Exact Diagonalization Of The Fractional Quantum Hall Many-Body Hamiltonian In The Lowest Landau Level

by

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Abstract: For a gaussian interaction $V(x,y) = \lambda\, e^{-\frac{x^2+y^2}{2r^2}}$ with long range $r >> \ell_B$, ℓ_B the magnetic length, we rigorously prove that the eigenvalues of the finite volume Hamiltonian $H_{N,\mathrm{LL}} = P_{\mathrm{LL}} H_N P_{\mathrm{LL}}$, $H_N = \sum_{i=1}^N \bigl(-i\hbar\nabla_{x_i} - eA(x_i)\bigr)^2 + \sum_{i,j;\, i\neq j} V(x_i - x_j)$, $\mathrm{rot}A = (0,0,B)$, and P_{LL} the projection onto the lowest Landau level, are given by the following set: Let M be the number of flux quanta flowing through the sample such that $\nu = N/M$ is the filling factor. Then each eigenvalue is given by $E = E(n_1,\cdots,n_N) = \sum_{i,j=1;i\neq j}^N W(n_i - n_j)$. Here $n_i \in \{1,2,\cdots,M\}$, $n_1 < \cdots < n_N$ and the function W is given by $W(n) = \lambda \sum_{j\in\mathbb{Z}} e^{-\frac{1}{2r^2}(L\frac{n}{M} - jL)^2}$ if the system is kept in a volume $[0,L]^2$. The eigenstates are also explicitely given.

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In this paper we consider the two dimensional many electron system in finite volume in a constant magnetic field $\vec{B}=(0,0,B)$ described by the Hamiltonian

$$H_N=\sum_{i=1}^{N}\left(\tfrac{\hbar}{i}\nabla_i-eA(\vec{x}_i)\right)^2+\sum_{\substack{i,j=1\\ i\neq j}}V(\vec{x}_i-\vec{x}_j) \tag{1}$$

We restrict to the completely spin polarized case and neglect the Zeemann energy. The electron-electron interaction is assumed to be a gaussian,

$$V(x,y)=\lambda\, e^{-\frac{x^2+y^2}{2r^2}} \tag{2}$$

which is long range, $r>>\ell_B$, ℓ_B being the magnetic length. The only approximation we will use is (see (33,35) below)

$$\textstyle\int ds\, ds'\, h_n(s)\, h_{n'}(s')\, e^{-\frac{\ell_B^2}{2r^2}(s-s')^2}\;\approx\;\int ds\, ds'\, h_n(s)\, h_{n'}(s') \tag{3}$$

where $h_n(s)=c_nH_n(s)\,e^{-\frac{s^2}{2}}$ denotes the normalized Hermite function. With this approximation, the Hamiltonian $P_{\rm LL}H_NP_{\rm LL}$, $P_{\rm LL}$ being the projection onto the lowest Landau level, can be exactly diagonalized. There is the following

Theorem: *Let H_N be the Hamiltonian (1) in finite volume $[0,L_x]\times[0,L_y]$ (with magnetic boundary conditions (11), see below), let $A(x,y)=(-By,0,0)$ and let the interaction be gaussian with long range,*

$$V(x,y)=\lambda\, e^{-\frac{x^2+y^2}{2r^2}}\,,\qquad r>>\ell_B\,. \tag{4}$$

Let $P_{\rm LL}:\mathcal{F}_N\to\mathcal{F}_N^{\rm LL}$ be the projection onto the lowest Landau level, where $\mathcal{F}_N$ is the antisymmetric N-particle Fock space and $\mathcal{F}_N^{\rm LL}$ is the antisymmetric Fock space spanned by the eigenfunctions of the lowest Landau level. Then, with the approximation (3), the Hamiltonian $H_{N,\rm LL}=P_{\rm LL}H_NP_{\rm LL}$ becomes exactly diagonalizable. Let M be the number of flux quanta flowing through $[0,L_x]\times[0,L_y]$ such that $\nu=N/M$ is the filling factor. Then the eigenstates and eigenvalues are labelled by N-tupels $(n_1,\cdots,n_N)$, $n_1<\cdots<n_N$ and $n_i\in\{1,2,\cdots,M\}$ for all i,

$$H_{N,\rm LL}\Psi_{n_1\cdots n_N}=(\varepsilon_0N+E_{n_1\cdots n_N})\Psi_{n_1\cdots n_N} \tag{5}$$

where $\varepsilon_0=\hbar eB/(2m)$ and

$$E_{n_1\cdots n_N}=\sum_{\substack{i,j=1\\ i\neq j}}^{N}W(n_i-n_j)\,,\qquad W(n)=\lambda\sum_{j\in\mathbb{Z}}e^{-\frac{1}{2r^2}\left(L_x\frac{n}{M}-jL_x\right)^2} \tag{6}$$

and the normalized eigenstates are given by $\Psi_{n_1\cdots n_N}=\phi_{n_1}\wedge\cdots\wedge\phi_{n_N}$ *where*

$$\begin{aligned}
\phi_n(x,y) &= \frac{\pi^{-\frac14}}{\sqrt{\ell_B L_y}}\sum_{s=-\infty}^{\infty} e^{-\frac{1}{2\ell_B^2}\left(x-\frac{n}{M}L_x-sL_x\right)^2} e^{i(x-\frac{n}{M}L_x-sL_x)y/\ell_B^2} &(7)\\
&= \frac{1}{\sqrt{M}}\frac{\pi^{-\frac14}}{\sqrt{\ell_B L_x}}\sum_{r=-\infty}^{\infty} e^{-\frac{1}{2\ell_B^2}\left(y-\frac{r}{M}L_y\right)^2} e^{i\left(x-\frac{n}{M}L_x\right)\frac{r}{M}L_y/\ell_B^2} &(8)
\end{aligned}$$

Proof: We proceed in four steps: Review of the unperturbed single body problem, projection onto the lowest Landau level using fermionic annihilation and creation operators, implementation of the approximation (3) and finally diagonalization.

(i) Single Body Eigenfunctions:

We compute in finite volume with a rectangular geometry $[0,L_x]\times[0,L_y]$ and with the Landau gauge

$$A(x,y)=(-By,0,0) \tag{9}$$

This setup also has been used in the paper of Koma [1]. The eigenfunctions $\varphi=\varphi_{n,k}$ of the unperturbed single body hamiltonian

$$H_0=\left(\tfrac{\hbar}{i}\nabla-eA(\vec{x})\right)^2 \tag{10}$$

with magnetic boundary conditions

$$\varphi(x+L_x,y)=\varphi(x,y)\,,\qquad \varphi(x,y+L_y)=e^{ixL_y/\ell_B^2}\varphi(x,y), \tag{11}$$

see [1], are labelled by quantum numbers $n=0,1,2,\cdots$, the Landau level index, and momenta

$$k=\tfrac{2\pi}{L_x}m,\qquad m=0,1,2,\cdots,M-1 \tag{12}$$

Here M is the number of flux quanta flowing through the sample. Because of flux quantization, this has to be a natural number,

$$M=\frac{L_xL_y}{2\pi\ell_B^2}=\frac{L_xL_yB}{h/e}=\frac{\Phi}{\phi_0}\in\mathbb{N} \tag{13}$$

The magnetic length is given by

$$\ell_B^2=\frac{\hbar}{eB} \tag{14}$$

and a flux quantum is given by

$$\phi_0=2\pi\frac{\hbar}{e}=2,07\cdot10^{-11}\mathrm{T\ cm}^2 \tag{15}$$

The energy eigenvalues are given by

$$H_0\varphi_{n,k} = \varepsilon_n\,\varphi_{n,k}\,, \qquad \varepsilon_n = \hbar\tfrac{eB}{m}\left(n+\tfrac{1}{2}\right) \tag{16}$$

and have an M-fold degeneracy. The fraction

$$\nu := \frac{N}{M} \tag{17}$$

is the filling factor of the system, if N denotes the number of electrons. The explicit form of the normalized finite volume eigenstates in asymmetric gauge is [1]

$$\varphi_{n,k}(x,y) = \tfrac{1}{\sqrt{L_x\ell_B}} \sum_{j=-\infty}^{\infty} e^{i(k+jK)x}\, h_{n,k}(y-jL_y) \tag{18}$$

where $K := L_y/\ell_B^2$ and

$$h_{n,k}(y) = h_n\big((y-y_k)/\ell_B\big) \tag{19}$$

$y_k = \ell_B^2 k$ and h_n is the normalized Hermite function,

$$h_n(y) = c_n\, H_n(y)\, e^{\frac{-y^2}{2}}\,, \qquad c_n = \tfrac{1}{\sqrt[4]{\pi}}\tfrac{1}{\sqrt{2^n n!}} \tag{20}$$

(ii) Projection onto the Lowest Landau Level

To project H_N onto the lowest Landau level, we rewrite H_N in terms of fermionic annihilation and creation operators

$$\begin{aligned} H_N \;=\; & \Big\{ \int d^2x\, \psi^+(\vec{x}) \left(\tfrac{\hbar}{i}\nabla - eA(\vec{x})\right)^2 \psi(\vec{x}) \\ & \quad + \int d^2x\, d^2x' \psi^+(\vec{x})\psi^+(\vec{x}\,')V(\vec{x}-\vec{x}\,')\psi(\vec{x}\,')\psi(x) \Big\}\Big|_{\mathcal{F}_N} \end{aligned} \tag{21}$$

where $\mathcal{F}_N$ is the antisymmetric N-particle Fock-space. We consider the completely spin polarized case in which only one spin direction (say $\psi = \psi_\uparrow$) contributes and we neglect the Zeeman energy. Introducing $a_{n,k}$, $a^+_{n,k}$ according to

$$\psi(\vec{x}) = \sum_{n,k} \varphi_{n,k}(\vec{x})a_{n,k}\,, \qquad \psi^+(\vec{x}) = \sum_{n,k} \bar{\varphi}_{n,k}(\vec{x})a^+_{n,k} \tag{22}$$

$$a_{n,k} = \textstyle\int d^2x\, \bar{\varphi}_{n,k}(\vec{x})\psi(\vec{x})\,, \qquad a^+_{n,k} = \int d^2x\, \varphi_{n,k}(\vec{x})\psi^+(\vec{x}) \tag{23}$$

the $a_{n,k}$ obey the canonical anticommutation relations

$$\{a_{n,k}, a^+_{n',k'}\} = \delta_{n,n'}\delta_{k,k'} \tag{24}$$

and (21) becomes, if $H = \oplus_N H_N$,

$$H \;=\; H_{\rm kin} + H_{\rm int} \tag{25}$$

where

$$H_{\rm kin} = \sum_{n,k} \varepsilon_n \, a^+_{n,k} a_{n,k} \tag{26}$$

The interacting part becomes

$$\begin{aligned} H_{\rm int} &= \sum_{\substack{n,k\\ n',k'}} \int d^2x\, d^2x' \psi^+(\vec{x})\psi^+(\vec{x}\,')\bar{\varphi}_{n,k}(\vec{x})\, \langle nk|V|\overline{n'k'}\rangle\, \varphi_{n',k'}(\vec{x}\,')\psi(\vec{x}\,')\psi(\vec{x}) \qquad (27)\\ &= \sum_{\substack{n,k\\ n',k'}} \sum_{\substack{n_1,\cdots,n_4\\ l_1,\cdots,l_4}} (\overline{n_1 l_1}; n_2 l_2; \overline{nk})\, \langle nk|V|\overline{n'k'}\rangle\, (n'k'; \overline{n_3 l_3}; n_4 l_4)\, a^+_{n_1,l_1} a^+_{n_3,l_3} a_{n_4,l_4} a_{n_2,l_2} \end{aligned}$$

where we used the notation

$$\langle nk|V|\overline{n'k'}\rangle \;:=\; \int d^2x \int d^2x' \varphi_{n,k}(\vec{x})\, V(\vec{x}-\vec{x}\,')\, \bar{\varphi}_{n',k'}(\vec{x}\,') \tag{28}$$

$$(\overline{n_1 l_1}; n_2 l_2; \overline{nk}) \;:=\; \int d^2x\, \bar{\varphi}_{n_1,l_1}(\vec{x})\, \varphi_{n_2,l_2}(\vec{x})\, \bar{\varphi}_{n,k}(\vec{x}) \tag{29}$$

Now we consider systems with fillings

$$\nu \;=\; \frac{N}{M} \;<\; 1 \tag{30}$$

and restrict the electrons to the lowest Landau level. Since the kinetic energy is constant, we consider only the interacting part,

$$\begin{aligned} H_{\rm LL} &:= P_{\rm LL} H_{\rm int} P_{\rm LL} \\ &= \sum_{\substack{n,k\\ n',k'}} \sum_{l_1,\cdots,l_4} (\overline{0 l_1}; 0 l_2; \overline{nk})\, \langle nk|V|\overline{n'k'}\rangle\, (n'k'; \overline{0 l_3}; 0 l_4)\, a^+_{l_1} a^+_{l_3} a_{l_4} a_{l_2} \qquad (31) \end{aligned}$$

where we abbreviated

$$a_l := a_{0,l}\,, \quad a^+_l := a^+_{0,l} \tag{32}$$

(iii) The Approximation

The matrix element $\langle nk|V|\overline{n'k'}\rangle$ is computed in the appendix. For a gaussian interaction (2) the exact result is

$$\langle n,k|V|\overline{n',k'}\rangle = \sqrt{2\pi}\ell_B\, \delta_{k,k'}\, \lambda\, r\, [e^{-\frac{r^2}{2}k^2}]_M \int ds \int ds' h_n(s)\, h_{n'}(s')\, e^{-\frac{\ell_B^2}{2r^2}(s-s')^2} \tag{33}$$

where, if $k = 2\pi m/L_x$,

$$[e^{-\frac{r^2}{2}k^2}]_M := \sum_{j=-\infty}^{\infty} e^{-\frac{r^2}{2}(k-jK)^2} = \sum_{j=-\infty}^{\infty} e^{-\frac{r^2}{2}\left[\frac{2\pi}{L_x}(m-jM)\right]^2} \tag{34}$$

is an M-periodic function (as a function of m). For a long range interaction $r >> \ell_B$, we may approximate this by

$$\begin{aligned} \langle n,k|V|\overline{n',k'}\rangle &\approx \sqrt{2\pi}\ell_B\,\delta_{k,k'}\,\lambda\,r\,[e^{-\frac{r^2}{2}k^2}]_M \int ds\, h_n(s)\int ds'\,h_{n'}(s') \\ &=: \delta_{k,k'}\,v_k\int ds\,h_n(s)\int ds'\,h_{n'}(s') \end{aligned} \tag{35}$$

Then $H_{\rm LL}$ becomes

$$\begin{aligned} H_{\rm LL} &= \sum_{\substack{n,n'\\k}}\sum_{l_1,\cdots,l_4} (\overline{0l_1};0l_2;\overline{nk})\,v_k\int h_n(s)ds\int h_{n'}(s')ds'\,(n'k;\overline{0l_3};0l_4)\,a^+_{l_1}a^+_{l_3}a_{l_4}a_{l_2} \\ &= \sum_{k}\sum_{l_1,\cdots,l_4} (\overline{0l_1};0l_2;\overline{1_yk})\,v_k\,(1_yk;\overline{0l_3};0l_4)\,a^+_{l_1}a^+_{l_3}a_{l_4}a_{l_2} \end{aligned} \tag{36}$$

Here we used that

$$\sum_{n=0}^{\infty} h_n(y)\int h_n(s)ds = 1 \tag{37}$$

which is a consequence of $\sum_{n=0}^{\infty} h_n(y)\,h_n(s) = \delta(y-s)$. Thus

$$\begin{aligned} &\sum_{n=0}^{\infty}\bar\varphi_{n,k}(x,y)\int h_n(s)ds \\ &\quad= \tfrac{1}{\sqrt{L_x\ell_B}}\sum_{j=-\infty}^{\infty} e^{-i(k+jK)x}\sum_{n=0}^{\infty} h_n\big((y-y_k-j\,L_y)/\ell_B\big)\int h_n(s)ds \\ &\quad= \tfrac{1}{\sqrt{L_x\ell_B}}\sum_{j=-\infty}^{\infty} e^{-i(k+jK)x} \end{aligned} \tag{38}$$

and (36) follows if we define

$$(\overline{0l_1};0l_2;\overline{1_yk}) := \int dxdy\,\bar\varphi_{0,l_1}(x,y)\,\varphi_{0,l_2}(x,y)\tfrac{1}{\sqrt{L_x\ell_B}}\sum_{j=-\infty}^{\infty} e^{-i(k+jK)x} \tag{39}$$

These matrix elements are also computed in the appendix and the result is

$$(\overline{0l_1};0l_2;\overline{1_yk}) = \delta^M_{m,m_2-m_1}\,\tfrac{1}{\sqrt{L_x\ell_B}}\,[e^{-\frac{\ell_B^2}{4}(l_1-l_2)^2}]_M \tag{40}$$

if $k = \frac{2\pi}{L_x} m$, $l_j = \frac{2\pi}{L_x} m_j$ and $\delta^M_{m_1,m_2} = 1$ iff $m_1 = m_2 \bmod M$. In the following we write, by a slight abuse of notation, also $\delta^M_{l,l'}$ if $l = \frac{2\pi}{L_x} m$. Then the Hamiltonian (36) becomes

$$\begin{aligned} H_{\mathrm{LL}} &= \tfrac{1}{L_x \ell_B} \sum_k \sum_{l_1,\cdots,l_4} \delta^M_{k,l_2-l_1} \, [e^{-\frac{\ell_B^2}{4}(l_1-l_2)^2}]_M \, v_k \, \delta^M_{k,l_3-l_4} \, [e^{-\frac{\ell_B^2}{4}(l_3-l_4)^2}]_M \, a^+_{l_1} a^+_{l_3} a^+_{l_4} a_{l_2} \\ &= \tfrac{1}{L_x} \sum_{l_1,\cdots,l_4} \delta^M_{l_2-l_1,l_3-l_4} \, w_{l_2-l_1} \, a^+_{l_1} a^+_{l_3} a_{l_4} a_{l_2} \end{aligned} \tag{41}$$

where the interaction is given by

$$w_k := \sqrt{2\pi}\, \lambda \, r \, [e^{-\frac{r^2}{2}k^2}]_M [e^{-\frac{\ell_B^2}{4}k^2}]^2_M \tag{42}$$

(iv) Diagonalization

Apparently (41) looks like a usual one dimensional many body Hamiltonian in momentum space. Thus, since the kinetic energy is constant, we can easily diagonalize it by taking the Fourier transform. For $1 \leq n \leq M$ let

$$\psi_n := \tfrac{1}{\sqrt{M}} \sum_{m=1}^M e^{2\pi i \frac{nm}{M}} a_m, \qquad \psi^+_n = \tfrac{1}{\sqrt{M}} \sum_{m=1}^M e^{-2\pi i \frac{nm}{M}} a^+_m \tag{43}$$

or

$$a_m = \tfrac{1}{\sqrt{M}} \sum_{n=1}^M e^{-2\pi i \frac{nm}{M}} \psi_n, \qquad a^+_m = \tfrac{1}{\sqrt{M}} \sum_{n=1}^M e^{2\pi i \frac{nm}{M}} \psi^+_n \tag{44}$$

Substituting this in (41), we get

$$H_{\mathrm{LL}} = \sum_{n,n'} \psi^+_n \psi^+_{n'} \, W(n-n') \, \psi_{n'} \psi_n \tag{45}$$

with an interaction

$$W(n) = \tfrac{1}{L_x} \sum_{m=1}^M e^{2\pi i \frac{nm}{M}} w_m \tag{46}$$

where $w_m \equiv w_k$ is given by (42), $k = 2\pi m / L_x$. The N-particle eigenstates of (45) are labelled by N-tupels $(n_1, \cdots, n_N)$ where $1 \leq n_j \leq M$ and $n_1 < n_2 < \cdots < n_N$ and are given by

$$\begin{aligned} \Psi_{n_1 \cdots n_N} &= \psi^+_{n_1} \psi^+_{n_2} \cdots \psi^+_{n_N} |\mathbf{1}\rangle \\ &= \tfrac{1}{M^{N/2}} \sum_{j_1 \cdots j_N} e^{-\frac{2\pi i}{M}(n_1 j_1 + \cdots n_N j_N)} \, a^+_{j_1} \cdots a^+_{j_N} |\mathbf{1}\rangle \\ &= \tfrac{1}{M^{N/2}} \sum_{j_1 \cdots j_N} e^{-\frac{2\pi i}{M}(n_1 j_1 + \cdots n_N j_N)} \, \varphi_{0 j_1} \wedge \cdots \wedge \varphi_{0 j_N} \\ &= \phi_{n_1} \wedge \cdots \wedge \phi_{n_N} \end{aligned} \tag{47}$$

if we define

$$\phi_n(x,y) := \tfrac{1}{\sqrt{M}} \sum_{j=1}^{M} e^{-2\pi i \frac{nj}{M}} \varphi_{0j}(x,y) \tag{48}$$

The energy eigenvalues are

$$H_{\rm LL} \Psi_{n_1 \cdots n_N} = E_{n_1 \cdots n_N} \Psi_{n_1 \cdots n_N} \tag{49}$$

where

$$E_{n_1 \cdots n_N} = \sum_{\substack{i,j=1 \\ i \neq j}}^{N} W(n_i - n_j) \tag{50}$$

The Fourier sums in (46) and (48) can be performed with the Poisson summation formula. This is done in Lemma 3 in the appendix. If we approximate $w_k \approx \sqrt{2\pi}\,\lambda\, r\, [e^{-\frac{r^2}{2}k^2}]_M$, since by assumption $r >> \ell_B$, we find for this w_k

$$\begin{aligned} W(n) &= \lambda \sum_{j \in \mathbb{Z}} e^{-\frac{1}{2r^2}\left(L_x \frac{n}{M} - j L_x\right)^2} \\ &= \lambda \sum_{j \in \mathbb{Z}} e^{-\frac{1}{2r^2}\left(\ell_B^2 \frac{2\pi n}{L_y} - j L_x\right)^2} \end{aligned} \tag{51}$$

Thus the theorem is proven ■

We close with some comments. First, the approximation (3) looks quite innocent. However, by reviewing the computations one finds that it is actually equivalent to the approximation $V(x,y) = \lambda\, e^{-\frac{x^2+y^2}{2r^2}} \approx \lambda\, e^{-\frac{x^2}{2r^2}}$. Thus we substitute an integrable interaction by a non integrable, non symmetric one. As a consequence, the total energy is no longer proportional to the volume (if the density is kept fixed) but grows as $L \times L^2$. Second, one may speculate that for fillings $\nu = 1/q$ the ground states are labelled by the N-tupels $(n_1, \cdots, n_N) = \big(j, j+q, j+2q, \cdots, j+(N-1)q\big)$ which have a q-fold degeneracy, $1 \leq j \leq q$. A q-fold degeneracy for fillings $\nu = p/q$, p, q without common devisor, follows already from general symmetry considerations (see [1] or [2]). One may also conjecture that there is a gap for rational fillings, that is

$$\Delta(\nu) := \lim_{\substack{N,M\to\infty \\ N/M=\nu}} \big(E_1(N,M) - E_0(N,M)\big) \begin{cases} > 0 & \text{if } \nu \in \mathbb{Q} \\ = 0 & \text{if } \nu \notin \mathbb{Q} \end{cases} \tag{52}$$

Here E_0 is the lowest and E_1 the second lowest eigenvalue in finite volume. However, we find it hard to imagine that the energy (6) distinguishes between even and odd denominators q. In other words, it is questionable whether the model (1,2) favours the observed fractional quantum Hall fillings (see [3, 4, 5] for an overview).

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Appendix

Lemma 1: *Let* $V(x,y) := \lambda\, e^{-\frac{x^2+y^2}{2r^2}}$. *Then one has*

$$\langle nk|V|\overline{n'k'}\rangle = \sqrt{2\pi}\ell_B\,\delta_{k,k'}\,\lambda\, r\,[e^{-\frac{r^2}{2}k^2}]_M \int ds \int ds'\, h_n(s)\, h_{n'}(s')\, e^{-\frac{\ell_B^2}{2r^2}(s-s')^2} \tag{53}$$

where, if $k = 2\pi m/L_x$,

$$[e^{-\frac{r^2}{2}k^2}]_M := \sum_{j=-\infty}^{\infty} e^{-\frac{r^2}{2}(k-jK)^2} = \sum_{j=-\infty}^{\infty} e^{-\frac{r^2}{2}\left[\frac{2\pi}{L_x}(m-jM)\right]^2} \tag{54}$$

is an M*-periodic function (as a function of* m*).*

Proof: We have

$$\langle n,k|V|\overline{n',k'}\rangle = \int d^2x\, d^2x'\, \varphi_{n,k}(\vec{x})\, V(\vec{x}-\vec{x}')\, \bar{\varphi}_{n',k'}(\vec{x}')$$

$$\begin{aligned}
&= \frac{1}{L_x\ell_B}\sum_{j,j'}\int dxdx'dydy'\, e^{i(k-jK)x-i(k'-j'K)x'} h_{n,k}(y-jL_y)\, h_{n',k'}(y'-j'L_y)\, V(\vec{x}-\vec{x}') \\
&= \frac{1}{L_x\ell_B}\sum_{j,j'}\int dxdx'dydy'\, e^{i(k-jK)(x-x')} e^{i(k-jK-k'+j'K)x'} h_n\big((y-y_k-jL_y)/\ell_B\big)\,\times \\
&\qquad h_{n'}\big((y'-y_{k'}-j'L_y)/\ell_B\big)\,\lambda\, e^{-\frac{(x-x')^2}{2r^2}} e^{-\frac{(y-y')^2}{2r^2}}
\end{aligned} \tag{55}$$

The x'-integral gives $L_x\,\delta_{m-jM,m'-j'M} = L_x\,\delta_{m,m'}\delta_{j,j'}$ if $k = 2\pi m/L_x$, $k' = 2\pi m'/L_x$, $0 \le m, m' \le M-1$. Thus we get

$$\begin{aligned}
\langle n,k|V|\overline{n',k'}\rangle &= \frac{\lambda}{\ell_B}\,\delta_{k,k'}\sum_j \int dx\, e^{i(k-jK)x} e^{-\frac{x^2}{2r^2}} \int dydy' h_n\big((y-y_k-jL_y)/\ell_B\big)\,\times \\
&\qquad h_{n'}\big((y'-y_k-jL_y)/\ell_B\big)\, e^{-\frac{(y-y')^2}{2r^2}} \\
&= \sqrt{2\pi}\ell_B\lambda r\,\delta_{k,k'}\sum_j e^{-\frac{r^2}{2}(k-jK)^2}\int dsds' h_n(s)\, h_{n'}(s')\, e^{-\frac{\ell_B^2}{2r^2}(s-s')^2} \\
&= \sqrt{2\pi}\ell_B\lambda r\,\delta_{k,k'}\,[e^{-\frac{r^2}{2}k^2}]_M \int dsds' h_n(s)\, h_{n'}(s')\, e^{-\frac{\ell_B^2}{2r^2}(s-s')^2}
\end{aligned} \tag{56}$$

and the lemma follows ■

Lemma 2: *The matrix elements (39) are given by*

$$(\overline{0,l_1};0,l_2;\overline{1_y,k}) = \delta^M_{m,m_2-m_1}\frac{1}{\sqrt{L_x\ell_B}}\,[e^{-\frac{\ell_B^2}{4}(l_1-l_2)^2}]_M \tag{57}$$

if $k=\frac{2\pi}{L_x}m,\ l_j=\frac{2\pi}{L_x}m_j$ *and* $\delta^M_{m_1,m_2}=1$ *iff* $m_1=m_2 \ mod\ M$.

Proof: We have

$$(\overline{0,l_1};0,l_2;\overline{1_y,k}) = \frac{1}{\sqrt{L_x\ell_B}^3}\sum_{j_1,j_2,j}\int dxdy\, e^{-i(l_1+j_1K)x}e^{i(l_2+j_2K)x}e^{-i(k+jK)x}h_{0,l_1}(y)\,h_{0,l_2}(y) \tag{58}$$

The plane waves combine to

$$\exp\left[i\tfrac{2\pi}{L_x}(m_2+j_2M-m_1-j_1M-m-jM)x\right] \tag{59}$$

and the x-integral gives a volume factor L_x times a Kroenecker delta which is one iff

$$m_2+j_2M-m_1-j_1M-m-jM=0 \tag{60}$$

or

$$\begin{aligned} m=m_2-m_1 \quad &\wedge \quad j=j_2-j_1 \quad \text{if}\ \ m_2\geq m_1\\ m=m_2-m_1+M \quad &\wedge \quad j=j_2-j_1-1 \quad \text{if}\ \ m_2<m_1 \end{aligned} \tag{61}$$

Thus (58) becomes

$$\begin{aligned}
(\overline{0,l_1};0,l_2;\overline{1_y,k}) &=\\
&\delta^M_{m,m_2-m_1}\frac{1}{\sqrt{\ell_B}^3}\frac{1}{\sqrt{L_x}}\sum_{j_1,j_2}\int_0^{L_y}dy\,h_0\big((y-y_{l_1}-j_1L_y)/\ell_B\big)\,h_0\big((y-y_{l_2}-j_2L_y)/\ell_B\big)\\
=\ &\delta^M_{m,m_2-m_1}\frac{1}{\sqrt{\ell_B}^3}\frac{1}{\sqrt{L_x}}\sum_{j_1,j_2}\int_0^{L_y}dy\,h_0\big((y-y_{l_1}-j_1L_y)/\ell_B\big)\times\\
&\qquad\qquad h_0\big((y-y_{l_2}-j_1L_y+(j_1-j_2)L_y)/\ell_B\big)\\
=\ &\delta^M_{m,m_2-m_1}\frac{1}{\sqrt{\ell_B}^3}\frac{1}{\sqrt{L_x}}\sum_{j_1,j}\int_0^{L_y}dy\,h_0\big((y-y_{l_1}-j_1L_y)/\ell_B\big)\,h_0\big((y-y_{l_2}-j_1L_y+jL_y)/\ell_B\big)\\
=\ &\delta^M_{m,m_2-m_1}\frac{1}{\sqrt{\ell_B}^3}\frac{1}{\sqrt{L_x}}\sum_{j}\int_{-\infty}^{\infty}dy\,h_0\big((y-y_{l_1})/\ell_B\big)\,h_0\big((y-y_{l_2}+jL_y)/\ell_B\big)\\
=\ &\delta^M_{m,m_2-m_1}\frac{1}{\sqrt{L_x\ell_B}}\sum_j e^{-\frac{1}{4\ell_B^2}(y_{l_1}-y_{l_2}+jL_y)^2}\\
=\ &\delta^M_{m,m_2-m_1}\frac{1}{\sqrt{L_x\ell_B}}\sum_j e^{-\frac{\ell_B^2}{4}(l_1-l_2+jK)^2} &(62)\\
=\ &\delta^M_{m,m_2-m_1}\frac{1}{\sqrt{L_x\ell_B}}\,[e^{-\frac{\ell_B^2}{4}(l_1-l_2)^2}]_M &(63)
\end{aligned}$$

where $\delta^M_{m,m'}$ equals one iff $m = m'$ mod M and equals zero otherwise ■

Lemma 3: (i) *For $m \in \mathbb{Z}$ let*

$$v_m = [e^{-\frac{r^2}{2}k^2}]_M = \sum_{j\in\mathbb{Z}} e^{-\frac{r^2}{2}(\frac{2\pi}{L_x})^2(m-jM)^2} \tag{64}$$

and let $V(n) = \frac{1}{L_x}\sum_{m=1}^M e^{2\pi i\frac{nm}{M}} v_m$. Then

$$V(n) = \frac{1}{\sqrt{2\pi}\,r}\sum_{j\in\mathbb{Z}} e^{-\frac{1}{2r^2}\left(L_x\frac{n}{M} - jL_x\right)^2} \tag{65}$$

(ii) *Let $k = 2\pi m/L_x$ and let $\varphi_{0,m} \equiv \varphi_{0,k}$ be the single body eigenfunction (18). Then*

$$\frac{1}{\sqrt{M}}\sum_{m=1}^{M} e^{-2\pi i\frac{nm}{M}}\varphi_{0,m}(x,y) \;=\; \frac{\pi^{-\frac14}}{\sqrt{\ell_B L_y}}\sum_{s=-\infty}^{\infty} e^{-\frac{1}{2\ell_B^2}\left(x-\frac{n}{M}L_x - sL_x\right)^2} e^{i\frac{(x-\frac{n}{M}L_x - sL_x)y}{\ell_B^2}} \tag{66}$$

$$= \frac{1}{\sqrt{M}}\frac{\pi^{-\frac14}}{\sqrt{\ell_B L_x}}\sum_{r=-\infty}^{\infty} e^{-\frac{1}{2\ell_B^2}\left(y-\frac{r}{M}L_y\right)^2} e^{i\left(x-\frac{n}{M}L_x\right)\frac{r}{M}L_y/\ell_B^2} \tag{67}$$

Proof: (i) We have

$$[e^{-\frac{r^2}{2}k^2}]_M \;=\; \sum_{j\in\mathbb{Z}} e^{-\frac{r^2}{2}\frac{M^2}{L_x^2}\left(2\pi\frac{m}{M} - 2\pi j\right)^2} \tag{68}$$

We use the following formula which is obtained from the Poisson summation theorem

$$\sum_{j\in\mathbb{Z}} e^{-\frac{1}{2t}(x-2\pi j)^2} = \sqrt{\tfrac{t}{2\pi}}\sum_{j\in\mathbb{Z}} e^{-\frac{t}{2}j^2} e^{ijx} \tag{69}$$

with

$$x = 2\pi\frac{m}{M}\,, \qquad t = \frac{L_x^2}{r^2M^2} \tag{70}$$

Then

$$v_m \;=\; \frac{1}{\sqrt{2\pi}\,r}\,\frac{L_x}{M}\sum_{j\in\mathbb{Z}} e^{-\frac{1}{2r^2}\frac{L_x^2}{M^2}j^2} e^{-2\pi i\frac{jm}{M}} \tag{71}$$

and $V(n)$ becomes

$$\begin{aligned}
V(n) &= \tfrac{1}{L_x}\sum_{m=1}^{M} e^{2\pi i\frac{nm}{M}}\tfrac{1}{\sqrt{2\pi}\,r}\,\tfrac{L_x}{M}\,\sum_{j\in\mathbb{Z}} e^{-\frac{1}{2r^2}\frac{L_x^2}{M^2}j^2}e^{-2\pi i\frac{jm}{M}} \\
&= \tfrac{1}{\sqrt{2\pi}\,r}\,\tfrac{1}{M}\,\sum_{j\in\mathbb{Z}} e^{-\frac{1}{2r^2}\frac{L_x^2}{M^2}j^2}\sum_{m=1}^{M} e^{2\pi i\frac{(n-j)m}{M}} \\
&= \tfrac{1}{\sqrt{2\pi}\,r}\,\tfrac{1}{M}\,\sum_{j\in\mathbb{Z}} e^{-\frac{1}{2r^2}\frac{L_x^2}{M^2}j^2}\,M\delta_{n,j}^{M} \\
&= \tfrac{1}{\sqrt{2\pi}\,r}\,\sum_{s\in\mathbb{Z}} e^{-\frac{1}{2r^2}\frac{L_x^2}{M^2}(n-sM)^2} \qquad (72)
\end{aligned}$$

which proves part (i).
(ii) According to (18) we have

$$\begin{aligned}
&\tfrac{1}{\sqrt{M}}\sum_{m=1}^{M} e^{-2\pi i\frac{nm}{M}}\varphi_{0,m}(x,y) = \\
&\qquad \tfrac{1}{\sqrt{M}}\sum_{m=1}^{M} e^{-2\pi i\frac{nm}{M}}\tfrac{1}{\sqrt{L_x\ell_B}}\sum_{j=-\infty}^{\infty} e^{i\frac{2\pi}{L_x}(m+jM)x}h_{0,k}(y-jL_y) \\
&= \tfrac{\pi^{-\frac14}}{\sqrt{ML_x\ell_B}}\sum_{j=-\infty}^{\infty}\sum_{m=1}^{M} e^{i\frac{2\pi}{L_x}(x-\frac{n}{M}L_x)m}e^{i\frac{2\pi}{L_x}jMx}e^{-\frac{1}{2\ell_B^2}\left(y-\ell_B^2\frac{2\pi}{L_x}m-jL_y\right)^2} \\
&= \tfrac{\pi^{-\frac14}}{\sqrt{ML_x\ell_B}}\sum_{m=1}^{M}\sum_{j=-\infty}^{\infty} e^{i\frac{2\pi}{L_x}(x-\frac{n}{M}L_x)m}e^{i\frac{L_y}{\ell_B^2}jx}\tfrac{1}{\sqrt{2\pi}}\int dq\, e^{-\frac{q^2}{2}}e^{iq\left(\frac{y}{\ell_B}-\ell_B\frac{2\pi}{L_x}m-j\frac{L_y}{\ell_B}\right)} \\
&= \tfrac{\pi^{-\frac14}}{\sqrt{ML_x\ell_B}}\tfrac{1}{\sqrt{2\pi}}\int dq\, e^{-\frac{q^2}{2}}e^{iq\frac{y}{\ell_B}}\sum_{m=1}^{M} e^{i\frac{2\pi}{L_x}(x-\frac{n}{M}L_x-q\ell_B)m}\sum_{j=-\infty}^{\infty} e^{i\left(\frac{x}{\ell_B}-q\right)\frac{L_y}{\ell_B}j} \\
&= \tfrac{\pi^{-\frac14}}{\sqrt{ML_x\ell_B}}\tfrac{1}{\sqrt{2\pi}}\int dq\, e^{-\frac{q^2}{2}}e^{iq\frac{y}{\ell_B}}\sum_{m=1}^{M} e^{i\frac{2\pi}{L_x}(x-\frac{n}{M}L_x-q\ell_B)m}\sum_{r=-\infty}^{\infty} 2\pi\,\delta\left((\tfrac{x}{\ell_B}-q)\tfrac{L_y}{\ell_B}-2\pi r\right) \quad (73)
\end{aligned}$$

The delta-function forces q to take values

$$q = \tfrac{x}{\ell_B} - 2\pi r\tfrac{\ell_B}{L_y} \qquad (74)$$

which gives

$$x - \tfrac{n}{M}L_x - q\ell_B = -\tfrac{n}{M}L_x + 2\pi r\tfrac{\ell_B^2}{L_y} = \tfrac{r-n}{M}\,L_x \qquad (75)$$

Therefore the m-sum in (73) becomes

$$\sum_{m=1}^{M} e^{i\frac{2\pi}{L_x}(x-\frac{n}{M}L_x-q\ell_B)m} = \sum_{m=1}^{M} e^{2\pi i\frac{(r-n)m}{M}} = M\,\delta_{r,n}^{M} \qquad (76)$$

and we get

$$
\begin{aligned}
&\frac{1}{\sqrt{M}}\sum_{m=1}^{M} e^{-2\pi i\frac{nm}{M}}\varphi_{0,m}(x,y) = \\
&\qquad \frac{\pi^{-\frac{1}{4}}\sqrt{M}}{\sqrt{L_x\ell_B}}\sqrt{2\pi}\frac{\ell_B}{L_y}\sum_{r=-\infty}^{\infty} e^{-\frac{1}{2\ell_B^2}\left(x-r\frac{2\pi\ell_B^2}{L_y}\right)^2} e^{i\left(\frac{x}{\ell_B}-2\pi r\frac{\ell_B}{L_y}\right)\frac{y}{\ell_B}}\,\delta^M_{r,n} \\
&= \frac{\pi^{-\frac{1}{4}}}{\sqrt{\ell_B L_y}}\sum_{s=-\infty}^{\infty} e^{-\frac{1}{2\ell_B^2}\left(x-(n+sM)\frac{2\pi\ell_B^2}{L_y}\right)^2} e^{i\left(\frac{x}{\ell_B}-2\pi(n+sM)\frac{\ell_B}{L_y}\right)\frac{y}{\ell_B}} \\
&= \frac{\pi^{-\frac{1}{4}}}{\sqrt{\ell_B L_y}}\sum_{s=-\infty}^{\infty} e^{-\frac{1}{2\ell_B^2}\left(x-\frac{n}{M}L_x-sL_x\right)^2} e^{i\frac{(x-\frac{n}{M}L_x-sL_x)y}{\ell_B^2}} \qquad (77)
\end{aligned}
$$

This proves (66). (67) is obtained directly from (18) by putting $r = m + jM$. ■

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