The Periodic Multidimensional Schrodinger Operator, Part 1, Asymptotic Formulae for Eigenvalues

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Abstract

In this paper we obtain asymptotic formulas of arbitrary order for the eigenvalues of the d-dimensional periodic Schrodinger operator $L(q) = -\Delta + q(x)$ for arbitrary dimension d. The eigenvalues, whose asymptotic formulas are connected with Sturm-Liouville operators, are considered in detail.

1 Introduction

In this paper we consider the Schrodinger operator

$$L(q) = -\Delta + q(x),\ x \in R^d,\ d \geq 2 \tag{1}$$

with a periodic (relative to a lattice Ω) potential $q(x) \in W_2^l(F)$, where $l \geq 6(3^d(d+1)^2) + d$, $F \equiv R^d/\Omega$ is a fundamental domain of Ω, and without loss of generality we assume that the measure $\mu(F)$ of F is 1. Let $L_t(q)$ be the operator generated in F by (1) and the quasiperiodic conditions:

$$u(x+\omega) = e^{i(t,\omega)}u(x),\ \forall \omega \in \Omega, \tag{2}$$

where $t \in F^* \equiv R^d/\Gamma$ and Γ is the lattice dual to Ω, that is, Γ is the set of all vectors $\gamma \in R^d$ satisfying $(\gamma, \omega) \in 2\pi Z$ for all $\omega \in \Omega$. The operator $L_t(q)$ has a discrete spectrum denoted by $\Lambda_1(t) \leq \Lambda_2(t) \leq ....$These functions are called band functions of the operator L and they form the spectrum $Spec(L)$ of the operator L (see [1]), that is, $Spec(L) = \{\Lambda_n(t) : t \in F^*, n = 1, 2, ...\}$. The corresponding eigenfunctions $\Psi_{1,t}(x),\ \Psi_{2,t}(x), ...,$ of L_t are known as Block functions. In the case $q(x) = 0$ these eigenvalues and eigenfunctions are $\mid \gamma + t \mid^2$ and $e^{i(\gamma+t,x)}$ respectively, for $\gamma \in \Gamma$.

In papers [2-6] for the first time the eigenvalues $|\gamma + t|^2$, for big $\gamma \in \Gamma$, were divided into two groups: non-resonance ones and resonance ones and for the perturbations of each group various asymptotic formulae were obtained. To describe the non-resonance and resonance eigenvalues $|\gamma + t|^2$ of the order of ρ^2 (written as $|\gamma + t|^2 \sim \rho^2$ which means that $c_1\rho^2 < |\gamma + t|^2 < c_2\rho^2$,where by c_i for $i = 1, 2, ...$ we denote the positive, independent of ρ, constants, whose exact values are inessential) for big paramater ρ we write the potential $q(x) \in W_2^l$ in the form

$$q(x) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} e^{i(\gamma_1,x)} + O(\rho^{-p\alpha}), \tag{3}$$

where $\Gamma(\rho^\alpha) = \{\gamma \in \Gamma : 0 < \mid \gamma \mid < \rho^\alpha)\}$, $p = l - d$, $q_0 = 0$, $\alpha = \frac{1}{q}$,
$q = 4(3^d(d + 1)) + 1$. Note that $q(x) \in W_2^l$ means that
$\sum_\gamma \mid q_\gamma \mid^2 (1+ \mid \gamma \mid^{2l}) < \infty$. If $l \geq d$ then

$$\sum_\gamma \mid q_\gamma \mid < c_3,\ \ \sup \mid \sum_{\gamma \notin \Gamma(\rho^\alpha)} q_\gamma e^{i(\gamma,x)} \mid \leq \sum_{|\gamma| \geq \rho^\alpha} \mid q_\gamma \mid = O(\rho^{-p\alpha}), \tag{4}$$

i.e., (3) holds. Define $\alpha_k = 3^k\alpha$, for $k = 1, 2, ..., d-1$, and introduce the sets
$V_{\gamma_1}(\rho^{\alpha_1}) \equiv \{x \in R^d : \mid\mid x \mid^2 - \mid x + \gamma_1 \mid^2 \mid < \rho^{\alpha_1}\}$,

$$E_1(\rho^{\alpha_1}, p) \equiv \bigcup_{\gamma_1 \in \Gamma(p\rho^\alpha)} V_{\gamma_1}(\rho^{\alpha_1}),\ \ U(\rho^{\alpha_1}, p) \equiv R^d \backslash E_1(\rho^{\alpha_1}, p)$$

$$E_k(\rho^{\alpha_k}, p) \equiv \bigcup_{\gamma_1,\gamma_2,...,\gamma_k \in \Gamma(p\rho^\alpha)} (\cap_{i=1}^k V_{\gamma_i}(\rho^{\alpha_k})),$$

where the intersection $\cap_{i=1}^k V_{\gamma_i}$ in the definition of E_k is taken over $\gamma_1, \gamma_2, ..., \gamma_k$ which are linearly independent and the length of γ_i is not greater than the length of the other vectors in $\Gamma \cap \gamma_i R$ (see Remark 1). The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain and the eigenvalue $\mid \gamma + t \mid^2$ is called a non-resonance eigenvalue if $\gamma + t \in U(\rho^{\alpha_1}, p)$, i.e., if $\gamma + t$ is far from hyperplanes $\{x : \mid x \mid^2 = \mid x + \gamma_1 \mid^2\}$ for all $\gamma_1 \in \Gamma(p\rho^{\alpha})$. The domains $V_{\gamma_1}(\rho^{\alpha_1})$ for $\gamma_1 \in \Gamma(p\rho^{\alpha})$ are called resonance domains and the eigenvalue $\mid \gamma + t \mid^2$ is called a resonance eigenvalue if $\gamma + t \in V_{\gamma_1}(\rho^{\alpha_1})$. We proved that for every $\gamma + t \in U(\rho^{\alpha_1}, p)$ there exists an eigenvalue Λ_N of the operator $L_t(q)$ satisfying the following formulae

$$\Lambda_N = \mid \gamma + t \mid^2 + F_{k-1}(\gamma + t) + O(\rho^{-6k\alpha}), \tag{5}$$

for $k = 1, 2, ..., [\frac{1}{6}(p - \frac{1}{2}q(d-1))]$, where F_{k-1} explicitly expresed by the potential and eigenvalues of $L_t(0)$ and $[a]$ denotes the integer part of a. Moreover we renumbered the eigenvalues $\Lambda_N(t)$ by $\Lambda_{N(\gamma+t)}(t) \equiv \Lambda(\gamma + t)$ if

$$\mid b(N, \gamma) \mid = \max_{\gamma \in \Gamma} \mid b(N, \gamma) \mid, \; b(N, \gamma) = (\Psi_{N,t}(x), e^{i(\gamma + t, x)}), \tag{6}$$

where the max is taken under condition (11) (in the case when the max is attained for several γ, then the selection from them is arbitrary), $\Psi_{N,t}(x)$ is a normalized eigenfunction of $L_t(q)$ corresponding to $\Lambda_N(t)$, and proved:

(a) if $\gamma + t$ is in non-resonance domain then $\Lambda(\gamma + t)$ satisfies (5).

(b) if $\gamma + t \in E_s \backslash E_{s+1}$, where $s = 1, 2, ..., d-1$, then
$\Lambda(\gamma + t) = \lambda_j(\gamma + t) + O(\rho^{-k\alpha})$,

where λ_j is an eigenvalue of the matrix $C(\gamma + t)$ (more precisely about this see Theorem 1(c)). In case $s = 1$, $k = 1$ this matrix corresponds to Sturm-Liouville operator (see [3-5]). Every big eigenvalue of the operator $L_t(q)$ for all value of quasimomentum t satisfies one of these formulae (see Theorem 1(d)), since big $\gamma + t$ satisfies either (a) or (b) (see (36)).

In papers [8,9], [10,11], and [13,14] using different methods asymptotic formulas for eigenvalues of the two and three dimensional operator $L_t(q)$ were obtained. In [12] asymptotic formulae for nonresonance eigenvalues of $L_0(q)$ were obtained.

In this paper we consider in detail the single resonance domains

$V'_{\gamma_1}(\rho^{\alpha_1}) \equiv V_{\gamma_1}(\rho^{\alpha_1}) \backslash E_2$, i.e., the part of the resonance domains $V_{\gamma_1}(\rho^{\alpha_1})$ which does not contain the intersection of two resonance domains. Namely, for this case we obtain asymptotic formulas of arbitrary order for the eigenvalues of the d-dimensional periodic Schrodinger operator $L(q)$ for arbitrary dimension d. This case is connected with Sturm-Lionville operators. In the next papers (in Part 3) which use this we will constructively determine a family of spectral invariants by given Floquet spectrum. Then (in Part 4) by this spectral invariants we constructively determine and give the rules for the algorithmic calculations of the potential $q(x)$. Thus the new results about asymptotic formulas are the results corresponding to the single resonance case (section 3). The nonresonance case, more precisely the results of section 2 (see Theorem 1) is considered in [6]. However in that paper these results are written only briefly. The enlarged variant is written in [7] which can not be used as a reference. Here we write the nonresonance case in an improved and enlarged form and in such form that can easily be used in section 3. Moreover it helps when reading the resonance case and it will be used very much in Part 2, where Bloch functions and isoenergetic surfaces are considered.

In this paper by $\mid A \mid$ we denote the number of elements of the set A. Moreover we use the following obvious fact. If $a \sim \rho$ then the number of elements of the set $\{\gamma + t : \gamma \in \Gamma\}$ satisfying $\mid\mid \gamma + t \mid -a \mid < 1$ is less than $c_\Gamma \rho^{d-1}$, where c_Γ is a constant depending on Γ. Therefore the number of eigenvalues of $L_t(q)$ lying in $(a^2 - \rho, a^2 + \rho)$ is less than $c_\Gamma \rho^{d-1}$.

2 Asymptotic Formulae for Eigenvalues

The asymptotic formulae for nonresonance eigenvalues were obtained in [4-6] by following iteration of the well-known formula

$$(\Lambda_N - \mid \gamma + t \mid^2) b(N, \gamma) = (\Psi_{N,t}(x) q(x), e^{i(\gamma + t, x)}). \tag{7}$$

We call it binding formula for $L_t(q(x))$ and $L_t(0)$. Putting the decomposition (3) of $q(x)$ into (7), we get

$$(\Lambda_N - \mid \gamma + t \mid^2) b(N, \gamma) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} b(N, \gamma - \gamma_1) + O(\rho^{-p\alpha}). \tag{8}$$

¿From the relations (7), (8) it follows that

$$b(N, \gamma') = \frac{(\Psi_{N,t} q(x), e^{i(\gamma' + t, x)})}{\Lambda_N - \mid \gamma' + t \mid^2} = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} \frac{q_{\gamma_1} b(N, \gamma' - \gamma_1)}{\Lambda_N - \mid \gamma' + t \mid^2} + O(\rho^{-p\alpha}) \tag{9}$$

for every vector $\gamma^{'}\in\Gamma$ satisfying the inequality

$$|\Lambda_N-|\gamma^{'}+t|^2|>\frac{1}{2}\rho^{\alpha_1}, \tag{10}$$

which is called the iterability condition. If

$$|\Lambda_N-|\gamma+t|^2|<\frac{1}{2}\rho^{\alpha_1}, \tag{11}$$

and $|\gamma+t|^2$ is a non-resonance eigenvalue, i.e., $\gamma+t\in U(\rho^{\alpha_1},p)$ then

$$||\gamma+t|^2-|\gamma-\gamma_1+t|^2|>\rho^{\alpha_1}, |\Lambda_N-|\gamma-\gamma_1+t|^2|>\frac{1}{2}\rho^{\alpha_1}, \tag{12}$$

for all $\gamma_1\in\Gamma(p\rho^{\alpha})$. Hence the vector $\gamma-\gamma_1$ satisfies the iterability condition (10). Therefore, in (9) one can replace $\gamma^{'}$ by $\gamma-\gamma_1$ and write

$$b(N,\gamma-\gamma_1)=\sum_{\gamma_2\in\Gamma(\rho^{\alpha})}\frac{q_{\gamma_2}b(N,\gamma-\gamma_1-\gamma_2)}{\Lambda_N-|\gamma-\gamma_1+t|^2}+O(\rho^{-p\alpha}).$$

Putting it into right side of (8) and isolating the terms with the coefficient $b(N,\gamma)$, i.e., isolating the case $\gamma_1+\gamma_2=0$ we get

$$(\Lambda_N-|\gamma+t|^2)b(N,\gamma)=\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha})}\frac{q_{\gamma_1}q_{\gamma_2}b(N,\gamma-\gamma_1-\gamma_2)}{\Lambda_N-|\gamma-\gamma_1+t|^2}+O(\rho^{-p\alpha})=$$

$$\sum_{\gamma_1\in\Gamma(\rho^{\alpha})}\frac{|q_{\gamma_1}|^2\,b(N,\gamma)}{\Lambda_N-|\gamma-\gamma_1+t|^2}+\sum_{\gamma_1,\gamma_2\in\Gamma(\rho^{\alpha})}\frac{q_{\gamma_1}q_{\gamma_2}b(N,\gamma-\gamma_1-\gamma_2)}{\Lambda_N-|\gamma-\gamma_1+t|^2}+O(\rho^{-p\alpha}).$$

The last sum is taken under the additional condition $\gamma_1+\gamma_2\neq 0$. Repeating this process $p_1\equiv[\frac{p-1}{2}]$ times, i.e., in last formula replacing $b(N,\gamma^{'})$ for $\gamma^{'}=\gamma-\gamma_1-\gamma_2$, and then for $\gamma^{'}=\gamma-\gamma_1-\gamma_2-\gamma_3, ...,$ by its expression from (9) and isolating each time the terms with coefficient $b(N,\gamma)$ we obtain

$$(\Lambda_N-|\gamma+t|^2)b(N,\gamma)=A_{p_1}(\Lambda_N,\gamma+t)b(N,\gamma)+C_{p_1}+O(\rho^{-p\alpha}), \tag{13}$$

where $A_{p_1}(\Lambda_N,\gamma+t)=\sum_{k=1}^{p_1}S_k(\Lambda_N,\gamma+t)$,

$$S_k(\Lambda_N,\gamma+t)=\sum_{\gamma_1,...,\gamma_k\in\Gamma(\rho^{\alpha})}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_k}q_{-\gamma_1-\gamma_2-...-\gamma_k}}{\prod_{j=1}^{k}(\Lambda_N-|\gamma+t-\sum_{i=1}^{j}\gamma_i|^2)},$$

$$C_{p_1} = \sum_{\gamma_1,...,\gamma_{p_1+1}\in\Gamma(\rho^\alpha)} \frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{p_1+1}}b(N,\gamma-\gamma_1-\gamma_2-...-\gamma_{p_1+1})}{\prod_{j=1}^{p_1}(\Lambda_N- \mid \gamma+t-\sum_{i=1}^{j}\gamma_i \mid^2)}.$$

Here the sums in the expressions of S_k and C_{p_1} are taken under the additional conditions $\gamma_1+\gamma_2+...+\gamma_s\neq 0$ for $s=1,2,...,k$ and for $s=1,2,...,p_1+1$ respectively. These conditions and the inclutions $\gamma_i\in\Gamma(\rho^\alpha)$, for $i=1,2,...,p_1$ give the relation $\sum_{i=1}^{j}\gamma_i\in\Gamma(p\rho^\alpha)$ for the sum in the denominators of the fractions in expressions of S_k and C_{p_1}. Therefore from the relation (12) it follows that the absolute values of the denominators of the fractions in expressions of S_k and C_{p_1} more than $(\frac{1}{2}\rho^{\alpha_1})^k$ and $(\frac{1}{2}\rho^{\alpha_1})^{p_1}$ respectively. Hence the first inequality in (4) and $p_1\alpha_1\geq p\alpha$ (see definition of p_1) give

$$C_{p_1}=O(\rho^{-p_1\alpha_1})=O(\rho^{-p\alpha}),\ S_k(\Lambda_N,\gamma+t)=O(\rho^{-k\alpha_1}),\forall k=1,2,...,p_1. \tag{14}$$

Taking into account that for Λ_N we only used condition (11) we obtain

$$S_k(a,\gamma+t)=O(\rho^{-k\alpha_1}),\forall k=1,2,...,p_1 \tag{15}$$

for all a satisfying $\mid a- \mid\gamma+t\mid^2\mid<\frac{1}{2}\rho^{\alpha_1}$. Thus finding the number N such that Λ_N is close to $\mid\gamma+t\mid^2$ and $b(N,\gamma)$ is not very small then dividing both sides of (13) by $b(N,\gamma)$ we can get the asymptotic formulas for Λ_N (see Theorem 1(a)). These formulas show that in the nonresonance case the eigenvalue of the perturbed operator $L_t(q(x))$ is close to the eigenvalue of the unperturbed operator $L_t(0)$. However in Theorem 1(c) we prove that if $\mid\gamma+t\mid^2$ is resonance eigenvalue, say, if $\gamma+t\in(\cap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k}))\backslash E_{k+1}$ for $k\geq 1$,where $\gamma_1,\gamma_2,...,\gamma_k$ are linearly independent vectors of $\Gamma(p\rho^\alpha)$ and the length of γ_i is not more than the length of other vectors in $\Gamma\cap\gamma_iR$, then the corresponding eigenvalue of $L_t(q(x))$ is close to the eigenvalue of the matrix constructed as follows. Introduse the sets

$$\begin{aligned} B_k(\gamma_1,\gamma_2,...,\gamma_k) &= \{b: b=\sum_{i=1}^{k}n_i\gamma_i, n_i\in Z,\mid b\mid<\frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}\}, \\ B_k(\gamma+t) &= \gamma+t+B_k\equiv\{\gamma+t+b: b\in B_k,\}, \\ B_k(\gamma+t,p_1) &= B_k(\gamma+t)+\Gamma(p_1\rho^\alpha). \end{aligned}$$

Denote by h_i+t for $i=1,2,...,b_k$ the vectors of $B_k(\gamma+t,p_1)$, where

$b_k\equiv b_k(\gamma_1,\gamma_2,...,\gamma_k)$ is the number of the vectors of $B_k(\gamma+t,p_1)$. Define the matrix $C(\gamma+t,\gamma_1,\gamma_2,...,\gamma_k)=(c_{i,j})$ by the formula

$c_{i,j}=q_{h_i-h_j},\forall i\neq j, c_{i,i}=\mid h_i+t\mid^2,\ i,j=1,2,...,b_k.$

Theorem 1 (a) *Suppose $\gamma+t\in U(\rho^{\alpha_1},p)$, $\mid\gamma\mid\sim\rho$. If (11) and*

$$\mid b(N,\gamma)\mid>c_4\rho^{-c\alpha} \tag{16}$$

hold then Λ_N satisfies formulas (5), for $k=1,2,...,[\frac{1}{6}(p-c)]$, where

$$F_s=O(\rho^{-2\alpha_1}),\forall s=0,1,..., \tag{17}$$

and $F_0=0$, $F_s=A_s(\mid\gamma+t\mid^2+F_{s-1},\gamma+t)$, for $s=1,2,....$

(b) *For $\gamma+t\in U(\rho^{\alpha_1},p)$, $\mid\gamma\mid\sim\rho$ there exists an eigenvalue Λ_N of $L_t(q(x))$ satisfying (5).*

(c)*Suppose $\mid\gamma\mid\sim\rho$, $\gamma+t\in(\cap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k}))\backslash E_{k+1}$, where $1\leq k\leq d-1$. If (11) and (16) hold then there is an indeks j such that*

$$\Lambda_N=\lambda_j(\gamma+t)+O(\rho^{-(p-c-\frac{1}{4}d3^d)\alpha}), \tag{18}$$

where $\lambda_1(\gamma+t)\leq\lambda_2(\gamma+t)\leq...\leq\lambda_{b_k}(\gamma+t)$ are the eigenvalues of the matrix $C(\gamma+t,\gamma_1,\gamma_2,...,\gamma_k)$.

(d) *Every eigenvalue $\Lambda_N(t)\equiv\Lambda(\gamma+t)\sim\rho^2$ (see (6) for the definition of $\Lambda(\gamma+t)$) of the operator $L_t(q(x))$ satisfies either (5) or (18) for $c=\frac{q(d-1)}{2}$, namely $\Lambda(\gamma+t)$ satisfies (5) for $\gamma+t\in U(\rho^{\alpha_1},p)$ and satisfies (18) for $\gamma+t\in E_k\backslash E_{k+1}$, where $k=1,2,...,d-1$.*

Proof (a) To prove (5) in case $k=1$ we divide both side of (13) by $b(N,\gamma)$ and use the estimations (14). Then we obtain

$$\Lambda_N-\mid\gamma+t\mid^2=S_1(\Lambda_N,\gamma+t)+O(\rho^{-2\alpha_1})=\mid\gamma+t\mid^2+O(\rho^{-\alpha_1}) \tag{19}$$

In $S_1(\Lambda_N,\gamma+t)$ replasing Λ_N by $\mid\gamma+t\mid^2+O(\rho^{-\alpha_1})$ and using

$$\frac{1}{\mid\gamma+t\mid^2+O(\rho^{-\alpha_1})-\mid\gamma-\gamma_1+t\mid^2}-\frac{1}{\mid\gamma+t\mid^2-\mid\gamma-\gamma_1+t\mid^2}=O(\rho^{-3\alpha_1}),$$

which follows from the first inequality in (12), we get

$$S_1(\mid\gamma+t\mid^2+O(\rho^{-\alpha_1}),\gamma+t)=\sum_{\gamma_1\in\Gamma(\rho^\alpha)}\frac{\mid q_{\gamma_1}\mid^2}{(\mid\gamma+t\mid^2-\mid\gamma-\gamma_1+t\mid^2)}+O(\rho^{-3\alpha_1}).$$

In the sum for S_1 grouping the terms with indeks γ_1 and $-\gamma_1$ and using the obvious relations, $\mid q_{\gamma_1}\mid^2=\mid q_{-\gamma_1}\mid^2$, $\sum_{\gamma_1}\mid\gamma_1\mid^2\mid q_{\gamma_1}\mid^2\leq\infty$,

$$(\mid\gamma+t\mid^2-\mid\gamma-\gamma_1+t\mid^2)^{-1}+(\mid\gamma+t\mid^2-\mid\gamma+\gamma_1+t\mid^2)^{-1}=\mid\gamma_1\mid^2O(\rho^{-2\alpha_1})$$

we get $S_1(\Lambda_N,\gamma+t)=O(\rho^{-2\alpha_1}), S_1(\mid\gamma+t\mid^2+O(\rho^{-\alpha_1}),\gamma+t)=O(\rho^{-2\alpha_1})$. This, (19), and $\alpha_1=3\alpha$ imply that formula (5) for $k=1$ holds and $F_0=0$. Hence (17) for $s=0$ is also proved. Moreover by (15) we have

$S_k(\mid\gamma+t\mid^2+O(\rho^{-\alpha_1}),\gamma+t)=O(\rho^{-2\alpha_1})$ for $k=1,2,...$. Therefore (17) for arbitrary s follows from the definition of F_s by induction. Now we prove (5) for all k by induction. If (5) is true for $k=j$, then putting the expression from (5) (for $k=j$), for Λ_N into $A_{p_1}(\Lambda_N,\gamma+t)$ in (13) and dividing both sides of (13) by $b(N,\gamma)$ we get

$$\begin{aligned}\Lambda_N &= \mid\gamma+t\mid^2+A_{p_1}(\mid\gamma+t\mid^2+F_{j-1}+O(\rho^{-2j\alpha_1}),\gamma+t)+O(\rho^{-(p-c)\alpha})=\\ \{A_{p_1}(&\mid \gamma+t\mid^2+F_{j-1}+O(\rho^{-2j\alpha_1}),\gamma+t)-A_{p_1}(\mid\gamma+t\mid^2+F_{j-1},\gamma+t)\}+\\ A_{p_1}(&\mid \gamma+t\mid^2+F_{j-1},\gamma+t)+O(\rho^{-(p-c)\alpha})\end{aligned}$$

To prove (a) for $k=j+1$ we need to show that the expression in curly brackets is equal to $O(\rho^{-2(j+1)\alpha_1})$. It can be cheked by using (4), (12), (17) and the obvious relation

$$\begin{aligned}&\frac{1}{\prod_{j=1}^{s}(\mid\gamma+t\mid^2+F_{j-1}+O(\rho^{-2j\alpha_1})-\mid\gamma+t-\sum_{i=1}^{s}\gamma_i\mid^2)}-\\ &\frac{1}{\prod_{j=1}^{s}(\mid\gamma+t\mid^2+F_{j-1}-\mid\gamma+t-\sum_{i=1}^{s}\gamma_i\mid^2)}\\ =\;&\frac{1}{\prod_{j=1}^{s}(\mid\gamma+t\mid^2+F_{j-1}-\mid\gamma+t-\sum_{i=1}^{s}\gamma_i\mid^2)}\left(\frac{1}{1-O(\rho^{-(2j+1)\alpha_1})}-1\right)\end{aligned}$$

$=O(\rho^{-2(j+1)\alpha_1}), \forall s=1,2,...,p_1$.

(b) Let A be the set of indices N satisfying (11). Using (7) and Bessel inequality, we obtain

$$\sum_{N\notin A}\mid b(N,\gamma)\mid^2=\sum_{N\notin A}\mid\frac{(\Psi_N(x),(q(x)e^{i(\gamma+t,x)})}{\Lambda_N-\mid\gamma+t\mid^2}\mid^2=O(\rho^{-2\alpha_1})$$

Hence by the Parseval equality we have $\sum_{N\in A}\mid b(N,\gamma)\mid^2=1-O(\rho^{-2\alpha_1})$. This together with $\mid A\mid<c_\Gamma\rho^{d-1}=c_\Gamma\rho^{(d-1)q\alpha}$ (see the introduction) implies that there exsists a number N satisfying $\mid b(N,\gamma)\mid>\frac{1}{2}(c_\Gamma)^{-1}\rho^{-\frac{(d-1)q}{2}\alpha}$, that is, (16) holds, for $c=\frac{(d-1)q}{2}$. Thus Λ_N satisfies (5) according to (a).

(c)Writing the equation (8) for all $h_i+t\in B_k(\gamma+t,p_1)$, we obtain

$$(\Lambda_N- \mid h_i+t\mid^2)b(N,h_i)=\sum_{\gamma^{'}\in\Gamma(\rho^{\alpha})}q_{\gamma^{'}}b(N,h_i-\gamma^{'})+O(\rho^{-p\alpha}), \tag{20}$$

for $i=1,2,...,b_k$. First we show that if $(h_i-\gamma^{'}+t)\notin B_k(\gamma+t,p_1)$ then

$$\begin{aligned} b(N,h_i-\gamma^{'}) &= \sum_{\gamma_1,...,\gamma_{p_1-1}\in\Gamma(\rho^{\alpha})}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{p_1}}b(N,h_i-\gamma^{'}-\sum_{i=1}^{p_1}\gamma_i)}{\prod_{j=0}^{p_1-1}(\Lambda_N- \mid h_i-\gamma^{'}+t-\sum_{i=1}^{j}\gamma_i\mid^2)} \\ +O(\rho^{-p\alpha}) &= O(\rho^{-p\alpha}). \end{aligned} \tag{21}$$

For this, we apply the formula (9) p_1 times and use the inequality

$$\mid \Lambda_N- \mid h_i-\gamma^{'}-\gamma_1-\gamma_2-...-\gamma_s+t\mid^2\mid>\frac{1}{6}\rho^{\alpha_{k+1}}, \tag{22}$$

for $h_i+t\in B_k(\gamma+t,p_1)$, $(h_i-\gamma^{'}+t)\notin B_k(\gamma+t,p_1)$, and $\gamma_j\in\Gamma(\rho^{\alpha})$, where $j=1,2,...,s$ and $s=0,1,...,p_1-1$. So now we prove (22). The relations $p>2p_1$, $h_i+t\in B_k(\gamma+t,p_1)$, $(h_i-\gamma^{'}+t)\notin B_k(\gamma+t,p_1)$ and

$\mid\gamma^{'}\mid,\mid\gamma_1\mid,...,\mid\gamma_{p_1-1}\mid<\rho^{\alpha}$ imply that

$a_s\equiv h_i-\gamma^{'}-\gamma_1-\gamma_2-...-\gamma_s+t\in B_k(\gamma+t,p)\backslash B_k(\gamma+t)$ for all $s=0,1,...,p_1-1$. Since $B_k(\gamma+t,p)=\gamma+t+B_k+\Gamma(p\rho^{\alpha})$ we have a decomposition $a_s=\gamma+t+b+a$, where $b\in B_k, a\in\Gamma(p\rho^{\alpha})$, that is,

$$\mid b\mid<\frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}},\mid a\mid<p\rho^{\alpha}. \tag{23}$$

Using this notation we see that (22) follows from (11) and

$$\mid\mid\gamma+t+a+b\mid^2-\mid\gamma+t\mid^2\mid>\frac{1}{5}\rho^{\alpha_{k+1}} \tag{24}$$

To prove (24) we consider two cases:

Case 1. $a\in P\equiv Span\{\gamma_1,\gamma_2,...,\gamma_k\}$. Then the relations $a+b\in P$, $\gamma+t+a+b\notin\gamma+t+B_k$ imply that $a+b\in P\backslash B_k$,i.e., $\mid a+b\mid\geq\frac{1}{2}\ \rho^{\frac{1}{2}\alpha_{k+1}}$. Consider orthogonal decomposition $\gamma+t=x+v$ of $\gamma+t$, where $v\in P$ and $x\perp P$. It is not hard to show that (see Remark 1)

$$\mid v\mid=O(\rho^{(k-1)\alpha+\alpha_k}). \tag{25}$$

It together with above inequality for $\mid a+b\mid$ and the relations

$\alpha_{k+1} > 2(\alpha_k + (k-1)\alpha)$, $\gamma + t = x + v$, $(x, v) = (x, a) = (x, b) = 0$,

$$\mid \gamma + t + a + b \mid^2 - \mid \gamma + t \mid^2 = \mid a + b + v \mid^2 - \mid v \mid^2 \tag{26}$$

give us the estimation (24).

Case 2. $a \notin P$. First we show that

$$\mid\mid \gamma + t + a \mid^2 - \mid \gamma + t \mid^2 \mid \geq \rho^{\alpha_{k+1}}. \tag{27}$$

Suppose that this inequality does not hold. Then $\gamma + t \in V_a(\rho^{\alpha_{k+1}})$. On the other hand $\gamma + t \in \cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_{k+1}})$. Therefore we have the contradiction $\gamma + t \in E_{k+1}$. Now the difference $\mid \gamma + t + a + b \mid^2 - \mid \gamma + t \mid^2$ we write as the sum of the differences $d_1 \equiv \mid \gamma + t + a + b \mid^2 - \mid \gamma + t + b \mid^2$ and $d_2 \equiv \mid \gamma + t + b \mid^2 - \mid \gamma + t \mid^2$. Since $d_1 = \mid \gamma + t + a \mid^2 - \mid \gamma + t \mid^2 + 2\langle a, b \rangle$, it follows from the inequalities (27), (23) that $\mid d_1 \mid > \frac{2}{3}\rho^{\alpha_{k+1}}$. Taking $a = 0$ in (26) we have $d_2 = \mid b + v \mid^2 - \mid v \mid^2$. Therefore (25) and (23) imply that
$\mid d_2 \mid < \frac{1}{3}\rho^{\alpha_{k+1}}$, $\mid d_1 \mid - \mid d_2 \mid > \frac{1}{3}\rho^{\alpha_{k+1}}$, that is, (24) holds.

Thus in all cases (24) and hence (22) is proved. Therefore relation (21) follows from (4) and (22), since $p_1\alpha_{k+1} \geq p_1\alpha_2 > p\alpha$. Hence (20) has form

$$(\Lambda_N - \mid h_i + t \mid^2) b(N, h_i) = \sum_{\gamma'} q_{\gamma'} b(N, h_i - \gamma') + O(\rho^{-p\alpha}), i = 1, 2, ..., b_k,$$

where the sum is taken under the conditions $\gamma' \in \Gamma(\rho^\alpha)$ and
$h_i - \gamma' + t \in B_k(\gamma + t, p_2)$. It can be written as

$$(C - \Lambda_N I)(b(N, h_1), b(N, h_2), ...b(N, h_{b_k})) = O(\rho^{-p\alpha}),$$

where the rigth side of this system is a vector having the norm
$\mid\mid O(\rho^{-p\alpha}) \mid\mid = O(\sqrt{b_k}\rho^{-p\alpha})$. Now by (16) we have

$$\rho^{-c\alpha} < (\sum_{h_i} \mid b(N, h_i) \mid^2)^{\frac{1}{2}} \leq \mid\mid (C - \Lambda_N I)^{-1} \mid\mid \sqrt{b_k} c_5 \rho^{-p\alpha}, \tag{28}$$

$$\max_{i=1,2,...,b_k} \mid \Lambda_N - \lambda_i \mid^{-1} = \mid\mid (C - \Lambda_N I)^{-1} \mid\mid > c_5^{-1} b_k^{-\frac{1}{2}} \rho^{-c\alpha + p\alpha}. \tag{29}$$

Using that $\mid B_k \mid = O(\rho^{\frac{k}{2}\alpha_{k+1}})$, $\mid \Gamma(p_1\rho^\alpha) \mid = O(\rho^{d\alpha})$ and $d\alpha < \frac{1}{2}\alpha_d$ we get

$$b_k = O(\rho^{d\alpha + \frac{k}{2}\alpha_{k+1}}) = O(\rho^{\frac{d}{2}\alpha_d}) = O(\rho^{\frac{d}{2}3^d\alpha}), \forall k = 1, 2, ..., d-1 \tag{30}$$

Thus the formula (18) follows from (29) and (30).

(d) Let $\Lambda_N \equiv \Lambda(\gamma+t)$ be any eigenvalue of order ρ^2 of the operator $L_t(q(x))$. Denote by K the set of all vectors $\gamma \in \Gamma$ satisfying (11). From (7), arguing as in the proof of (b), we obtain $\sum_{\gamma\in K} \mid b(N,\gamma)\mid^2 = 1 - O(\rho^{-2\alpha_1})$. Since $\mid K \mid = O(\rho^{d-1})$, from definition of $\Lambda(\gamma+t)$ we get

$\mid b(N,\gamma)\mid > \frac{1}{2}(c_\Gamma)^{-1}\rho^{-\frac{(d-1)q}{2}\alpha}$, that is, condition (16) for $c = \frac{(d-1)q}{2}$ holds. Now the proof follows from (a) and (c) since either $\gamma+t \in U(\rho^{\alpha_1},p)$ or $\gamma+t \in E_k\backslash E_{k+1}$ for $k = 1,2,...,d-1$ (see (36)). The Theorem is proved.$\diamondsuit$

Remark 2 *Here we note that the nonresonance domain $U(c_6\rho^{\alpha_1},p)$ has an asymptotically full measure on R^d in the sence that $\frac{\mu(U\cap B(\rho))}{\mu(B(\rho))}$ or $\frac{\mu(U\cap(R(2\rho)\backslash R(\rho)))}{\mu(R(2\rho)\backslash R(\rho))}$ tends to 1 as ρ tends to infinity, where $B(\rho) = \{x \in R^d :\mid x \mid = \rho\}$ and $R(\rho) = \{x \in R^d :\mid x \mid < \rho\}$. (Note that for the different types of the measures of the subset A of R^d we use the same notation $\mu(A)$, because it will be clear which measure we mean.) Similarly, we prove that the single resonanse domain $V_b(\rho^{\alpha_1})\backslash E_2$ has an asymptotically full measure in $V_b(\rho^{\alpha_1})$. Without loss of generality we may consider the resonance domain $V_b(c_6\rho^{\alpha_1})$ for vector $b \in \Gamma(p\rho^\alpha)$ whose length is not greater than the length of other vectors in $\Gamma\cap bR$, becouse $V_{sb}(c_6\rho^{\alpha_k}) \subset V_b(c_6\rho^{\alpha_k})$, for all integers $s > 1$. Indeed if $sb \in \Gamma(p\rho^\alpha)$ and $x \in V_{sb}(c_6\rho^{\alpha_k})$ then using $\alpha_k = 3^k\alpha > 2\alpha$, we get*

$$\mid 2(x,sb) + \mid sb\mid^2 \mid < c_6\rho^{\alpha_k}, \mid 2s(x,b)\mid < \frac{4c_6}{3}\rho^{\alpha_k}, \mid 2(x,b)\mid < \frac{2c_6}{3}\rho^{\alpha_k},$$

and $x \in V_b(c_6\rho^{\alpha_k})$. It is clear and well known that $B(\rho)\cap V_b(c_6\rho^{\alpha_1})$ is the part of sphere $B(\rho)$ which is contained between two parallel hyperplanes

$\{x :\mid x\mid^2 - \mid x+b\mid^2 = -c_6\rho^{\alpha_1}\}$and $\{x :\mid x\mid^2 - \mid x+b\mid^2 = c_6\rho^{\alpha_1}\}$. The distance of these hyperplanes from origin are $\frac{c_6\rho^{\alpha_1}}{|b|}$. This, the relation $\mid \Gamma(p\rho^\alpha)\mid = O(\rho^{d\alpha})$, and $\alpha_1 + d\alpha < 1-\alpha$ give

$$\mu(B(\rho)\cap V_b(c_6\rho^{\alpha_1})) \sim \frac{\rho^{\alpha_1+d-2}}{\mid b\mid},\ \mu(E_1\cap B(\rho)) = O(\rho^{d-1-\alpha}) \tag{31}$$

$$\mu(U(c_6\rho^{\alpha_1},p)\cap B(\rho)) = (1+O(\rho^{-\alpha}))\mu(B(\rho)), \tag{32}$$

$$\mu((R(2\rho)\backslash R(\rho))\cap U) = (1+O(\rho^{-\alpha}))\mu(R(2\rho)\backslash R(\rho)). \tag{33}$$

Now we consider $E \equiv (\cap_{i=1}^k V_{\gamma_i}(\rho^{\alpha_k}))\cap B(\rho)$. We turn the coordinate axis so that $Span\{\gamma_1,\gamma_2,...,\gamma_k\}$ coincides with the span of the vectors

$e_1 = (1,0,0,...,0)$, $e_2 = (0,1,0,...,0)$, ..., e_k. *Then* $\gamma_s = \sum_{i=1}^{k} \gamma_{s,i} e_i$ *for* $s = 1,2,...,k$. *Therefore the relation* $x \in \cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ *implies that*

$$\sum_{i=1}^{k} \gamma_{s,i} x_i = O(\rho^{\alpha_k}), s = 1,2,...,k;\ x_n = \frac{\det(b_{j,i}^n)}{\det(\gamma_{j,i})} ,n = 1,2,...,k, \qquad (34)$$

where $x = (x_1, x_2, ..., x_d), \gamma_j = (\gamma_{j,1}, \gamma_{j,2}, ..., \gamma_{j,k}, 0, 0, ..., 0)$, $b_{j,i}^n = \gamma_{j,i}$ *for* $n \neq j$ *and* $b_{j,i}^n = O(\rho^{\alpha_k})$ *for* $n = j$. *Taking into account that the determinant* $\det(\gamma_{j,i})$ *of the system is a volume of the parallelepiped*

$\{\sum_{i=1}^{k} b_i \gamma_i : b_i \in [0,1], i = 1,2,...,k\}$, *hence greater than some constant* c *and using that* $|\gamma_{j,i}| < p\rho^{\alpha}$ *(since* $\gamma_j \in \Gamma(p\rho^{\alpha})$ *) we get the estimations*

$$x_n = O(\rho^{\alpha_k + (k-1)\alpha}) ,\forall n = 1,2,...,k \qquad (35)$$

¿From this for projection v *of* $x \in \cap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ *onto* $Span\{\gamma_1, \gamma_2, ..., \gamma_k\}$ *we obtain estimation (25). Moreover, note that if* $k = d$, *then* $v = x$ *and by (35) we have* $|x| = O(\rho^{\alpha_d + (d-1)\alpha})$ *which is imposible, becouse* $\alpha_d + (d-1)\alpha < 1$, *and* $x \in B(\rho)$. *It means that the intersection of* d *resonance domains lying in* $\{|x| > \rho_0\}$ *for* $\rho_0 \gg 1$ *is empty set. Therefore we have*

$$R^d \cap \{|x| > \rho_0\} = (U \cup (\cup_{s=1}^{d-1}(E_s \backslash E_{s+1}))) \cap \{|x| > \rho_0\} \qquad (36)$$

Now consider the case $k < d$. *If* $x \in E$ *then it follows from (35) that, there exists index* $i > k$ *such that* $|x_i| > \frac{\rho}{d}$. *Define*

$E(+i) = \{x \in E, x_i > \frac{\rho}{d}\}, E(-i) = \{x \in E, x_i < -\frac{\rho}{d}\}$. *It is clear that* $E \subset (\cup_i E(+i)) \cup (\cup_i E(-i))$. *Using the formula*

$$\mu(E(+i)) = \int\limits_{\Pr(E(+i)),} \frac{|x|}{x_i} dx_1 dx_2 ... dx_{i-1} dx_{i+1} ... dx_d, \qquad (37)$$

where $\Pr(E(+i))$ *is the projection of* $E(+i)$ *onto hyperplane* $x_i = 0$, *and the formula* $\mu(\Pr(E(+i))) = O(\rho^{k(\alpha_k + (k-1)\alpha) + (d-1-k)})$ *which follows from (35), and doing the same estimation for* $\mu(\Pr(E(-i))$, *we get*

$$\mu(E) = O(\rho^{k(\alpha_k + (k-1)\alpha) + (d-1-k)}). \qquad (38)$$

In the case $k = 2$ *using* $|\Gamma(p\rho^{\alpha})| = O(\rho^{d\alpha})$, *we obtain*

$\mu((\cup_{\gamma_2 \in \Gamma(p\rho^{\alpha})}(V_b(\rho^{\alpha_2}) \cap V_{\gamma_2}(\rho^{\alpha_2})) \cap B(\rho)) = O(\rho^{2(\alpha_2 + \alpha) + (d-3) + d\alpha})$. *Using this, the first relation in (31),* $b \in \Gamma(p\rho^{\alpha})$ *and*

$2(\alpha_2 + \alpha) + (d-3) + d\alpha < \alpha_1 + d - 2 - \alpha$, *we see that* $V_b^{'}(\rho^{\alpha_1}) \equiv V_b(\rho^{\alpha_1}) \backslash E_2$ *has an asymptotically full measure on* $V_b(\rho^{\alpha_1})$.

3 Asymptotic Formulae for the Eigenvalues in the Single Resonance Domains

To obtain the asymptotic formulas for the pertubation of the nonresonance eigenvalue $\mid \gamma + t \mid^2$ we used binding formula (7) for the unpertubed operator $L_t(0)$ and the perturbed operator $L_t(q(x))$. To obtain the asymptotic formulae for pertubation of the resonance eigenvalue $\mid \gamma + t \mid^2$ in the single resonance case $\gamma + t \in V_\delta^{'}(\rho^{\alpha_1}) \equiv V_\delta(\rho^{\alpha_1}) \backslash E_2$ for $\delta \in \Gamma(p\rho^\alpha)$ we take the operator$L_t(q^\delta(x))$ for an unperturbed operator, where

$$q^\delta(x) = \sum_{n \in Z} q_{n\delta} e^{in(\delta,x)} = \sum_{n\delta \in \Gamma(\rho^\alpha)} q_{n\delta} e^{ins} + O(\rho^{-(l-1)\alpha}) = Q(s), \tag{39}$$

and $s = (\delta, x)$. Therefore instead of binding formula (7) for $L_t(q(x))$ and $L_t(0)$ we use the binding formula for $L_t(q(x))$ and $L_t(q^\delta(x))$ (see (46)). For this in formula (7) we replace the pertubation $q(x)$ by a pertubation $q(x) - q^\delta(x)$, the eigenvalues $\mid \gamma + t \mid^2$ and eigenfunctions $e^{i(\gamma + t, x)}$ of $L_t(0)$ by the eigenvalues and eigenfunctions of the operator $L_t(q^\delta(x))$ respectively. So we need to calculate the eigenvalues and eigenfunctions of the operator $L_t(q^\delta(x))$. Let Ω_δ be the sublattice $\{h \in \Omega : (h, \delta) = 0\}$ of Ω in the hyperplane

$H_\delta = \{x \in R^d : (x, \delta) = 0\}$, $\Gamma_\delta \equiv \{a \in H_\delta : (a, k) \in 2\pi Z, \forall\ k \in \Omega_\delta\}$ be the dual lattice of Ω_δ, and $F_\delta = H_\delta / \Gamma_\delta$ be the fundanental domain of Γ_δ. Then $t \in F^*$ has a unique decomposition

$$t = a + \tau + \mid \delta \mid^{-2} (t, \delta)\delta,\ a \in \Gamma_\delta,\ \tau \in F_\delta. \tag{40}$$

Define the sets $\Omega^{'}$ and $\Gamma^{'}$ by $\Omega^{'} = \{h + l\delta^* : h \in \Omega_\delta, l \in Z\}$, and by

$\Gamma^{'} = \{b + (p - (2\pi)^{-1}(b, \delta^*))\delta : b \in \Gamma_\delta, p \in Z\}$,

where δ^* is the element of Ω satisfying $(\delta^*, \delta) = 2\pi$.

Lemma 3 (a) *The following relations hold,* $\Omega = \Omega^{'}$, $\Gamma = \Gamma^{'}$.

(b) *The eigenvalues and eigenfunctions of the operator* $L_t(q^\delta(x))$ *are*

$$\lambda_{j,\beta}(v, \tau) = \mid \beta + \tau \mid^2 + \mu_j(v(\beta, t)), \Phi_{j,\beta}(x) = e^{i(\beta + \tau, x)} \varphi_{j, v(\beta,t))}(s),$$

for $j \in Z$, $\beta \in \Gamma_\delta$, *where* $v(\beta, t)$ *is the fractional part of*

$\mid \delta \mid^{-2} (t, \delta) - (2\pi)^{-1}(\beta - a, \delta^*)$, τ *and* a *are uniquely determined from decomposition (40),* $s = (\delta, x)$, *and* $\mu_j(v(\beta, t)), \varphi_{j,v(\beta,t)}(s)$ *are eigenvalues and* *corresponding normalized eigenfunctions of the operator* $T_{v(\beta,t)}(Q(s))$ *generated by the boundary value problem*

$$-\mid \delta \mid^2 y''(s) + Q(s)y(s) = \mu y(s),\ \ y(s+2\pi) = e^{i2\pi v}y(s).$$

Here for simplicity of the notation instead of $v(\beta,t)$ *we write* $v(\beta)$ *(or* v*) if* t *(or* t *and* β*), for which we consider* $v(\beta,t)$*, is unambiguous. Moreover we number the eigenvalues* $\mu_j(v)$ *of* $T_v(Q(s))$ *by* $j \in Z$ *and so that*

$$\mid \mu_j(v) - \mid (j+v)\delta \mid^2 \mid \leq \sup \mid Q(s) \mid, \tag{41}$$

in accordance with a convenient numaration of the eigenvalues $\mid (j+v)\delta \mid^2$ *of* $T_v(0)$ *by all integers* $j \in Z$ *.*

Proof (a) For every vector ω of the lattice Ω taking $h = \omega - (2\pi)^{-1}(\omega,\delta)\delta^*$, using the relations $(\omega,\delta) \equiv 2\pi l \in 2\pi Z$, and $(\delta^*,\delta) = 2\pi$ we see that $h \in \Omega$ and $(h,\delta) = 0$,i.e., $h \in \Omega_\delta$. Hence $\Omega \subset \Omega^{'}$. Now for every vector γ of the lattice Γ define b by $b = \gamma - \mid \delta \mid^{-2} (\gamma,\delta)\delta$. It is not hard to verify that $b \in H_\delta$ and $(b,\omega) = (\gamma,\omega) \in 2\pi Z$ for $\omega \in \Omega_\delta \subset \Omega$. Therefore $b \in \Gamma_\delta$. Moreover $(b,\delta^*) = (\gamma,\delta^*) - 2\pi(\gamma,\delta) \mid \delta \mid^{-2}$. Since $(\gamma,\delta^*) \in 2\pi Z$, that is, $(\gamma,\delta^*) = 2\pi p$ for some integer p, we have $(\gamma,\delta) \mid \delta \mid^{-2} = p - (2\pi)^{-1}(b,\delta^*)$. Therefore we obtain an orthogonal decomposition

$$\gamma = b + \langle \gamma, \frac{\delta}{\mid \delta \mid} \rangle \frac{\delta}{\mid \delta \mid} = b + (p - (2\pi)^{-1}(b,\delta^*))\delta \tag{42}$$

of $\gamma \in \Gamma$, where $b \in \Gamma_\delta$, and $p \in Z$. Hence $\Gamma \subset \Gamma^{'}$. On the other hand if $b \in \Gamma_\delta$, $h \in \Omega_\delta$ and $p, l \in Z$ then $(h + l\delta^*, b + (p - (2\pi)^{-1}(b,\delta^*))\delta) = (h,b) + 2\pi lp \in 2\pi Z$. Thus we have the relations (see definition of the sets $\Omega^{'}, \Gamma^{'}$)

$$\Omega \subset \Omega^{'}, \Gamma \subset \Gamma^{'}, (\omega^{'},\gamma^{'}) \in 2\pi Z, \forall \omega^{'} \in \Omega^{'}, \forall \gamma^{'} \in \Gamma^{'}. \tag{43}$$

Since Ω is the set of all vectors $\omega \in R^d$ satisfying $(\omega,\gamma) \in 2\pi Z$, for all $\gamma \in \Gamma$, and Γ is the set of all vectors $\gamma \in R^d$ satisfying $(\omega,\gamma) \in 2\pi Z$, for all $\omega \in \Omega$ the relations in (43) imply $\Omega^{'} \subset \Omega$, $\Gamma^{'} \subset \Gamma$ and then the validity of (a).

(b) Since $\beta + \tau$ is orthogonal to δ, turning the coordinate axis so that δ coincides with one of the coordinate axis, and taking into account that the Laplace operator is invariant under rotation, one can easily verify that

$$(-\Delta + q^\delta(x))\Phi_{j,\beta}(x) = \lambda_{j,\beta}\Phi_{j,\beta}(x)$$

Now using the relation$(\delta,\omega)=2\pi l$, where $\omega\in\Omega$, $l\in Z$, and the definitions of $\Phi_{j,\beta}(x)$, $\varphi_{j,v}(\delta,x)$ we obtain

$$\Phi_{j,\beta}(x+\omega)=e^{i(\beta+\tau,x+\omega)}\varphi_{j,v}(\delta,x+\omega)=\Phi_{j,\beta}(x)e^{i(\beta+\tau,\omega)+i2\pi lv(\beta,t)}.$$

Replasing τ and ω by $t-a-\mid\delta\mid^{-2}(t,\delta)\delta$ and $h+l\delta^{*}$,where $h\in\Omega_{\delta}, l\in Z$,(see (40) and the first equality of (a)) respectively, and then using
$(h,\delta)=0$, $(\delta^{*},\delta)=2\pi$ one can easily verify that
$(\beta+\tau,\omega)=(t,\omega)+(\beta-a,h)-2\pi l[\mid\delta\mid^{-2}(t,\delta)-(2\pi)^{-1}(\beta-a,\delta^{*})]$. From this using that $(\beta-a,h)\in 2\pi Z$, (because $\beta-a\in\Gamma_{\delta}$, $h\in\Omega_{\delta}$), and $v(\beta,t)$ is a fractional part of the expression in the last square bracket, we infer $\Phi_{j,\beta}(x+\omega)=e^{i(t,\omega)}\Phi_{j,\beta}(x)$. Thus $\Phi_{j,\beta}(x)$ is a eigenfunction of $L_t(q^{\delta}(x))$.

Now we prove that the system $\{\Phi_{j,\beta}(x): j\in Z,\beta\in\Gamma_{\delta}\}$ is a total system of the eigenfunctions. Assume the converse. Then there exists $f(x)\in L_2(F)$ which is orthogonal to all elements of this system. For $t\in F^{*}$ and $\gamma\in\Gamma$ the orthogonal decompositions (40), (42) of t, γ, and definition of $v(\beta,t)$ give the orthogonal decomposition

$$\gamma+t=\beta+\tau+(j+v)\delta \tag{44}$$

of $\gamma+t$, where $\beta\in\Gamma_{\delta},\tau\in F_{\delta}$, $j\in Z$, and $v=v(\beta,t)$. Since $e^{i(j+v)s}$ can be decomposed by ortogonal basis $\varphi_{j,v(\beta,t))}(s)$ for $j\in Z$, the $e^{i(\gamma+t,x)}=e^{i(\beta+\tau,x)}e^{i(j+v)s}$ (see (44)) can be decomposed by $\Phi_{j,\beta}(x)=e^{i(\beta+\tau,x)}\varphi_{j,v(\beta,t))}(s)$ for $j\in Z$. Then the assumption that $f(x)$ orthogonal to all $\Phi_{j,\beta}(x)$ implies that $f(x)$ orthogonal to $e^{i(\gamma+t,x)}$, where γ is arbitrary element of Γ. This controdicts the basisness of the system $\{e^{i(\gamma+t,x)}:\gamma\in\Gamma\ \}\diamondsuit$

Thus in section 2 we proved that if $\gamma+t\notin V_{\delta}(\rho^{\alpha_1})$ for all $\delta\in\Gamma(p\rho^{\alpha})$ then there is an eigenvalue Λ_N of $L_t(q(x))$ which is clous to the eigenvalue $\mid\gamma+t\mid^2$ of $L_t(0)$. If $\gamma+t\in V_{\delta}(\rho^{\alpha_1})\backslash E_2$, then using (44) and $\alpha_1=3\alpha$, we get

$$\mid(j+v)\delta\mid<\frac{\rho^{\alpha_1}}{\mid 2\delta\mid}+\mid\delta\mid,\ \ \mid j\delta\mid<\frac{\rho^{\alpha_1}}{\mid 2\delta\mid}+\mid 2\delta\mid\equiv r_1,\ r_1>2\rho^{\alpha}. \tag{45}$$

To the eigenvalue $\mid\gamma+t\mid^2=\mid\beta+\tau\mid^2+\mid(j+v)\delta\mid^2$ (see (44) and Lemma 2) of $L_t(0)$ corresponds the eigenvalue $\lambda_{j,\beta}(v,\tau)=\mid\beta+\tau\mid^2+\mu_j(v)$ of $L_t(q^{\delta}(x))$. Now we prove that there is an eigenvalue Λ_N of $L_t(q(x))$ which is clous to $\lambda_{j,\beta}(v,\tau)$. As we discussed above for this we use the formula

$$(\Lambda_N-\lambda_{j,\beta})(\Psi_{N,t}(x),\Phi_{j,\beta}(x))=(\Psi_{N,t}(x),(q(x)-q^{\delta}(x))\Phi_{j,\beta}(x)) \tag{46}$$

called binding formula for $L_t(q(x))$ and $L_t(q^\delta(x))$. Now as in the nonresonance case we decompose $(q(x)-q^\delta(x))\Phi_{j,\beta}(x)$ by the $\Phi_{j',\beta'}(x)$ and put this decomposition into (46). Let us find this decomposition. Writing the decomposition (42) for every $\gamma_1 \in \Gamma(\rho^\alpha)$ and using (3), we get
$\gamma_1 = \beta_1 + (n_1 - (2\pi)^{-1}(\beta_1, \delta^*))\delta,\ e^{i(\gamma_1,x)} = e^{i(\beta_1,x)}e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}$,

$$q(x) - Q(s) = \sum_{(n_1,\beta_1)\in\Gamma'(\rho^\alpha)} c(n_1,\beta_1)e^{i(\beta_1,x)}e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s} + O(\rho^{-p\alpha}),$$

$$\begin{aligned} & (q(x)-Q(s))\Phi_{j,\beta}(x) \\ = & \sum_{(n_1,\beta_1)\in\Gamma'(\rho^\alpha)} c(n_1,\beta_1)e^{i(\beta_1+\beta+\tau,x)}e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j,v(\beta)}(s) + O(\rho^{-p\alpha}), \end{aligned} \tag{47}$$

where $c(n_1,\beta_1) = q_{\gamma_1}$,

$\Gamma'(\rho^\alpha) = \{(n_1,\beta_1) : \beta_1 \in \Gamma_\delta\backslash\{0\}, n_1 \in Z, \beta_1 + (n_1 - (2\pi)^{-1}(\beta_1,\delta^*))\delta \in \Gamma(\rho^\alpha)\}.$

Note that if $(n_1,\beta_1) \in \Gamma'(\rho^\alpha)$ then $\mid \beta_1 + (n_1-(2\pi)^{-1}(\beta_1,\delta^*))\delta \mid < \rho^\alpha$ and

$$\mid \beta_1 \mid < \rho^\alpha, \ \ \mid (n_1-(2\pi)^{-1}(\beta_1,\delta^*))\delta \mid < \rho^\alpha < \frac{1}{2}r_1, \tag{48}$$

becouse β_1 orthogonal to δ and $r_1 > 2\rho^\alpha$ (see (45)). To decompose the right side of (47) by $\Phi_{j',\beta'}(x)$ we use the following lemma about eigenfunctions and eigenvalues of the operator $T_v(Q(s))$.

Lemma 4 (a) *If integers j,m satisfy the inequalities $\mid m \mid > 2 \mid j \mid$, $\mid m\delta \mid \geq 2r$, where $r \geq r_1$ and r_1 is defined in (45), then*

$$(\varphi_{j,v}(s), e^{i(m+v)s}) = O(\mid m\delta \mid^{-l-1}) = O(\rho^{-(l+1)\alpha}), \tag{49}$$
$$(\varphi_{m,v}, e^{i(j+v)s}) = O(\mid m\delta \mid^{-l-1}) \tag{50}$$

(b) *There is $\varepsilon(\rho) > 0$ such that $\varepsilon(\rho) \to 0$ as $\rho \to \infty$ and*

$$\mid \mu_j(v) - \mu_{j'}(v) \mid > 2(\ln\rho)^{-1},\ \forall v \in A(\varepsilon(\rho)),\ \forall j' \neq j,$$

where $A(\varepsilon(\rho)) = (\varepsilon(\rho), \frac{1}{2} - \varepsilon(\rho)) \cup (\frac{1}{2} + \varepsilon(\rho), 1 - \varepsilon(\rho))$.

Proof (a). We use the binding formula for $T_v(Q(s))$ and $T_v(0)$ which is

$$(\mu_j(v)- \mid (m+t)\delta \mid^2)(\varphi_{j,v}(s), e^{i(m+v)s}) = (\varphi_{j,v}(s)Q(s), e^{i(m+v)s}), \qquad (51)$$

and the depomposition

$$Q(s) = \sum_{|l_1|<\frac{|m|}{2l}} q_{l_1\delta}e^{il_1 s} + O(\mid m\delta \mid^{-(l-1)}) \qquad (52)$$

Moreover we take into account that for integers $l_1, l_2, ..., l_k, j, m$, where $k \equiv [\frac{l}{2}]$, satisfying the conditions $\mid l_i \mid < \frac{|m|}{2l}$, $\mid j \mid < \frac{|m|}{2}$, the inequalities $\mid m+v-l_1-l_2-...-l_q \mid - \mid j \mid > \frac{1}{5} \mid m \mid$, $\mid m \mid - \mid j+v-l_1-l_2-...-l_q \mid > \frac{1}{5} \mid m \mid$ hold for all $q = 0, 1, ..., k$. Therefore by (41) we have

$$(\mid \mu_j - \mid (m-l_1-l_2-...-l_q+v)\delta \mid^2 \mid)^{-1} = O(\mid m\delta \mid^{-2}), \qquad (53)$$
$$(\mid \mu_m - \mid (j-l_1-l_2-...-l_q+v)\delta \mid^2 \mid)^{-1} = O(\mid m\delta \mid^{-2}), \qquad (54)$$

for $q = 0, 1, ..., k$. Iterating (51) k times, by using (52), (53) we get

$$(\varphi_j, e^{i(m+v)s}) = \sum_{|l_1\delta|,|l_2\delta|,...,|l_{k+1}\delta|<\frac{|m\delta|}{2l}} q_{l_1\delta}q_{l_2\delta}...q_{l_{k+1}\delta} \times \qquad (55)$$

$$\frac{(\varphi_j, e^{i(m-l_1-l_2-...-l_{k+1}+v)s})}{\prod_{q=0}^{k}(\mu_j - \mid (m-l_1-l_2-...-l_q+v)\delta \mid^2)} + O(\mid m\delta \mid^{-l-1}).$$

Now (49) follows from (53), (55), and the first inequality in (4). Formula (50) can be proved in the same way by using (54) instead of (53). Note that in (49), and (50) instead of $O(\mid m\delta \mid^{-l-1})$ we can write $O(\rho^{-(l+1)\alpha})$, since $\mid m\delta \mid \geq r \geq r_1 > 2\rho^\alpha$ (see (45)).

(b). During the proof of (b) we number the eigenvalue $\mu_j(v)$ of $T_v(Q(s))$ in nondecreasing order, i.e., $\mu_1(v) \leq \mu_2(v) \leq ....$ It is well-known that the spectrum of Hill's operator $T(Q(s))$ consists of the intervals $[\mu_{2j-1}(0), \mu_{2j-1}(\frac{1}{2})], [\mu_{2j}(\frac{1}{2}), \mu_{2j}(1)]$ for $j = 1, 2, ...$ The length of the j-th interval Δ_j of the spectrum tends to infinity as j tends to infinity. The distance between neighbouring intervals, that is the length of gaps in spectrum, tends to zero. The eigenvalues $\mu_{2j-1}(v)$ and $\mu_{2j}(v)$ are increasing continuous functions in the intervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ respectively and $\mu_j(1+v) =$

$\mu_j(v)=\mu_j(1-v)$. Since $(\ln\rho)^{-1}\to 0$ as $\rho\to\infty$, the length of intervals Δ_j are sufficienly more than $(\ln\rho)^{-1}$ for $\rho\gg 1$ and there are numbers $\varepsilon_j^{'}(\rho),\varepsilon_j^{''}(\rho)$ in $(0,\frac{1}{2})$ such that

$$\begin{aligned}\mu_{2j-1}(\varepsilon_{2j-1}^{'}(\rho)) &= \mu_{2j-1}(0)+c, \mu_{2j-1}(\frac{1}{2}-\varepsilon_j^{''}(\rho))=\mu_{2j-1}(\frac{1}{2})-c, \quad (56)\\ \mu_{2j}(\frac{1}{2}+\varepsilon_{2j}^{'}(\rho)) &= \mu_{2j}(\frac{1}{2})+c, \mu_{2j}(1-\varepsilon_j^{''}(\rho))=\mu_{2j}(1)-c,\end{aligned}$$

where $c=(\ln\rho)^{-1}$. Denote $\varepsilon^{'}(\rho)=\sup_j\varepsilon_j^{'}(\rho)$, $\varepsilon^{''}(\rho)=\sup_j\varepsilon_j^{''}(\rho)$,

$\varepsilon(\rho)=\max\{\varepsilon^{'}(\rho),\ \varepsilon^{''}(\rho)\}$. To prove that $\varepsilon(\rho)\to 0$ as $\rho\to\infty$ we show that both $\varepsilon^{'}(\rho)$ and $\varepsilon^{''}(\rho)$ tends to zero as $\rho\to\infty$. If $\rho_1<\rho_2$ then

$\varepsilon_j^{'}(\rho_2)<\varepsilon_j^{'}(\rho_1)$ and $\varepsilon^{'}(\rho_2)<\varepsilon^{'}(\rho_1)$, since $\mu_{2j-1}(v)$ and $\mu_{2j}(v)$ are increasing functions in intervals $(0,\frac{1}{2})$ and $(\frac{1}{2},1)$ respectively. Hence

$\varepsilon^{'}(\rho)\to a\in[0,\frac{1}{2}]$ as $\rho\to\infty$. Suppose that $a>0$. Then there is sequence $\rho_k\to\infty$ as $k\to\infty$ such that $\varepsilon^{'}(\rho_k)>\frac{a}{2}$ for all k. This implies that there is a sequence $\{i_k\}$ and without loss of generality it can be assumed that there is a sequence $\{2j_k-1\}$ of odd numbers suct that $\varepsilon_{2j_k-1}^{'}(\rho_k)>\frac{a}{4}$ for all k. Since $\mu_{2j-1}(v)$ increases in $(0,\frac{1}{2})$ and

$\mu_{2j_k-1}(\varepsilon_{2j_k-1}^{'}(\rho_k))-\mu_{2j_k-1}(0)=(\ln\rho_k)^{-1}$ we have

$\mid\mu_{2j_k-1}(\frac{a}{4})-\mu_{2j_k-1}(0)\mid\leq(\ln\rho_k)^{-1}\to 0$ as $k\to\infty$ which contradicts to the well-known asymptotic formulas for eigenvalues $\mu_j(v)$, for $v=0$ and $v=\frac{a}{4}$, where $a\in(0,\frac{1}{2}]$. Thus we proved that $\varepsilon^{'}(\rho)\to 0$ as $\rho\to\infty$. In the same way we prove this for $\varepsilon^{''}(\rho)$, and hence for $\varepsilon(\rho)$. Now suppose $v\in A(\varepsilon(\rho))$. Using (56), the definition of $\varepsilon(\rho)$, and taking into account that $\mu_{2j-1}(v)$ and $\mu_{2j}(v)$ increase in $(0,\frac{1}{2})$ and $(\frac{1}{2},1)$ respectively, we obtain that the eigenvalues $\mu_1(v),\mu_2(v),...,$ are contained in the intervals

$[\mu_{2j-1}(0)+(\ln\rho)^{-1},\mu_{2j-1}(\frac{1}{2})-(\ln\rho)^{-1}],[\mu_{2j}(\frac{1}{2})+(\ln\rho)^{-1},\mu_{2j}(1)-(\ln\rho)^{-1}]$

for $j=1,2,...,$and in each interval there exists a unique eigenvalue of T_v. Therefore the distance between eigenvalues of T_v for $v\in A(\varepsilon(\rho))$ is not less than the distance between these intervals which is not less than $2(\ln\rho)^{-1}\diamondsuit$

Lemma 5 *(a) If* $\mid j\delta\mid<r$ *and* $(n_1,\beta_1)\in\Gamma^{'}(\rho^\alpha)$, *then*

$$e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j,v(\beta)}(s)$$

$$=\sum_{|j_1\delta|<9r}a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)\varphi_{j+j_1,v(\beta+\beta_1)}(s)+O(\rho^{-(l-1)\alpha}),\quad (57)$$

where r *is defined in Lemma 3(a), and*
$a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)=(e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j,v(\beta)}(s),\varphi_{j+j_1,v(\beta+\beta_1)}(s))$.

Proof Since $e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j,v(\beta)}(s)$ is equal to its Fourier series with the orthonormal basis $\varphi_{j+j_1,v(\beta+\beta_1)}(s)$ it suffices to show that

$\sum_{j_1:|j_1\delta|\geq 9r} | a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1) |= O(\rho^{-(l-1)\alpha})$. For this we prove

$$| a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1) |= O(| j_1\delta |^{-l}), \tag{58}$$

for all j_1 satisfying $| j_1\delta |\geq 9r$, since $r\geq r_1>\rho^{\alpha}$ (see the last inequality in (45)). Decomposing $\varphi_{j,v(\beta)}$ over $e^{i(m+v)s}$ we have

$$e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_j(s)=\sum_{m\in Z}(\varphi_j,e^{i(m+v)s})e^{i(m+n+v(\beta+\beta_1))s}, \tag{59}$$

where $n\in Z$ and $| n\delta |< r$ (see the last inequality in (48)). Using (59) and the decomposition $\varphi_{j+j_1}(s)=\sum_{m\in Z}(\varphi_{j+j_1},e^{i(m+v(\beta+\beta_1))s})e^{i(m+v(\beta+\beta_1))s}$ we get

$$a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1)=\sum_{m\in Z}(\varphi_j,e^{i(m-n+v)s})(\varphi_{j+j_1},e^{i(m+v(\beta+\beta_1))s}) \tag{60}$$

Consider the cases $| m\delta |>\frac{1}{3}| j_1\delta |\geq 3r$ and $| m |\leq\frac{1}{3}| j_1 |$. In the first case we have $| (m-n)\delta |>2r$. Therefore (49) implies that

$$\begin{aligned}(\varphi_{j,v}(s),e^{i(m-n+v)s}) &= O(| m\delta |^{-l-1}),\\ \sum_{|m|>\frac{1}{3}|j_1|} | & (\varphi_j,e^{i(m-n+v)s}) |= O(| j_1\delta |^{-l}).\end{aligned}$$

Since $| j_1\delta |\geq 9r$, $| j\delta |<r$, in the second case $| j_1+j |>2| m |$ and by (50),

$$\begin{aligned}(\varphi_{j+j_1},e^{i(m+v(\beta+\beta_1))s}) &= O(| (j_1+j)\delta |^{-(l-1)})=O(| j_1\delta |^{-l-1}),\\ \sum_{|m|\leq\frac{1}{3}|j_1|} | & (\varphi_{j+j_1},e^{i(m+v(\beta+\beta_1))s}) |= O(| j_1\delta |^{-l}).\end{aligned}$$

These estimations for these two cases together with (60) give (58)$\diamondsuit$

Now (47) and (57) yield

$$(q(x)-Q(s))\Phi_{j',\beta'}(x)=O(\rho^{-p\alpha})+\sum_{(n_1,j_1,\beta_1)\in G(\rho^{\alpha},9r)}c(n_1,\beta_1)\times$$

$$a(n_1,\beta_1,j,\beta^{'},j^{'}+j_{1,}\beta^{'}+\beta_1)e^{i(\beta_1+\beta^{'}+\tau,x)}\varphi_{j^{'}+j_1,v(\beta^{'}+\beta_1)}(s) \tag{61}$$

for every $j^{'}$ satisfying $\mid j^{'}\delta \mid< r$, where

$G(\rho^{\alpha},9r)=\{(n,j,\beta):\mid j\delta \mid< 9r,(n,\beta)\in \Gamma^{'}(\rho^{\alpha}),\beta\neq 0\}$. In (61) the multiplicant $e^{i(\beta_1+\beta^{'}+\tau,x)}\varphi_{j^{'}+j_1,v(\beta+\beta_1)}(s)=\Phi_{j^{'}+j_1,\beta^{'}+\beta_1}(x)$ does not depend on n_1. Its coefficient is

$$\sum_{n_1:(n_1,\beta_1)\in\Gamma^{'}(\rho^{\alpha})} c(n_1,\beta_1)a(n_1,\beta_1,j^{'},\beta^{'},j^{'}+j_{1,}\beta^{'}+\beta_1)\equiv\overline{A(j^{'},\beta^{'},j^{'}+j_{1,}\beta^{'}+\beta_1)}. \tag{62}$$

Lemma 6 *If $\mid \beta^{'}\mid\sim\rho$ and $\mid j^{'}\delta \mid< r$, then*

$$(q(x)-Q(s))\Phi_{j^{'},\beta^{'}}(x) \tag{63}$$

$$=\sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)}\overline{A(j^{'},\beta^{'},j^{'}+j_{1,}\beta^{'}+\beta_1)}\Phi_{j^{'}+j_{1,}\beta^{'}+\beta_1}(x)+O(\rho^{-p\alpha}),$$

where $Q(\rho^{\alpha},9r)=\{(j,\beta):\mid j\delta \mid< 9r,\ 0<\mid \beta \mid<\rho^{\alpha}\}$. Moreover there exists c_7 which does not depend on $(j^{'},\beta^{'})$ and

$$\sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)}\mid A(j^{'},\beta^{'},j^{'}+j_{1,}\beta^{'}+\beta_1)\mid< c_7. \tag{64}$$

Proof Since (63) follows from (61), (62) we need to prove (64). If

$$\sum_{j_1}\mid a(n_1,\beta_1,j^{'},\beta^{'},j^{'}+j_{1,}\beta^{'}+\beta_1)\mid< c_7(c_3)^{-1}, \tag{65}$$

then by using $\sum_{(n_1,\beta_1)\in\Gamma^{'}(\rho^{\alpha})}\mid c(n_1,\beta_1)\mid\leq\sum_{\gamma}\mid q_{\gamma}\mid< c_3$ (see definition of $c(n_1,\beta_1)$ and (4)), and (62), we get (64). The inequality (65) follows from decomposition (60) and the inequalities

$$\sum_{m\in Z}\mid\ (\varphi_{j^{'}},e^{i(m-n+v(\beta^{'})s})\mid< c_8, \tag{66}$$

$$\sum_{j_1\in Z}\mid\ (\varphi_{j^{'}+j_1},e^{i(m+v(\beta_1+\beta^{'}))s})\mid< c_9. \tag{67}$$

Now we prove (66) and (67). Since the distance between numbers

$\mid v\delta\mid^2,\mid(1+v)\delta\mid^2,...,$ for $v\in[0,1]$ and similarly the distance between numbers $\mid(-1+v)\delta\mid^2,\mid(-2+v)\delta\mid^2,...,$ is not less than c_{10}, it follows from (41) that the number of elements of the sets

$\{m :\mid (m-n+v(\beta^{'}))\delta \mid^2\in [\mu_{j'}(v(\beta^{'}))-1, \mu_{j'}(v(\beta^{'}))+1]\} \equiv A$ and $\{j_1 : \mu_{j'+j_1}(v(\beta_1+\beta^{'})) \in [\mid (m+v(\beta_1+\beta^{'}))\delta \mid^2 -1, \mid (m+v)\delta \mid^2 +1]\} \equiv B$ is less than c_{11}. Now in (66) and (67) isolating the term with $m \in A$ and $j_1 \in B$ respectively, appling (51) to other terms and then using

$$\sum_{m\notin A} \frac{1}{\mid \mu_{j'}(v^{'}) - \mid (m-n+v^{'})\delta \mid^2\mid} < c_{12},$$
$$\sum_{j_1\notin B} \frac{1}{\mid \mu_{j'+j_1}(v_1^{'}) - \mid (m+v_1^{'})\delta \mid^2\mid} < c_{12}$$

we get (66), (67) and hence (64). Clearly the constants $c_{12}, c_{11}, c_{10}, c_9, c_8$ may be chosen independently of $(j^{'}, \beta^{'})$. Therefore c_7 does not depend on $(j^{'}, \beta^{'})\diamondsuit$

To find the eigenvalue Λ_N which is close to $\lambda_{j,\beta}$,where $\mid j\delta \mid< r_1$ (see (45)) we are going to iterate (46) by using (63). Now we give a brief scheme of this iteration. Replasing in (46) the (j,β) by $(j^{'}, \beta^{'})$ and using (63) we get

$$(\Lambda_N - \lambda_{j',\beta'})b(N, j^{'}, \beta^{'}) = (\Psi_N(x), (q(x)-Q(s))\Phi_{j',\beta'}(x)) = O(\rho^{-p\alpha})$$

$$+ \sum_{(j_1,\beta_1)\in Q(\rho^\alpha, 9r)} A(j^{'}, \beta^{'}, j^{'}+j_1, \beta^{'}+\beta_1)b(N, j^{'}+j_1, \beta^{'}+\beta_1), \qquad (68)$$

where $b(N,j,\beta) = (\Psi_N(x), \Phi_{j,\beta}(x))$. If the condition

$$\mid \Lambda_N - \lambda_{j',\beta'} \mid > i(\rho) > 0, \qquad (69)$$

called the iterability condition for $(N, j^{'}, \beta^{'})$, holds then (68) can be written in the form

$$b(N, j^{'}, \beta^{'}) = \frac{(\Psi_N(x), (q(x)-Q(s))\Phi_{j',\beta'}(x))}{\Lambda_N - \lambda_{j',\beta'}} = O(\rho^{-p\alpha}(i(\rho))^{-1}) + \qquad (70)$$

$$\sum_{(j_1,\beta_1)\in Q(\rho^\alpha, 9r)} \frac{A(j^{'}, \beta^{'}, j^{'}+j_1, \beta^{'}+\beta_1)b(N, j^{'}+j_1, \beta^{'}+\beta_1)}{\Lambda_N - \lambda_{j',\beta'}}$$

Now we give the iteration of (46), for $\mid j\delta \mid< r_1$, by using (68) and (70). For this in (68) instead of $j^{'}$, $\beta^{'}$ and r taking j, β , and r_1, we get

$$(\Lambda_N - \lambda_{j,\beta})b(N,j,\beta) = (\Psi_N(x)(q(x)-Q(s)), \Phi_{j,\beta}(x)) = O(\rho^{-p\alpha}) \qquad (71)$$

$$+\sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)} A(j,\beta,j+j_1,\beta+\beta_1)b(N,j+j_1,\beta+\beta_1)$$

In order to use formula (70) for $b(N,j+j_1,\beta+\beta_1)$, that is, for $j^{'}=j+j_1$ and $\beta^{'}=\beta+\beta_1$ we will prove that there is a number N such that

$$\mid \Lambda_N-\lambda_{j+j_1,\beta+\beta_1}\mid>\frac{1}{2}\rho^{\alpha_2}. \qquad (72)$$

Then $(N,\ j+j_1,\ \beta+\beta_1)$ (where $\mid (j+j_1)\delta\mid<10r_1\equiv r_2$, since $\mid j\delta\mid<r_1$,and $\mid j_1\delta\mid<9r_1$) satisfies the condition (69) for $i(\rho)=\frac{1}{2}\rho^{\alpha_2}$. This means that, in formula (70) the pair $(j^{'},\beta^{'})$ can be replased by the pair $(j+j_1,\beta+\beta_1)$. Then formula (70) takes the form (instead of r and $i(\rho)$ in (70) we should take r_2 and $\frac{1}{2}\rho^{\alpha_2}$ respectively)

$$b(N,j+j_1,\beta_1+\beta)=\sum_{(j_2,\beta_2)\in Q(\rho^\alpha,9r_2)} b(N,j+j_1+j_2,\beta+\beta_1+\beta_2)\times$$

$$\frac{A(j+j_1,\beta+\beta_1,j+j_1+j_2,\beta+\beta_1+\beta_2)}{\Lambda_N-\lambda_{j+j_1,\beta_1+\beta}}+O(\rho^{-p\alpha-\alpha_2}).$$

Putting the obtained formula for $b(N,j+j_1,\beta_1+\beta)$ into (71) we obtain

$$(\Lambda_N-\lambda_{j,\beta})b(N,j,\beta)=O(\rho^{-p\alpha})+$$

$$\sum_{(j_1,\beta_1),(j_2,\beta_2)}\frac{A(j,\beta,j+j_1,\beta+\beta_1)A(j+j_1,\beta+\beta_1,j^2,\beta^2)b(N,j^2,\beta^2),}{\Lambda_N-\lambda_{j+j_1,\beta+\beta_1}} \qquad (73)$$

where $j^2=j+j_1+j_2,\beta^2=\beta+\beta_1+\beta_2$ and the sum in (73) is taken under the conditions $(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)$, and $(j_2,\beta_2)\in Q(\rho^\alpha,9r_2)$. Thus we got one times iteration of (46). It will give first term of asymptotic of Λ_N. For this we find the index N such that $b(N,j,\beta)$ is not very small (see Lemma 7) and (72) is satisfied, i.e., the denominator of the fraction in (73) is a big number (see Lemma 6). Then dividing both sides of (73) by $b(N,j,\beta)$ we get the asymptotic formula for Λ_N (we do this in Theorem 8).

Lemma 7 *Let* $\gamma + t \equiv \beta + \tau + (j + v)\delta \in V_\delta^{'}(\rho^{\alpha_1})$ *(see (44)),* $\mid \gamma + t \mid\sim \rho$, *and* $(j_1, \beta_1) \in Q(\rho^\alpha, 9r_1)$, $(j_k, \beta_k) \in Q(\rho^\alpha, 9r_k)$, *where* r_1 *is defined in (45) and* $r_k = 10r_{k-1}$, *for* $k = 2, 3, ..., p - 1$. *Then*

$$\mid\ j\delta \mid= O(\rho^{\alpha_1}),\quad \mid j_k\delta \mid= O(\rho^{\alpha_1}),\quad \mid \beta_k \mid< \rho^\alpha, \forall k, \tag{74}$$

$$\mid\ \lambda_{j,\beta}(v, \tau) - \lambda_{j^{'},\beta} \mid> 2(\ln \rho)^{-1},\ \forall j^{'} \neq j,\ \forall v(\beta) \in A(\varepsilon(\rho)). \tag{75}$$

If $\mid j^{'}\delta \mid< \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}, \mid \beta^{'} - \beta \mid< (p - 1)\rho^\alpha,\ \beta^{'} \in \Gamma_\delta,\ j^k = j + j_1 + ... + j_k$, $\beta^k = \beta + \beta_1 + ... + \beta_k$, *where* $k = 1, 2, ..., p - 1$, *then*

$$\mid \lambda_{j,\beta} - \lambda_{j^{'},\beta^{'}} \mid> \frac{5}{9}\rho^{\alpha_2},\ \forall\beta^{'} \neq \beta, \tag{76}$$

$$\mid \lambda_{j,\beta}(v, \tau) - \lambda_{j^k,\beta^k} \mid> \frac{5}{9}\rho^{\alpha_2},\ \forall\beta^k \neq \beta,\ v(\beta) \in A(\varepsilon(\rho)). \tag{77}$$

Proof The inequalities in (74) follows from (45) and from the definition of $Q(\rho^\alpha, 9r_k)$. Inequality (75) is a consequence of Lemma 3(b). Inequality (77) follows from (76) and (74). It remains to prove (76). Since

$$\mid \lambda_{j,\beta} - \lambda_{j^{'},\beta^{'}} \mid\geq\mid\mid \beta^{'} + \tau \mid^2 - \mid \beta + \tau \mid^2\mid - \mid \mu_j - \mu_{j^{'}} \mid, \tag{78}$$

it is enough to prove the next two inequalities $\mid \mu_j - \mu_{j^{'}} \mid< \frac{1}{3}\rho^{\alpha_2}$, and

$$\mid\mid \beta + \tau \mid^2 - \mid \beta^{'} + \tau \mid^2\mid> \frac{8}{9}\rho^{\alpha_2}. \tag{79}$$

The first inequality follows from relations $\mid j^{'}\delta \mid< \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}, \mid j\delta \mid= O(\rho^{\alpha_1})$ (see assupmtion of this lemma and (74)) and (41), becouse $\alpha_2 > 2\alpha_1$. Now we prove (79). The conditions $\mid \beta^{'} - \beta \mid< (p - 1)\rho^\alpha, \mid \delta \mid< \rho^\alpha$

imply that there is $n \in Z$, such that

$\gamma^{'} \equiv \beta^{'} - \beta + (n + (2\pi)^{-1}(\beta^{'} - \beta, \delta^*))\delta \in \Gamma(p\rho^\alpha)$. Since $\beta^{'} - \beta$ is nonzero element of H_δ we have $\gamma^{'} \in \Gamma(p\rho^\alpha)\backslash\delta R$. It together with the inclusion

$\gamma+t = \beta+\tau+(j+v)\delta \in V_\delta^{'}(\rho^{\alpha_1}) = V_\delta(\rho^{\alpha_1})\backslash E_2$ (see definition of E_2) imply that $\gamma+t \notin V_{\gamma^{'}}(\rho^{\alpha_1})$, that is, $\mid\mid \gamma+t \mid^2 - \mid \gamma+t+\gamma^{'} \mid^2\mid\geq \rho^{\alpha_2}$. From this using the orthogonal decompositions of $\gamma + t, \gamma^{'}$, taking into account that $\beta, \tau, \beta^{'}$ orthogonal to δ, $\mid j\delta \mid= O(\rho^{\alpha_1})$ (see (74)), $\mid n+(2\pi)^{-1}(\beta^{'}-\beta, \delta^*))\delta \mid= O(\rho^\alpha)$, $\alpha_2 > 2\alpha$, we obtain (79)$\diamondsuit$

Lemma 8 *Suppose* $h_1(x), h_2(x), ..., h_m(x) \in L_2(F)$, *where* $m = p_1 - 1$,

$p_1 = [\frac{p_1-1}{2}]$. *Then for every eigenvalue* $\lambda_{j,\beta} \sim \rho^2$ *of the operator* $L_t(q^\delta(x))$ *there exists an eigenvalue* Λ_N *and a corresponding normalized eigenfunction* Ψ_N *of the operator* $L_t(q(x))$ *such that*

(i) $\mid \lambda_{j,\beta} - \Lambda_N \mid \leq 2M$, *where* $M = \sup \mid q(x) \mid$,

(ii) $\mid b(N,j,\beta) \mid > \frac{1}{2}(c_\Gamma)^{-\frac{1}{2}}\rho^{-\frac{1}{2}(d-1)q\alpha}$,

(iii) $\mid b(N,j,\beta) \mid^2 > \frac{1}{2m}\sum_{i=1}^m (\Psi_N, \frac{h_i}{||h_i||}) \mid^2 > \frac{1}{2m} \mid (\Psi_N, \frac{h_i}{||h_i||}) \mid^2$, $\forall i$.

Proof Let A, B, C be the set of indexes N satisfying $(i), (ii), (iii)$ respectively. Using (46) and Bessel inequality we obtain

$$\begin{aligned}\sum_{N \notin A} \mid \ & b(N,j,\beta) \mid^2 = \sum_{N \notin A} \mid \frac{(\Psi_N(x), (q(x) - Q(s))\Phi_{j,\beta}(x))}{\Lambda_N - \lambda_{j,\beta}} \mid^2 \\ < \ & (2M)^{-2} \mid\mid q(x) - Q(s))\Phi_{j,\beta}(x)) \mid\mid^2 \leq \frac{1}{4}\end{aligned}$$

Using $\mid A \mid < c_\Gamma \rho^{(d-1)q\alpha}$, we get $\sum_{N \in A \backslash B} \mid b(N,j,\beta) \mid^2 < \frac{1}{4}$. From these inequalities by using Parseval equality we obtain

$$\sum_{N \in A} \mid b(N,j,\beta) \mid^2 \geq \frac{3}{4}, \quad \sum_{N \in A \cap B} \mid b(N,j,\beta) \mid^2 > \frac{1}{2} \tag{80}$$

Now it suffices to show that there exists $N \in A \cap B$ satisfying (iii). Assume the converse. Then for all $N \in A \cap B$ the condition (iii) does not hold. This, Bessel inequality, and (80) give the contradiction

$$\begin{aligned}\frac{1}{2} \ & < \sum_{N \in A \cap B} \mid b(N,j,\beta) \mid^2 < \frac{1}{2m}\sum_{i=1}^m \sum_{N \in A} \mid (\Psi_N, \frac{h_i}{\mid\mid h_i \mid\mid}) \mid^2 \\ & \leq \frac{1}{2m}\sum_{i=1}^m \mid\mid \frac{h_i}{\mid\mid h_i \mid\mid} \mid\mid^2 = \frac{1}{2} \diamondsuit\end{aligned}$$

Theorem 9 *For every eigenvalue* $\lambda_{j,\beta}(v,\tau) \sim \rho^2$ *of* $L_t(q^\delta(x))$ *with* $\beta + \tau + (j+v)\delta \in V'_\delta(\rho^{\alpha_1})$ *there exists an eigenvalue* $\Lambda_N \equiv \Lambda_N(\lambda_{j,\beta}(v,\tau))$ *of* $L_t(q(x))$ *satisfying*

$$\Lambda_N = \lambda_{j,\beta} + O(\rho^{-\alpha_2}) \tag{81}$$

Proof By Lemma 7 there is an eigenvalue Λ_N satisfying (i)-(iii) for

$$h_i(x) = \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1),(j_2,\beta_2)\in Q(\rho^\alpha,9r_2)} \frac{\overline{A(j,\beta,j^1,\beta^1)}\,\overline{A(j^1,\beta^1,j^2,\beta^2)}\Phi_{j^2,\beta^2}(x)}{(\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1})^i},$$

where $i = 1,2,...,m = p_1 - 1$. Since $\beta_1 \neq 0$, inequality (76) and condition (i) of lemma 7 yield (72). Hence in brief notations $a = \lambda_{j,\beta}$, $z = \lambda_{j+j_1,\beta+\beta_1}$ we have the relations $\mid \Lambda_N - a \mid < 2M, \mid z - a \mid > \frac{1}{2}\rho^{\alpha_2}$. Using

$$\frac{1}{\Lambda_N - z} = -\sum_{i=1}^{\infty}\frac{(\Lambda_N - a)^{i-1}}{(z-a)^i} = -\sum_{i=1}^{m}\frac{(\Lambda_N - a)^{i-1}}{(z-a)^i} + O(\rho^{-p_1\alpha_2})$$

and $p_1\alpha_2 > p\alpha$ we see that formula (73) can be written as

$$(\Lambda_N - \lambda_{j,\beta})b(N,j,\beta,) = \sum_{i=1}^{m}(\Lambda_N - a)^{i-1}(\Psi_N, \frac{h_i}{\mid\mid h_i \mid\mid}) \mid\mid h_i \mid\mid + O(\rho^{-p\alpha}).$$

Dividing both sides by $b(N,j,\beta)$ and using conditions $(ii), (iii)$ we get

$$\mid (\Lambda_N - \lambda_{j,\beta}) \mid < (2m)^{\frac{1}{2}}\sum_{i=1}^{m} \mid \Lambda_N - a \mid^{i-1} \mid\mid h_i \mid\mid + O(\rho^{-\alpha_2}) \qquad (82)$$

On the other hand (64) and (77), for $k = 1$, imply that $\mid\mid h_i \mid\mid = O(\rho^{-\alpha_2})$, for all i. Therefore (82) and $\mid \Lambda_N - a \mid < 2M$, yield the proof of the theorem$\diamondsuit$

It follow from formulas (75), (77), and (81) that

$$\mid \quad \Lambda_N(\lambda_{j,\beta}) - \lambda_{j^k,\beta}(v,\tau) \mid > (\ln\rho)^{-1}, \forall\ v(\beta) \in A(\varepsilon(\rho)) \qquad (83)$$

$$\mid \quad \Lambda_N(\lambda_{j,\beta}) - \lambda_{j^k,\beta^k} \mid > \frac{1}{2}\rho^{\alpha_2},\ \forall \beta^k \neq \beta, \qquad (84)$$

for $(j_k,\beta_k) \in Q(\rho^\alpha, 9r_k)$, $k = 1,2,...,p-1$. Hence the triple (N, j^k, β^k) for $k = 1,2,...,p-1$ satisfy the iterability condition (69) for either $i(\rho) = (\ln\rho)^{-1}$ (in case $\beta^k = \beta$) or $i(\rho) = \frac{1}{2}\rho^{\alpha_2}$ (in case $\beta^k \neq \beta$) . Therefore to obtain the other terms of the asymptotic formula for Λ_N we iterate the formula (73) $2p_1$ ($2p_1 \leq p-1$) times as follows. We isolate the terms (in the right side of (73)) with multiplicant $b(N,j,\beta)$, i.e.,the case $(j^2,\beta^2) = (j,\beta)$, apply the formula (70) for $b(N,j^2,\beta^2)$ when $(j^2,\beta^2) \neq (j,\beta)$ and use (72). Repeating these process $2p_1$ times, i.e., every time isolating the terms with multiplicant $b(N,j,\beta)$ and appling (70) to the other terms we obtain

$$(\Lambda_N-\lambda_{j,\beta})b(N,j,\beta)=(\sum_{k=1}^{2p_1}S_k^{'}(\Lambda_N,\lambda_{j,\beta}))b(N,j,\beta)+C_{2p_1}^{'}+O(\rho^{-p\alpha}),\quad (85)$$

where $S_1'(\Lambda_N,\lambda_{j,\beta})=\sum\frac{A(j,\beta,j+j_1,\beta+\beta_1)A(j+j_1,\beta+\beta_1,,j,\beta)}{\Lambda_N-\lambda_{j+j_1,\beta+\beta_1}}$,

$$S_k'(\Lambda_N,\lambda_{j,\beta})=\sum(\prod_{i=1}^{k}\frac{A(j_,^{i-1}\beta^{i-1},j^i,\beta^i)}{(\Lambda_N-\lambda_{j^i,\beta^i})})A(j^k,\beta^k,j,\beta),$$

$$C_k'=\sum(\prod_{i=1}^{k}\frac{A(j_,^{i-1}\beta^{i-1},j^i,\beta^i)}{(\Lambda_N-\lambda_{j^i,\beta^i})})A(j_,^k\beta^k,j^{k+1},\beta^{k+1})b(N,j^{k+1},\beta^{k+1})$$

Here $j^0=j$, $\beta^0=\beta$ and the sum for S_k', and $C_k^{'}$ are taken under the conditions $(j_i,\beta_i)\in Q(\rho^\alpha,9r_i)$, $(j^i,\beta^i)\neq(j,\beta)$, for $i=2,3,...,k$ and for $i=2,3,...,k+1$ respectively. Since $\beta_k\neq 0$ for every integer k, the relation $\beta_1+...+\beta_i=0$ implies that $\beta_1+...+\beta_{i\pm1}\neq 0$. Therefore the number of the multiplicants $\Lambda_N-\lambda_{j^i,\beta^i}$ in the denominators of $S_k^{'}$ and $C_{2p_1}^{'}$ satisfying (84) is not less than $\frac{k}{2}$ and p_1 respectively. Hence using (64) and $p_1\alpha_2>p\alpha$, we obtain

$$\begin{aligned} C_{2p_1}^{'} &= O((\rho^{-\alpha_2}\ln\rho)^{p_1})=O(\rho^{-p\alpha}),\ S_1^{'}(\Lambda_N,\lambda_{j,\beta})=O(\rho^{-\alpha_2}) \quad (86)\\ S_k^{'}(\Lambda_N,\lambda_{j,\beta}) &= O((\rho^{-\alpha_2}\ln\rho)^{\frac{k}{2}}),\forall k=2,3,...,2p_1. \end{aligned}$$

Note that if a real number a satisfies $\mid a-\lambda_{j,\beta}\mid<(\ln\rho)^{-1}$ then, by (75), (77), in (83), (84), hence in (86) instead of Λ_N we can write a and get

$$S_1^{'}(a,\lambda_{j,\beta})=O(\rho^{-\alpha_2}),S_k^{'}(a,\lambda_{j,\beta})=O(\rho^{-\alpha_2}\ln\rho)^{\frac{k}{2}}),\forall k=2,3,...,2p_1.\quad (87)$$

Theorem 10 *For every eigenvalue $\lambda_{j,\beta}(v,\tau)\sim\rho^2$ of the operator $L_t(q^\delta(x))$ such that $\beta+\tau+(j+v)\delta\in V_\delta^{'}(\rho^{\alpha_1})$, $v(\beta)\in A(\varepsilon(\rho))$ there exists an eigenvalue Λ_N of $L_t(q(x))$ satisfying the formulas*

$$\Lambda_N=\lambda_{j,\beta}+E_{k-1}(\lambda_{j,\beta})+O(\rho^{-k\alpha_2}(\ln\rho)^{2k}),\quad (88)$$

where $E_0=0$, $E_s=\sum_{k=1}^{2p_2}S_k^{'}(\lambda_{j,\beta}+E_{s-1},\lambda_{j,\beta})$, for $s=1,2,...$, and

$$E_{k-1}(\lambda_{j,\beta})=O(\rho^{-\alpha_2}(\ln\rho)),\quad (89)$$

for $k=1,2,...,[\frac{1}{9}(p-\frac{1}{2}q(d-1)]$

Proof By Theorem 8 formula (88) for the case $k=1$ is proved and $E_0=0$. Hence (89) for $k=1$ is also proved. The proof of (89), for arbitrary k, follows from (87) and definition of E_s by induction. Now we prove (88) by induction. Assume that (88) is true for $k=s<[\frac{1}{9}(p-\frac{1}{2}q(d-1)]$,i.e.,

$\Lambda_N=\lambda_{j,\beta}+E_{s-1}+O(\rho^{-s\alpha_2})$. Putting this expression for Λ_N into $\sum_{k=1}^{2p_1}S_k^{'}(\Lambda_N,\lambda_{j,\beta})$, dividing both sides of (85) by $b(N,j,\beta)$, using (86) and Lemma 7 we get

$\Lambda_N=\lambda_{j,\beta}+\sum_{k=1}^{2p_1}S_k^{'}(\lambda_{j,\beta}+E_{s-1}+O(\rho^{-s\alpha_2}(\ln\rho)^{2s}),\lambda_{j,\beta})+O(\rho^{-(p-\frac{1}{2}q(d-1))\alpha})$

$$= O(\rho^{-(p-\frac{1}{2}q(d-1))\alpha})+\lambda_{j,\beta}+\sum_{k=1}^{2p_1}S_k^{'}(\lambda_{j,\beta}+E_{s-1},\lambda_{j,\beta})+$$

$$\{\sum_{k=1}^{2p_1}S_k^{'}(\lambda_{j,\beta}+E_{s-1}+O(\rho^{-s\alpha_2}(\ln\rho)^{2s}),\lambda_{j,\beta})-\sum_{k=1}^{2p_1}S_k^{'}(\lambda_{j,\beta}+E_{s-1},\lambda_{j,\beta})\}$$

To prove (88) for $k=s+1$ we need to show that the expression in the curly brackets is equal to $O((\rho^{-\alpha_2}\ln\rho)^{s+1})$. This can easily be checked by using the estimations (64), (89), (75), (77) and the obvious relation

$$\frac{1}{\prod_{i=1}^{n}(\lambda_{j,\beta}+E_{s-1}+O(\rho^{-s\alpha_2}(\ln\rho)^{2s})-\lambda_{j^i,\beta^i})}-\frac{1}{\prod_{i=1}^{n}(\lambda_{j,\beta}+E_{s-1}-\lambda_{j^i,\beta^i})}$$
$$=\frac{1}{\prod_{i=1}^{n}(\lambda_{j,\beta}+E_{s-1}-\lambda_{j^i,\beta^i})}(\frac{1}{1+O(\rho^{-s\alpha_2}(\ln\rho)^{2s}\ln\rho)}-1)$$

$=O(\rho^{-(s+1)\alpha_2}(\ln\rho)^{2(s+1)})$ for all $n=1,2,...,2p_1$. The theorem is proved$\diamondsuit$

Remark 11 *Here we note some properties of the known parts* $\mid\gamma+t\mid^2+F_k(\gamma+t)$, $\lambda_j(\gamma+t)\equiv\mid\gamma+t\mid^2+r_j(\gamma+t)$ *(see Theorem 1) and* $\mu_j(v)+\mid\beta+\tau\mid^2+E_k$ *(see Theorem 9) of the eigenvalues of* $L_t(q(x))$. *Denoting* $\gamma+t$ *by* x *, where* $\mid\gamma+t\mid\sim\rho$, *we prove*

$$\frac{\partial F_k(x)}{\partial x_i}=O(\rho^{-2\alpha_1+\alpha}),\forall i=1,2,...,d.;\forall k, \qquad (90)$$

$$\frac{\partial(E_k(\mu_j(v)+\mid\beta+\tau\mid^2))}{\partial\tau_i}=O(\rho^{-2\alpha_2+\alpha}\ln\rho),\forall i=1,2,...,d-1, \qquad (91)$$

$$\mid r_i(x)-r_i(x^{'})\mid\leq 2\rho^{\frac{1}{2}\alpha_d}\mid x-x^{'}\mid,\forall i \qquad (92)$$

where $\tau=(\tau_1,\tau_2,...,\tau_{d-1}),k=1,2,...,$ *and* $v(\beta)\in A(\varepsilon(\rho))$.

First we prove (90) and (91) by induction with respect to k. For $k=1$ the formula (90) follows from (4) and

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|x|^2-|x-\gamma_1|^2}\right)=\frac{-2\gamma_1(i)}{(|x|^2-|x-\gamma_1|^2)^2}=O(\rho^{-2\alpha_1+\alpha}), \qquad (93)$$

where $\gamma_1(i)$ is the i-th component of the vector $\gamma_1\in\Gamma(p\rho^\alpha)$ hence is equal to $O(\rho^\alpha)$. Now suppose that (90) holds for $k=s$. Using this and (17), replacing $|x|^2$ by $|x|^2+F_s(x)$ in (93) and evaluting as above we obtain

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|x|^2+F_s-|x-\gamma_1|^2}\right)=\frac{-2\gamma_1(i)+\frac{\partial F_s(x)}{\partial x_i}}{(|x|^2+F_s-|x-\gamma_1|^2)^2}=O(\rho^{-2\alpha_1+\alpha}).$$

This formula together with the definition of F_k give (90) for $k=s+1$.

To prove (91) for $k=1$ we calculate the derivatives of the expression

$H(\beta^k,\tau)\equiv(\mu_j+|\beta+\tau|^2-\mu_{j^k}-|\beta^k+\tau|^2)^{-1}$. Since μ_j,and $\mu_{j'}$ do not depend on τ_i the function $H(\beta^k,\tau)$, for $\beta^k=\beta$ do not depend on τ_i and by lemma 3 $H(\beta,\tau)=O(\ln\rho)$. For $\beta^k\neq\beta$ using (77), and equality

$|\beta^k-\beta|=|\beta_1+\beta_2...+\beta_i|=O(\rho^\alpha)$ (see last inequality in (74)) we obtain that the derivatives of $H(\beta^k,\tau)$ is equal to $O(\rho^{-2\alpha_2+\alpha})$. Therefore using (64) and the definition of E_1, by direct calculation, we get (91) for $k=1$. Now suppose that (91) holds for $k=s-1$. Using this, replacing $\mu_j+|\beta+\tau|^2$ by $\mu_j+|\beta+\tau|^2+E_{s-1}$ in $H(\beta^k,\tau)$, arguing as above we get (91) for $k=s$.

Now we prove (92). Here $r_1(x)\le r_2(x)\le...\le r_{b_k}(x)$ are the eigenvalue of the matrix $C(x)-|x|^2I\equiv C'(x)$, where $C(x)$ is defined in Section 2. Clearly only diagonal elements of the matrix $C'(x)$ depend on x and they are $|x|^2-|x-b_i|^2=2(x,b_i)-|b_i|^2$, where

$b_i=h_i+t-\gamma-t\in B_k+\Gamma(p_1\rho^\alpha)$. Hence $|b_i|<\rho^{\frac{1}{2}\alpha_d}$. Therefore $C'(x)-C'(x')=(a_{i,j})$, where $a_{i,i}=2(x-x',b_i), a_{i,j}=0$ for $i\neq j$ and $\|C'(x)-C'(x')\|\le 2\rho^{\frac{1}{2}\alpha_d}|x-x'|$ from which follows (92).

Remark 12 *Here we consider some properties of the lattice Γ_δ which will be used in Part 2. Let $\gamma_1,\gamma_2,...,\gamma_d$ be a fixed basis of Γ, i.e.,*

$\Gamma=\{\sum_{i=1}^d n_i\gamma_i : n_i\in Z\}$, and $b_1,b_2,...,b_d$ be the projections of $\gamma_1,\gamma_2,...,\gamma_d$ into H_δ (see (42). Then by Lemma 2 (a) we have $\Gamma_\delta=A$, where

$A=\{\sum_{i=1}^d n_ib_i : n_i\in Z\}$. Consider two cases.

Case 1. $\delta\in\gamma_iZ$. Without loss of generality we assume that $i=d$. Then $b_d=0$ and it is obvious that $\Gamma_\delta=\Gamma'_\delta$, where $\Gamma'_\delta\equiv\{\sum_{i=1}^{d-1}n_ib_i : n_i\in Z\}$ is

$d-1$ dimensional lattice of H_δ, since the independence of system $b_1, b_2, ..., b_{d-1}$ follows from independence of $\gamma_1, \gamma_2, ..., \gamma_{d-1}, \delta$.

Case 2. *$\delta \notin \gamma_i Z$ for $i = 1, 2, ..., d$. Then $\delta = \sum_{i=1}^{d} n_i \gamma_i$, where at least two of n_i , for $i = 1, 2, ..., d$ is not zero. Without loss of generality we assume that $n_1 \neq 0$ and $n_d \neq 0$. Since $\gamma_1, \gamma_2, ..., \gamma_{d-1}, \delta$ are linearly independent (because $n_d \neq 0$), the system $b_1, b_2, ..., b_{d-1}$ is also linearly independent. On the other hand using that the projection of δ into H_δ is zero we get*

$b_d = \sum_{i=1}^{d-1} \frac{n_i}{n_d} b_i$. It together with $\Gamma_\delta = A$ implies that Γ_δ is sublattice of

$\frac{1}{n_d}\Gamma'_\delta = \{\frac{1}{n_d}\sum_{i=1}^{d-1} n_i b_i : n_i \in Z\}$. So we have $\Gamma'_\delta \subset \Gamma_\delta \subset \frac{1}{n_d}\Gamma'_\delta$ which implies that $H_\delta/\frac{1}{n_d}\Gamma'_\delta \subset H_\delta/\Gamma_\delta \subset H_\delta/\Gamma'_\delta$. From this taking into account that $\mid b_i \mid < \mid \gamma_i \mid < c_{13}$, for $i = 1, 2, ..., d-1$, where c_{13} depend only on the fixed basis $\gamma_1, \gamma_2, ..., \gamma_d$ of Γ, we infer $d_\delta \equiv \sup_{\tau,\tau' \in F_\delta} \mid \tau - \tau' \mid = O(1)$, and

$$\mu(F_\delta) \geq \mu(\{\frac{1}{n_d}\sum_{i=1}^{d-1} t_i b_i : t_i \in [0,1]\} = n_d^{1-d}\mu(P), \tag{94}$$

where $P = \{\sum_{i=1}^{d-1} t_i b_i : t_i \in [0,1]\}$. On the other hand the set

$\{y + t\frac{\delta}{|\delta|} : y \in P, t \in [- \mid \delta \mid -c_{13}, \mid \delta \mid +c_{13}]\}$, having measure

$2(\mid \delta \mid +c_{13})\mu(P)$, contains the parallelepiped $\{\sum_{i=1}^{d-1} t_i\gamma_i + t_d\delta : t_i \in [0,1]\}$ (genereted by vectors $\gamma_1, \gamma_2, ..., \gamma_{d-1}, \delta$ of Γ) having measure no less than $\mu(F^*)$. Therefore $\mu(P) \geq (2(\mid \delta \mid +c_{13}))^{-1}\mu(F^*)$. This together with (94) gives $\mu(F_\delta) > c_{14}\rho^{-d\alpha}$, since $n_d = O(\rho^\alpha)$ due to the fact that $\mid \delta \mid < p\rho^\alpha$.

References

[1] M.S.P. Eastham, "The Spectral Theory of Periodic Differential Equations," Scotting Academic Press, Edinburgh, 1973.

[2] O.A. Veliev, On the Spectrum of the Schrodinger Operator with Periodic Potential, Dokl. Akad. Nauk SSSR 268 (1983), no.6, 1289-1292.

[3] O.A. Veliev and S.A. Molchanov, Structure of the Spectrum of the Periodic Schrödinger Operator on a Euclidean torus, Funktsional. Anal. i Prilozhen 19 (1985), no3, 86-87.

[4] O.A. Veliev, Asumptotic Formulas for the Eigenvalues of the Multidimensional Schrödinger Operator and Periodic Differential Operators, Preprint Inst. Prys. AN Azerb. SSR, no.156, 1985.

[5] O.A. Veliev, Asumptotic Formulas for the Eigenvalues of the Periodic Schrödinger Operator and the Bethe-Sommerfeld Conjecture, Funktsional. Anal. i Prilozhen 21(1987), no. 2, 1-15.

[6] O.A. Veliev, The Spectrum of Multidimensional Periodic Operators, Teor. Funktsi. i Funktsional. Anal. i Prilozhen 49 (1988), 17-34.

[7] O.A. Veliev, Asumptotic Formulas for the Eigenvalues and Periodic Differential Operators, Doctor of Sciences Thesis, 1989.

[8] Yu. E. Karpeshina, Pertubation Theory Formulae for the Schrodinger Operator with a Non-smooth Periodic Potential, Math. USSR_Sb.71 (1992), 101-123.

[9] Yu. E. Karpeshina, Pertubation series for the Schrodinger Operator with a Periodic Potential near Planes of diffraction, Communication in Analysis and Geometry 4 (1996), 3, 339-413.

[10] J. Feldman, H. Knorrer, E. Trubowitz, The Perturbatively Stable Spectrum of the Periodic Schrödinger Operator, Invent. Math. 100 (1990), 259-300.

[11] J. Feldman, H. Knorrer, E. Trubowitz, The Perturbatively Unstable Spectrum of the Periodic Schrödinger Operator, Comment. Math. Helvetica 66 (1991), 557-579.

[12] L. Friedlanger, On the Spectrum for the Periodic Problem for the Schrodinger Operator, Communication in Partial Differential Equations 15 (1990), 1631-1647.

[13] O.A. Veliev, Asymptotic Formulae for Resonance Eigenvalues and Inverse Problem for Multidimensional Periodic Schrödinger Operator, Preprint ETH. Zurich. April (1996).

[14] O.A. Veliev, Inverse Problem for the Multidimensional Periodic Schrodinger Operator, *Proceeding of the Azerbaijan Mathematical Society* **2** (1996), 14-32.