = (\theta_x)^2 + \theta^2 - \theta + e^{-\tau}\theta_{xx}. \tag{5.4}$$

Now we accept the usual stabilization assumption and look for the asymptotic behaviour of the evolution of this equation as $\tau \to \infty$ among its stationary solutions, that is we drop the term θ_τ. We also consider the term with θ_{xx} as an asymptotically (exponentially) small perturbation, though discarding it is nontrivial matter since the perturbation is *singular*, i.e., it is of the *second order* unlike the *first-order autonomous operator*. Summing up, we look at the equation

$$(\theta_x)^2 + \theta^2 - \theta = 0, \tag{5.5}$$

which is a quadratic equation of the *HJ type*. The degeneracy of a second-order diffusion equation into a first-order HJ equation at the asymptotic level is a main feature of blow-up problems, encountered over and over. The blow-up function following from (5.2), $v_*(x,t) = (T-t)^{-1}\cos^2(x/2)$, is not a solutions of the semilinear heat equation (5.3), it satisfies the HJ equation $v_t = (v_x)^2 + v^2$. It is an *approximate* self-similar solution, which does not satisfy the equation but describes correctly its asymptotic behaviour.

A wide number of techniques and results are at our disposal concerning the proof of the stabilization conjecture, due to *Lyapunov*, *La Salle*, *Intersection Comparison* and so on. There are two elementary solutions of equation (5.5), both constant: $\theta = 0$ and $\theta = 1$; it happens that the latter is unstable, hence it must be rejected, while $\theta = 0$ is stable and could be a candidate. It happens that it is not the correct solution. The chosen stationary solution is precisely $G(x) = \{\cos^2(x/2),\ x \in [-\pi,\pi];\quad 0 \text{ otherwise}\}$. The justification of this choice is the object of [88] and depends on the special properties (estimates) of the asymptotic profiles (technically, the ω-limits) that are inherited from the parabolic form of the original problem. The asymptotic result relies then on a general stability theorem for dynamical systems with asymptotically small perturbations from [87], which makes it possible to pass to the limit in the *singular perturbation problem* (5.4). More difficult singular perturbation problems occur for the quasilinear models like

$$u_t = \nabla\cdot(\ln^\sigma(1+u)\nabla u) + (1+u)\ln^\beta(1+u), \quad \sigma > 0,\ \beta > 1.$$

Such an asymptotic study for all the three case of single point, regional and global blow-up was performed in [92]. A detailed analysis of such singular limits is presented in the survey paper [91] and in the book [96].

6. **Continuation after blow-up**

An important question that we want to discuss at some length is the possibility of continuation of the solutions of a blow-up problem **after** the blow-up time; this will force us to examine which reasonable physical or mathematical options are at our disposal if classical continuation fails. In the best of cases the experts in evolution problems aim at defining a continuous semigroup of maps in a certain space which will represent the time evolution of the physical problem, for all time if possible. Baras and Cohen addressed this problem in 1987, [13], for the semilinear heat equations (1.7) $u_t = \Delta u + f(u)$, posed for $\Omega \subset \mathbb{R}^N$ and $t > 0$ with exponent $p > 1$ in the framework of nonnegative solutions. They proved that for $f(u) = u^p$ in the *subcritical Sobolev parameter range* $1 < p < (N+2)/(N-2)$, if $N \geq 3$ ($1 < p < \infty$ if $N = 1,2$) a continuation defined in a natural or physical way is not possible, because it leads to the conclusion that

$$u(x,t) = \infty \quad \text{for all}\quad x \in \Omega,\ t > T. \tag{6.1}$$

They labeled the phenomenon *complete blow-up*. For further work see [126], [127] and their references. Brezis posed the problem of finding equations where a nontrivial natural continuation exists, i.e., the existence of reaction-diffusion equations with *incomplete blow-up*. This problem has been addressed by the authors in three works [90], [93] and [94]. The first two deal with one-dimensional problems. The results show that the continuation after blow-up is a relatively simple phenomenon in 1D. In contrast, the situation in several dimensions studied in [94] is much more complex, and we have found new forms of blow-up described by a kind of blow-up solutions that we call peaking solutions, which blow-up only at one point and one instant of time, with a classical continuation afterwards.

Concept of continuation. Proper (minimal) solutions. In the study of blow-up for reaction-diffusion equations with linear diffusion the continuation after blow-up can be studied by writing the standard integration by parts formula

$$u(t) = S_t u_0 + \int_0^t S_{t-s} f(u(s))\, ds, \tag{6.2}$$

cf. [13], where S_t is the semigroup generated by the linear equation part, and investigating when and where the formula makes sense for $t > T$. This method cannot be used in the presence of nonlinear diffusion. Following [13] we may introduce a sequence of approximate problems that admit solutions globally in time. Namely, we assume that the reactive term $f(u)$, source of the explosive event, is replaced by a nicer term $f_n(u)$ where f_n is a smooth function with not more than the linear growth in $u \gg 1$ and the monotone convergence $f_n \nearrow f$ holds, so that the corresponding solution of the initial-value problem u_n is globally defined in time and, by the standard comparison, the sequence $\{u_n\}$ is monotone increasing in n. Then, by *monotonicity*, we can pass to the limit and using the *Maximum Principle* obtain a function

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{6.3}$$

This method works for quasilinear equations and also allows also to treat unbounded initial data by approximating them in a monotone increasing manner with bounded ones. It can also work for fully nonlinear equations. It produces a "solution", finite or infinite, that we have called in [94] the *proper solution* of the initial-value problem to distinguish it from other possible methods of constructing a *limit solution*. We proved that this definition does not depend on the approximations performed on the reaction term and the data, and also that it coincides with the classical or weak solution as long as it exists and is bounded, or with the *minimal* nonnegative solution in cases of nonuniqueness. Complete blow-up amounts to $u \equiv \infty$ for $t > T$. In the sequel continuation will be discussed in the framework of proper solutions.

When the solution blows up at time $T > 0$, we can define the blow-up set in the usual way, cf. (2.1), (2.2), or also as the set of points where the proper solution becomes infinite for $t = T^-$. In general we can define a similar set,

$$B[u](t) = \{x \in \Omega : \lim_{n\to\infty} u_n(x,t) = \infty\}$$

for every $t > 0$. While for $0 < t < T$ the set obtained in this way is empty, for $t > T$ it will be the whole domain if blow-up is *complete*. The set is interesting when blow-up is incomplete since it will be a subset of Ω, generally nonvoid for $t > T$. Using the motivation coming from combustion problems, such a set is called then the *burnt zone*.

Characterization of complete blow-up in 1D. In a first work, [90], we have studied the more general one-dimensional reaction-diffusion model (3.1). Under suitable generic assumptions on ϕ, f and the initial data u_0 we get a rather complete characterization of the nontrivial continuation of solutions of the Cauchy problem after blow-up in terms the properties of the constitutive nonlinearities ϕ and f (and not on u_0). A natural problem is to find conditions for complete/incomplete blow-up in terms of the constitutive functions ϕ and f. In principle, the alternative will also depend on the initial data.

More precisely, we consider equation (3.1), where ϕ and f are real functions which are defined and positive for $u > 0$. The term $\phi(u)_{xx}$ represents nonlinear diffusion, possibly degenerate or singular. We assume that $\phi \in C([0,\infty)) \cap C^1(0,\infty)$, with $\phi'(u) > 0$ for $u > 0$ and $\phi(0) = 0$. The reaction term $f(u)$ is assumed to be positive for $u > 0$; for convenience we may also assume that it is continuous. We recall that finite-time blow-up occurs whenever $f(u)$ is super-linear for large u, as is well-known. We will precise the necessary condition in a moment. We study the Cauchy problem $u(x,0) = u_0(x),\ x \in \mathbb{R}$, with bounded, continuous, nonnegative and nontrivial initial data. We will introduce in a moment further restrictions on the class of data which we will be discussed.

(1) Flat and bell-shaped data. There is a first simple choice of initial data which very much simplifies the problem, namely, the class of nontrivial and constant data, that we shall call *flat data*. It is clear that for flat initial data the solution will be flat for all time as long as it exists, and consequently blow-up is flat, hence complete, when it occurs. Precisely, blow-up for positive constant initial data occurs if and only if the integral

$$I_1(u) = \int_1^u ds/f(s) \tag{6.4}$$

converges at infinity, $I_1(\infty) < \infty$. This follows from the ODE (1.2) satisfied by all spatially flat solutions $u(t)$. The problem of blow-up and continuation is thus completely solved and the result does not depend on the data.

We want to avoid such trivial situations and still keep some geometrical simplicity. Therefore, we make in the sequel the assumption, typical in the blow-up literature, that the data are *bell-shaped*, i.e., that u_0 has only one maximum point and goes to zero at infinity. It will be clear from the proofs that the main results generalize to data with several humps or not going to zero.

(2) Integral conditions and main results. Let us begin by stating the conditions which determine the form of continuation for bell-shaped data. The presence of complete or incomplete blow-up depends on the behaviour at infinity of three integrals. The first condition, **(B1)**, is $I_1(\infty) < \infty$. For non-flat initial data the necessity of this condition for blow-up follows from the Maximum Principle. Therefore, we will always assume (B1). We will see that for bell-shaped data it is not always sufficient for blow-up, ϕ has a say!

The second integral that comes into play is

$$I_2(u) = \int_1^u f(s)\phi'(s)\,ds, \tag{6.5}$$

which allows to measure the relative influence of diffusion and reaction, and is reflected in our proofs in the existence of a certain type of blow-up traveling waves through the boundedness of the ratio

$$F(u) = I_2(u)/u^2 \tag{6.6}$$

as $u \to \infty$ (condition **(B2)** in [90]). Finally, we need to control

$$I_3(u) = \int_1^\infty [\phi'(s)/s]\,ds, \tag{6.7}$$

which affects only the strength of the diffusion nonlinearity. The boundedness of this integral (condition **(B3)**) is equivalent to the property of finite speed of propagation for large values of u, reflected in the existence of traveling waves which become infinite at a finite distance. In order to get an intuitive idea of what these conditions mean we may consider the case of power-like nonlinearities, equation (4.1) in 1D with $m > 0$ and $p > 0$. Then (B1) holds for $p > 1$, (B2) for $m + p \le 2$ and (B3) for $m < 1$. In terms of these integrals we can formulate the following blow-up results.

Theorem 6.1. (Global continuation) *Let u be a proper solution to the above problem under the stated general conditions. If (B1), (B2) and (B3) hold, then u can be continued in a non-trivial way for all times $t > 0$ (i.e., $u(\cdot, t) \not\equiv \infty$ for all $t > 0$) even if u blows up at a time $T < \infty$.*

In fact, the result says that the *burnt zone* $B[u](t)$ is a bounded subset of $I\!R^n$ or the empty set. (B1)-(B3) are valid for equation (4.1) in 1D precisely when $p > 1$ and $p+m \le 2$. In particular, (B3) excludes linear diffusion, probably the main reason why the phenomenon of nontrivial continuation after blow-up in such a simple type of equation was unnoticed. The opposite situation happens when (B2) fails.

Theorem 6.2. (Complete blow-up) *Let u be a solution of equation (3.1) under the above general conditions and assume that it blows up at time $0 < T < \infty$. If (B2) does not hold than $u \equiv \infty$ for $t > T$.*

The proofs given in [90] extend to solutions with quite general initial data, but then we need some additional conditions on the approach to blow-up in the statement of the results. We have decided to present here the more restricted but cleaner version. On the other hand, for classes of data which do not go to zero at infinity ($u_0(x) \ge a > 0$ for all large $|x|$) incomplete blow-up implies continuation for a finite time, after which the solution necessarily undergoes complete blow-up. The occurrence of this second stage is a simple consequence of the Maximum Principle (comparison with flat subsolutions).

On the other hand, the reader will observe that we do not have still a complete characterization since the assumptions of the two results are not complementary of each other, due to the presence of condition (B3). They would combine into a complete results if it could be eliminated. This is almost true: under a number of light extra assumptions (B1) and (B2) imply (B3). It is for instance true for the 1D power equation (4.1). Besides, if condition (B2) is strengthened into (B2')

$$\phi'(u)f(u)/u \le C \text{ for all large } u,$$

then we have $\phi'(u)/u \le C/f(u)$, hence (B1) + (B2') imply (B3). The implication follows also when f is monotone. However, we have shown in [90] that there exist "pathological" choices of ϕ and f for which (B1) and (B2) hold but (B3) does not. Equations in the *pathological class* behave in a very special way.

Theorem 6.3. *If (B1) and (B2) hold and (B3) does not, then all the solutions with flat initial data* $u_0 \equiv \text{const} > 0$ *blow-up in finite time, while no solution with bell-shaped compactly supported data does.*

Our results are true not only for solutions to the Cauchy problem, but can also be directly applied to initial-boundary value problems in bounded spatial domains with Dirichlet or Neumann boundary conditions. In fact, the analysis of complete/incomplete blow-up is local in the sense that the behaviour for $t > T$ depends only on the behaviour of the solution in a small neighbourhood of a given blow-up point, thus being independent of the boundary conditions.

(3) The Method of traveling Waves. A well-known method of investigation of blow-up phenomena is the *Method of Stationary States* [80], Chapter 7 in [154], whereby the intersection comparison analysis of the family of stationary solutions allows to conclude important features of the quite different blow-up solutions by means of smart comparisons. This method describes the blow-up behaviour as $t \to T^-$. See other applications in [88] and [92]. In the behaviour for $t > T$ and $t \to T^+$, if the phenomenon of complete blow-up occurs, a kind of sudden avalanche happens near the blow-up points: a solution which is everywhere finite for $t < T$ becomes everywhere infinite for $t > T$, thus implying infinite speeds of propagation near the blow-up set. In the incomplete blow-up the singular propagation for $t \ge T$ is assumed to be finite. As a consequence of those reasons, the analysis proposed and performed in [90] for $t \ge T$ to prove the above results is based on the *Method of traveling Waves* (so that we deal with *moving* singularities). In a first step we perform a careful analysis of the existence and properties of family of the traveling wave solutions (TW's), and a subsequent step allows to prove the blow-up results using those waves.

(3a) Classification of the traveling Waves. Therefore, we first study TW solutions of the standard form $V(x,t) = \theta(\xi), \quad \text{with } \xi = x - \lambda t + a$, where λ is the speed parameter, $a \in I\!R$ is arbitrary, and the profile function $\theta \ge 0$ solves a nonlinear ODE

$$\phi(\theta)'' + \lambda\theta' + f(\theta) = 0,$$

where $' = d/d\xi$. To investigate it, we carry out a phase-plane analysis. A complete analysis of the asymptotic behaviour of the family $\{\theta\}$ can be done in terms of the conditions (B2) and (B3), (B1) is always assumed. Then the crucial point is the behaviour of the envelope of the family for large values of the parameter λ.

The first concern of the ODE analysis is the existence of TW solutions with a monotone profiles, versus the existence of profiles which have a hump with a finite maximum. Without loss of generality (up to a symmetry, $x \mapsto -x$) we may assume that $\lambda > 0$. The existence of monotone traveling waves is characterized as follows.

Theorem 6.4. *There exists a monotone traveling wave for some $\lambda_0 > 0$ if and only if condition (B2) is satisfied, and then such waves exist for all $\lambda \geq \lambda_0$.*

If f is so large with respect to ϕ that (B2) does not hold, then the non-monotone profiles will become increasingly tall and narrow as $\lambda \to \infty$, forming *spikes*. On the other hand, if (B2) holds the monotone profile will tend to infinity as ξ decreases. We still have to investigate in that case the existence of *singular (blow-up) traveling waves*, i.e., of profiles which reach infinity at a finite ξ_0 (a vertical asymptote). We have to make a calculation of the total ξ-range of the orbits we have constructed. We get

Theorem 6.5. *There exist singular traveling waves for some $\lambda > 0$ if and only if conditions (B2) and (B3) hold, and then they exist for all large λ.*

If a monotone traveling wave is not singular, it is defined for all $\xi \ll -1$ and reaches the singular level $\theta = \infty$ at $\xi = -\infty$. The existence of such waves is the main feature of the pathological class.
(3b) Geometric Step. The second part of the proofs is a geometric analysis based on the comparison of the given solution $u(x,t)$ and the family of traveling waves $\mathcal{B}$. Complete blow up happens by means of an *avalanche phenomenon* which is rigorously established using the technique of Intersection Comparison with tall spikes which travel very fast and are shown to lie below the solution near the blow-up points. By the Maximum Principle they push the high-intensity points of u from below in a complete range of directions, thus creating the result $u \equiv \infty$ after blow-up. In the other two results standard comparison may be used. Detailed proofs can be found in [90].
(4) Blow-up free boundaries. Finally, incomplete blow-up leads to a two-phase problem: while in the space-time region $Q_1 = B[u](t) \times I\!R_+$ the proper solution $u(x,t)$ is infinite, in the complement $Q_2 = \{I\!R \setminus B[u](t)\} \times I\!R_+$ the solution is finite and the equation holds. These two regions are the *fresh* and the burnt zones. The separation between them is a kind of *combustion front*, mathematically a *singular (blow-up) free boundary*. Though much is known about free boundaries and combustion fronts, these precise objects have special features worth investigating. The singular free boundary problem for the 1D equation (4.1) with $m > 1 > p$ and $m + p = 2$ was studied in [93]. It is important to mention that unlike the classical regularity and analyticity results on the free boundaries for the PMEs or other similar ones, the singular blow-up interfaces are *essentially* non-analytic. Moreover, there exists a class of initial data such that for the blow-up interfaces $s(t)$ the optimal regularity for $t > T(u_0)$ is $C^{1,1}$ ($s'(t)$ is Lipschitz continuous) but **not** C^2. In this class the analytic continuation of the solution beyond blow-up time $t > T$ fails to be a proper minimal continuation starting from a moment $t = T_1 > T$, at which the second derivative $s''(t)$ is discontinuous, [93].

For the equation with the p-Laplace operator

$$u_t = (|u_x|^\sigma u_x)_x + u^p \quad (\sigma > -1),$$

complete blow-up occurs if and only if $p > 1/(\sigma+1)$ [90], in the critical case $p = 1/(\sigma+1) > 1$ ($\sigma \in (-1, 0)$) the blow-up interfaces exhibit eventual C^2 discontinuity [93].

7. On transient blow-up patterns in several dimensions

Consider now equation (4.1) for $x \in I\!R^n$ and $t > 0$ in the *supercritical Sobolev range*,

$$p > p_s = m\,(n+2)/(n-2) \qquad (n \geq 3,\ m > m_c = (n-2)/n). \tag{7.1}$$

The novelty consists of a new type of radially symmetric blow-up, in the form of a solution which has a momentary single-point blow-up peak at $t = T > 0$ and then evolves immediately into a classical bounded function for the rest of time $t > T$, hence a transient form of blow-up. The existence of such solutions was established by the authors in [94] in the framework of proper solutions by means of a self-similar construction both for $t > T$ and $t > T$, and they were called **peaking solutions**. A precedent is found in the work of Lacey and Tzanetis [127] for the exponential equation (1.6), but there the matching of both developments is formal. Peaking solutions are a transient form of incomplete blow-up. *Incomplete* blow-up patterns are known to be structurally *unstable*, Sect. 14.1 [94]. Stable self-similar patterns exhibit *complete* blow-up which is expected to be a *stable* mode, Sect. 14.2 [94]. Our main result can be stated as follows.

Theorem 7.1. *There exists a number $p_p > p_s$ depending on m, n such that for $p \in (p_s, p_p)$ there exist proper (minimal) solutions globally defined in time, which blow up at finite $T > 0$ and are bounded at all other times. For $n \leq 10$ we can take $p_p = \infty$. Our solutions satisfy $u(x,t) \leq S(x)$ for all $t \geq T$, where S is the corresponding singular stationary solution.*

In dimensions $n \geq 11$, denoting $N = n - 10 > 0$, we have a finite value for p_p given by the impressive formula

$$p_p = 1 + \frac{3m + [(m-1)^2N^2 + 2(m-1)(5m-4)N + 9m^2]^{1/2}}{N}. \tag{7.2}$$

This new exponent was first introduced in [129] for $m = 1$ where it takes the simpler form $p_p = 1 + 6/(n-10)$. Observe that $p_p > p_u = m[1 + 4/(n - 4 - 2\sqrt{n-1})]$, $\quad n \geq 11$; p_u is an important *critical exponent* which is responsible for the uniqueness of a solution with the singular initial data

$$S(x) = c_s|x|^{-2/(p-m)}, \quad c_s = [2m(p-m)^{-1}(n - 2 - 2m(p-m)^{-1})]^{1/(p-m)}.$$

$S(x)$ is defined if $n \geq 3$ and $p > p_{st} = mn/(n-2)$, another *critical exponent*. Observe that $S(x)$ is locally integrable and moreover $S^p \in L^1_{\rm loc}(I\!R^n)$. The subindex "$p$" in the exponent p_p refers to "peaking". The solution we construct has a self-similar form both for $t < T$ and $t > T$. This means that

$$u(x,t) = (T-t)^\alpha \theta_1(|x|(T-t)^{-\beta}), \quad u(x,t) = (t-T)^\alpha \theta_2(|x|(t-T)^{-\beta}),$$

resp. for $t < T$ and $t > T$, with $\alpha = -1/(p-1)$ and $\beta = (p-m)/2(p-1)$ and θ_1 and θ_2 suitable one-dimensional profile functions such that

$$\lim_{s\to\infty} \theta_1(s)s^{2/(p-m)} = \lim_{s\to\infty} \theta_2(s)s^{2/(p-m)} > 0.$$

The construction relies on an ODE stability study for the singular stationary solution S, which happens to be completely different for $p < p_u$ and $p \geq p_u$. Moreover, for $p < p_u$ we can construct infinitely many different peaking solutions, while for $p \geq p_u$ only a finite number can be shown to exist by our method. We claim that at least for self-similar solutions the above value p_p is optimal for the theorem, see an example below.

The peaking solutions have the weakest possible form of blow-up. In fact, it is easy a posteriori to check that $u \in C([0,\infty) : L^r_{\rm loc}(I\!R^n))$ for every $1 < r < n(p-m)/2$, a number that for $p > p_s$ is larger than $2mn/(n-2)$. They are also *weak solutions* of the equation in the standard sense of integration by parts.

Related to this result let us mention that in [140] it is explained that for $p \geq p_s$ equation (1.7) posed in a bounded, smooth, convex domain Ω with Dirichlet boundary condition and a special choice of nonnegative initial data admits a *global* L^1*-solution* $\tilde u(x,t)$ which is *not bounded* in $L^\infty(\Omega)$. It follows from the result on complete blow-up in the critical Sobolev case [94], Sect. 5, that for $p = p_s$ in the case where $u_0(|x|)$ is decreasing, the solution $\tilde u(|x|,t)$ does not blow-up in finite time (otherwise, blow-up is complete and no global continuation exists), so that $\tilde u(t) \in L^\infty(I\!R^n)$ for all $t > 0$ and $\sup_x \tilde u(x,t) \to \infty$ as $t \to \infty$. We prove that in the range $p \in (p_s, p_p)$ such symmetric solutions $\tilde u$ do blow-up in finite time.

Blowing-up self-similar solutions for $p > p_s$. We first construct self-similar blow-up solutions for $t < T$ as a first step into the proof of existence of global peaking solutions. Thus, we consider solutions of the form

$$u_*(r,t) = (T-t)^{-1/(p-1)}\theta(\eta), \quad \eta = r/(T-t)^\beta, \tag{7.3}$$

with $\beta = (p-m)/2(p-1) > 0$. Then the function $\theta(\eta) \geq 0$ satisfies the ODE

$$\eta^{1-n}(\eta^{n-1}(\theta^m)')' - \beta\theta'\eta - \theta/(p-1) + \theta^p = 0, \quad \eta > 0, \tag{7.4}$$

together with the symmetry condition $\theta'(0) = 0$. One can see that (7.4) admits the flat solution $\theta \equiv k = (p-1)^{-1/(p-1)}$, and the singular stationary solution $S(\eta)$ also satisfies it. Existence of a nontrivial solution $\theta > 0$ in the subcritical range $1 < p < p_s$ has been established in [154], Chapter 4, $p = p_s$ was studied in Sect. 14 in [94]. We first give an example of an explicit solution in the semilinear case.

Example ([154], p. 287). Let $m = 1$ and $p = 2$. For all dimensions $6 < n < 16$ (we can take n a real number), equation (7.4) admits the explicit solution

$$\theta(\eta) = A/(a+\eta^2)^2 + B/(a+\eta^2),$$

where $a = 2[10D - (n+14)] > 0$, $A = 24a$, $B = 24(D-2) > 0$, with $D = (1+n/2)^{1/2}$. As $\eta \to \infty$, $\theta(\eta) = B\eta^{-2}(1+o(1))$, with $B < c_s = 2(n-4)$ if $n < 16$. Moreover, we have that $p_s = (n+2)/(n-2) < p = 2$ provided that $n > 6$, whence the lower bound on the dimension. Besides, $p = 2 < p_p$ if $N < 16$. This means that in these circumstances,

p_p is an optimal upper bound and we have an explicit solution with an exponent p larger than p_u if the dimension satisfies $10+4\sqrt{2} \le n < 16$.

Let us state a general result on the solvability of the problem (7.4) in both ranges $p_s < p < p_u$ and $p_u \le p < p_p$. Recall that now we are looking for "small" profiles $\theta(\eta)$ corresponding to *incomplete* blow-up.

Theorem 7.2. *Let $m > 1$ and $p_s < p < p_u$. Then (7.4) admits an infinite sequence of positive radial solutions $\{\theta_M(\eta),\ M = 1, 2, ...\}$ satisfying as $\eta \to \infty$.*

$$\theta_M(\eta) = c_M \eta^{-2/(p-m)}(1 + o(1)), \quad \textit{with } \ c_M \in (0, c_s). \tag{7.5}$$

The upper range of p is dealt as follows.

Theorem 7.3. *If $p_u \le p < p_p$ then there exists at least one solution $\theta_1(\eta) > 0$ satisfying (7.5) with $c_1 \in (0, c_s)$.*

The proof relies on a detailed ODE analysis of the intersection properties of the solutions to (7.3) with respect to the singular stationary profile S, [94], Sect. 12.

Self-similar continuation after blow-up. We now show that the self-similar solutions constructed above can be continued in a nontrivial way as proper solutions past T. As a first result we have [94], Sect. 13.

Theorem 7.4. *The blow-up self-similar solutions, constructed in the previous two theorems have a nontrivial proper continuation for $t > T$ and indeed $u(r, t) \le S(x)$ in $I\!R^n \times (T, \infty)$. They are also self-similar for $t > T$.*

The self-similar continuation for $t > T$ has the standard self-similar form

$$u(r, t) = (t - T)^{-1/(p-1)} f(\zeta), \quad \zeta = r/(t - T)^{\beta}. \tag{7.6}$$

Then $f \ge 0$ solves equation the ODE

$$\zeta^{1-n}(\zeta^{n-1}(f^m)')' + \beta f'\zeta + f/(p - 1) + f^p = 0, \quad \zeta > 0; \quad f'(0) = 0.$$

In particular, we have the following results.

Theorem 7.5. *Let $p_s < p < p_u$. Then the proper continuation for $t > T$ of the blow-up self-similar solution coincides with the self-similar solution (7.6) with a specially chosen profile $f(\zeta)$.*

The analysis in the case $p \ge p_u$ gives a similar result by a slightly different argument. Observe that the proper solution with stationary singular data S does not go down with time in this case.

Theorem 7.6. *Let $p_u \le p < p_p$. Then the proper continuation of the blow-up self-similar solution (7.3) with the profile $\theta_1(\eta)$ satisfying (7.5) is self-similar and bounded for $t > T$.*

8. **Extinction**

Maybe the simplest evolution process on which extinction problems are modeled is given by the ODE

$$u_t = -f(u), \quad f(u) = u^p,$$

where $u \geq 0$ is sought, and we impose initial conditions $u(0) = a > 0$. While for $p \geq 1$ we obtain a curve $t \mapsto u(t)$ which is defined and positive-valued for all $t > 0$, for $p < 1$ the evolution is described by the rule $u(t) = [a^{1-p} - (1-p)t]^{1/(1-p)}$, which holds up to a moment $T = a^{1-p}/(1-p)$, at which time the solution vanishes. For $p > 0$ we define in a natural way $f(0) = 0$ and the solution curve is continued to the interval $T \leq t < \infty$ as $u = 0$ with a continuous derivative u_t at $t = T$. Difficulties arise for $p = 0$ where we would like to define $f(0) = 0$, and continue u in the same way as before, now with a discontinuous derivative. Finally, for $p < 0$ the derivative u_t blows up as $t \to T^-$ and the singularity at the extinction point is unavoidable.

The problem is more interesting when we add a spatial structure and consider equations of the diffusion-absorption form, like

$$u_t = \Delta u - f(u), \quad \text{or} \quad u_t = \Delta\phi(u) - f(u), \tag{8.1}$$

for suitable ϕ and f, for instance power functions. If $f(u)/u$ is singular at $u = 0$ then it can happen that $u \to 0$ at a certain time $T > 0$ at a number of points (*quenching points*) even if the initial data are given by a strictly positive function. This phenomenon is called *extinction* or *quenching* and the resulting zero-set in the (x, t) plane is called the *dead core*. These problems arise in chemical processes. Although extinction can be reduced to blow-up for a different parabolic equation by the transformation $u \mapsto 1/u$, the transformation is not usually convenient and it is better to study the singular behaviour directly. A detailed survey on existence, uniqueness and regularity for degenerate parabolic equations with absorption is given in [115]. A general account of extinction problems can be found in [131].

A very important problem in this theory consists in deciding whether the solution can be continued after quenching starts with two coexisting regions, the dead zone where $u = 0$ and the active zone where $u > 0$ (*incomplete extinction, standard quenching*), or otherwise the dead zone covers the whole spatial domain for $t > T$ (*complete extinction*). Sections 5 and 6 of the paper [90] are devoted to the study of complete/incomplete extinction for the quasilinear equation (3.2) with a singular absorption term satisfying $f(u) \to -\infty$ as $u \to 0$. The same technique used in [90] for the characterization of Continuation after Blow-up in reaction-diffusion in 1D allows to solve the problem of Continuation after Extinction for the general quasilinear heat equations with absorption (3.2), which was also open for a long time. The generalization of the results on continuation after extinction to several dimensions is done in [94]. In particular for the quasilinear diffusion-absorption equation with power nonlinearities

$$u_t = (u^m)_{xx} - u^p$$

there are two parameter zones. For $p \le -m$ extinction is always *complete* (no proper maximal continuation $u \not\equiv 0$ for $t > T$ exists). If $p > -m$, such a maximal (weak) continuation exists and the Cauchy problem with compactly supported initial data makes sense. The asymptotic behaviour of such extinction vanishing behaviour was established to be approximately self-similar [89] in the *critical case* $p = 2-m$. The interface propagation in this case is studied in [86], where analyticity is proved and a *countable spectrum* of *turning* and *inflection* interface patterns is constructed with the local spatial shape governed by *Kummer's* polynomials. It turned out that the singular interface propagation is the most interesting in the *subcritical range* $p \in (-m, 2-m)$, where the interface equation was proved to be of the second order [85], unlike the first-order Darcy's law for the PME. In the semilinear case $m = 1$, $p < 1$, the asymptotic extinction behaviour is completely different and is similar to that of blow-up for the semilinear heat equation (1.7) [107]. In the equation with gradient-dependent diffusivity

$$u_t = (|u_x|^\sigma u_x)_x - u^p \quad (\sigma > -1),$$

incomplete extinction exists if and only if $p > -1$, [90], Sect. 7. For semilinear equations like

$$u_t = u_{xx} - u^{-\beta}, \quad \beta > 0,$$

the structure of extinction or *quenching* singularity patterns is similar to those in the blow-up for (1.7), see [53]. For quasilinear diffusion operators like $(u^m)_{xx}$ such a behaviour can be self-similar (nonlinear patterns) as in the blow-up for quasilinear heat equations. For further work on extinction see also [36], [49], [53], [50], [51], [118], [148].

Finite-time extinction due to the singularity of the equation and boundary conditions (without a zero-order absorbing term) was first proved in [152] ($n = 1$) for the fast diffusion equation $u_t = \Delta u^m$, $m \in (0,1)$, in a bounded domain Ω with $u = 0$ on $\partial\Omega$. Separate asymptotic behaviour was obtained in [21] for $m > (n-2)/(n+2)$. In the Cauchy problem in $I\!R^n \times (0,T)$ for $m \in (0,(n-2)/n)$ there exists a self-similar solution of second kind and it is asymptotically stable for symmetric solutions [83].

9. **On singularities in mean curvature and curve shortening flows**

Often, such singularity formation has much in common with extinction or blow-up for nonlinear parabolic equations. For instance, equation

$$u_t = u_{xx}/(1+u_x^2) - (n-2)/u, \quad u > 0 \ (n \ge 3),$$

describes the evolution of rotationally symmetric hypersurfaces with the equation $r = u(x,t)$ moving by *mean curvature* in $I\!R^n$ and admits finite time "flat" quenching see [160] and references therein. The notation u_x^2 means $(u_x)^2$. A classification of semilinear patterns (similar to semilinear heat equations exhibiting *Hermite* polynomial space structure) is available in [5].

In the classical *curve shortening flows* we have $C_t = k\mathbf{n}$, where $C : S^1 \times [0,T) \to I\!R^2$ is a family of immersed curves on the plane parameterized by arc length s, k is the curvature and $\mathbf{n}$ is the unit normal. Since $k\mathbf{n} \equiv C_{ss}$ we get a parabolic quasilinear heat equation with $2m\pi$-periodic boundary conditions in the angle θ that the tangent to $C(t)$ makes with a fixed line:

$$k_t = k^2 k_{\theta\theta} + k^3$$

with a superlinear term k^3. On the other hand, the n-dimensional version $u_t = u^2(\Delta u + u)$ describes *diffusion in a force-free magnetic field*. For convex curves, it shrinks to a point in infinite time with eventual circle form with power rate, $k = O((T-t)^{-1/2})$, [64]. For non-convex curves the singularity structure is more complicated, cusp singularities have fast blow-up at the rate [6]

$$\max_\theta k(\theta, t) = (T-t)^{-1/2}[\ln|\ln(T-t)|]^{1/2}(1+o(1))$$

(see [154], p. 308). Cf. the rate of self-focusing for the 2D cubic NLS equation, Section 15 below. Localization of blow-up solutions with Dirichlet boundary conditions $u(\pm a, t) = 0$, $a > \pi/2$, was first proved in [59] where the localization length was proved to be π for rather arbitrary initial data $u_0 \geq 0$. Curve shortening flows with the equation $V = k + f(y)$ ($f(y)$ is a given function on the $\{x, y\}$-plane, V denotes the upward velocity of $C(t) = \{y = u(x, t)\}$) reduces to the quasilinear equation

$$u_t = u_{xx}/(1+u_x^2) + f(u)(1+u_x^2)^{1/2}$$

which admits blow-up of u_x, see [98].

10. **On blow-up due to nonlinear boundary conditions**

Blow-up can be generated by nonlinear Neumann type boundary conditions. For instance, for the heat equation $u_t = \Delta u$ in $\Omega \times (0, T)$ this would be

$$\partial u/\partial n = g(u) \quad \text{on } \partial\Omega \times (0, T), \tag{10.1}$$

where $\partial u/\partial n$ denotes the outward normal derivative on the smooth boundary $\partial\Omega$ and $g(u)$ is a nonnegative function with a superlinear growth as $u \to \infty$. For the quasilinear equations in (1.8) the corresponding Neumann conditions are

$$\partial(\phi(u))/\partial n = g(u), \quad (|\nabla u|^{m-2}\nabla u) \cdot n = g(u) \quad \text{on } \partial\Omega \times (0, T).$$

The same six questions on the blow-up behaviour apply. Depending on type of nonlinearity $g(u)$ and the diffusivity operator in the equation, the asymptotic behaviour can be asymptotically self-similar or approximate self-similar. In particular, the effect of degeneracy into a quadratic HJ equations occurs for weakly nonlinear functions $g(u)$ and $\phi'(u)$. Notice that since the nonlinear operator generating the blow-up event is prescribed on a fixed boundary $\partial\Omega$ (which has zero measure in $I\!R^n$), the blow-up problems with nonlinear flux conditions are usually easier than those where volumetric source terms entering the equations, as the ones we discussed above, where the singularity location is a priori unknown. In fact, problems like that exhibit many features of the boundary blow-up with a prescribed blow-up functions of the boundary $\partial\Omega$:

$$u(x, t) = f(t) \to \infty \quad \text{as } t \to T^-.$$

Once appropriate estimates of $u(x,t)$ on $\partial\Omega$ are available, the results on localization and blow-up patterns follow. The existence of localized blow-up functions was proved for a general heat conduction equations $u_t = (k(u)u_x)_x$, $k(u) > 0$ for $u > 0$, regardless the behaviour of $k(u)$ as $u \to \infty$, see Chapters 3, 5, 6 in [154] and references therein. We refer to the survey paper [132] and the analysis of critical Fujita exponents for different quasilinear equations in 1D [81], see also surveys of recent results in [82, 48].

11. **Fully nonlinear and nonlocal parabolic equations**

There are only a few results in that direction though fully nonlinear models are popular in the applications in recent years. Energy estimates are applied in [133] to prove blow-up for equations like $Q(t,u,u_t)+A(t,u) = F(t,u)$ admitting a conservation law. The critical Fujita exponent p_c was calculated in [82], Sect. 4, for the fully nonlinear equation in $I\!R \times I\!R_+$ with *dual porous medium operator*

$$u_t = |u_{xx}|^{m-1}u_{xx} + u^p,$$

where $m > 1$ and $p > 1$. It is $p_c = 1+1/\mu > 1$ where $\mu = \mu(m) > 0$ is the sharp exponent of the minimal L^∞ decay rate $O(t^{-\mu})$ for $t \gg 1$ of compactly supported solutions of the equation without reaction (it comes from self-similarity of *second kind* [19]). Blow-up and a "continuous" spectrum of blow-up patterns for the equation in *damage mechanics* $u_t = g(u)(\lambda^2 u_{xx} + u)_+, \quad g(u) \sim u^\alpha \ \ ((\cdot)_+ = \max\{\cdot, 0\})$, were constructed in [23], a model by Barenblatt (1986), see references therein. A fully nonlinear model from *detonation theory*

$$u_t = f(cuu_{xx}) - \frac{1}{2}(u_x)^2 + \ln u, \quad f(s) = \ln\left(\frac{e^s-1}{s}\right),\ c \in (0,1),$$

introduced by Buckmaster and Ludford (1986), describing the instability of the *Zel'dovich-von Neuman-Doering square wave*, was studied in [74]. It is shown that, given positive initial data $u_0(x) > 0$, single point extinction in finite time occurs with the asymptotic behaviour as $t \to T^-$ described by a semilinear HJ equation. Once extinction happens, the problem does not admit a non-trivial continuation beyond, so that $u(t) \equiv 0$ for any $t > T$ (the limits $t \to \pm T$ do not coincide).

An effective Fourier method to prove blow-up for equations with *nonlocal nonlinearities* (and higher-order equations, see below) was proposed in [146], see an application in [30], [29] for (nonlocal) parabolic equations on a subspace with nonuniform blow-up. On the other hand, comparison techniques also apply [82]. Self-similar patterns with global blow-up were constructed in [150] for the nonlocal *b-l model of the propagation of turbulent bursts* (Barenblatt, 1983)

$$u_t = l(t)(u^{1/2}u_x)_x + ku^{3/2}/l(t), \quad k > 0,$$

where $l(t) = \operatorname{meas}\operatorname{supp} u(\cdot,t)$ of a compactly supported solution $u \geq 0$; regional blow-up with a special $k = k_*$ is described by a separable solution similar to the local equation (4.1), $p = m$. A list of references on blow-up for a special class of nonlocal parabolic equations is given in [82] and in [161].

12. **On blow-up in free boundary problems**

A classical example of evolution blow-up singularities in free boundary problems is the *waiting-time singularity* for the one-dimensional PME equation $u_t = (u^m)_{xx}$, $m > 1$. Given compactly supported and nonnegative initial data this equation admits a unique weak solution whose interface may exhibit a stationary interface until a finite waiting time, as first indicated by Kalashnikov (1967), see [115]. At the waiting time the interface may exhibit a singularity in the form of a *corner point* which was studied in [9], or a milder singularity [8]. In terms of the solution blow-up occurs for the second derivative $(u^{m-1})_{xx}$. For $n > 1$ the PME admits a radial *focusing* solution, so-called *Graveleau solution*, which is self-similar of the second kind and represents a spherical zero-hole filled in finite time. It has blow-up for the first derivatives and for the speed of the interface, and it is a stable pattern in the context of radial solutions, see [7]. Similar results hold for the p-Laplacian equation, cf. [10].

Singularities in supercooled *Stefan problems* are known for thirty years [157]. Blow-up can be "non-essential" with a continuation beyond [46] or otherwise complete, see further references in Sect. 6 of the detailed survey [143]. A classification of blow-up free boundary patterns was done in [105]. A cusp formation mechanism and a formal construction of generic patterns in two and three dimensions are given in [168]. Cusp formation occurs also for the 2-dimensional Hele-Shaw model with suction, cf. [110], [111]. Complex-variable methods are quite important in studying these flows and their singularities, [109].

Finite time extinction may occur for flows in porous media, where the model consists of the heat equation supplied with the free-boundary conditions $u = 0$, $\partial u/\partial n = -1$, so-called *Florin problem*, [55]. The problem is also related to the propagation of equi-diffusional flames. A radial self-similar pattern was constructed in [34]. Such a pattern is stable in radial geometry as proved in [75], where the corresponding *focusing problem* was studied. See further details on this problem in [164].

13. **On blow-up in parabolic systems**

For the *potential* weakly coupled parabolic systems $Pu_t = -Au + \mathcal{F}(u)$, first results on blow-up were obtained in [130], see generalizations to quasilinear systems $Pu_t = AG(Au) + \mathcal{F}(u)$ [134] and $Pu_t = -AG(u) + \mathcal{F}(u)$ [66]. The study of blow-up in more general (non-potential) parabolic systems with the Maximum Principle

$$u_t = \Delta u^{\nu+1} + v^p, \quad v_t = \Delta v^{\mu+1} + u^q, \tag{13.2}$$

were initiated in [78]. In particular, a sufficient condition of *global* blow-up was proved, the method applies to more general systems, see [154], p. 463.

For the semilinear system (13.2) with $\nu = \mu = 0$ the critical Fujita exponent was calculated in [44], see a complete classification in [45], and single point blow-up was proved in a particular case $\mu = \nu = 0$, $p = q$ in [57]. In the subcritical case, optimal L^∞ bounds and blow-up patterns were obtained in [3]. For essentially quasilinear systems, excluding typical questions on the critical Fujita exponents and some important, but particular estimates (we do not pretend to complete expressing of a series of recent results in that direction), the blow-up patterns are far away from being understand well though it is clear that self-similar patterns can exist as for a single quasilinear equation. In such systems blow-up in variables u and v can be distinct, say, single-point in u and global in v, see p. 472 in [154]. Self-similar blow-up patterns (unstable) exist for a system with reaction-absorption terms

$$u_t = (u^\sigma u_x)_x + v^\nu u^\beta, \quad v_t = -v^\nu u^\varepsilon$$

(the concentration diffusion is negligible in the second equation), are available in Chapter 7 in [154], where a list of related references is available. Related systems with nonlinear flux conditions are treated in [149], [151]. Finite time extinction occurs in the *b-ε model of turbulence* (based on ideas of Kolmogorov and Prandtl):

$$b_t = \alpha((b^2/\varepsilon)b_x)_x - \varepsilon, \quad \varepsilon_t = \beta((b^2/\varepsilon)\varepsilon_x)_x - \gamma\varepsilon^2/b.$$

For $\alpha = \beta = 1$ extinction of nonnegative solutions occurs for $\gamma \in (0,1)$ [24] (the asymptotic behaviour is not self-similar).

14. **On blow-up in higher-order parabolic equations**

The concavity energy methods prove blow-up for potential semilinear and quasilinear parabolic equations of arbitrary order [130], [134], [66], and the Fourier method [146] also applies. Blow-up in the phenomenological *Cahn-Hilliard equation*

$$u_t + \gamma u_{xxxx} = (-u^3 + \gamma_1 u^2 - u)_{xx}$$

was proved in [43] by energy analysis. Much less is known about the asymptotic blow-up behaviour. Blow-up patterns on linear subspaces invariant under nonlinear operators [73] were constructed in [71], [73] for the modified fourth-order *Kuramoto-Sivashinskii equation* (KS)

$$u_t = -u_{xxxx} - u_{xx} + (u_x)^2 + u^2.$$

Blow-up for the following *modified KS equation*:

$$u_t = -u_{xxxx} - u_{xx} + (1-\lambda)(u_x)^2 \pm \lambda(u_{xx})^2, \quad \lambda \in (0,1],$$

was proved in [20] in 1D and 2D by the Fourier method [146], and a formal analysis of self-similar blow-up is presented. See further references there on blow-up for other equations including nonlocal ones. Finite time extinction singularities of initially strictly positive solutions are known for the *thin film* (or *lubrication*) equations

$$u_t + (u^n u_{xxx})_x = 0,$$

see the proof in [18] for $n \in (0,1/2)$. A formal asymptotic analysis of extinction patterns can be found in [22].

15. **Blow-up in other nonlinear PDEs**

Though this is the type of equations we will discuss in these notes, there is no reason in principle to restrict the scope of the investigation, and much work has been done in recent fifty years to understand the blow-up properties of HJ equations (conservation laws), nonlinear wave equations, nonlinear Scrödinger equations, Navier-Stokes equations and different mixed systems of such equations of interest in the applied sciences. A survey of the main results of the 70's and the 80's on blow-up for nonlinear PDEs of different types can be found in [78], [132], see comments to Chapter 4 in [154]. Below we present a few key references to some types of nonlinear PDEs. It is curious that for several types of equations some of the blow-up problems, which are very hard for parabolic equations (like the measure of blow-up sets, countable spectra of asymptotic blow-up patterns, etc.) become easier due to a natural finite propagation of blow-up free boundaries.

The mathematical history of blow-up for *semilinear wave equations* like

$$u_{tt} = \Delta u + |u|^p, \quad p > 1,$$

is longer than for parabolic equations, see [120], where Euler-Poisson-Darboux equation was also studied. There exists a critical Fujita exponent first obtained nor $n = 3$ by F. John (1979), see references in [132] and [78]. Regularity of blow-up free boundaries was studied in [32], [33], [60]. Among recent results, we would like to mention proofs of blow-up in *nonlinear nonlocal Kirchhoff equation* in a bounded domain [144]

$$u'' + M(\|A^{1/2}u\|^2)Au + \delta u' = f(u), \;\; A = \Delta, \;\; M(r) = a + br^\gamma, \;\; \gamma > 0,$$

(see also a general setting in [142]); *nonlinear Pochhammer-Chree equation* [135]

$$u_{tt} - u_{xxtt} - f(u)_{xx} = 0$$

($f(u) = u + u^2$ corresponds to the *regularized Boussinesq equation*); *periodic Camassa-Holm equation* [37]

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

Blow-up patterns for the quasilinear wave equation

$$u_{tt} - \Delta u = u_t u_{tt} \quad \text{in } I\!R^2 \times I\!R,$$

were described in [1]. Little is known on blow-up patterns for quasilinear hyperbolic equations of the form

$$u_{tt} = \Delta\phi(u) + |u|^{p-1}u,$$

e.g., $\phi(u) = |u|^{m-1}u, \; m > 1$ (the speed of propagation on the singular level $\{u \to \infty\}$ can be infinite). Such equations admit blow-up exact solutions on linear subspaces invariant under nonlinear operators [73], which represent interesting blow-up patterns. Similar blow-up patterns for *Boussinesq-type equations* like

$$u_{tt} = -u_{xxxx} + \alpha uu_{xx} + \beta(u_x)^2 + \gamma u^2,$$

were considered in [71], [73]. Regularity of blow-up interfaces for the *Hamilton-Jacobi equations*

$$u_t + H(D_x u) = F(u)$$

was established in [61]. Blow-up for the *generalized Korteweg-de Vries equation*

$$u_t + u^p u_x + u_{xxx} = 0 \quad (p \text{ is natural})$$

(and for the fifth-order equation with the derivative u_{xxxxx}) were studied in [25]. In the *nonlinear Schrödinger equation* (NLS)

$$iu_t + \Delta u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \quad p \geq 1 + 4/n,$$

blow-up is known from 60s (papers by Akhmanov, Sukhorukov and Khohlov on nonlinear optics; first proofs were due to Talanov, 1966, and Zakharov, 1967), see references in [78], [132]. The generic blow-up behaviour was predicted to be self-similar in the supercritical range $p > 1 + 4/n$ and can be very delicate for the critical exponent $p = 1 + 4/n$: for the cubic equation with $p = 3$ in 2D, L^∞ rates may develop as $t \to T^-$ like $(T-t)^{-1/2}[|\ln(T-t)|]^{1/2}$ (predicted by Talanov, 1965) or $(T-t)^{-1/2}[\ln|\ln(T-t)|]^{1/2}$ (a similar Zakharov's result is known), see a formal analysis in [128], references therein and a survey on blow-up in the NLS in [139]. Self-similar blow-up for the *Zakharov system* [174]

$$iu_t = -\Delta u + nu, \quad \mu_0 n_{tt} = \Delta n + \Delta|u|^2 \quad (\mu_0 > 0),$$

describing the collapse of Lengmuir waves in plasma, was studied in the [101], see also [100], where further references are available. Let us mention some recent results on blow-up in *Navier-Stokes equations*. In the *compressible* case blow-up was established in [173]. For the *incompressible* equations in $\mathbb{R}^2$, blow-up patterns (free-boundary plane jets) were constructed in [95] by means of a free boundary problem for a nonlocal semilinear parabolic equation and the first pattern was proved to be stable in the current "linear" geometry. Cusp formation for two-dimensional slow viscous (Stokes) flows is studied in [112]. Blow-up for *Prandtl's equation* was proved in [42]. See related references therein.

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