DYNAMICAL UPPER BOUNDS ON WAVEPACKET SPREADING

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Abstract. We derive a general upper bound on the spreading rate of wavepackets in the framework of Schrödinger time evolution. Our result consists of showing that a portion of the wavepacket cannot escape outside a ball whose size grows dynamically in time, where the rate of this growth is determined by properties of the spectral measure and by spatial properties of solutions of an associated time independent Schrödinger equation. We also derive a new lower bound on the spreading rate, which is strongly connected with our upper bound. We apply these new bounds to the Fibonacci Hamiltonian—the most studied one-dimensional model of quasicrystals. As a result, we obtain for this model upper and lower dynamical bounds establishing wavepacket spreading rates which are intermediate between ballistic transport and localization. The bounds have the same qualitative behavior in the limit of large coupling.

1. Introduction

Let H be a self adjoint operator on a separable Hilbert space $\mathcal{H}$. The time-dependent Schrödinger equation of quantum mechanics, $i\partial_t\psi = H\psi$, leads to a unitary dynamical evolution in $\mathcal{H}$, given by

$$\psi(t) = e^{-iHt}\psi. \tag{1.1}$$

We are primarily interested here in cases where H is a Schrödinger operator of the form $\Delta + V$ on a space such as $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$ and where the initial vector ψ is a localized wavepacket. Under the evolution (1.1), the wavepacket will often spread with time and it is the nature of this spreading which is our main object of interest here. This question, "What determines the spreading of a wavepacket" [24], has been an active field of research over the last 15 years and there is by now a considerable body of literature devoted to it ([1, 2, 3, 4, 9, 10, 14, 15, 16, 17, 18, 19, 20, 24, 25, 27, 28, 29, 30, 33, 34, 41, 42, 44] is just a partial list). We note that there are some situations, such as some systems with absolutely continuous spectrum which can be studied by scattering theory [32], where good understanding of the dynamics exists and is much older. The primary focus of the more recent studies has been on greater generality and, in particular, on systems exhibiting spectral phenomena

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such as singular continuous spectrum and dense point spectrum. In particular, singular continuous spectrum tends to occur for basic models of quasicrystals (see, e.g., [12, 40]) and the study of wavepacket spreading in such models is important to the understanding of transport properties in such media.

A major focus of many recent studies has been to elucidate the relations between wavepacket spreading rates and continuity properties of spectral measures. Recall that each initial vector ψ has a spectral measure μ_ψ, defined as the unique Borel measure obeying $\langle f(H)\psi, \psi\rangle = \int_{\sigma(H)} f(x)\, d\mu_\psi(x)$ for any measurable function f. Here we use $\langle \cdot, \cdot \rangle$ to denote the scalar product in $\mathcal{H}$. A major discovery of Guarneri [16, 17], which has also been extended by several other authors [2, 4, 9, 19, 20, 27], is that appropriately measured continuity properties of the spectral measure μ_ψ imply fairly general lower bounds on the spreading rate of the wavepacket. Recently, Kiselev-Last [25] (also see [24] for a related heuristic result and [42] for a recent extension) proved a variant of Guarneri's bound which blends continuity properties of the spectral measure with the spatial decay rate of generalized eigenfunctions. This bound is generally stronger than what one can obtain from properties of the spectral measure alone and it is also of somewhat broader applicability. We note that while there is some pure theoretical interest in relating wavepacket dynamics to such things as continuity properties of spectral measures and spatial properties of eigenfunctions, the interest in the above bounds is more than purely theoretical. Properties of spectral measures and eigenfunctions can be studied for concrete models of interest and the above discussed bounds have been used to obtain dynamical results in cases where there is currently no alternative approach to study the dynamics. In particular, this approach has been used to obtain dynamical results for a number of quasiperiodic operators [11, 13, 23] and for operators with decaying potentials [25].

The situation with putting upper bounds on wavepacket spreading rates is much more problematic. There is a fairly general argument of Simon [33] showing that for a broad class of Schrödinger operators (in particular, every operator of the form $\Delta + V$ on $\ell^2(\mathbb{Z}^d)$) wavepacket spreading rates cannot be more than ballistic (namely, linear in time). For the regime of dynamical localization (namely, situations where the wavepacket does not spread beyond a finite region of space and so, in particular, the spectrum must be pure point) there is a fairly good understanding [14, 15, 41] of how suitable spatial properties of eigenvectors (or of spectral projections) are related to dynamics and how to specify necessary and sufficient conditions for the occurrence of complete dynamical localization. For problems with continuous spectrum, however, it had been a noted open problem to get any results beyond Simon's ballistic bound, namely, to be able to specify conditions that would guarantee wavepacket spreading rates that are slower than ballistic.

The main aim of the current paper is to provide a certain solution to this problem. What we are able to do, in essence, is to specify conditions that would guarantee that *some* portion of the wavepacket does not spread too fast (namely, it remains within a box whose size grows with time at an appropriate rate). We cannot control the entire wavepacket and thus we cannot bound the growth rate of quantities that are sensitive to small, fast spreading portions of the wavepacket, such as moments of the position operator. Nevertheless, we believe that our result is in many ways a natural complement to some of the Guarneri type lower bounds discussed above. We note that while Guarneri type bounds, such as the variants obtained in [25, 27], have been often formulated in terms of moments of the position operator, such bounds are usually obtained by controlling only a portion of the wavepacket. It so happens that in order to establish a *lower* bound on the growth rate of moments of the position operator it suffices to show that *some* portion of the wavepacket is spreading at an appropriate rate, whereas bounding such quantities from above would require tight control of the entire wavepacket—a much more delicate task. Moreover, we will see that our upper bound has a strong connection with the Kiselev-Last [25] lower bound discussed above. While we formulate things somewhat differently here, the basic information going into our upper bound is, roughly speaking, the same combination of spectral measure continuity and spatial behavior of generalized eigenfunctions which goes into that lower bound. Indeed, we also formulate here a new stronger variant of this lower bound showing that, in essence, we have good understanding of how the spreading rate of the slowest moving portion of the wavepacket is determined. The behavior of the fastest moving portion remains outside our scope of control and, in particular, establishing sub-ballistic upper bounds on the growth rate of moments of the position operator remains an interesting open problem.

From here on we study tridiagonal operators of the form

$$(Hu)(n) = a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n) \tag{1.2}$$

(also called Jacobi matrices) defined on $\ell^2(\mathbb{Z}_+)$ or $\ell^2(\mathbb{Z})$. Here $b(n)$ and $a(n)$ are real numbers, and $a(n) \neq 0$ for any n. Moreover, we assume that $\sum_{n=\pm 1}^{\pm\infty} |a(n)|^{-1} = \infty$, which is sufficient to ensure that these operators are self-adjoint (see, e.g., [5]). We note that discrete one-dimensional Schrödinger operators of the form $\Delta + V$ on $\ell^2(\mathbb{Z}_+)$ or $\ell^2(\mathbb{Z})$ are a special case of such tridiagonal operators (obtained by setting $a(n) \equiv 1$ and $b(n) = V(n)$). While such one-dimensional operators will be our main interest here, we discuss in Section 7 below how our results are also applicable to more general problems and, in particular, how our dynamical upper bound is also applicable to multidimensional Schrödinger operators of the form $\Delta + V$ on $\ell^2(\mathbb{Z}^d)$.

We will look at the evolution of the initial vector $\psi = \delta_1$ (where $\delta_n(k) = \delta_{nk}$), but the results can also be recast for other ℓ^2 vectors.

Let $u_\theta(n, z)$ be the solution of the equation

$$a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n) = zu(n), \tag{1.3}$$

$z \in \mathbb{C}$, satisfying $u_\theta(0, z) = \sin\theta$, $u_\theta(1, z) = \cos\theta$. Given a function $\phi : \mathbb{Z}_+ \to \mathbb{C}$, we define for any $L \geq 0$,

$$\|\phi\|_L^2 = \sum_{n=1}^{\lfloor L \rfloor} |\phi(n)|^2 + (L - \lfloor L \rfloor)|\phi(\lfloor L \rfloor + 1)|^2, \tag{1.4}$$

where we use the convention that the sum is zero if the upper limit is less than the lower one. Similarly, for a function $\phi : \mathbb{Z} \to \mathbb{C}$, we define

$$\|\phi\|^2_{L_1,L_2} = \sum_{n=-\lfloor L_1 \rfloor}^{\lfloor L_2 \rfloor} |\phi(n)|^2 + (L_1 - \lfloor L_1 \rfloor)|\phi(-\lfloor L_1 \rfloor - 1)|^2 + (L_2 - \lfloor L_2 \rfloor)|\phi(\lfloor L_2 \rfloor + 1)|^2.$$

Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x. While the extension to real L is not essential, it will be technically convenient. Consider first the half-line problem, namely, the operator on $\ell^2(\mathbb{Z}_+)$. Without loss of generality, we restrict consideration to a Dirichlet boundary condition ($\theta = 0$), where the operator (1.2) is defined on functions vanishing for $n \leq 0$. We denote the resulting self-adjoint operator by H_+. We also assume here $a(0) = 1$. Note that the operator H_+ itself is independent of $a(0)$. The choice of $a(0)$ is only effecting the definition of the solutions $u_\theta(n, z)$ for $\theta \neq 0$. Given $\epsilon > 0$ and an energy $E \in \mathbb{R}$, we define the length scale $L^+_\epsilon(E)$ by

$$\max_\theta \|u_\theta(n, E)\|_{L^+_\epsilon(E)} \cdot \min_\theta \|u_\theta(n, E)\|_{L^+_\epsilon(E)} = \epsilon^{-1}. \tag{1.5}$$

The left-hand side is monotonely increasing as the region of summation expands and so $L^+_\epsilon(E)$ is well-defined for every $\epsilon > 0$. We have $L^+_\epsilon(E) \to 1$ as $\epsilon \to \infty$ and $L^+_\epsilon(E) \to \infty$ as $\epsilon \to 0$. Let $m_+(z)$ be the Weyl-Titchmarsh m function corresponding to the operator H_+,

$$m_+(z) = \langle (H_+ - z)^{-1}\delta_1, \delta_1 \rangle$$

The spectral measure μ^+ of the operator H_+ is defined by the equality

$$m_+(z) = \int_{\mathbb{R}} \frac{d\mu^+(x)}{x - z},$$

and coincides with μ_{δ_1}, the spectral measure of the vector δ_1. As in previous works on the subject, we deal with time-averaged quantities. For any function $A(t)$ of time, we denote

$$\langle A(t) \rangle_T = \tfrac{2}{T} \int_0^\infty e^{-2t/T} A(t)\, dt.$$

This type of averaging is slightly different from the Cesàro averaging used in many previous works and it is more convenient for what we do here. Note that the difference is not very significant. In particular, power law behaviors in T must be the same for both kinds of average.

Our first result is the following

Theorem 1.1. *Let H_+ be the half-line operator defined on $\ell^2(\mathbb{Z}_+)$ by (1.2) and a Dirichlet boundary condition. Let the characteristic scale $L^+_{T^{-1}}(E)$ be defined by (1.5). Then for any $T > 0$ and $L > 1$, we have*

$$\langle \|e^{-iH_+t}\delta_1\|^2_L \rangle_T > C\mu^+\left(\left\{E \mid L^+_{T^{-1}}(E) \leq L\right\}\right), \tag{1.6}$$

where C is a universal positive constant.

Remarks. 1. The expression on the left hand side of (1.6) gives the averaged norm of the portion of the wavepacket remaining in a ball of size L. Theorem 1.1 shows that this norm is bounded from below by a quantity proportional to the norm of the spectral projection of δ_1 on the set of energies where $L^+_{T^{-1}}(E) \leq L$. Thus, if we choose the size L to be greater than a certain scale which depends on time and properties of the solutions u_θ, we are guaranteed to have a significant portion of the wavepacket remaining (on average) in a ball of this size.
2. While the universal constant C can be explicitly estimated from our proof below, we made no real effort to obtain an optimal value for it. Our technique can only yield a number which is significantly smaller than 1 and thus (1.6) can control the location of only a portion of the wavepacket.

The next criterion is a simple corollary of Theorem 1.1. It relates an upper bound on wavepacket spreading to the growth of the norms of transfer matrices. Let us denote by $\Phi(n, E)$ the transfer matrix from the site 0 to n :

$$\Phi(n, E) = \begin{pmatrix} u_0(n+1, E) & u_{\pi/2}(n+1, E) \\ u_0(n, E) & u_{\pi/2}(n, E) \end{pmatrix}.$$

We also denote, for $L \geq 1$,

$$\|\Phi(E)\|_L^2 = \sum_{n=1}^{\lfloor L \rfloor - 1} \|\Phi(n, E)\|^2 + (L - \lfloor L \rfloor)\|\Phi(\lfloor L \rfloor, E)\|^2,$$

where $\|\Phi(n, E)\|$ is the usual operator norm of the matrix $\Phi(n, E)$. Define $\tilde{L}^+_\epsilon(E)$ by

$$\|\Phi(E)\|_{\tilde{L}^+_\epsilon(E)} = 2\|\Phi(1, E)^{-1}\|\epsilon^{-1}. \tag{1.7}$$

Then we have the same result for a new characteristic scale:

Corollary 1.2. *Let H_+ be the half-line operator defined on $\ell^2(\mathbb{Z}_+)$ by (1.2) and a Dirichlet boundary condition. Let the characteristic scale $\tilde{L}^+_{T^{-1}}(E)$ be defined by (1.7). Then for any $T > 0$ and $L \geq 2$, we have*

$$\langle \|e^{-iH_+t}\delta_1\|_L^2 \rangle_T > C\mu^+ \left(\left\{ E \mid \tilde{L}^+_{T^{-1}}(E) \leq L \right\} \right), \tag{1.8}$$

where C is a universal positive constant.

Remark. By the Landauer formula [37], the resistance $\rho(n)$ of a sample of size n is given by

$$\rho(n) = \tfrac{1}{2}\left[\tfrac{1}{2}\mathrm{Tr}\left(\Phi^t(n, E)\Phi(n, E)\right) - 1\right]$$

(where Φ^t is the transpose of Φ). Hence, the growth rate (with L) of $\|\Phi(E)\|_L^2$ is connected with the growth rate of the sum

$$\sum_{n=1}^{L} \rho(n).$$

Corollary 1.2 can thus be viewed as a confirmation of a physically reasonable statement that higher resistance leads to slower transport.

We will show below that $L_\epsilon^+(E) \le \tilde{L}_\epsilon^+(E)$, and hence Corollary 1.2 is strictly weaker than Theorem 1.1. However, it may be more straightforward to apply it in some situations.

In addition, we derive a new lower bound on wavepacket spreading which is also related to the characteristic length scale $L_\epsilon^+(E)$.

Theorem 1.3. *Let H_+ be the half-line operator defined on $\ell^2(\mathbb{Z}_+)$ by (1.2) with a Dirichlet boundary condition. Denote by P_S the spectral projection on some Borel set S. Then for every $T > 0$ and $L > 0$,*

$$\langle \| e^{-iH_+t} P_S \delta_1 \|_L^2 \rangle_T \le C \int_S \frac{\|u_0(n,E)\|_L^2}{\|u_0(n,E)\|_{L_{T^{-1}}^+(E)}^2} d\mu^+(E), \tag{1.9}$$

where C is a universal constant.

Roughly, Theorem 1.3 shows that if L is such that the ratio

$$\|u_0(n,E)\|_L^2 / \|u_0(n,E)\|_{L_{T^{-1}}^+}^2$$

is small for $E \in S$, then most of the portion of the wavepacket which is supported on energies within the set S leaves the ball of radius L by the time T.

While we defined the scales $L_\epsilon^+(E)$ purely in terms of solutions of the equation (1.3), we will see below that the quantities which enter in this definition also determine the local scaling properties of the spectral measure. This connection can be turned around to control $L_\epsilon^+(E)$ using partial information on solution behavior along with continuity properties of the spectral measure. In particular, the meaning of Theorem 1.3 and its connection with the Kiselev-Last lower bound [25] can be clarified by noting that it implies the following. Let $\alpha(E)$ be a measurable function such that for a.e. E w.r.t. μ^+,

$$D^{\alpha(E)} \mu^+(E) = \limsup_{\epsilon \to 0} \frac{\mu^+(E+\epsilon, E-\epsilon)}{(2\epsilon)^{\alpha(E)}} < \infty. \tag{1.10}$$

Let $\gamma(E)$ be a measurable function such that for a.e. E w.r.t. μ^+,

$$\limsup_{L \to \infty} L^{-\gamma(E)} \|u_0(n,E)\|_L^2 < \infty. \tag{1.11}$$

We will show that Theorem 1.3 implies:

Theorem 1.4. *Let $\alpha(E)$, $\gamma(E)$ satisfy (1.10), (1.11) and let $\eta(E) = \alpha(E)/\gamma(E)$. Let $S \subset \mathbb{R}$ be a Borel set and assume that for all energies $E \in S$, $\eta(E) \ge b > 0$. Then for any $g > 0$, there exists a constant C_g such that for all $T > 0$,*

$$\langle \| e^{-iH_+t} P_S \delta_1 \|_{C_g T^b}^2 \rangle_T \le 1 - \mu^+\left(\{E \mid \eta(E) \ge b\}\right) + g. \tag{1.12}$$

Remarks. 1. As usual [16, 27], from the estimate (1.12) follow lower bounds on the growth rate of moments of the position operator.
2. Theorem 1.4 is a local version (and thus also a somewhat stronger variant) of

Theorem 1.2 of [25]. It can be proven directly by a slight modification to the proofs in [25] (or see [42]), in which case it comes out naturally as a multidimensional variant, where the exponent $\gamma(E)$ is connected with decay of generalized eigenfunctions in the multidimensional space. We include this theorem here, showing that it follows from Theorem 1.3, mainly to illustrate the connection of Theorem 1.3 with this type of results.

It is often convenient to filter scaling behaviors in terms of explicitly defined scaling exponents and to formulate relations between scaling behaviors as inequalities between such exponents. It is thus natural in our context to define

$$\overline{\beta} = \lim_{\delta \to 0} \limsup_{T \to \infty} \frac{\log \left(\inf \left\{ L \mid \langle \| e^{-iH_+ t} \delta_1 \|_L^2 \rangle_T > \delta \right\} \right)}{\log T},$$

$$\underline{\beta} = \lim_{\delta \to 0} \liminf_{T \to \infty} \frac{\log \left(\inf \left\{ L \mid \langle \| e^{-iH_+ t} \delta_1 \|_L^2 \rangle_T > \delta \right\} \right)}{\log T}.$$

$\overline{\beta}$ and $\underline{\beta}$ are the upper and lower spreading rates associated with the slowest spreading portion of the wavepacket. Local exponents for the asymptotic scaling behavior of the scales $L_\epsilon^+(E)$ are given by

$$\overline{\lambda}(E) = \limsup_{\epsilon \to 0} \frac{\log L_\epsilon^+(E)}{\log \epsilon^{-1}}, \qquad \underline{\lambda}(E) = \liminf_{\epsilon \to 0} \frac{\log L_\epsilon^+(E)}{\log \epsilon^{-1}}.$$

Theorem 1.1 immediately implies

$$\overline{\beta} \le \mu^+\text{-ess}\inf \overline{\lambda}(E).$$

Moreover, we see that in problems where the solutions $u_0(n, E)$ behave nicely enough to ensure that $\|u_0(n, E)\|_{L_1} / \|u_0(n, E)\|_{L_2}$ is small whenever L_1/L_2 is small, Theorem 1.3 would imply

$$\mu^+\text{-ess}\inf \underline{\lambda}(E) \le \underline{\beta}.$$

These inequalities are particularly interesting in cases where the problem exhibits nicely scaling power law behaviors so that $u_0(n, E)$ behaves as described above and also $\mu^+\text{-ess}\inf \underline{\lambda}(E) = \mu^+\text{-ess}\inf \overline{\lambda}(E)$. If this happens, we see that we also have $\underline{\beta} = \overline{\beta}$ and the spreading rate of the slowest spreading portion of the wavepacket is precisely given by the slowest growth rate of the scales $L_\epsilon^+(E)$ with respect to the spectral measure μ^+.

Our second main goal in this paper is to apply the new dynamical upper bounds to the Fibonacci Hamiltonian, the most studied of all one-dimensional models of quasicrystals. For this application we need a whole-line (namely, $\ell^2(\mathbb{Z})$) version of Theorem 1.1. To formulate this whole-line version, note that the scales $L_\epsilon^-(E)$ and $\tilde{L}_\epsilon^-(E)$ can be defined in a way directly analogous to (1.5), (1.7), involving the same kind of sums but taken over the negative half-line. Notice that the negative half-axis in our setting is $(\dots, -1, 0)$, so that the summation in the analog of definition (1.5) will now start (or rather end) at 0 instead of -1. Let μ be the spectral measure of the whole-line operator H corresponding to δ_1, defined by

$$M(z) = \langle (H-z)^{-1}\delta_1, \delta_1\rangle = \int_{\mathbb{R}} \frac{d\mu(x)}{x-z}.$$

Theorem 1.5. *Let H be a self-adjoint operator of the form (1.2) on $\ell^2(\mathbb{Z})$. Let the characteristic scales $L^{\pm}_{T^{-1}}(E)$ be defined by (1.5). Then for any $T>0$, $L_1, L_2 > 1$, we have*

$$\langle \|e^{-iHt}\delta_1\|^2_{L_1,L_2}\rangle_T > C\mu\left(\{E \mid L^-_{T^{-1}}(E) \le L_1 \text{ and } L^+_{T^{-1}}(E) \le L_2\}\right), \tag{1.13}$$

where C is a universal positive constant. If $L_1, L_2 > 2$, the same result holds if $L^{\pm}_{T^{-1}}(E)$ is replaced in the statement with $\tilde{L}^{\pm}_{T^{-1}}(E)$ defined by (1.7).

The Fibonacci Hamiltonian H_λ is defined by

$$(H_\lambda u)(n) = u(n+1) + u(n-1) + \lambda V(n)u(n),$$

where $V(n) = \lfloor (n+1)\omega\rfloor - \lfloor n\omega\rfloor$ and $\omega = (\sqrt{5}-1)/2$ is the golden mean. It is known [6, 38, 39] that for every $\lambda > 0$, H_λ has purely singular continuous spectrum, and moreover, its spectrum (as a set) is a Cantor set of zero Lebesgue measure. Lower bounds on wavepacket spreading rates for H_λ were recently shown in [23]. We are going to show both upper and lower bounds for the dynamics of H_λ which imply that the spreading rate is intermediate between ballistic ($\sim T$ at time T) and localized ($\sim T^0$). To the best of our knowledge, this is the first proof of such behavior in an explicit model of this type.

Theorem 1.6. *Let H_λ be the Fibonacci Hamiltonian. Then*

i) *There exists a constant $G>0$ such that for sufficiently large λ,*

$$\left\langle \|e^{-iH_\lambda t}\delta_1\|^2_{C(T+1)^{p_1(\lambda)},C(T+1)^{p_1(\lambda)}}\right\rangle_T \ge G \qquad \forall T>0, \tag{1.14}$$

where $p_1(\lambda) = C_1(\log\lambda)^{-1}(1+O((\lambda\log\lambda)^{-1}))$.

ii) *Given any $g>0$, there exists a constant C_g such that for every coupling $\lambda > 0$ and every time $T>0$,*

$$\left\langle \|e^{-iH_\lambda t}\delta_1\|^2_{C_gT^{p_2(\lambda)},C_gT^{p_2(\lambda)}}\right\rangle_T \le g, \tag{1.15}$$

where the positive function $p_2(\lambda)$ satisfies $p_2(\lambda) = C_2(\log\lambda)^{-1}(1+O((\log\lambda)^{-1}))$ for large λ.

Remarks. 1. The first part of Theorem 1.6 is an upper bound on wavepacket spreading for large coupling, which shows that on the average, there is a portion of the wavepacket remaining in a ball of the radius $\sim T^{p_1(\lambda)}$ at time T. The second part of the Theorem provides a lower bound on wavepacket spreading, showing that on the average, only an arbitrarily small part of the wavepacket remains in a ball of radius $\sim T^{p_2(\lambda)}$ at time T. The constants $p_1(\lambda)$ and $p_2(\lambda)$ have the same (logarithmic in λ) asymptotic behavior for large coupling, up to a constant factor in front of the main term. This implies that the logarithmic law is precise and cannot be improved in the estimates (1.14), (1.15).

2. It will follow from the proof below that $\lambda > 8$ is sufficient to get a nontrivial upper bound ($p_1(\lambda) < 1$ in (1.14)). This range is not optimal and can be improved by additional technical effort; however our current methods do not allow to extend the bound all the way to $\lambda = 0$. Moreover, if λ is so small that $p_1(\lambda) \geq 1$, then there is little interest since this would constitute a ballistic (or worse) bound.
3. We remark that Sinai, in a recent paper [36], studied anomalous transport (in terms of moments of the position operator) for an almost periodic potential in a different setting (in fact, in a pure point regime).

Although we only treat discrete operators in this paper, results parallel to Theorems 1.1, 1.2, 1.3, 1.4, 1.5 also hold in continuous settings by direct translation of the arguments given here.

The rest of this paper is organized as follows. In Section 2 we prove some auxiliary facts relating the scale $L_\epsilon^+(E)$ and the m function. In Section 3 we derive the upper bounds on wavepacket spreading. In particular, the proofs of Theorem 1.1, Theorem 1.5 and Corollary 1.2 appear there. In Section 4 we show the new lower bounds, Theorems 1.3 and 1.4. In Sections 5 and 6 we treat the Fibonacci Hamiltonian and prove Theorem 1.6. Finally, In Section 7 we discuss how our results are applicable to multidimensional problems.

2. Bounds on the m function

We are going to prove a series of auxiliary estimates relating the behavior of solutions $u_{0,\pi/2}$ and the half-line m functions. The line of the argument follows Jitomirskaya-Last extension of subordinacy theory [22], but there will be an essential modification that will be crucial for the derivation of the dynamical criteria.

Recall the following simple example of variation of parameters which will prove extremely useful:

Lemma 2.1. *Suppose $w(n,z), w(n,E)$ solve (1.3) for spectral parameters z, E respectively. Assume that $w(0,z) = w(0,E)$ and $w(1,z) = w(1,E)$. Then for $n \geq 0$,*

$$w(n,z) = w(n,E) + (z-E)\sum_{m=1}^{n} k(n,m;E)w(m,z) \tag{2.1}$$

where

$$k(n,m;E) = u_0(n,E)u_{\pi/2}(m,E) - u_{\pi/2}(n,E)u_0(m,E)$$

and with the convention that the sum is zero if the upper limit is less than the lower one.

Proof. It is a direct computation to check that the right-hand side satisfies (1.3) with the spectral parameter z. Hence, on both sides we have solutions of (1.3) which coincide at the sites $0, 1$. This implies the equality. ☐

Equation (2.1) has the form

$$w(n,z) = w(n,E) + (z-E)(K(E)w)(n,z),$$

where $K(E)$ is an integral operator with the kernel $k(n,m;E)$. As introduced in (1.4), $\|\cdot\|_L$ defines a norm on an $\lfloor L\rfloor+1$-dimensional Hilbert space (except when L is an integer, in which case the dimension is L). The corresponding inner product is

$$\langle\phi_1,\phi_2\rangle_L=\sum_{n=1}^{\lfloor L\rfloor}\phi_1(n)\phi_2(n)+(L-\lfloor L\rfloor)\phi_1(\lfloor L\rfloor+1)\phi_2(\lfloor L\rfloor+1). \tag{2.2}$$

We wish to estimate the norm of the operator $K(E)$ acting in this space:

Lemma 2.2. *The Hilbert-Schmidt norm of $K(E)$ is given by*

$$|||K(E)|||_L^2=\|u_0\|_L^2\|u_{\pi/2}\|_L^2-|\langle u_0,u_{\pi/2}\rangle_L|^2 \tag{2.3}$$

$$=\max_\theta\|u_\theta(n,E)\|_L^2\cdot\min_\theta\|u_\theta(n,E)\|_L^2. \tag{2.4}$$

where $u_0,u_{\pi/2}$ are the solutions at energy E. In particular, this gives an upper bound on the operator norm of K.

Proof. Recall that the Hilbert-Schmidt norm of an integral operator is equal to the L^2 norm of its kernel. For integer L,

$$|||K(E)|||_L^2=\sum_{n,m=1}^{L}|k(n,m;E)|^2=\|u_0\|_L^2\|u_{\pi/2}\|_L^2-|\langle u_0,u_{\pi/2}\rangle_L|^2$$

can be shown fairly easily. Non-integer L merely require a more lengthy computation.

To show the second equality, consider the 2×2 matrix

$$Q=\begin{bmatrix}\|u_0\|_L^2 & \langle u_0,u_{\pi/2}\rangle_L\\ \langle u_{\pi/2},u_0\rangle_L & \|u_{\pi/2}\|_L^2\end{bmatrix}.$$

It is self-adjoint and positive, and if $\vec e_\theta=(\cos\theta,\sin\theta)$, then

$$\vec e_\theta Q\vec e_\theta^{\,t}=\|u_\theta\|_L^2.$$

The right-hand side of (2.3) is the determinant of Q and so the product of the eigenvalues of Q. By the minimax characterization, these eigenvalues are given by the factors in (2.4). ☐

Remark. Notice that if the maximum of $\|u_\theta(n,E)\|_L$ is achieved at θ_0, the minimum is achieved at the orthogonal boundary condition $\theta_0^\perp$ ($u_{\theta_0^\perp}$ satisfies $u_{\theta_0^\perp}(0,E)=\cos\theta_0$, $u_{\theta_0^\perp}(1,E)=-\sin\theta_0$). This is because the eigenvectors of the self-adjoint matrix Q are orthogonal.

From (2.4) we know that $|||K(E)|||_L$ is strictly increasing. Hence $\epsilon|||K(E)|||_L=1$ determines L as a function of ϵ and E. As in the introduction, we denote this length scale by $L_\epsilon^+(E)$. The main result of this section is an estimate for the norm of the Weyl solution u_+ over this length scale. Before stating and proving this result, let us recall some facts about the Weyl theory of Jacobi matrices (see, e.g., [5]): For each $z\in\mathbb{C}\setminus\mathbb{R}$, there exists a unique solution of (1.3) which is square summable and obeys $u_+(0,z)=1$. One may write this solution in the form

$$u_+(n,z)=u_{\pi/2}(n,z)-m_+(z)u_0(n,z),$$

and so define the Weyl m-function $m_+(z)$. As we will discuss a little more in the next section, $m_+(z)$ captures all of the spectral information about H_+. For the moment, however, we merely need the observation that

$$a(0)\mathrm{Im}\, m_+(z) = \mathrm{Im}\, z \sum_{n=1}^{\infty} |u_+(n,z)|^2, \tag{2.5}$$

which follows from summation by parts. Recall that we assume $a(0) = 1$.

Theorem 2.3. *Fix $E \in \mathbb{R}$ and $\epsilon > 0$, then we have*

$$2 - \sqrt{3} \leq \frac{\|u_0\|_{L_\epsilon^+} |m_+(E+i\epsilon)|}{\|u_{\pi/2}\|_{L_\epsilon^+}} \leq 2 + \sqrt{3}. \tag{2.6}$$

Moreover,

$$\epsilon \|u_+\|^2_{L_\epsilon^+} \geq \frac{|m_+(E+i\epsilon)|}{4\epsilon \|u_0\|_{L_\epsilon^+} \|u_{\pi/2}\|_{L_\epsilon^+}} \geq \tfrac{2-\sqrt{3}}{16} \mathrm{Im}\, m_+(E+i\epsilon). \tag{2.7}$$

In both formulae, $u_0, u_{\pi/2}$ are the solutions at energy E.

Remark. The bounds (2.6) are very similar to those proved by Jitomirskaya-Last [22]. The difference is that the scale L_ϵ^+ is given in [22] by the condition

$$\epsilon \|u_0\|_{L_\epsilon^+} \|u_{\pi/2}\|_{L_\epsilon^+} = 1.$$

The scale L_ϵ^+ which we define by

$$|||K|||_{L_\epsilon^+} = \max_\theta \|u_\theta(n,E)\|^2_{L_\epsilon^+} \cdot \min_\theta \|u_\theta(n,E)\|^2_{L_\epsilon^+} = \epsilon^{-1} \tag{2.8}$$

is larger (compare (2.3)), and the constants appearing in (2.6) are sharper than those in [22]. The scale defined by (2.8) might be less convenient for dimensional spectral analysis, since its definition is more involved. However, an improvement contained in (2.8) will be quite crucial for the proof of dynamical criteria in the next section.

Proof. To shorten the formulae which will follow, we introduce

$$\beta = \frac{\langle u_0, u_{\pi/2}\rangle_{L_\epsilon^+}}{\|u_0\|_{L_\epsilon^+} \|u_{\pi/2}\|_{L_\epsilon^+}} \quad \text{and} \quad \zeta = \sqrt{1-\beta^2}.$$

Notice that Cauchy-Schwarz says $\zeta > 0$. At times we shall also write m_+ for $m_+(E+i\epsilon)$. Notice that by Lemma 2.2, (2.3) and (1.5), the scale $L_\epsilon^+(E)$ given by (1.5) is chosen exactly in a way to ensure that $\|K(E)\|_L \leq \epsilon^{-1}$ if $L \leq L_\epsilon^+(E)$. Lemmas 2.1,2.2 and the definition of L_ϵ^+ combine to show that

$$4\|u_+(n, E+i\epsilon)\|^2_{L_\epsilon^+} \geq \|u_{\pi/2}(n,E) - m_+(E+i\epsilon) u_0(n,E)\|^2_{L_\epsilon^+}. \tag{2.9}$$

Using (2.5), we see that

$$4\epsilon^{-1} \mathrm{Im}\, m_+ \geq 4\|u_+\|^2_{L_\epsilon^+} \geq \|u_{\pi/2}\|^2_{L_\epsilon^+} + |m_+|^2 \cdot \|u_0\|^2_{L_\epsilon^+} - 2\mathrm{Re}\, m_+ \langle u_0, u_{\pi/2}\rangle_{L_\epsilon^+}.$$

The definitions of β, ζ and L_ϵ^+ are such that $\epsilon\zeta \|u_0\|_{L_\epsilon^+} \|u_{\pi/2}\|_{L_\epsilon^+} = 1$. Thus,

$$\mathrm{Im}\, m_+ \geq \epsilon \|u_+\|^2_{L_\epsilon^+} \geq \frac{1}{4\zeta} \left\{ \frac{\|u_{\pi/2}\|_{L_\epsilon^+}}{\|u_0\|_{L_\epsilon^+}} + |m_+(E+i\epsilon)|^2 \frac{\|u_0\|_{L_\epsilon^+}}{\|u_{\pi/2}\|_{L_\epsilon^+}} - 2\beta \mathrm{Re}\, m_+ \right\}. \tag{2.10}$$

This equation implies both claims of the theorem. To prove (2.6), notice that $|\pm\beta + i\zeta| = 1$ and so

$$-2|\beta \mathrm{Re}\,(m_+)| - 4\zeta \mathrm{Im}\, m_+ \geq -2|m_+| - 2\mathrm{Re}\left[(\pm\beta + i\zeta)m_+\right] \geq -4|m_+|,$$

where the sign is chosen to make $\pm\beta\mathrm{Re}\,(m_+) \geq 0$. Applying this to (2.10), leads to

$$\frac{\|u_0\|_{L_\epsilon^+}}{\|u_{\pi/2}\|_{L_\epsilon^+}}|m_+(E+i\epsilon)|^2 - 4|m_+(E+i\epsilon)| + \frac{\|u_{\pi/2}\|_{L_\epsilon^+}}{\|u_0\|_{L_\epsilon^+}} \leq 0,$$

which is equivalent to (2.6). To prove (2.7), use (2.10) to obtain

$$\begin{aligned}\epsilon\|u_+(n,E+i\epsilon)\|^2_{L_\epsilon^+} &\geq \frac{|m_+|}{4\zeta}\left\{\frac{\|u_{\pi/2}\|_{L_\epsilon^+}}{|m_+|\|u_0\|_{L_\epsilon^+}} + \frac{\|u_0\|_{L_\epsilon^+}|m_+|}{\|u_{\pi/2}\|_{L_\epsilon^+}} - 2\beta\frac{\mathrm{Re}\, m_+}{|m_+|}\right\} \\ &\geq \frac{|m_+|}{4\zeta}(2-2|\beta|) = \frac{\zeta|m_+|}{2(1+|\beta|)} \\ &\geq \tfrac{1}{4}\zeta|m_+(E+i\epsilon)|.\end{aligned}$$

(We used $x + x^{-1} \geq 2$ in the second step.) This proves the left-hand inequality in (2.7), because $\epsilon\zeta\|u_0\|_{L_\epsilon^+}\|u_{\pi/2}\|_{L_\epsilon^+} = 1$. To complete the proof, we need to show the right-hand inequality. From (2.9) and (2.5) we infer that

$$4\mathrm{Im}\, m_+ \geq \epsilon(\mathrm{Im}\, m_+)^2\|u_0\|^2_{L_\epsilon^+},$$

and from (2.6),

$$\|u_0\|^2_{L_\epsilon^+} \geq (2-\sqrt{3})|m_+(E+i\epsilon)|^{-1}\|u_0\|_{L_\epsilon^+}\|u_{\pi/2}\|_{L_\epsilon^+}.$$

Combining these two gives the right-hand side of (2.7). □

The following variant of the inequality (2.7) in Theorem 2.3 will prove useful in the next section.

Proposition 2.4. *Suppose $E, E' \in \mathbb{R}$, $\epsilon > 0$ and $|E - E'| < \epsilon$. Then there exists a universal constant C, so that*

$$\epsilon\|u_+(n,E'+i\epsilon)\|^2_{L_\epsilon^+(E)} \geq C\mathrm{Im}\, m_+(E'+i\epsilon). \tag{2.11}$$

Remark. It is important that while the bound in (2.11) involves m_+ and the solution u_+ at the energy $E' + i\epsilon$, the scale L_ϵ^+ is defined at the energy E.

Proof. This result is a direct corollary of the proof of Theorem 2.3. All statements of Theorem 2.3 hold with $E+i\epsilon$ replaced by $E'+i\epsilon$ and adjusted constants. Specifically, the only change introduced by replacing $u_+(n,E+i\epsilon)$ by $u_+(n,E'+i\epsilon)$, $|E-E'| < \epsilon$, is a change in the constant in (2.9) from 4 to $(1+\sqrt{2})^2 \leq 6$. Following through the proof with the constant 6 shows that C may be chosen to be $(3-2\sqrt{2})/36$. □

3. An upper bound on wavepacket spreading

We begin by relating the dynamical quantity we need to estimate to a solution of equation (1.3). As before,

$$u_+(n,z) = u_{\pi/2}(n,z) - m_+(z)u_0(n,z)$$

denotes the unique $\ell^2(\mathbb{Z}_+)$ solution for $z \in \mathbb{C} \setminus \mathbb{R}$ with $u_+(0,z) = 1$.

Lemma 3.1. *For every $z \in \mathbb{C} \setminus \mathbb{R}$ and $n \geq 1$,*

$$\langle (H_+ - z)^{-1}\delta_1, \delta_n\rangle = -u_+(n,z). \tag{3.1}$$

Proof. It's easy to see from the definition of the resolvent of H_+ that $\langle (H_+ - z)^{-1}\delta_1, \delta_n\rangle$, $n \geq 1$, is an $\ell^2(\mathbb{Z}_+)$ solution of (1.3). It thus follow from the uniqueness of the Weyl solution that it must be a multiple of $u_+(n,z)$. The Lemma is thus implied by the definition of $m_+(z)$ and our normalization of $u_+(n,z)$. □

Lemma 3.2. *For any $T > 0$,*

$$\langle |\langle e^{-iH_+t}\delta_1, \delta_n\rangle|^2\rangle_T = \frac{1}{\pi T}\int\limits_{\mathbb{R}} |u_+(n, E + \frac{i}{T})|^2 \, dE. \tag{3.2}$$

Proof. Recall that

$$\langle |\langle e^{-iH_+t}\delta_1, \delta_n\rangle|^2\rangle_T = \frac{2}{T}\int\limits_0^\infty e^{-2t/T}|\langle e^{-iH_+t}\delta_1, \delta_n\rangle|^2 \, dt.$$

The right-hand side in the above equality is a constant times the square of the L^2 norm of the function $e^{-t/T}\int e^{-iE't}\, d\mu_{1,n}(E')$, where the complex measure $\mu_{1,n}$ is defined by

$$\langle (H_+ - z)^{-1}\delta_1, \delta_n\rangle = \int_{\mathbb{R}} \frac{d\mu_{1,n}(x)}{x - z}.$$

The Fourier transform of this function is equal to

$$\frac{i}{2\pi}\int\limits_{\mathbb{R}} \frac{d\mu_{1,n}(E')}{(E - E') + (i/T)} = -\frac{i}{2\pi}\langle (H_+ - E - \frac{i}{T})^{-1}\delta_1, \delta_n\rangle = \frac{i}{2\pi}u_+(n, E + \frac{i}{T})$$

by (3.1). Now (3.2) follows from Parseval's equality for the Fourier transform. □

Lemma 3.2 implies,

$$\langle \|e^{-iH_+t}\delta_1\|_L^2\rangle_T = \frac{1}{\pi T}\int\limits_{\mathbb{R}} \|u_+(n, E + \frac{i}{T})\|_L^2 \, dE. \tag{3.3}$$

Given a set $S \subset \mathbb{R}$, let us denote by S_ϵ the ϵ-neighborhood of the set S. We now complete the proof of the half-line dynamical bound.

Proof of Theorem 1.1. Let

$$S = \left\{E \mid L^+_{T^{-1}}(E) \leq L\right\}.$$

By Lemma 3.2, to prove Theorem 1.1 it suffices to bound the right-hand side of (3.3) from below by $C\mu^+(S)$. Hence, it suffices to estimate from below the integral

$$\epsilon \int_{S_\epsilon} \|u_+(n, E+i\epsilon)\|_L^2 \, dE,$$

where $L \geq L_\epsilon^+(E)$ for $E \in S$ (we think of $\epsilon = T^{-1}$). By the definition of S_ϵ, for every $E' \in S_\epsilon$ there is $E \in S$ such that $|E - E'| < \epsilon$. Applying Proposition 2.4, we obtain

$$\epsilon \int_{S_\epsilon} \|u_+(n, E+i\epsilon)\|_L^2 \, dE \geq C \int_{S_\epsilon} \operatorname{Im} m_+(E+i\epsilon) \, dE. \tag{3.4}$$

By the Fubini theorem, the left-hand side in (3.4) is equal to

$$C \int_{\mathbb{R}} d\mu_+(x) \int_{S_\epsilon} \frac{\epsilon \, dE}{(x-E)^2+\epsilon^2} \geq C \int_S d\mu_+(x) \int_{-\epsilon}^{\epsilon} \frac{\epsilon \, dE'}{(E')^2+\epsilon^2} \geq C\mu_+(S).$$

□

The following Lemma shows that Theorem 1.1 implies Corollary 1.2.

Lemma 3.3. *For every $E \in \mathbb{R}$ and $L \geq 2$,*

$$4\|\Phi(1,E)^{-1}\|^2 \left(\max_\theta \|u_\theta(n,E)\|_L^2\right) \left(\min_\theta \|u_\theta(n,E)\|_L^2\right) \geq \|\Phi(E)\|_L^2. \tag{3.5}$$

Proof. A direct computation using the definition of $\Phi(n, E)$ shows that for any θ,

$$\|\Phi(n,E)\|^2 \leq (u_\theta(n,E))^2 + (u_\theta(n+1,E))^2 + (u_{\theta^\perp}(n,E))^2 + (u_{\theta^\perp}(n+1,E))^2.$$

Thus,

$$\|\Phi(E)\|_L^2 \leq 2(\|u_\theta(n,E)\|_L^2 + \|u_{\theta^\perp}(n,E)\|_L^2) \leq 4\left(\max_\theta \|u_\theta(n,E)\|_L^2\right).$$

For $L \geq 2$, we have for any θ, $\|u_\theta\|_L \geq \|u_\theta\|_2 \geq \|\Phi(1,E)^{-1}\|^{-1}$, and we thus see that (3.5) holds. □

Proof of Corollary 1.2. Since $\tilde{L}_\epsilon^+(E)$ was defined in (1.7) by

$$\|\Phi(E)\|_{\tilde{L}_\epsilon^+(E)} = 2\|\Phi(1,E)^{-1}\|\epsilon^{-1},$$

we see from (1.5) and Lemma 3.3, that $\tilde{L}_\epsilon^+(E) \geq L_\epsilon^+(E)$ whenever $L_\epsilon^+(E) \geq 2$. Therefore, Corollary 1.2 follows immediately from Theorem 1.1. □

It remains to consider the whole line case. Denote by $u_-(n,z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, a solution of (1.3) which belongs to $\ell^2(\mathbb{Z}_-)$, $Z_- = \{0, -1, \dots, \}$, and is normalized by

$$u_-(n,z) = u_{\pi/2}(n,z) + m_-(z)u_0(n,z).$$

Recall the notation H for the operator defined by (1.2) on $\ell^2(\mathbb{Z})$.

Proof of Theorem 1.5. As in Lemma 3.2, we can show

$$\langle \|e^{-iHt}\delta_1\|^2_{L_1,L_2}\rangle_T = \frac{1}{\pi T}\int_{\mathbb{R}} \|G(1,n,E+\frac{i}{T})\|^2_{L_1,L_2} dE,$$

where

$$G(1,n,z) = \langle (H-z)^{-1}\delta_1, \delta_n\rangle.$$

One can easily verify that

$$G(1,n,z) = \begin{cases} \frac{m_-(z)}{a(0)(m_+(z)+m_-(z))} u_+(n,z) & n \geq 1 \\ \frac{-m_+(z)}{a(0)(m_+(z)+m_-(z))} u_-(n,z) & n < 1. \end{cases}$$

In particular,

$$G(1,1,z) = M(z) = \frac{-m_+(z)m_-(z)}{a(0)(m_+(z)+m_-(z))},$$

and the spectral measure μ of H corresponding to the vector δ_1 satisfies

$$M(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x-z}.$$

Now define

$$S = \left\{E \mid L^+_{T^{-1}}(E) \leq L_2 \text{ and } L^-_{T^{-1}}(E) \leq L_1\right\}.$$

Repeating the same arguments that led us to the proof of Theorem 1.1, we get ($\epsilon = T^{-1}$)

$$\epsilon\|G(1,n,E+i\epsilon)\|^2_{-1,L_2} \geq C\int_{S_\epsilon} \frac{|m_-(E+i\epsilon)|^2}{|m_+(E+i\epsilon)+m_-(E+i\epsilon)|^2}\operatorname{Im} m_+(E+i\epsilon)\, dE$$

and similarly

$$\epsilon\|G(1,n,E+i\epsilon)\|^2_{L_1,0} \geq C\int_{S_\epsilon} \frac{|m_+(E+i\epsilon)|^2}{|m_+(E+i\epsilon)+m_-(E+i\epsilon)|^2}\operatorname{Im} m_-(E+i\epsilon)\, dE.$$

Combining these two inequalities, we get

$$\epsilon\|G(1,n,E+i\epsilon)\|^2_{L_1,L_2} \geq C\int_{S_\epsilon} \operatorname{Im} M(E+i\epsilon)\, dE \geq C\mu(S),$$

similarly to the last step in the proof of Theorem 1.1. □

4. A lower bound on wavepacket spreading

Before treating our main application, the Fibonacci Hamiltonian, we pause to prove the general lower bound on dynamics given by Theorem 1.3.

Lemma 4.1. *Let P_S be a spectral projection of the operator H_+ on a measurable set S. Then*

$$\left(e^{-iH_+t}P_S\delta_1\right)(n) = \int\limits_S e^{-iEt}u_0(n,E)d\mu^+(E). \tag{4.1}$$

Proof. Since δ_1 is a cyclic vector for H_+, it follows from the spectral theorem that H_+ is unitarily equivalent to multiplication by the parameter on $L^2(\mathbb{R}, d\mu^+)$. $u_0(n,E)$, $n \geq 1$, is known (see, e.g., [5]) to be the representation of δ_n in this space. □

Proof of Theorem 1.3. Notice that according to Lemma 4.1,

$$\begin{aligned}
&\langle \|e^{-iH_+t}P_S\delta_1\|_L^2\rangle_T \\
&\leq \frac{2}{T}\int\limits_0^\infty e^{-2t/T}\sum_{n\leq L}\int\limits_S e^{iEt}u_0(n,E)\,d\mu^+(E)\int\limits_S e^{-iE't}u_0(n,E')\,d\mu^+(E')\,dt \\
&= \sum_{n\leq L}\int\limits_S\int\limits_S u_0(n,E')u_0(n,E)\frac{4T^{-2}}{(E-E')^2+4T^{-2}}\,d\mu^+(E)\,d\mu^+(E') \\
&\leq \int\limits_S\left(\int\limits_S \frac{4T^{-2}}{(E-E')^2+4T^{-2}}\,d\mu^+(E')\right)\|u_0(n,E)\|_L^2\,d\mu^+(E). \qquad (4.2)
\end{aligned}$$

We used Cauchy-Schwartz in the last step with respect to the product measure $d\mu^+(E)d\mu^+(E')$. Notice that the term in the brackets in (4.2) does not exceed (setting $\epsilon = T^{-1}$)

$$2\epsilon\mathrm{Im}\, m_+(E+2i\epsilon) \leq 4\epsilon\mathrm{Im}\, m_+(E+i\epsilon).$$

By the estimates (2.6) and (2.7) of Theorem 2.3, we have

(4.3)

$$\epsilon\mathrm{Im}\, m_+(E+i\epsilon)\|u_0(n,E)\|_{L_\epsilon^+}^2 \leq (2+\sqrt{3})\frac{\epsilon\mathrm{Im}\, m_+(E+i\epsilon)\|u_0\|_{L_\epsilon^+}\|u_{\pi/2}\|_{L_\epsilon^+}}{|m_+(E+i\epsilon)|} \leq C.$$

Combining (4.2) and (4.3), we obtain

$$\langle \|e^{-iH_+t}P_S\delta_1\|_L^2\rangle_T \leq C\int\limits_S \frac{\|u_0(n,E)\|_L^2}{\|u_0(n,E)\|_{L_\epsilon^+}^2}d\mu^+(E),$$

which is exactly what we wanted to show. □

Proof of Theorem 1.4. By the extension of subordinacy theory due to Jitomirskaya and Last [22],

$$\liminf_{L\to\infty}\frac{\|u_0\|_L^{2-\alpha}}{\|u_{\pi/2}\|_L^\alpha} = 0 \Leftrightarrow D^\alpha\mu^+(E) = \infty.$$

Therefore, for a.e. E, we have with some $C_1(E) > 0$

$$\|u_0\|_L^{\frac{2-\alpha(E)}{\alpha(E)}} \geq C_1(E)\|u_{\pi/2}\|_L. \tag{4.4}$$

By (4.4), the definition (1.5) of L_ϵ^+ and relation (2.3),

$$\epsilon^{-1} \le \|u_0\|_{L_\epsilon^+} \|u_{\pi/2}\|_{L_\epsilon^+} \le C_1^{-1}(E) \|u_0\|_{L_\epsilon^+}^{\frac{2}{\alpha(E)}}.$$

Therefore,

$$\|u_0\|_{L_\epsilon^+}^2 \ge C_1^{-1}(E) \epsilon^{-\alpha(E)}. \tag{4.5}$$

On the other hand, by the definition of $\gamma(E)$, we have

$$\|u_0\|_L^2 \le C_2(E) L^{\gamma(E)}.$$

Denote

$$S = \{E \mid \eta(E) \ge b\}$$

(recall $\eta(E) = \alpha(E)/\gamma(E)$). Let $S_1 \subset S$ be the set such that for $E \in S_1$, $C_1(E)C_2(E) \le C$ with some uniform constant C. Clearly, S_1 may be chosen so that $\|P_{S\setminus S_1}\delta_1\|^2 = \mu^+(S \setminus S_1)$ is as small as we want by adjusting C. Then from (1.9) of Theorem 1.3 we find (with C denoting different universal constants in different places)

$$\begin{aligned} \langle \|e^{-iH_+t} P_{S_1}\delta_1\|_L^2\rangle_T &\le C \int\limits_{S_1} \frac{\|u_0(n,E)\|_L^2}{\|u_0(n,E)\|_{L_\epsilon^+}^2} d\mu^+(E) \\ &\le C \int\limits_{S_1} T^{-\alpha(E)} L^{\gamma(E)} d\mu^+(E). \end{aligned} \tag{4.6}$$

Choosing $L = C_g T^b$, we can make the left hand side in (4.6) arbitrarily small by changing C_g. It remains to observe that

$$\begin{aligned} \langle \|e^{-iH_+t}\delta_1\|_L^2\rangle_T &\le \langle (\|e^{-iH_+t}P_{S_1}\delta_1\|_L + \|e^{-iH_+t}(I-P_{S_1})\delta_1\|_L)^2\rangle_T \\ &\le \left(\sqrt{\langle \|e^{-iH_+t}P_{S_1}\delta_1\|_L^2\rangle_T} + \|(I-P_{S_1})\delta_1\|\right)^2. \end{aligned}$$

We claim that by choosing S_1 and then C_g the last expression can be made smaller than $1 - \mu^+(S) + g$ for any $g > 0$. Indeed, first choose S_1 so that

$$\|(I-P_{S_1})\delta_1\|^2 < 1 - \mu^+(S) + (g/2).$$

Then choose C_g so that

$$\langle \|e^{-iH_+t}P_{S_1}\delta_1\|_L^2\rangle_T \le (g/5)$$

(assuming g is small). This completes the proof, demonstrating that the part of the wavepacket corresponding to the energies in S leaves (on the average) a ball of the size $\sim T^b$ at time T. □

5. The Fibonacci Hamiltonian: bounds on traces

Let us recall from the Introduction that the Fibonacci Hamiltonian is the discrete Schrödinger operator

$$[H_\lambda u](n) = u(n+1) + u(n-1) + \lambda V(n)u(n) \tag{5.1}$$

acting on $\ell^2(\mathbb{Z})$ with the Fibonacci potential $V(n) = \lfloor (n+1)\omega \rfloor - \lfloor n\omega \rfloor$. Here ω is the golden ratio, $(\sqrt{5}-1)/2$. The most important property of this potential is the substitution rule [38],

$$V(q_k + n) = V(n) \quad \text{ for } n = 1, ..., q_k \text{ and } k \geq 3 \tag{5.2}$$

where q_k denote the Fibonacci numbers, $q_0 = 1, q_1 = 1, q_k = q_{k-1} + q_{k-2}$.
For $k \geq 1$ we define by Φ_k the transfer matrix $\Phi(q_k, E)$:

$$\begin{bmatrix} u(q_k+1) \\ u(q_k) \end{bmatrix} = \Phi_k \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

for all $u(n)$ satisfying $H_\lambda u = Eu$. It is convenient to make the additional definitions

$$\Phi_{-1} = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix}, \quad \Phi_0 = \begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix}.$$

From the substitution rule it follows that $\Phi_{k+1} = \Phi_{k-1}\Phi_k$, $k \geq 0$ [26]. Let us denote $x_k(E) = \operatorname{Tr} \Phi_k$. With more work, one obtains the trace map and trace invariant [26]:

$$x_{k+1} = x_k x_{k-1} - x_{k-2} \tag{5.3}$$

$$x_{k+1}^2 + x_k^2 + x_{k-1}^2 - x_{k+1}x_k x_{k-1} = 4 + \lambda^2. \tag{5.4}$$

We denote by σ_k the spectrum of the periodic potential with period q_k coinciding with $V(n)$ for $n = 1, \ldots, q_k$. By the Bloch theorem, σ_k is a set of intervals (bands) for which $x_k(E) \in [-2, 2]$ (see, e.g. [43]). Moreover, in each band $x_k(E)$ varies monotonically in $[-2, 2]$ and takes values ± 2 at the ends. Due to the relations (5.3), (5.4), the traces are among the most convenient objects of study in the Fibonacci model. Since x_k describe the spectrum of periodic approximants, there is also a natural relation to the spectrum of the limiting Fibonacci Hamiltonian. The following properties of the traces and their relation to the spectrum of H_λ are quite useful:

Proposition 5.1. i) *The spectrum of H_λ coincides with the set of energies for which the sequence $x_k(E)$ is bounded.*

ii) *If $|x_k(E)| > 2$, $x_{k+1}(E) > 2$ for some k, then the sequence $x_n(E)$ is unbounded.*

iii) *If $\lambda > 4$, there cannot exist E, k such that $|x_k(E)| \leq 2$, $|x_{k+1}(E)| \leq 2$ and $|x_{k+2}(E)| \leq 2$.*

Proof. The first and second statements have been proved by Sütő [38]. The third statement is a direct consequence of the trace invariant (5.4). □

Our goal in this section is to prove a lower bound on the derivative $|x_k'(E)|$ for energies in the spectrum of H_λ. In the next section we show that this bound translates directly into the lower bound on the growth of $\|\Phi(n, E)\|_L^2$ that we need for the upper bounds on dynamics. The results of this section are contained, in a somewhat

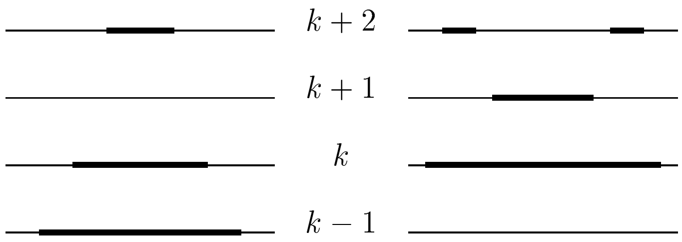

FIGURE 1. Types of bands in σ_k : left, a type A band; right, a type B band.

different form, in the preprint of Raymond [31], who used them to derive an upper bound on the Hausdorff dimension of the spectrum (which alone does not imply any upper bounds on dynamics, see [25]). For the sake of completeness, we present here a simplified version of the argument given in [31]. The main result of this section is

Proposition 5.2. *Assume that the coupling λ is sufficiently large ($\lambda > 8$ will do). Then for every E in the spectrum of H_λ, the derivative of the trace $x_k(E)$ satisfies*

$$|x_k'(E)| \geq \xi(\lambda)^{k/2}, \tag{5.5}$$

where $\xi(\lambda) > 1$ and

$$\xi(\lambda) = \lambda(1 + O(\lambda^{-1}))$$

in the large coupling regime.

Remark. We do not attempt to get the optimal range of values of λ. Instead, we opt for the clarity of exposition and freely assume that λ is large enough. The arguments presented below lead to nontrivial dynamical bounds for $\lambda > 8$. With more technical effort, this value can be reduced, but remains far from zero.

Definition. We call a band $I_k \subset \sigma_k$ a type A band if $I_k \subset \sigma_{k-1}$ (and consequently by Proposition 5.1 $I_k \cap (\sigma_{k+1} \cup \sigma_{k-2}) = \emptyset$). We call a band $I_k \subset \sigma_k$ a type B band if $I_k \subset \sigma_{k-2}$ (and so $I_k \cap \sigma_{k-1} = \emptyset$).

See Figure 1 for an illustration. Notice that by the definition of Φ_{-1} and Φ_0, $\sigma_{-1} = \mathbb{R}$, $\sigma_0 = [-2, 2]$ and $\sigma_1 = [\lambda - 2, \lambda + 2]$. Hence, σ_0 consists of one band of type A, and σ_1 consists of one band of type B. The following Lemma allows to determine inductively the structure of the spectrum of H_λ.

Lemma 5.3. *Assume that $\lambda > 4$. Then for any $k \geq 0$:*

i) *Every type A band $I_k \subset \sigma_k$ contains exactly one type B band $I_{k+2} \subset \sigma_{k+2}$, and no other bands from σ_{k+1}, σ_{k+2}.*
ii) *Every type B band $I_k \subset \sigma_k$ contains exactly one type A band $I_{k+1} \subset \sigma_{k+1}$ and two type B bands from σ_{k+2}, positioned around I_{k+1}.*

Proof. Consider a type A band $I_k \subset \sigma_k$. By definition, $I_k \subset \sigma_{k-1}$, and so $|x_{k+1}| > 2$ on I_k by Proposition 5.1 iii). This implies $I_k \cap \sigma_{k+1} = \emptyset$. On I_k, x_k changes monotonically from -2 to 2, in particular, there is a unique $E_1 \in I_k$ such that $x_k(E_1) = 0$. By the trace map (5.3), we have $|x_{k+2}(E_1)| = |x_{k-1}(E_1)| \leq 2$, so $I_k \cap \sigma_{k+2} \neq \emptyset$. Notice also that when $x_k = \pm 2$, $|x_{k+2}| \geq 2|x_{k+1}| - |x_{k-1}| > 2$, so all possible bands of σ_{k+2} intersecting I_k lie inside I_k. Moreover, in each band $I_{k+2} \subset I_k$ of σ_{k+2}, x_{k+2} changes from -2 to 2, and so by intermediate value theorem there exists $E_2 \in I_{k+2}$ such that $x_{k+2}(E_2) = -x_{k-1}(E_2)$. Then by (5.3), $x_k(E_2)x_{k+1}(E_2) = 0$, and since $|x_{k+1}(E)| > 2$ in I_k, $x_k(E_2) = 0$. Hence, every band $I_{k+2} \subset I_k$ contains an energy where $x_k = 0$; by monotonicity, there is only one such band.

Now consider a type B band $I_k \subset \sigma_{k-2}$. When $x_k(E) = 0$, we have $|x_{k+1}| = |x_{k-2}| \leq 2$, and so $I_k \cap \sigma_{k+1} \neq \emptyset$. As in the above argument, $|x_{k+1}| > 2$ when $|x_k| = 2$, so all bands of σ_{k+1} intersecting I_k lie strictly inside. Moreover, by same argument as above, any band $I_{k+1} \subset \sigma_{k+1}$ inside I_k must contain an energy E where $x_k(E) = 0$. Hence, there is a unique such band $I_{k+1} \subset I_k$. Next consider σ_{k+2}. Iterating the trace map, we find $x_{k+2} = (x_k^2 - 1)x_{k-1} - x_{k-2}x_k$. When $x_k = \pm 1$, $|x_{k+2}| < 2$. Also, if $x_k = \pm 2$, $|x_{k+2}| \geq 3|x_{k-1}| - 2|x_{k-2}| > 2$. Therefore, there are at least two bands of σ_{k+2} lying strictly inside I_k to the right and to the left from I_{k+1} (bands of $\sigma_{k+2} \subset I_k$ cannot intersect I_{k+1} by iii) of Proposition 5.1). It remains to show that there are only two such bands. Using (5.3), it is easy to verify that

$$(x_k \pm 1)(x_{k+2} \pm x_{k-2}) = (x_k^2 - 1)(x_{k+1} \pm x_{k-1}). \tag{5.6}$$

Consider a band $I_{k+2} \subset \sigma_{k+2}$ lying in I_k. For $E \in I_{k+2}$, $x_{k+1}(E)$, $x_{k-1}(E)$ have fixed signs (both satisfy $|x_{k\pm 1}(E)| > 2$ there). Pick a sign in (5.6) so that $x_{k+1} \pm x_{k-1} \neq 0$. By the intermediate value theorem, there exists the energy $E \in I_{k+2}$ where $x_{k+2} \pm x_{k-2} = 0$. At this energy, we must have $x_k^2 - 1 = 0$. But there are only two energies in I_k where $x_k(E) = \pm 1$. Hence, there are at most two bands of σ_{k+2} in I_k. ☐

For the proof of the key lemma, we need the following auxiliary result.

Lemma 5.4. *Let the functions $f_\pm(x, y, \lambda)$ be defined by*

$$f_\pm(x, y, \lambda) = \tfrac{1}{2}\Big[xy \pm \sqrt{4\lambda^2 + (4 - x^2)(4 - y^2)}\Big]. \tag{5.7}$$

Assume that $\lambda \geq 4$. Then for $-2 \leq x, y \leq 2$,

$$\left|\frac{\partial f_\pm}{\partial x}(x, y, \lambda)\right|, \left|\frac{\partial f_\pm}{\partial y}(x, y, \lambda)\right| \leq 1. \tag{5.8}$$

Proof. Since $f_-(x, y, \lambda) = -f_+(x, -y, \lambda)$, and $f_+(x, y, \lambda) = f_+(y, x, \lambda)$, it suffices to show the bound for

$$\frac{\partial f_+}{\partial x}(x, y, \lambda) = \frac{1}{2}\left(y - \frac{x(4 - y^2)}{\sqrt{4\lambda^2 + (4 - x^2)(4 - y^2)}}\right).$$

Notice that

$$\frac{\partial^2 f_+}{\partial x \partial y}(x, y, \lambda) = \frac{1}{2}\left(1 + \frac{8xy\lambda^2 + xy(4 - x^2)(4 - y^2)}{(4\lambda^2 + (4 - x^2)(4 - y^2))^{3/2}}\right) \geq 0$$

for $\lambda > 4$ and $|x|, |y| \leq 2$. Thus,

$$\max_{|x|,|y| \leq 2}\left|\frac{\partial f_+}{\partial x}(x, y, \lambda)\right| = \max_{|x| \leq 2, y = \pm 2}\left|\frac{\partial f_+}{\partial x}(x, y, \lambda)\right| = 1.$$

☐

The Proposition 5.2 follows immediately from Lemma 5.3 and

Lemma 5.5. *Assume that the coupling λ is sufficiently large. Then there exists a number $\xi(\lambda) > 1$ such that, given any (type-A) band $I_{k+1} \subset \sigma_{k+1}$ lying in the band $I_k \subset \sigma_k$, we have*

$$\left| \frac{x'_{k+1}(E)}{x'_k(E)} \right| \geq \xi(\lambda)$$

for $E \in I_{k+1}$. Similarly, for any (type-B) band $I_{k+2} \subset \sigma_{k+2}$ lying in I_k,

$$\left| \frac{x'_{k+2}(E)}{x'_k(E)} \right| \geq \xi(\lambda)$$

for $E \in I_{k+2}$. Moreover, $\xi(\lambda) = \lambda(1 + O(\lambda^{-1}))$ in the large coupling regime.

Proof. The proof is by induction. The induction is well-founded since $x'_{-1} = 0$, $x'_0 = 1$, $x'_1 = 1$. There are three cases to consider.
1. First, consider the case where $I_{k+1} \subset \sigma_{k+1}$ lies in $I_k \subset \sigma_k$. Then by Proposition 5.1 $I_k \cap \sigma_{k-1} = \emptyset$, and $I_k \subset \sigma_{k-2}$. Differentiating the equality $x_{k+1} = x_k x_{k-1} - x_{k-2}$ and dividing by x'_k, we find

$$\frac{x'_{k+1}}{x'_k} = x_{k-1} + \frac{x_k x'_{k-1}}{x'_k} - \frac{x'_{k-2}}{x'_k}. \tag{5.9}$$

From the trace invariant (5.4)

$$x_k^2 + x_{k-1}^2 + x_{k-2}^2 - x_k x_{k-1} x_{k-2} = 4 + \lambda^2$$

we get

$$x_{k-1} = f_{\pm}(x_k, x_{k-2}, \lambda) = \tfrac{1}{2}\left(x_k x_{k-2} \pm \sqrt{4\lambda^2 + (4 - x_{k-2}^2)(4 - x_k^2)} \right) \geq \lambda - 2$$

since $|x_k|, |x_{k-2}| \leq 2$. Therefore this term in (5.9) is large. To estimate the remaining terms, notice that

$$x'_{k-1} = \frac{\partial f_{\pm}}{\partial x}(x_k, x_{k-2}, \lambda) x'_k + \frac{\partial f_{\pm}}{\partial y}(x_k, x_{k-2}, \lambda) x'_{k-2}$$

(where $\pm$ means that either plus or minus may occur). Applying Lemma 5.4, we derive from (5.9) that

$$\left| \frac{x'_{k+1}}{x'_k} \right| \geq \lambda - 2 - 3\left| \frac{x'_{k-2}}{x'_k} \right| - |x_k|.$$

By the induction assumption, $|x'_{k-2}/x'_k| \leq \xi(\lambda)^{-1}$, so the induction step holds true provided that

$$\xi(\lambda) \leq \lambda - 4 - 3\xi(\lambda)^{-1}.$$

The maximal $\xi(\lambda)$ we can take under this condition is

$$\xi(\lambda) = \tfrac{1}{2}\left((\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12} \right) = \lambda + O(1).$$

Next, consider $I_{k+2} \subset \sigma_{k+2}$ lying in I_k. Here we have to distinguish two scenarios.
2. $I_{k+2} \cap \sigma_{k-1} = \emptyset$, $I_{k+2} \subset \sigma_{k-2}$. Given

$$x'_{k+2} = x_{k+1}x'_k + x'_{k+1}x_k - x'_{k-1},$$
$$x'_{k+1} = x_k x'_{k-1} + x'_k x_{k-1} - x'_{k-2}$$

we find

$$\frac{x'_{k+2}}{x'_k} = 2x_{k+1} - x_{k-2} + (x_k^2 - 1)\frac{x'_{k-1}}{x'_k} - x_k \frac{x'_{k-2}}{x'_k}.$$

Similarly to the previous argument,

$$|x_{k+1}| = |f_\pm(x_{k+2}, x_k, \lambda)| \geq \lambda - 2.$$

Also,

$$x'_{k-1} = \frac{\partial f_\pm}{\partial x}(x_k, x_{k-2}, \lambda)x'_k + \frac{\partial f_\pm}{\partial y}(x_k, x_{k-2}, \lambda)x'_{k-2}.$$

Using Lemma 5.4, the bounds $|x_k|, |x_{k-2}| \leq 2$, and the induction assumption, we arrive at

$$\left|\frac{x'_{k+2}}{x'_k}\right| \geq 2\lambda - 9 - 5\xi(\lambda)^{-1}.$$

Solving the quadratic inequality, we find that the maximal $\xi(\lambda)$ for which induction step goes through in this case is

$$\xi(\lambda) = \tfrac{1}{2}\left((2\lambda - 9) + \sqrt{(2\lambda - 9)^2 - 20}\right) = 2\lambda + O(1).$$

3. $I_{k+2} \subset \sigma_{k-1}$, $I_{k+2} \cap \sigma_{k-2} = \emptyset$. In this case we obtain

$$\frac{x'_{k+2}}{x'_k} = x_{k+1} + x_k \frac{x'_{k+1}}{x'_k} - \frac{x'_{k-1}}{x'_k}.$$

As before, $x_{k+1} \geq \lambda - 2$. Also

$$x'_{k+1} = \frac{\partial f_\pm}{\partial x}(x_k, x_{k-1}, \lambda)x'_k + \frac{\partial f_\pm}{\partial y}(x_k, x_{k-1}, \lambda)x'_{k-1}.$$

This leads to

$$\left|\frac{x'_{k+2}}{x'_k}\right| \geq \lambda - 4 - 3\xi(\lambda)^{-1}.$$

Hence the induction step works with the same $\xi(\lambda)$ as in the first case.

From Lemma 5.3 it follows that we have considered all possible situations.

□

Remark. In particular, the given argument shows that Lemma 5.5 (and hence Proposition 5.2) holds with $\xi(\lambda) > 1$ for $\lambda \geq 8$.

6. The Fibonacci Hamiltonian: dynamical bounds

Our first goal is to relate the derivatives of traces with the information required by the dynamical criterion of Theorem 1.5. The following formula is well-known (see, e.g., [43])

$$x_k'(E) = \sum_{n=1}^{q_k} \Big\{ u_{\pi/2}(q_k, E) u_0^2(n, E) - u_0(q_k + 1, E) u_{\pi/2}^2(n, E) + \\ + \big[u_{\pi/2}(q_k + 1, E) - u_0(q_k, E)\big] u_0(n, E) u_{\pi/2}(n, E) \Big\}$$

and can be derived using variation of parameters. Therefore,

$$x_k'(E) \le 4 \|\Phi(n, E)\|_{q_k+1}^3$$

(one can show that $x_k'(E) \le 4(\lambda + O(1)) \|\Phi(n, E)\|_{q_k+1}^2$ for E in the spectrum of H_λ, but we choose to avoid the technicalities). Hence, Proposition 5.2 implies

$$\|\Phi(n, E)\|_{q_k+1}^3 \ge \frac{1}{4} \xi(\lambda)^{k/2}. \tag{6.1}$$

As ω is the golden ratio $(\sqrt{5} - 1)/2$,

$$q_k = [\omega^{-k} - (-\omega)^k]/\sqrt{5}$$

for $k \ge 1$ (see, e.g., [21]). So for large k, $q_k \sim \omega^{-k}/\sqrt{5}$. Consider $q_k \le L \le q_{k+1}$. Since $q_{k+1}/q_k \le 2$ for all k, (6.1) implies

$$\|\Phi(n, E)\|_L^2 \ge C q_k^{\zeta_1} \ge C' L^{\zeta_1} \quad \text{with } \zeta_1 = \frac{\log \xi(\lambda)}{3 \log(\omega^{-1})}. \tag{6.2}$$

Later we will derive an upper bound on this quantity of the form CL^{ζ_2} with ζ_2 also depending logarithmically on λ.

Now we complete the proof of the upper dynamical bound in Theorem 1.6.

Proof. By the symmetry of Fibonacci potential ($V(-n) = V(n-1)$ for $n \ge 2$, see [38]) the bound identical to (6.2) also holds for the negative semi-axis. These bounds allow us to define the characteristic scales $\tilde{L}_{T^{-1}}^{\pm}(E)$. Consider the case of $\tilde{L}_{T^{-1}}^{+}(E)$ (the other one is analogous). By (1.7), $\tilde{L}_{T^{-1}}^{+}(E)$ is determined by the equality

$$\|\Phi(n, E)\|_{\tilde{L}_{T^{-1}}^{+}(E)}^2 = 4 \|\Phi(1, E)^{-1}\|^2 T^2.$$

From (6.2) we get

$$\tilde{L}_{T^{-1}}^{+}(E)^{\frac{\log \xi(\lambda)}{3 \log(\omega^{-1})}} \le C T^2,$$

and since $\xi(\lambda) = \lambda + O(1)$ we can take

$$\tilde{L}_{T^{-1}}^{+}(E) = C T^{\frac{6 \log(\omega^{-1})}{\log \xi(\lambda)}}.$$

This proves the first part of Theorem 1.6. $\square$

It remains to prove the second part of Theorem 1.6, involving the lower bound on dynamics. The lower bound on dynamics for the Fibonacci Hamiltonian has been proved recently in [23]. We present a sketch of the argument here, making explicit the behavior of the bound in the large coupling regime. The lower dynamical bound will follow from the continuity estimate for the spectrum. The idea is to study the behavior of solutions; as soon as appropriate bounds are available, one could apply the reasoning of Jitomirskaya-Last [23], a recent result of Damanik, Killip and Lenz [13] relating bounds on solutions to the α-continuity of the whole line operator, or Theorem 1.3.

Lemma 6.1. *For every energy E in the spectrum of H_λ, every solution $u(n,E)$ of the equation $(H_\lambda - E)u(n,E) = 0$ satisfies*

$$\|u(n,E)\|_L \geq CL^\kappa, \tag{6.3}$$

where $\kappa = \log \frac{\sqrt{17}}{20\log(\omega^{-1})}$ is independent of λ.

Proof. The proof follows closely Proposition 10 of [23]. As shown by Sütő [38], the Fibonacci potential obeys $V(q_n + l) = V(l)$ for $n \geq 3$ and $1 \leq l \leq q_n$. For n large and $1 \leq l \leq q_{n-2}$, we also get $V(2q_n + l) = V(q_{n+1} + q_{n-2} + l) = V(q_{n-2} + l) = V(l)$, so that

$$V(l) = V(q_n + l) = V(2q_n + l). \tag{6.4}$$

By Lemma 1 of [38], we have for any 2×2 matrix B with $\det B = 1$,

$$\max\{|\mathrm{Tr}\, B|\|B\Psi\|, \|B^2\Psi\|\} \geq \frac{1}{2}\|\Psi\|$$

for any 2-vector Ψ. Therefore,

$$\|B\Psi\|^2 + \|B^2\Psi\|^2 > \frac{1}{4\max(1, |\mathrm{Tr}\, B|^2)}\|\Psi\|^2. \tag{6.5}$$

Let $\Phi(m,k)$ denote the transfer matrix that takes $(u(m+1), u(m))^T$ to $(u(k+1), u(k))^T$. By (6.4), for $1 \leq l \leq q_{n-2}$ we have $\Phi(l, q_n + l) = \Phi(q_n + l, 2q_n + l)$, and, moreover, $\mathrm{Tr}\, \Phi(l, q_n + l) = x_n$. Thus, (6.5) implies

$$|u(q_n+l+1)|^2 + |u(q_n+l)|^2 + |u(2q_n+l+1)|^2 + |u(2q_n+l)|^2 > \frac{1}{4\max(1, x_n^2)}(|u(l+1)|^2 + |u(l)|^2)$$

for $1 \leq l \leq q_{n-2}$, and

$$|u(q_{n+1}+l+1)|^2 + |u(q_{n+1}+l)|^2 + |u(2q_{n+1}+l+1)|^2 + |u(2q_{n+1}+l)|^2 > \frac{1}{4\max(1, x_{n+1}^2)}(|u(l+1)|^2 + |u(l)|^2)$$

for $1 \leq l \leq q_{n-1}$. Notice that for any E in the spectrum of H_λ, for every n either $|x_n|$ or $|x_{n+1}|$ is less than 2. By combining the above estimates, we easily deduce that

$$\|u(n,E)\|_{q_{n+5}} \geq \frac{\sqrt{17}}{4}\|u(n,E)\|_{q_n} \tag{6.6}$$

for large n. The estimate (6.3) follows directly from (6.6). $\square$

On the other hand, Iochum and Testard [21] have shown that

$$\|\Phi(n,E)\| \le \left(d^{\frac{\log\sqrt{5}}{\log(\omega^{-1})}}\right)^{\log n}.$$

The number d is defined as a product $d = ab^2$, where $b = 2c+1$, $c = \sup_n |x_n|$, and $a = \max(c,2)$. For λ large, one can check using (5.4) that $c \le \lambda + 2$. Hence, $d \le (\lambda+2)(2\lambda+5)^2$. Therefore we conclude that

$$\sum_{n=1}^{L} \|\Phi(n,E)\|^2 \le CL^{\zeta_2(\lambda)}, \tag{6.7}$$

where

$$\zeta_2(\lambda) = \frac{6\log\sqrt{5}}{\log(\omega^{-1})}(\log\lambda + O(1)).$$

Recall that a measure μ is called α-continuous if it does not give weight to sets of zero α-dimensional Hausdorff measure.

Proposition 6.2. *The spectral measure μ of the operator H_λ is $\alpha(\lambda)$-continuous for every $\lambda > 0$, with $\alpha(\lambda)$ satisfying*

$$\alpha(\lambda) \ge \frac{2\kappa}{\kappa + \zeta_2(\lambda)} = C(\log\lambda)^{-1}(1 + O((\log\lambda)^{-1})). \tag{6.8}$$

Proof. Given the estimates (6.3) and (6.7), for the Fibonacci operator H_λ the Proposition follows along the lines of [23]. Alternatively, one may use Theorem 1 of [13] which shows for any discrete Schrödinger operator H on whole axis that if every solution of $(H-E)u = 0$ obeys

$$C_1(E)L^{q_1} \le \|u(n,E)\|_L \le C_2(E)L^{q_2}$$

for $E \in \Sigma$, then the spectrum of H is α-continuous in Σ with $\alpha = 2q_1/(q_1+q_2)$. $\square$

Now we complete the proof of Theorem 1.6.

Proof. The fact that α-continuity of the spectral measure implies lower bounds on dynamics is well-known. The original result is due to Guarneri [16, 17], with later contributions by many authors. Assume that the spectral measure μ^ϕ of the Schrödinger operator H corresponding to the vector ϕ is α-continuous. Then by Theorem 4.2 of [27], the measure μ^ϕ can be represented as a sum of mutually singular measures $\mu^\phi = \mu^{\phi_1} + \mu^{\phi_2}$, where μ^{ϕ_1} is uniformly α-Hölder continuous ($\mu^{\phi_1}(I) \le C|I|^\alpha$ for any interval I) while $\mu^{\phi_2}(\mathbb{R})$ can be made arbitrarily small. Moreover, it has been shown in [27], proof of Theorem 6.1, that

$$\tfrac{1}{a}\int_0^a \|e^{-iHt}\phi_1\|_L^2\, dt \le CLa^{-\alpha}. \tag{6.9}$$

(in dimension one). Multiplying (6.9) by a and integrating with respect to the kernel $\frac{2}{T^2}e^{-2a/T}$, we get

$$\tfrac{2}{T}\int_0^\infty e^{-2t/T}\|e^{-iHt}\phi_1\|_L^2\,dt \le C'LT^{-\alpha}.$$

In particular, given any $g>0$, we can choose ϕ_2 of sufficiently small norm and C_g so that for $L(T)=C_gT^\alpha$,

$$\tfrac{2}{T}\int_0^\infty e^{-2t/T}\|e^{-iHt}\phi\|_{L(T)}^2\,dt \le g. \tag{6.10}$$

Application of (6.10) and Proposition 6.2 completes the proof. □

7. Multidimensional problems

Our purpose in this section is to show how our results for tridiagonal operators can be applied to more general problems. We are particularly interested in discrete Schrödinger operators of the form $\Delta+V$ on $\ell^2(\mathbb{Z}^d)$, defined by

$$((\Delta+V)\psi)(n)=\sum_{|n-m|=1}\psi(m)+V(n)\psi(n), \tag{7.1}$$

where $\{V(n)\}_{n\in\mathbb{Z}^d}\subset\mathbb{R}$.

Given any self adjoint operator H on a separable Hilbert space $\mathcal{H}$ and a vector ψ in the domain of H, the cyclic subspace spanned by H and ψ is defined by $\mathcal{H}_\psi=\overline{\{f(H)\psi \mid f\in C_\infty(\mathbb{R})\}}$. Here $C_\infty(\mathbb{R})$ is the set of continuous (complex valued) functions on $\mathbb{R}$ vanishing at infinity and $\overline{\,\cdot\,}$ denotes norm closure in $\mathcal{H}$. $\mathcal{H}_\psi$ is an invariant subspace for H and by the spectral theorem, the restricted operator $H\restriction\mathcal{H}_\psi$ is unitarily equivalent to multiplication by the coordinate parameter on the space $L^2(\mathbb{R},d\mu_\psi)$, where μ_ψ is the spectral measure of ψ (and H). We assume that μ_ψ is not supported on a finite number of points and thus $\mathcal{H}_\psi$ is infinite dimensional. The evolution given by (1.1) is thus confined to $\mathcal{H}_\psi$ (namely, $\psi(t)\in\mathcal{H}_\psi$ for any t) and so to study the time evolution, it suffices to consider the restricted operator $H\restriction\mathcal{H}_\psi$.

For large classes of problems, the moment vectors $\{H^n\psi\}_{n=0}^\infty$ are well defined and the set of their finite linear combinations is dense in $\mathcal{H}_\psi$, namely,

$$\mathcal{H}_\psi=\overline{\operatorname{span}\{H^n\psi\}_{n=0}^\infty}. \tag{7.2}$$

We note that this holds for any ψ if H is bounded. If H is unbounded, then ψ must be chosen appropriately in order for this to occur (but such ψ's always exist, e.g., any ψ with a compactly supported spectral measure). We note that ψ must be in the domain of each of the moment operators H^n and thus its spectral measure must have sufficient decay at infinity to ensure finite moments. This necessary condition is not sufficient, though, and determining whether this property holds or not for a given ψ is in general a rather rich question, equivalent to the problem of whether the spectral measure of ψ can be determined from its moments. See [35] for more information. The important thing for us here is that the property (7.2) holds whenever H is of the form (7.1) on $\ell^2(\mathbb{Z}^d)$ and ψ is any vector which is compactly supported on $\mathbb{Z}^d$.

If the property (7.2) holds, then $\mathcal{H}_\psi$ has a natural orthonormal basis $\{\rho_n\}_{n=0}^\infty$, which is obtained by applying the Gram-Schmidt orthonormalization procedure to the set of moment vectors $\{H^n\psi\}_{n=0}^\infty$. Each ρ_n has the form $\rho_n = P_n(H)\psi$, where $P_n(\cdot)$ is a polynomial of degree n. The polynomials $\{P_n(\cdot)\}_{n=0}^\infty$ are known as the orthonormal polynomials of the spectral measure μ_ψ. The self-adjoint operator $H \restriction \mathcal{H}_\psi$ is known [35] to have a tridiagonal matrix representation in the basis $\{\rho_n\}_{n=0}^\infty$ and by choosing the phases of the ρ_n appropriately, this tridiagonal matrix can be made to have only real entries. Moreover, the off-diagonal entries cannot vanish. Thus, we see that $\mathcal{H}_\psi$ can be viewed as being $\ell^2(\mathbb{Z}_+)$, by identifying ρ_n with δ_{n+1}, and then $H \restriction \mathcal{H}_\psi$ has precisely the form (1.2) with a Dirichlet boundary condition. Theorems 1.1, 1.2, 1.3, and 1.4 are thus fully applicable in this general setting. We call the above tridiagonal representation of $H \restriction \mathcal{H}_\psi$ the orthonormal polynomial representation.

The problem, of course, is that when one is interested in wavepacket spreading dynamics, one is normally interested in how the spreading occurs in some natural coordinate space which is generally different from the orthonormal basis $\{\rho_n\}_{n=0}^\infty$. The ability to apply our results to such a coordinate space depends on the ability to relate the vectors ρ_n to it. Fortunately, for the case of an operator of the form (7.1) on $\ell^2(\mathbb{Z}^d)$ and a compactly supported initial vector ψ, an appropriate connection exists. For simplicity, take the initial vector ψ to be the delta function vector at the origin of $\mathbb{Z}^d$, namely $\psi = \delta_0$. Then one easily sees from (7.1) that $H^n\delta_0$, and thus also ρ_n, is supported in a ball of radius $n+1$ in $\mathbb{Z}^d$. Thus, any portion of the wavepacket which is supported on the vectors $\rho_0, ..., \rho_L$ must also be confined to a ball of radius $L+1$ in $\mathbb{Z}^d$. This means that any upper bound on the spreading of the wavepacket in the "coordinates" $\{\rho_n\}_{n=0}^\infty$ is immediately also bounding its spreading in $\mathbb{Z}^d$. In particular, Theorem 1.1 takes the following form.

Theorem 7.1. *Let H be an operator of the form (7.1) on $\ell^2(\mathbb{Z}^d)$. Let the characteristic scale $L^+_{T^{-1}}(E)$ be defined by (1.5) for the orthonormal polynomial representation of $H \restriction \mathcal{H}_{\delta_0}$. Then for any $T > 0$ and $L > 1$, we have*

$$\langle \|\chi_{L+1} e^{-iHt}\delta_0\|^2\rangle_T > C\mu_{\delta_0}\left(\left\{E \mid L^+_{T^{-1}}(E) \le L\right\}\right), \tag{7.3}$$

where C is some universal positive constant and χ_L denotes the orthogonal projection on a ball of radius L around the origin in $\mathbb{Z}^d$.

We note that the scale $L^+_{T^{-1}}(E)$ in Theorem 7.1 is defined through the orthonormal polynomial representation, a fact which may make it seem like a somewhat obscured quantity in the multidimensional context. It is interesting to note, however, that even in the multidimensional context, it is actually among the more straight forward quantities to compute. Each of the moment vectors $H^n\delta_0$ can be easily computed explicitly and it only depends on values of the potential within a ball of a corresponding radius n, and given the moment vectors up to some scale L, the corresponding $\rho_0, ..., \rho_L$ are obtained from them by the Gram-Schmidt procedure. The matrix elements of the (tridiagonal) orthonormal polynomial representation of $H \restriction \mathcal{H}_{\delta_0}$ are then easily obtained up to a corresponding scale L and from them one can explicitly compute solutions of equation (1.3) up to a corresponding scale. It is thus possible, for any E and T, to compute (or more precisely, to bound within two integer values) the scale $L^{+}_{T^{-1}}(E)$ by a straight forward computation consisting of finitely many steps and only depending on values of the potential within a ball of a corresponding radius. This should be contrasted with the fact that multidimensional representations of generalized eigenfunctions, such as the ones considered in the Kiselev-Last lower bound [25], depend on the entire potential and cannot be computed by a finite procedure.

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