Random walks on randomly oriented lattices[1]

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Abstract:
Simple random walks on various types of partially horizontally oriented regular lattices are considered. The horizontal orientations of the lattices can be of various types (deterministic or random) and depending on the nature of the orientation the asymptotic behaviour of the random walk is shown to be recurrent or transient. In particular, for randomly horizontally oriented lattices the random walk is almost surely transient.

1 Introduction

1.1 Motivations

Random walks are mathematical objects with important applications in many scientific disciplines and in particular in physics. Although the bulk of this paper is devoted to the probabilistic problems arising for a particular class of random walks, some indications on the physical interest of the objects we introduce will be given briefly in this subsection.

Beyond the original impetus for the study of random walks given by the seminal work of Einstein on diffusion — an informal but fascinating account of which can be found in chapter 5 of [8], — there was a revival of the physical interest for the subject in the early '80 because it allowed a rigorous and powerful representation of Green's functions in Euclidean (scalar) quantum field theory and statistical mechanics (see [3] for a review). This representation serves also as a rigorous basis for the numerical simulation of quantities of physical relevance in those two theories that remain otherwise inaccessible by the analytic computation. With respect to

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this latter aspect, the denumerable graph on which the random walk evolves is a discretised approximation of the continuum space(-time) manifold. The main drawback of the random walks on lattices is that they don't allow the study of quantum field theories more complicated than the scalar ones, like the gauge field theories or the fermionic field theories, or of quantum statistical physics because these theories are intrinsically non-commutative even in their Euclidean version. The discretised differential calculus for these theories becomes the study of differential forms on graded algebras, necessitating thus the introduction of oriented (directed) lattices as discretised versions of the space(-time) continuum [1].

Although random walks on oriented lattices are the relevant objects to study in the context of discretised gauge theories, their rigorous probabilistic study is still lacking. To the best of the authors knowledge, the only prior probabilistic work on the topic is a paragraph containing a side-result in the PhD thesis [4].

From a purely mathematical point of view, random walks on directed lattices present also very interesting features. For instance, simple random walks on undirected regular lattices (like $\mathbb{Z}^d$) are thoroughly studied and a vast literature establishes precise criteria for their transience or null recurrence properties. Not to mention but one result, the recurrence of the simple random walk on an undirected lattice is related to the convergence norm $r(\mathbf{P})$ of the transition matrix $\mathbf{P}$ and the latter is determined by the geometric properties of the graph through its transition isoperimetric number (see corollary 5.6 of [7]). We shall see that this is not any longer the case for directed graphs since we shall exhibit two different regular deterministic directed graphs having the same isoperimetric number the one being null recurrent and the other transient. This remark constitutes our main motivation for studying random walks on randomly oriented lattices. Since the choice of different deterministic orientation of the graph leaves enough room to have dramatically different asymptotic behaviour for the simple random walk defined on them, a natural question is how a random choice of orientations would affect the result.

1.2 Notations and definitions

An *oriented* (or equivalently *directed*) graph $\mathbb{G} = (\mathbb{V}, \mathbb{A})$ is the pair of a denumerable set $\mathbb{V}$ of vertices and a set $\mathbb{A} \subset \mathbb{V} \times \mathbb{V}$ of oriented edges. We exclude the presence of loops (*i.e.* edges $a = (\mathbf{v}, \mathbf{v})$ with $\mathbf{v} \in \mathbb{V}$). Multiple edges are also excluded by definition. The corresponding graph is then termed *simple*.

Range and a *source* functions, denoted respectively r and s, are defined as mappings $r, s : \mathbb{A} \to \mathbb{V}$, defined by $\mathbb{A} \ni a = (\mathbf{u}, \mathbf{v}) \mapsto r(a) = \mathbf{v} \in \mathbb{V}$ and $\mathbb{A} \ni a = (\mathbf{u}, \mathbf{v}) \mapsto s(a) = \mathbf{u} \in \mathbb{V}$. We can therefore define, for each vertex $\mathbf{v} \in \mathbb{V}$, its *inwards degree* $d^+_{\mathbf{v}} = \mathrm{card}\{a \in \mathbb{A} : r(a) = \mathbf{v}\}$ and its *outwards degree* $d^-_{\mathbf{v}} = \mathrm{card}\{a \in \mathbb{A} : s(a) = \mathbf{v}\}$. All the graphs we consider are *finitely transitive* in the sense for any two distinct vertices $\mathbf{u}, \mathbf{v} \in \mathbb{V}$, there is a *finite* sequence $(\mathbf{w}_0, \dots, \mathbf{w}_k)$ of vertices $\mathbf{w}_i \in \mathbb{V}$, for $i = 0, \dots, k$, $k \in \mathbb{N}^*$, with $\mathbf{w}_0 = \mathbf{u}$ and $\mathbf{w}_k = \mathbf{v}$, such that $(\mathbf{w}_i, \mathbf{w}_{i+1}) \in \mathbb{A}, \forall i = 0, \dots, k-1$. This proprerty implies in particular the *no sink condition*: $d^-_{\mathbf{v}} \geq 1$ for all $\mathbf{v} \in \mathbb{V}$. Notice that undirected graphs can be considered as directed ones verifying the condition that whenever an edge $(\mathbf{u}, \mathbf{v}) \in \mathbb{A}$ then the reverse edge

$(\mathbf{v}, \mathbf{u}) \in \mathbb{A}$. Therefore, when we speak about directed graphs in the sequel, we mean general graphs where some edges can be non-directed. However, we always consider graphs that are genuinely oriented in the sense that there exist vertices $\mathbf{u}$ and $\mathbf{v}$ with $(\mathbf{u}, \mathbf{v}) \in \mathbb{A}$ but $(\mathbf{v}, \mathbf{u}) \notin \mathbb{A}$.

Definition 1.1 [Simple random walk] Let $(\mathbb{V}, \mathbb{A})$ be an oriented graph. A *simple random walk* on $(\mathbb{V}, \mathbb{A})$ is a $\mathbb{V}$-valued Markov chain $(\mathbf{M}_n)_{n\in\mathbb{N}}$ with transition probability matrix $\mathbf{P}$ having as matrix elements

$$P(\mathbf{u}, \mathbf{v}) = \mathbb{P}(\mathbf{M}_{n+1} = \mathbf{v} | \mathbf{M}_n = \mathbf{u}) = \begin{cases} \frac{1}{d_{\mathbf{u}}^-} & \text{if } (\mathbf{u}, \mathbf{v}) \in \mathbb{A} \\ 0 & \text{otherwise.} \end{cases}$$

Remark: When the underlying graph is genuinely oriented, the Markov chain $(\mathbf{M}_n)_{n\in\mathbb{N}}$ *cannot be reversible*. Therefore, all the powerful techniques based on the analogy with electrical circuits (see [2, 9] for instance) do not apply. Notice moreover that the Markov operator of this chain *is not expressible in terms of the Laplace-Beltrami operator*; in the oriented case the Markov operator is expressible in terms of the Dirac operator of the lattice (roughly, the square root of the Laplace-Beltrami operator).

All the graphs that we shall consider in this paper are two-dimensional lattices, i.e. $\mathbb{V} = \mathbb{Z}^2$ and $\mathbb{A}$ is a subset of the set of nearest neighbours in $\mathbb{Z}^2$. We often write $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2$, with $\mathbb{V}_1$ and $\mathbb{V}_2$ isomorphic to $\mathbb{Z}$ when we wish to specify horizontal and vertical directions. In the latter notation, when necessary, we can distinguish between abscissas and ordinates of vertices $\mathbf{v} \in \mathbb{V}$ by writing $\mathbf{v} = (v_1, v_2)$.

Let $\boldsymbol{\epsilon} = (\epsilon_y)_{y\in\mathbb{V}_2}$ be a $\{-1, 1\}$-valued sequence of variables assigned to each ordinate. The sequence $\boldsymbol{\epsilon}$ can be deterministic or random as it will be specified later.

Definition 1.2 [$\boldsymbol{\epsilon}$-horizontally oriented lattice] Let $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2 = \mathbb{Z}^2$, with $\mathbb{V}_1$ and $\mathbb{V}_2$ isomorphic to $\mathbb{Z}$ and $\boldsymbol{\epsilon} = (\epsilon_y)_{y\in\mathbb{V}_2}$ be a sequence of $\{-1, 1\}$-valued variables assigned to each ordinate. We call *$\boldsymbol{\epsilon}$-horizontally oriented lattice* $\mathbb{G} = \mathbb{G}(\mathbb{V}, \boldsymbol{\epsilon})$, the directed graph with vertex set $\mathbb{V} = \mathbb{Z}^2$ and edge set $\mathbb{A}$ defined by the condition $(\mathbf{u}, \mathbf{v}) \in \mathbb{A}$ if, and only if, $\mathbf{u}$ and $\mathbf{v}$ are distinct vertices satisfying one of the following conditions:

1. either $v_1 = u_1$ and $v_2 = u_2 \pm 1$,
2. or $v_2 = u_2$ and $v_1 = u_1 + \epsilon_{u_2}$.

Remark: Notice that the $\boldsymbol{\epsilon}$-horizontally oriented lattice is regular; this means that the vertex degrees (both inwards and outwards) are constant $d_{\mathbf{v}}^- = d_{\mathbf{v}}^+ = d = 3$, $\forall \mathbf{v} \in \mathbb{V}$. The vertical directions of the graph are both-ways; the horizontal directions are one-way, the sign of ϵ_y determining whether the horizontal line at level y is left- or right-going.

Example 1.3 [Alternate lattice $\mathbb{L}$] In that case, $\boldsymbol{\epsilon}$ is the deterministic sequence $\epsilon_y = (-1)^y$ for $y \in \mathbb{V}_2$. The figure 1 depicts a part of this graph.

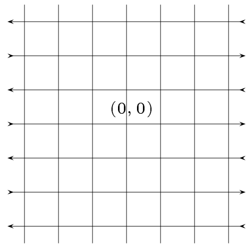

Figure 1: The alternately directed lattice $\mathbb{L}$ corresponding to the choice $\epsilon_y = (-1)^y$.

Example 1.4 [The half-plane one-way lattice $\mathbb{H}$] Here $\boldsymbol{\epsilon}$ is the deterministic sequence

$$\epsilon_y = \begin{cases} 1 & \text{if } \; y \geq 0 \\ -1 & \text{if } \; y < 0. \end{cases}$$

The figure 2 depicts a part of this graph.

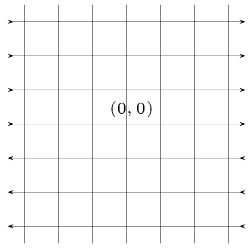

Figure 2: The half-plane one-way lattice $\mathbb{H}$ with $\epsilon_y = -1$, if $y < 0$ and $\epsilon_y = 1$, if $y \geq 0$.

Example 1.5 [The lattice with random horizontal orientations $\mathbb{O}_{\boldsymbol{\epsilon}}$] Here $\boldsymbol{\epsilon} = (\epsilon_y)_{y \in \mathbb{V}_2}$ is a sequence of Rademacher, *i.e.* $\{-1, 1\}$-valued symmetric Bernoulli random variables, that are independent for different values of y. The figure 3 depicts part of a realisation of this graph. The random sequence $\boldsymbol{\epsilon}$ is also termed the *environment of random horizontal directions*.

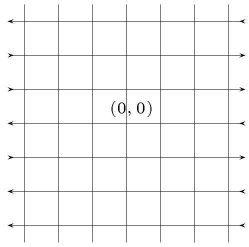

Figure 3: The randomly horizontally directed lattice $\mathbb{O}_{\boldsymbol{\epsilon}}$ with $(\epsilon_y)_{y \in \mathbb{Z}}$ an independent and identically distributed sequence of Rademacher random variables.

1.3 Results

The graphs defined previously are topologically non-trivial in the sense that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{y=-N}^{N} \epsilon_y = 0.$$

For the lattices $\mathbb{L}$ and $\mathbb{H}$ this is shown by a simple calculation and for the lattice $\mathbb{O}_{\boldsymbol{\epsilon}}$ this is an almost sure statement stemming from the independence of the sequence $\boldsymbol{\epsilon}$. The above condition guarantees the existence of infinitely many non trivial allowed loops having the origin as one of their vertices.

We are now ready to state our results.

Theorem 1.6 *The simple random walk on the alternate lattice* $\mathbb{L}$ *is recurrent.*

Remark: This result can be easily generalised to any lattice with periodically alternating horizontal directions (for every finite period).

Theorem 1.7 *The simple random walk on the half-plane one-way lattice* $\mathbb{H}$ *is transient.*

Remark: The result concerning transience in theorem 1.7 is robust. In particular, perturbing the orientation of any finite set of horizontal lines either by reversing the orientation of these lines or by transforming them into two-ways does not change the transient behaviour of the simple random walk. Therefore, the half-plane one-way lattice is so deeply in the transience region that the asymptotic behaviour of the simple random walk cannot be changed by simply modifying the transition probabilities along a lower dimensional manifold as was the case in [6] where the bulk behaviour is on the critical point and it can be changed by lower-dimensional perturbations.

Theorem 1.8 *For almost all realisations of the environment* $\boldsymbol{\epsilon}$*, the simple random walk on the randomly horizontally oriented lattice* $\mathbb{O}_{\boldsymbol{\epsilon}}$ *is transient and its speed is* 0.

2 Technical preliminaries

2.1 Embedding

We suppose that there is an abstract probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which are defined all the random variables we shall use. In particular, the Markov chain $(\mathbf{M}_n)_{n\in\mathbb{N}}$ is defined on this space and we denote $(\tilde{\mathcal{F}}_n)_{n\in\mathbb{N}}$ the natural filtration of the process $(\mathbf{M}_n)_{n\in\mathbb{N}}$. We assume a deterministic anchoring of the chain at the origin, *i.e.* $\mathbf{M}_0 = (0,0) \in \mathbb{V}$; obviously $\tilde{\mathcal{F}}_0 \subseteq \ldots \tilde{\mathcal{F}}_n \subseteq \tilde{\mathcal{F}}_{n+1} \subseteq \ldots \tilde{\mathcal{F}}_\infty \subseteq \mathcal{A}$. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ denote the unit vectors of $\mathbb{Z}^2$. We denote $(\tilde{\psi}_n)_{n\in\mathbb{N}}$ the sequence of the vertical projections of the increments of the Markov chain, more precisely,

$$\tilde{\psi}_{n+1} = (\mathbf{e}_2, \mathbf{M}_{n+1} - \mathbf{M}_n),$$

where $(\cdot,\cdot)$ denotes the Euclidean scalar product on $\mathbb{R}^2$ and $\mathbf{e}_1, \mathbf{e}_2$ the canonical basis of $\mathbb{R}^2$. The random variables $(\tilde{\psi}_n)_{n\in\mathbb{N}, y\in\mathbb{V}_2}$ form a sequence of independent $\{-1,0,1\}$-valued random variables, symmetrically distributed according to the law

$$\begin{aligned}\mathbb{P}(\tilde{\psi}_i = -1) &= \mathbb{P}(\tilde{\psi}_i = 1) = \frac{1}{d^-}\\ \mathbb{P}(\tilde{\psi}_i = 0) &= 1 - \frac{2}{d^-} = p,\end{aligned}$$

where $d^- = 3$ is the outwards degree of any vertex (recall that the graphs we consider are regular) and $q = 1 - p = 2/3$ represents the probability that the walk performs a vertical move.

Lemma 2.1 *On the event* $\{\mathbf{M}_n = \mathbf{u}\}$*, the Markov chain* $(\mathbf{M}_n)$ *verifies*

$$\mathbf{M}_{n+1} = \mathbf{u} + \epsilon_{u_2}\mathbf{e}_1 1\!\!1_{\{\tilde{\psi}_{n+1}=0\}} + \tilde{\psi}_{n+1}\mathbf{e}_2 1\!\!1_{\{\tilde{\psi}_{n+1}\neq 0\}}.$$

Proof: Obvious. □

Introduce now the infinite sequence of random times $(\tilde{\sigma}_k, \tilde{\tau}_k)_{k\in\mathbb{N}^*}$ by

$$\begin{aligned}\tilde{\sigma}_1 &= 1\\ \tilde{\tau}_1 &= \inf\{n \geq \tilde{\sigma}_1 : \tilde{\psi}_n \neq 0\} \geq \tilde{\sigma}_1\\ \tilde{\sigma}_2 &= \inf\{n \geq \tilde{\tau}_1 : \tilde{\psi}_n = 0\} > \tilde{\tau}_1\\ &\vdots\\ \tilde{\tau}_k &= \inf\{n \geq \tilde{\sigma}_k : \tilde{\psi}_n \neq 0\} > \tilde{\sigma}_k\\ \tilde{\sigma}_{k+1} &= \inf\{n \geq \tilde{\tau}_k : \tilde{\psi}_n = 0\} > \tilde{\tau}_k\\ &\vdots\end{aligned}$$

The idea of the embedding is to decompose the two-dimensional random walk $(\mathbf{M}_n)$ into a skeleton simple one-dimensional random walk — the vertical one (Y_n) —, a sequence of waiting times (ξ_n), and an embedded one-dimensional random walk with unbounded jumps — the horizontal one (X_n). In order to achieve the decomposition of the random walk, regroup the instants n into blocks separated by the symbol $|$ as follows:

$$\tilde{\sigma}_1 \ldots \tilde{\tau}_1 - 1 \mid \tilde{\tau}_1 \ldots \tilde{\sigma}_2 - 1 \mid \tilde{\sigma}_2 \ldots \tilde{\tau}_2 - 1 \mid \tilde{\tau}_2 \ldots \tilde{\sigma}_3 - 1 \mid \tilde{\sigma}_3 \ldots \tilde{\tau}_3 - 1 \mid \ldots$$

For the instants in the blocks starting with a $\tilde{\sigma}$ — notice that the leftmost one may be empty but all the other contain at least one instant — the walk performs horizontal moves, for blocks starting with a $\tilde{\tau}$ the walk performs vertical moves. More precisely, define a sequence of random sets of integers for $k \in \mathbb{N}^*$ by

$$\begin{aligned} I_k &= [\tilde{\sigma}_k, \tilde{\tau}_k - 1] \cap \mathbb{N} \\ J_k &= [\tilde{\tau}_k, \tilde{\sigma}_{k+1} - 1] \cap \mathbb{N}, \end{aligned}$$

with I_1 being the empty set when $\tilde{\tau}_1 = \tilde{\sigma}_1$. The random walk (M_n) performs horizontal moves when n is in a I_k for some $k \in \mathbb{N}^*$ and vertical moves when n is in a J_k for some $k \in \mathbb{N}^*$.

Shrink now the I sets and replace them by a waiting time. More precisely, if $I_1 \neq \emptyset$, define $\alpha = 1$ and $\tilde{\xi}_1 = \tilde{\tau}_1 - \tilde{\sigma}_1 = |I_1|$, where $|\cdot|$ denotes cardinality, else define $\alpha = 0$ and $\tilde{\xi}_0$ need not be defined. Then recursively, for $n \in \mathbb{N}^*$, define

$$\begin{aligned} \tilde{\xi}_{\sum_{i=1}^n |J_i| + \alpha + (n-1)} &= \tilde{\tau}_{n+1} - \tilde{\sigma}_n, \\ \tilde{\xi}_{\sum_{i=1}^n |J_i| + \alpha + (n-1) + k} &= 0, \ \forall k \in \{1, \ldots, |J_{n+1}|\}, \\ \psi_{\sum_{i=1}^n |J_i| + k} &= \tilde{\psi}_{\sum_{i=1}^n (|I_i| + |J_i|) + k} \in \{-1, 1\}, \ \forall k \in \{1, \ldots, |J_{n+1}|\}. \end{aligned} \tag{1}$$

Notice that $\tilde{\xi}_l$ cannot be non-zero for two consecutive indices since $|J_{n+1}| = \tilde{\sigma}_{n+2} - \tilde{\tau}_{n+2} \geq 1$ for every $n = 1, 2, \ldots$.

Lemma 2.2 *Given a realisation of the sequence $(\tilde{\psi}_n)_{n \in \mathbb{N}^*}$ the sequences $(\tilde{\xi}_k)_{k \in \mathbb{N}^*}$ and $(\psi_k)_{k \in \mathbb{N}^*}$ are uniquely determined and conversely.*

Proof: The previous construction proves the direct way of the lemma. To prove the converse, given the sequences $(\tilde{\xi}_k)_{k \in \mathbb{N}^*}$ and $(\psi_k)_{k \in \mathbb{N}^*}$, the sequence $(\tilde{\psi}_n)_{n \in \mathbb{N}^*}$ is obtained by inflating the time $\tilde{\xi}_k$ spent in waiting to reconstruct the intervals $(I_k, J_k)_{k \in \mathbb{N}^*}$. Then invert the relations (1) by assigning the value $\tilde{\psi}_i = 0$ whenever i belongs to an interval of I-type. $\square$

The figure (4) depicts an example of random walk on the lattice and the table (1) establishes the bijection between the various sequences.

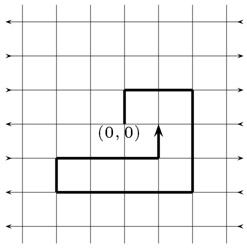

Figure 4: A realisation of the Markov chain $(\mathrm{M}_n)_{n=0,\ldots,15}$.

Lemma 2.3 *The sequence $(\psi_n)_{n \in \mathbb{N}^*}$ is an independent identically distributed sequence of symmetric Bernoulli $\{-1, 1\}$-valued random variables.*

Table 1: The reformulation of the random walk according to vertical and horizontal moves.

n	$\mathbf{M}$	ψ	$\tilde{\xi}$
0	$(0,0)$		0
1	$(0,1)$	1	2
2	$(1,1)$		
3	$(2,1)$		
4	$(2,0)$	-1	0
5	$(2,-1)$	-1	0
6	$(2,-2)$	-1	4
7	$(1,-2)$		
8	$(0,-2)$		
9	$(-1,-2)$		
10	$(-2,-2)$		
11	$(-2,-1)$	1	3
12	$(-1,-1)$		
13	$(0,-1)$		
14	$(1,-1)$		
15	$(1,0)$	1	

Proof: The independence follows from the independence of the $(\tilde{\psi}_n)_{n\in\mathbb{N}^*}$ sequence. For every $n \in \mathbb{N}^*$, the law of ψ_n is the conditional law of a $\tilde{\psi}_m$ with respect to the event $\{\tilde{\psi}_m \neq 0\}$. $\square$

Lemma 2.4 *The sequence $(\tilde{\xi}_n)_{n\in\mathbb{N}^*}$ is an independent, identically distributed sequence of $\mathbb{N}$-valued geometric random variables of parameters p and $q = 1 - p$ with*

$$\mathbb{P}(\tilde{\xi}_1 = \ell) = pq^{\ell}, \ \ \ell = 0, 1, 2, \dots .$$

Proof: The independence follows from the independence of the $(\tilde{\psi}_n)_{n\in\mathbb{N}^*}$ sequence. For every $m \in \mathbb{N}^*$, the variable $\tilde{\xi}_m$ is nothing else than the waiting time on the state 0 for the sequence $(\tilde{\psi}_n)_{n\in\mathbb{N}^*}$. $\square$

2.2 Basic definitions

Definition 2.5 Let $(\psi_n)_{n\in\mathbb{N}^*}$ be a sequence of independent, identically distributed, $\{-1, 1\}$-valued symmetric Bernoulli variables and

$$Y_n = \sum_{k=1}^{n} \psi_k, \ \ n = 1, 2, \dots$$

with Y_0, the simple $\mathbb{V}_2$-valued symmetric one-dimensional random walk. We call the process $(Y_n)_{n\in\mathbb{N}}$ the *vertical skeleton*. We denote by

$$\eta_n(y) = \sum_{k=0}^{n} 1\!\!1_{\{Y_k=y\}}, \quad n \in \mathbb{N}, y \in \mathbb{V}_2$$

its *occupation time* at level y.

Definition 2.6 Suppose the vertical skeleton and the environments of the orientations are given. Let $(\xi_n^{(y)})_{n\in\mathbb{N}^*, y\in\mathbb{V}_2}$ be a doubly infinite sequence of independent identically distributed $\mathbb{N}$-valued geometric random variables of parametres p and $q = 1-p$. Let $(\eta_n(y))$ be the occupation times of the vertical skeleton. We call *horizontally embedded* random walk the process $(X_n)_{n\in\mathbb{N}}$ with

$$X_n = \sum_{y\in\mathbb{V}_2} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}, \quad n \in \mathbb{N}.$$

Remark: The significance of the random variable X_n is the horizontal displacement after $n-1$ vertical moves of the skeleton (Y_l). Notice that the random walk (X_n) has unbounded (although integrable) increments. As a matter of fact, they are signed integer-valued geometric random variables.

Lemma 2.7 *Let*

$$T_n = n + \sum_{y\in\mathbb{V}_2} \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$$

be the instant just after the random walk $(\mathbf{M}_k)$ has performed its n^{th} vertical move (with the convention that the sum $\sum_i$ vanishes whenever $\eta_{n-1}(y) = 0$.) Then

$$\mathbf{M}_{T_n} = (X_n, Y_n).$$

Proof: Obvious. □

Define $\sigma_0 = 0$ and recursively, for $n = 1, 2, \ldots$, $\sigma_n = \inf\{k \geq \sigma_{n-1} : Y_k = 0\} > \sigma_{n-1}$, the n^{th} return to the origin for the vertical skeleton. Then obviously, $\mathbf{M}_{T_{\sigma_n}} = (X_{\sigma_n}, 0)$. To study the recurrence or the transience of $(\mathbf{M}_k)$, we must study how often $\mathbf{M}_k = (0,0)$. Now, $\mathbf{M}_k = (0,0)$ if and only if $X_k = 0$ and $Y_k = 0$. Since (Y_k) is a simple random walk, the event $\{Y_k = 0\}$ is realised only at the instants σ_n, $n = 0, 1, 2, \ldots$.

Lemma 2.8 *Let $\mathcal{F} = \sigma(\psi_i, i \in \mathbb{N})$ and $\mathcal{G} = \sigma(\epsilon_y, y \in \mathbb{V}_2)$. Denote (σ_n) the sequence of consecutive returns to 0 for the skeleton random walk $(Y_k)_{k\in\mathbb{N}}$ and Z a $\mathbb{N}$-valued random variable having the same distribution as ξ_1. Then*

$$\sum_{l=0}^{\infty} \mathbb{P}(\mathbf{M}_l = (0,0)|\mathcal{F} \vee \mathcal{G}) = \sum_{n=0}^{\infty} \mathbb{P}(I(X_{\sigma_n}, \epsilon_0 Z) \ni 0|\mathcal{F} \vee \mathcal{G}),$$

where, for $x \in \mathbb{Z}$, $z \in \mathbb{N}$, and $\epsilon = \pm 1$, $I(x, \epsilon z) = \{x, \ldots, x+z\}$ if $\epsilon = +1$ and $\{x-z, \ldots, x\}$ if $\epsilon = -1$.

Proof: For the process $(\mathbf{M}_l)$ to return to the origin, both horizontal and vertical components must be 0. Since $Y_{\sigma_n} = 0$ and only then $\mathbf{M}_{T_{\sigma_n}} = (X_{\sigma_n}, 0)$. For $k = T_{\sigma_n}, \ldots, T_{\sigma_{n+1}-1}$, the process $\mathbf{M}_k$ can as a matter of fact vanish only when k is in the first part of this discrete time interval, before the process performs any vertical move, namely if either $X_{\sigma_n} = 0$ or if the points X_{σ_n} and X_{σ_n+1} straddle the point 0. Now

$$\begin{aligned} X_{\sigma_n+1} - X_{\sigma_n} &= \sum_{y \in \mathbb{V}_2} \epsilon_y \left(\sum_{i=1}^{\eta_{\sigma_n}(y)} \xi_i^{(y)} - \sum_{i=1}^{\eta_{\sigma_n-1}(y)} \xi_i^{(y)} \right) \\ &= \epsilon_0 \xi^{(0)}_{\eta_{\sigma_n}(0)} \\ &\stackrel{\mathrm{d}}{=} \epsilon_0 Z. \end{aligned}$$

Therefore, the process $(\mathbf{M}_l)$ can vanish for $l \in \{T_{\sigma_n}, \ldots, T_{\sigma_{n+1}} - 1\}$ if, and only if, the point 0 belongs to the set of integers $I(X_{\sigma_n}, \epsilon_0 Z)$. $\square$

Remark: Since the random variable Z is almost surely finite (and even integrable), the recurrence/transience properties of the random walk $(\mathbf{M}_l)$ on the two-dimensional oriented lattice are essentially given by the recurrence/transience properties of the embedded random walk (X_{σ_n}) which is an one-dimensional random walk with unbounded jumps in a random scenery. Notice however that this situation is fundamentally different from the random walk in a random scenery studied in [5].

Although all the subsequent estimates for recurrence/transience of the process can be carried on the right hand side of the formula obtained in lemma 2.8, some can be simplified if we take advantage of the following

Lemma 2.9 *Let $\mathcal{F} = \sigma(\psi_i, i \in \mathbb{N})$ and $\mathcal{G} = \sigma(\epsilon_y, y \in \mathbb{V}_2)$. Denote (σ_n) the sequence of consecutive returns to 0 for the skeleton random walk $(Y_k)_{k \in \mathbb{N}}$. Then*

1. *If $\sum_{n=0}^{\infty} \mathbb{P}_0(X_{\sigma_n} = 0 | \mathcal{F} \vee \mathcal{G}) = \infty$ then $\sum_{l=0}^{\infty} \mathbb{P}(\mathbf{M}_l = (0,0) | \mathcal{F} \vee \mathcal{G}) = \infty$.*
2. *If $(X_{\sigma_n})_{n \in \mathbb{N}}$ is transient then $(M_n)_{n \in \mathbb{N}}$ is also transient.*

Proof: Notice that

$$\begin{aligned} \mathbb{P}(I(X_{\sigma_n}, \epsilon_0 Z) \ni 0 | \mathcal{F} \vee \mathcal{G}) &= p\mathbb{P}(X_{\sigma_n} = 0 | \mathcal{F} \vee \mathcal{G}) \\ &\quad + 1\!\!1_{\{\epsilon_0 = -1\}} \mathbb{P}(\cup_{x \in \mathbb{N}^*} \{X_{\sigma_n} = x; Z \geq x | \mathcal{F} \vee \mathcal{G}) \\ &\quad + 1\!\!1_{\{\epsilon_0 = 1\}} \mathbb{P}(\cup_{x \in \mathbb{N}^*} \{X_{\sigma_n} = -x; Z \geq x | \mathcal{F} \vee \mathcal{G}). \end{aligned}$$

In case 1. the result follows immediately from lemma 2.8. In case 2., since the process $(X_{\sigma_n})_{n \in \mathbb{N}}$ is transient, there exists a constant $C > 0$ such that for all $x \in \mathbb{Z}$, we have $\sum_{n \in \mathbb{N}} \mathbb{P}_0(X_{\sigma_n} =$

$x|\mathcal{F} \vee \mathcal{G}) \leq C < \infty$. Consequently,

$$\sum_{n\in\mathbb{N}} \mathbb{P}_0(\cup_{x\in\mathbb{N}^*}\{X_{\sigma_n} = x; Z \geq x\}|\mathcal{F} \vee \mathcal{G}) = \sum_{n\in\mathbb{N}}\sum_{x\geq 1} q^x \mathbb{P}_0(X_{\sigma_n} = x|\mathcal{F} \vee \mathcal{G}) \leq \frac{q}{1-q}C,$$

proving thus the transience of $(M_n)_{n\in\mathbb{N}}$. □

3 Proof of theorems 1.6 and 1.7

Let ξ be a geometric random variable equidistributed with $\xi_i^{(y)}$. Denote

$$\chi(\theta) = \mathbb{E}\exp(i\theta\xi) = \frac{p}{1 - q\exp(i\theta)} = r(\theta)\exp(i\alpha\theta), \;\; \theta \in [-\pi, \pi]$$

its characteristic function, where

$$r(\theta) = |\chi(\theta)| = \frac{p}{\sqrt{p^2 + 2q(1-\cos\theta)}} = r(-\theta)$$

and

$$\alpha(\theta) = \arctan\frac{q\sin\theta}{1 - q\cos\theta} = -\alpha(-\theta).$$

Notice that $r(\theta) < 1$ for $\theta \in [-\pi,$
$pi] \setminus \{0\}$. Recall that we denote $\mathcal{F} = \sigma(\psi_i, i \in \mathbb{N})$ and $\mathcal{G} = \sigma(\epsilon_y, y \in \mathbb{V}_2)$. Then

$$\begin{aligned}
\mathbb{E}\exp(i\theta X_n) &= \mathbb{E}\left(\mathbb{E}(\exp(i\theta X_n)|\mathcal{F} \vee \mathcal{G})\right) \\
&= \mathbb{E}\left(\mathbb{E}(\exp(i\theta\sum_{y\in\mathbb{V}_2}\epsilon_y\sum_{i=1}^{\eta_{n-1}(y)}\xi_i^{(y)}|\mathcal{F} \vee \mathcal{G})\right) \\
&= \mathbb{E}\left(\prod_{y\in\mathbb{V}_2}\chi(\theta\epsilon_y)^{\eta_{n-1}(y)}\right).
\end{aligned}$$

3.1 The random walk on the $\mathbb{L}$ lattice

Lemma 3.1 *For all $n \in \mathbb{N}^*$, the occupation time of the skeleton random walk verifies:*

$$\sum_{y\in\mathbb{V}_2}(-1)^y\eta_{\sigma_n-1}(y) = 0.$$

Proof: Using the strong Markov property of the process (Y_n), it is enough to show the above equality for $n = 1$. Let

$$H = \sum_{y\in\mathbb{V}_2}(-1)^y\eta_{\sigma_1-1}(y).$$

Since the skeleton walk is a simple one-dimensional walk, we know that $\sigma_1 < \infty$ almost surely. Hence,

$$H = \sum_{k\in\mathbb{N}} H 1\!\!1_{\{\sigma_1=2k\}} \equiv \sum_{k\in\mathbb{N}} H_{2k}.$$

Decompose the event $\{\sigma_1 = 2k\} = \{\sigma_1 = 2k; Y_1 = 1\} \cup \{\sigma_1 = 2k; Y_1 = -1\}$ and consider the trajectory

$$\omega_l = \begin{cases} l & \text{for } l = 0, \dots, k \\ 2k - l & \text{for } l = k+1, \dots, 2k. \end{cases}$$

Obviously this trajectory belongs to the event $A = \{\sigma_1 = 2k; Y_1 = 1\}$. On this trajectory,

$$H(\omega) = 1 + \sum_{l=1}^{k} (-1)^l + \sum_{l=k+1}^{2k-1} (-1)^{2k-l} = 0.$$

All other trajectories contributing to the event A are obtained from ω by applying successively elementary transformations of the following type: if a level $y > 2$ is a local maximum of a trajectory in A, reflect this local maximum with respect to the level $y-1$. The new trajectory is still in A and this operation modifies the occupation times of levels y and $y-2$ by

$$\begin{aligned} \eta_{\sigma_1-1}(y) &\leftarrow \eta_{\sigma_1-1}(y) - 1 \\ \eta_{\sigma_1-1}(y-2) &\leftarrow \eta_{\sigma_1-1}(y-2) + 1. \end{aligned}$$

The corresponding net modification in the value of H is $(-1)^y(-1) + (-1)^{y-2}(+1) = 0$. The proof if completed by symmetry for trajectories in the set $\{\sigma_1 = 2k; Y_1 = -1\}$. □

Proof of theorem 1.6: By lemma 2.8, taking expectations on both sides, we get the obvious minoration

$$\sum_{n\in\mathbb{N}} \mathbb{P}(\mathbf{M}_n = (0,0)) \geq \sum_{n\in\mathbb{N}} \mathbb{P}(X_{\sigma_n} = 0).$$

Moreover, at every moment that the skeleton random walk returns to the origin, the embedded random walk starts afresh so that the process $(X_{\sigma_n})_{n\in\mathbb{N}}$ verifies a renewal equation. Therefore, to show recurrence of the random walk $(\mathbf{M}_n)$, it is enough to show that $\sum_{n\in\mathbb{N}} \mathbb{P}(X_{\sigma_n} = 0) = \infty$.

Now, using parity properties of the modulus and angular part of the characteristic function we get

$$\begin{aligned} \mathbb{E}\exp(i\theta X_{\sigma_n}) &= \mathbb{E}\left(\prod_{y\in\mathbb{V}_2} \chi(\theta\epsilon_y)^{\eta_{\sigma_n-1}(y)}\right) \\ &= \mathbb{E}\left(r(\theta)^{\sum_{y\in\mathbb{V}_2}\eta_{\sigma_n-1}(y)} \exp(i\alpha(\theta)\sum_{y\in\mathbb{V}_2}\epsilon_y\eta_{\sigma_n-1}(y))\right) \\ &= \mathbb{E}\left(r(\theta)^{\sum_{y\in\mathbb{V}_2}\eta_{\sigma_n-1}(y)}\right), \end{aligned}$$

by the previous combinatorial lemma 3.1. Moreover,

$$\sum_{y\in\mathbb{V}_2}\eta_{n-1}(y)=\sum_{y\in\mathbb{V}_2}\sum_{l=0}^{n-1}1\!\!1_{\{Y_l=y\}}=n.$$

Hence, using again the renewal equation, $\mathbb{E}(\exp(i\theta X_{\sigma_n}))=\mathbb{E}(r(\theta)^{\sigma_n})=(\mathbb{E}r(\theta)^{\sigma_1})^n$. Now, σ_1 is the time of first return to the origin for a simple random walk, its generating function reads $f(s)=\mathbb{E}s^{\sigma_1}=\sum_{k=1}^{\infty}s^{2k}\mathbb{P}(\sigma_1=2k)=1-\sqrt{1-s^2}$ for $|s|\le 1$. Hence, finally,

$$\mathbb{E}(\exp(i\theta X_{\sigma_n}))=(1-\sqrt{1-r(\theta)^2})^n,$$

so that

$$\sum_{n=0}\mathbb{P}(X_{\sigma_n}=0)=\lim_{\epsilon\to 0}2\int_{\epsilon}^{\pi}\frac{1}{\sqrt{1-r(\theta)^2}}d\theta,$$

since for $\theta\in[\epsilon,\pi]$ the function $r(\theta)<1$. Now, for $\theta\to 0$, $\frac{1}{\sqrt{1-r(\theta)^2}}=\mathcal{O}(\frac{1}{|\theta|})$ and since $1/|\theta|$ is a non-integrable singularity at 0, $\sum_n\mathbb{P}(X_{\sigma_n}=0)=\infty$, proving thus the recurrence of $(\mathbf{M}_n)$.
□

3.2 The random walk on the $\mathbb{H}$ lattice

Lemma 3.2 *Let $\epsilon_y=1$ if $y\ge 0$ and $\epsilon_y=-1$ if $y<0$. Denote by $(\rho_k)_{k\in\mathbb{N}}$ a sequence of independent identically distributed Rademacher variables and $(\tau_k)_{k\in\mathbb{N}}$ a sequence of independent, identically distributed random variables, independent of the sequence $(\rho_k)_{k\in\mathbb{N}}$, such that $\tau_1\stackrel{d}{=}\sigma_1$, i.e. the random variables τ_k have the same law as the time of first return to the origin for the skeleton random walk. Then*

$$\sum_{y\in\mathbb{V}_2}\epsilon_y\eta_{\sigma_n-1}(y)\stackrel{d}{=}\sum_{k=1}^{n}\rho_k(\tau_k-1)+n.$$

Proof: We have

$$\begin{aligned}\sum_{y\in\mathbb{V}_2}\epsilon_y\eta_{\sigma_n-1}(y) &= \sum_{y\in\mathbb{V}_2}\epsilon_y\sum_{j=1}^{n}\sum_{k=\sigma_{j-1}}^{\sigma_j-1}1\!\!1_{\{Y_k=y\}}\\ &= \sum_{y\neq 0}\epsilon_y\sum_{j=1}^{n}\sum_{k=\sigma_{j-1}+1}^{\sigma_j-1}1\!\!1_{\{Y_k=y\}}+\epsilon_0 n.\end{aligned}$$

Now for every $j\in\{1,\dots,n\}$ the process Y_k has the same sign for all $k\in\{\sigma_{j-1}+1,\dots,\sigma_j-1\}$. Hence $\sum_{y\neq 0}\sum_{k=\sigma_{j-1}+1}^{\sigma_j-1}1\!\!1_{\{Y_k=y\}}=\sigma_j-\sigma_{j-1}-1\stackrel{\mathrm{d}}{=}\tau_j-1$. However, the contribution to the sum including the ϵ variables must be corrected by the sign of $Y_{\sigma_{j-1}+1} \stackrel{\text{d}}{=} Y_1$ and since the skeleton random walk is symmetric and strongly Markovian, we have finally

$$\sum_{y \in \mathbb{V}_2} \epsilon_y \eta_{\sigma_n - 1}(y) \stackrel{\text{d}}{=} \sum_{k=1}^{n} \rho_k(\tau_k - 1) + n.$$

□

Proposition 3.3 *For the embedded random walk, we have*

$$\mathbb{E}\left(\exp(i\theta X_{\sigma_n})\right) = g(\theta)^n,$$

where $g(\theta) = \frac{1}{2}\chi(\theta)\left[\left(1 - \sqrt{1-\chi(\theta)^2}\right)\exp(-i\alpha(\theta)) + \left(1 - \sqrt{1-\overline{\chi}(\theta)^2}\right)\exp(i\alpha(\theta))\right]$.

Proof: Denote $\mathcal{D} = \sigma(\rho_k, k \in \mathbb{N})$ the σ-algebra generated by the Rademacher variables. Then, using lemma 3.2, we have,

$$\begin{aligned}
\mathbb{E}\left(\exp(i\theta X_{\sigma_n})\right) &= \mathbb{E}\left(\mathbb{E}\left[\exp(i\theta X_{\sigma_n})|\mathcal{D}\right]\right) \\
&= \mathbb{E}\left(\mathbb{E}\left[r(\theta)^{\sum_{j=1}^{n}(\tau_j - 1)} \exp\left(i\alpha(\theta)(\sum_{j=1}^{n} \rho_j(\tau_j - 1) + n)\right)|\mathcal{D}\right]\right) \\
&= \chi(\theta)^n \mathbb{E}\left(\mathbb{E}(r(\theta)^{\sum_{j=1}^{n}\tau_j} \exp(i\alpha(\theta)\sum_{j=1}^{n}\rho_j(\tau_j - 1))|\mathcal{D})\right) \\
&= \chi(\theta)^n \mathbb{E}\left(\mathbb{E}\left[\prod_{j=1}^{n}(\chi(\theta\rho_j)^{\tau_j}\exp(-i\alpha(\theta)\rho_j))|\mathcal{D}\right]\right) \\
&= \chi(\theta)^n \prod_{j=1}^{n} \mathbb{E}\left[\left(1 - \sqrt{1 - \chi(\theta\rho_j)^2}\right)\exp(-i\alpha(\theta)\rho_j)\right] \\
&= g(\theta)^n.
\end{aligned}$$

□

Proposition 3.4 *The random walk* $(\mathbf{M}_n)_{n\in\mathbb{N}}$ *on the lattice* $\mathbb{H}$ *verifies*

$$\sum_{n=0}^{\infty} \mathbb{P}(\mathbf{M}_n = (0,0)) < \infty.$$

Proof: Recalling that $\epsilon_0 = 1$, from lemma 2.8 we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{P}(\mathbf{M}_n = (0,0)) &= \sum_{n=0}^{\infty} \mathbb{P}(I(X_{\sigma_n}, Z) \ni 0) \\
&= \sum_{n=0}^{\infty} \sum_{x \geq 0} \mathbb{P}(X_{\sigma_n} = -x)\mathbb{P}(Z \geq x).
\end{aligned}$$

From the proposition 3.3 we obtain

$$\mathbb{P}(X_{\sigma_n} = -x) = \int_{-\pi}^{\pi} \exp(i\theta x) g(\theta)^n d\theta,$$

so that

$$\begin{aligned} \sum_{x\geq 0} \mathbb{P}(X_{\sigma_n} = -x)\mathbb{P}(Z \geq x) &= \int_{-\pi}^{\pi} \left(\sum_{x\geq 0} \exp(i\theta x) pq^x \right) g(\theta)^n d\theta \\ &= \int_{-\pi}^{\pi} \frac{p}{1 - q\exp(i\theta)} g(\theta)^n d\theta. \end{aligned}$$

Therefore, since $|g(\theta)| < 1$ for $\theta \neq 0$,

$$\sum_{n=0}^{\infty} \mathbb{P}(\mathbf{M}_n = (0,0)) = \lim_{\epsilon\to 0} \int_{\epsilon}^{\pi} \left[2\,\mathrm{Re}\,\chi(\theta) \frac{1}{1-g(\theta)} \right] d\theta.$$

Now $\lim_{\theta\to 0^+} \frac{1-g(\theta)}{\sqrt{\theta}} = \frac{1}{\sqrt{2}}$, therefore for $\theta \to 0^+$, $(1-g(\theta))^{-1} = \mathcal{O}(\theta^{-1/2})$ that constitutes an integrable singularity, while the factor $\chi(\theta)$ does not change the singular behaviour at 0, proving thus the transience of the random walk. $\square$

4 Proof of the theorem 1.8

The main difficulty that arises when we deal with the $\mathbb{O}_\epsilon$ lattice stems from the fact that the embedded random walk X_n, which always satisfies the equation

$$X_n = \sum_{y\in\mathbb{V}_2} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)},$$

cannot any longer be split into independent parts when sampled on the moments σ_n of successive returns to the origin for the skeleton walk because the increment $X_{\sigma_{n+1}} - X_{\sigma_n}$ is not independent from the increment $X_{\sigma_n} - X_{\sigma_{n-1}}$, since they may share *the same* random variables ϵ_y for some y. Hence the embedded random walk does not verify a renewal equation and some new techniques are needed.

4.1 Technical estimates

Recall that all random variables are defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$; introduce the following sub-σ-algebras:

$$\begin{aligned} \mathcal{H} &= \sigma(\xi_i^{(y)}, i \in \mathbb{N}, y \in \mathbb{V}_2) \\ \mathcal{G} &= \sigma(\epsilon_y, y \in \mathbb{V}_2) \\ \mathcal{F}_n &= \sigma(\psi_i, i = 1, \ldots, n), \end{aligned}$$

with $\mathcal{F} \equiv \mathcal{F}_\infty$.

Introduce the sequence of events $A_n = A_{n,1} \cap A_{n,2}$ and B_n with

$$\begin{aligned} A_{n,1} &= \{\omega \in \Omega : \max_{0 \le k \le 2n} |Y_k| < n^{\frac{1}{2}+\delta_1}\} \text{ for some } \delta_1 > 0, \\ A_{n,2} &= \{\omega \in \Omega : \max_{y \in \mathbb{V}_2} \eta_{2n-1}(y) < n^{\frac{1}{2}+\delta_2}\} \text{ for some } \delta_2 > 0, \\ B_n &= \{\omega \in A_n : \left| \sum_{y \in \mathbb{V}_2} \epsilon_y \eta_{2n-1}(y) \right| > n^{\frac{1}{2}+\delta_3}\} \text{ for some } \delta_3 > 0. \end{aligned}$$

Obviously $A_{n,1}, A_{n,2}$ and hence A_n belong to $\mathcal{F}_{2n}$; moreover $B_n \subseteq A_n$ and $B_n \in \mathcal{F}_{2n} \vee \mathcal{G}$. We denote in the sequel generically $d_{n,i} = n^{\frac{1}{2}+\delta_i}$, for $i = 1, 2, 3$.

Since $B_n \subseteq A_n$ and both sets are $\mathcal{F}_{2n} \vee \mathcal{G}$-measurable, decomposing the unity as

$$1 = 1\!\!1_{B_n} + 1\!\!1_{A_n \setminus B_n} + 1\!\!1_{A_n^c},$$

we have

$$\begin{aligned} \mathbb{P}(X_{2n} = 0; Y_{2n} = 0 | \mathcal{F} \vee \mathcal{G}) &= 1\!\!1_{B_n} 1\!\!1_{\{Y_{2n}=0\}} \mathbb{P}(X_{2n} = 0 | \mathcal{F} \vee \mathcal{G}) \\ &\quad + 1\!\!1_{A_n \setminus B_n} 1\!\!1_{\{Y_{2n}=0\}} \mathbb{P}(X_{2n} = 0 | \mathcal{F} \vee \mathcal{G}) \\ &\quad + 1\!\!1_{A_n^c} 1\!\!1_{\{Y_{2n}=0\}} \mathbb{P}(X_{2n} = 0 | \mathcal{F} \vee \mathcal{G}), \end{aligned}$$

and taking expectations on both sides of the equality, we get

$$p_n = p_{n,1} + p_{n,2} + p_{n,3},$$

where

$$\begin{aligned} p_n &= \mathbb{P}(X_{2n} = 0; Y_{2n} = 0) \\ p_{n,1} &= \mathbb{P}(X_{2n} = 0; Y_{2n} = 0; B_n) \\ p_{n,2} &= \mathbb{P}(X_{2n} = 0; Y_{2n} = 0; A_n \setminus B_n) \\ p_{n,3} &= \mathbb{P}(X_{2n} = 0; Y_{2n} = 0; A_n^c). \end{aligned}$$

4.2 Proof of transience of the random walk on $\mathbb{O}_\epsilon$

The transience of the random walk $(\mathbf{M}_n)$ will be shown by establishing asymptotic estimates for the probabilities $p_{n,1}$, $p_{n,2}$, and $p_{n,3}$, for large n showing the summability of p_n and using lemma 2.9 to conclude.

Proposition 4.1 *For large n, there exist $\delta > 0$ and $c > 0$ such that*

$$p_{n,3} = \mathcal{O}(\exp(-cn^\delta)).$$

Proof: Write $A_n^c = A_{n,1}^c \cup A_{n,2}^c$.

We have $\mathbb{P}(A_{n,1}^c|Y_{2n}=0) = \mathbb{P}(\max_{0\le k\le 2n}|Y_k| \ge d_{n,1}|Y_{2n}=0)$ for $d_{n,1} = n^{\frac{1}{2}+\delta_1}$ and some $\delta_1 > 0$. Let $a_n = [d_{n,1}]$ and $R_n = \{a_n, a_n+1, \ldots, n\}$. With this notation,

$$\begin{aligned}
\mathbb{P}(A_{n,1}^c|Y_{2n}=0) &= \sum_{y\in R_n} \mathbb{P}(\max_{0\le k\le 2n}|Y_k| = y|Y_{2n}=0) \\
&\le 2\sum_{y\in R_n} \mathbb{P}_0(Y_{2n}=2y) \\
&= 2\mathbb{P}_0(Y_{2n}\ge 2a_n),
\end{aligned}$$

by the symmetry of the skeleton random walk and the reflection principle. The last probability is majorised by standard methods,

$$\begin{aligned}
\mathbb{P}(A_{n,1}^c|Y_{2n}=0) &\le 2\mathbb{P}_0(Y_{2n}\ge 2a_n) \\
&\le 2\inf_{t>0}\mathbb{P}_0(\exp(tY_{2n})\ge \exp(ta_n)) \\
&\le 2\inf_{t>0}\frac{(\cosh t)^{2n}}{\exp(2ta_n)} \\
&= 2\exp(-\frac{a_n^2}{n}) \\
&\le 2\exp(-n^{2\delta_1}).
\end{aligned}$$

In a similar way,

$$\mathbb{P}(A_{n,2}^c|Y_{2n}=0) = \mathbb{P}(\max_{y\in\mathbb{V}_2}(\eta_{2n-1}(y)\ge d_{n,2}|Y_{2n}=0),$$

with $d_{n,2} = n^{\frac{1}{2}+\delta_2}$ for some $\delta_2 > 0$. The conditional probability in the right hand side of the above equation can be trivially majorised as

$$\mathbb{P}(\max_{y\in\mathbb{V}_2}(\eta_{2n-1}(y)\ge d_{n,2}|Y_{2n}=0) \le \sum_{y\in\mathbb{V}_2}\frac{\mathbb{P}(\eta_{2n-1}(y)\ge d_{n,2})}{\mathbb{P}(Y_{2n}=0)}.$$

Now,

$$\mathbb{P}_0(\eta_{2n-1}(y)\ge d_{n,2}) \le \mathbb{P}_y(\sigma_{y,[d_{n,2}]}\le 2n)$$

where $\sigma_{y,k}$ denotes the time of k^{th} return to point y for the skeleton random walk. As a matter of fact, the occupation time, $\eta_{2n-1}(y)$, of level y for the skeleton random walk (Y_k) can exceed the threshold $[d_{n,2}]$ whenever the random walk (Y_k), starting at the origin, attains level y before time $2n-1$ and then returns to this level at least $[d_{n,2}]$ times before time $2n-1$. Therefore,

$$\begin{aligned}
\mathbb{P}_0(\eta_{2n-1}(y)\ge d_{n,2}) &\le \mathbb{P}_y\sigma_{y,[d_{n,2}]}\le 2n) \\
&\le \mathbb{P}_y(\exp(-t\sigma_{y,[d_{n,2}]})\ge \exp(-2nt)) \\
&\le \exp(2nt)\mathbb{E}_0(\exp(-t\sigma_{[d_{n,2}]})) \\
&= \exp(2nt)(1-\sqrt{1-\exp(-2t)})^{[d_{n,2}]}, \ \ \forall t>0 \\
&\le \inf_{t>0}\exp(2nt)(1-\sqrt{1-\exp(-2t)})^{[d_{n,2}]} \\
&= \exp(-cn^{\delta_2}),
\end{aligned}$$

for some positive constant c, uniformly in y. Using the well known estimate $\mathbb{P}_0(Y_{2n} = 0) = \mathcal{O}(n^{-\frac{1}{2}})$, and the fact that the sum $\sum_{y\in\mathbb{V}_2}$ is performed on the set $\{Y_{2n} = 0\}$, containing thus at most $2n + 1$ terms, we get the overall bound

$$\mathbb{P}(A^c_{n,2}|Y_{2n} = 0) \leq Cnn^{\frac{1}{2}} \exp(-cn^{\delta_2}).$$

Choosing finally $0 < \delta'_2 < \delta_2$ and $\delta = \min(2\delta_1, \delta'_2) > 0$ we conclude that

$$\mathbb{P}(A^c_n|Y_{2n} = 0) \leq 2\exp(-n^{2\delta-1}) + C\exp(-cn^{\delta'_2}) = \mathcal{O}(\exp(-cn^{\delta})).$$

□

Corollary 4.2 *We have* $\sum_{n\in\mathbb{N}} p_{n,3} < \infty$.

Recall that we have

$$X_{2n} = \sum_{y\in\mathbb{V}_2} \epsilon_y \sum_{i=1}^{\eta_{2n-1}(y)} \xi^{(y)}_i = \sum_{k=1}^{2n} \epsilon_{Y_k}\xi_k.$$

Introduce the random variables:

$$\begin{aligned} N_+ &= \sum_{k=1}^{2n} 1\!\!1_{\{\epsilon_{Y_k}=1\}} \\ N_- &= \sum_{k=1}^{2n} 1\!\!1_{\{\epsilon_{Y_k}=-1\}} \\ \Delta_n &= N_+ - N_- = \sum_{y\in\mathbb{V}_2} \epsilon_y\eta_{2n-1}(y). \end{aligned}$$

Proposition 4.3 *For large n, we have*

$$p_{n,1} = \mathcal{O}(\exp(-n^{\delta'}))$$

for any $\delta' \in]0, 2\delta_3[$.

Proof: Using ideas in the proof of lemma 2.9, it is enough to show that $\mathbb{P}(X_{2n} = 0; Y_{2n} = 0; B_n) = \mathcal{O}(\exp(-n^{\delta'}))$. Remark that N_+, N_-, and Δ_n are $\mathcal{F}_{2n}\vee\mathcal{G}$-measurable and $N_++N_- = 2n$. Denoting $m_1 = \mathbb{E}\xi_1$, $m_2 = \mathbb{E}(\xi^2_1)$ and $s^2 = m_2 - m^2_1$, we have

$$\begin{aligned} \mathbb{E}(X_{2n}|\mathcal{F}_{2n} \vee \mathcal{G}) &= m_1\Delta_n \\ \mathbb{E}(X^2_{2n}|\mathcal{F}_{2n} \vee \mathcal{G}) &= 2ns^2 + m^2_1\Delta^2_n \\ \mathrm{Var}(X_{2n}|\mathcal{F}_{2n} \vee \mathcal{G}) &= 2ns^2. \end{aligned}$$

For $t \in]-\infty, -\ln q[$, define the generating function for the random variable ξ_1, namely $\phi(t) = \mathbb{E}\exp(t\xi_1)$. Obviously, for small values of $|t|$, the generating function behaves like $\phi(t) = \exp(tm_1 + t^2s^2/2 + \mathcal{O}(t^3))$. Hence

$$\mathbb{E}(\exp(tX_{2n})|\mathcal{F}_{2n} \vee \mathcal{G}) = \phi(t)^{N_+}\phi(-t)^{N_-} = \exp(tm_1\Delta_n + t^2s^2n + \mathcal{O}(t^3n)).$$

Assume for the moment that $\Delta_n > d_{n,3}$. Using Markov inequality, we have for $t < 0$,

$$\begin{aligned} \mathbb{P}(X_{2n} = 0|\mathcal{F}_{2n} \vee \mathcal{G}) &\leq \mathbb{P}(X_{2n} \leq 0|\mathcal{F}_{2n} \vee \mathcal{G}) \\ &\leq \mathbb{E}(\exp(tX_{2n})|\mathcal{F}_{2n} \vee \mathcal{G}) \\ &= \exp(tm_1\Delta_n + t^2s^2n + \mathcal{O}(t^3n)). \end{aligned}$$

Now, choose $t = -\frac{m_1 n^{\delta_3 - 1/2}}{2s^2}$. Hence, on $\{\Delta_n > d_{n,3}\}$, we have

$$\mathbb{P}(X_{2n} = 0|\mathcal{F}_{2n} \vee \mathcal{G}) \leq \exp(-\frac{m_1^2}{4s^2}n^{2\delta_3} + \mathcal{O}(n^{3\delta_3 - 1/2})).$$

For the case $\{\Delta_n < -d_{n,3}\}$, we conclude similarly, majorising $\mathbb{P}(X_{2n} = 0|\mathcal{F}_{2n} \vee \mathcal{G}) \leq \exp(tm_1\Delta_n + t^2s^2n + \mathcal{O}(t^3n))$, for $t \in]0, -\ln q[$, and choosing for large n, $t = \frac{m_1 n^{\delta_3 - 1/2}}{2s^2}$.

Corollary 4.4 *We have* $\sum_{n\in\mathbb{N}} p_{n,1} < \infty$.

To conclude about transience, it remains to estimate the probability $p_{n,2} = \mathbb{P}(X_{2n} = 0; Y_{2n} = 0; A_n \setminus B_n)$.

Lemma 4.5 *On the set* $A_n \setminus B_n$*, we have*

$$\mathbb{P}(X_{2n} = 0|\mathcal{F} \vee \mathcal{G}) = \mathcal{O}(\sqrt{\frac{\ln n}{n}}).$$

Proof: Use the $\mathcal{F} \vee \mathcal{G}$-measurability of the variables $(\epsilon_y)_{y\in\mathbb{V}_2}$ and $(\eta_n(y))_{y\in\mathbb{V}_2, n\in\mathbb{N}}$ to express the conditional characteristic function of the variable X_{2n} as follows:

$$\chi_1(\theta) = \mathbb{E}(\exp(i\theta X_{2n})|\mathcal{F} \vee \mathcal{G}) = \prod_{y\in\mathbb{V}_2} \chi(\theta\epsilon_y)^{\eta_{2n-1}(y)}.$$

Hence,

$$\mathbb{P}(X_{2n} = 0|\mathcal{F} \vee \mathcal{G}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_1(\theta)d\theta.$$

Now use the decomposition of χ into a the modulus part, $r(\theta)$ — that is an even function of θ — and the angular part of $\alpha(\theta)$ and the fact that there is a constant $K < 1$ such that for $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$ we can bound $r(\theta) < K$ to majorise

$$\mathbb{P}(X_{2n} = 0|\mathcal{F} \vee \mathcal{G}) \leq \frac{1}{\pi}\int_0^{\pi/2} r(\theta)^{2n}d\theta + \mathcal{O}(K^n).$$

Fix $a_n = \sqrt{\frac{\ln n}{n}}$ and split the above integral over $[0, \pi/2] = [0, a_n] \cup [a_n, \pi/2]$. For the first part, we majorise the integrand by 1, so that

$$\int_0^{a_n} r(\theta)^{2n} d\theta \leq a_n.$$

For the second part, use the majorisation $r(\theta) \leq \exp(-\frac{3}{8}\theta^2)$ valid for $\theta \in]0, \pi/2]$ to estimate

$$\frac{1}{\pi} \int_{a_n}^{\pi/2} r(\theta)^{2n} d\theta = \mathcal{O}(n^{-3/4}).$$

Since the estimate of the first part dominates, the result follows. $\square$

Proposition 4.6 *For all $\delta_5 > 0$, and for large n*

$$\mathbb{P}(A_n \setminus B_n | \mathcal{F}) = \mathcal{O}(n^{-\frac{1}{4}+\delta_5}).$$

Proof: The required probability is an estimate, on the event A_n, of the conditional probability $\mathbb{P}(|\sum_{y \in \mathbb{V}_2} \zeta_y| \leq d_{n,3} | \mathcal{F})$, where we denote $\zeta_y = \epsilon_y \eta_{2n-1}(y)$. Extend the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ to carry an auxilliary variable G assumed to be centered Gaussian with variance $d_{n,3}^2$, (conditionally on $\mathcal{F}$) independent of the ζ_y's. Since both G and $\sum_{y \in \mathbb{V}_2} \zeta_y$ are (conditionally on $\mathcal{F}$) symmetric random variables and $[-d_{n,3}, d_{n,3}]$ is a symmetric set around 0, then by Anderson's inequality, there exists a positive constant c such that

$$\mathbb{P}(|\sum_{y \in \mathbb{V}_2} \zeta_y| \leq d_{n,3} | \mathcal{F}) \leq c \mathbb{P}(|\sum_{y \in \mathbb{V}_2} \zeta_y + G| \leq d_{n,3} | \mathcal{F}).$$

Let

$$\chi_2(t) = \mathbb{E}(\exp(it \sum_y \zeta_y) | \mathcal{F}) = \prod_y \cos(\eta_{2n-1}(y) t),$$

and

$$\chi_3(t) = \mathbb{E}(\exp(itG) | \mathcal{F}) = \exp(-t^2 d_{n,3}^2 / 2).$$

Therefore,

$$\mathbb{E}(\exp(it(\sum_y \zeta_y + G)) | \mathcal{F}) = \chi_2(t) \chi_3(t),$$

and using the Plancherel's formula,

$$\mathbb{P}(|\sum_{y \in \mathbb{V}_2} \zeta_y + G| \leq d_{n,3} | \mathcal{F}) = \frac{d_{n,3}}{\pi} \int \frac{\sin(t d_{n,3})}{t d_{n,3}} \chi_2(t) \chi_3(t) dt \leq C d_{n,3} I,$$

where

$$I = \int \prod_y \cos(\eta_{2n-1}(y)t) \exp(-t^2 d_{n,3}^2/2) dt.$$

Fix $b_n = \frac{n^{\delta_4}}{d_{n,3}}$, for some $\delta_4 > 0$ and split the integral defining I into $I_1 + I_2$, the first part being for $|t| \leq b_n$ and the second for $|t| > b_n$.

We have

$$\begin{aligned} I_2 &\leq C \int_{|t|>b_n} \exp(-t^2 d_{n,3}^2/2) \frac{dt}{2\pi} \\ &= \frac{C}{d_{n,3}} \int_{|s|>n^{\delta_4}} \exp(-s^2/2) \frac{ds}{2\pi} \\ &\leq 2 \frac{C}{d_{n,3}} \frac{1}{n^{\delta_4}} \frac{\exp(-n^{2\delta_4}/2)}{2\pi}, \end{aligned}$$

because the probability that a centred normal random variable of variance 1, whose density is denoted ϕ, exceeds a threshold $x > 0$ is majorised by $\frac{\phi(x)}{x}$.

For I_1 we get,

$$I_1 \leq \int_{|t|\leq b_n} \prod_y |\cos(\eta_{2n-1}(y)t)| dt.$$

Now, $\sum_y \frac{\eta_{2n-1}(y)}{2n} = 1$. Therefore, applying Hlder's inequality we obtain

$$\begin{aligned} I_1 &\leq \prod_y \left[\left(\int_{|t|\leq b_n} |\cos(\eta_{2n-1}(y)t)|^{\frac{2n}{\eta_{2n-1}(y)}} dt \right)^{\frac{\eta_{2n-1}(y)}{2n}} \right] \\ &\leq \sup_{y:\eta_{2n-1}(y)\neq 0} \int_{|t|\leq b_n} |\cos(\eta_{2n-1}(y)t)|^{\frac{2n}{\eta_{2n-1}(y)}} dt, \end{aligned}$$

because the terms in $|cos(\cdot)|$ in the integrand are less than 1 and for $x \in [0, 1]$ and $p \geq 1$ we have that $x^p \leq x$.

Now, on the set A_n and for every $y : \eta_{2n-1}(y) \neq 0$, we have $|\eta_{2n-1}(y) \neq 0| \leq b_n d_{n,2} = \frac{d_{n,2}}{d_{n,3}} n^{\delta_4}$ and we can always choose the parametres $\delta_2, \delta_3, \delta_4$ so that $\delta_2 + \delta_4 - \delta_3 = -\delta_5 < 0$. For those y,

$$\begin{aligned} I_1 &\leq 2b_n \int_{|v|\leq\pi/2} |\cos(\frac{\eta_{2n-1}(y)}{d_{n,3}} n^{\delta_4} v)|^{\frac{2n}{\eta_{2n-1}(y)}} dv \\ &\leq 2b_n \int_{|v|\leq\pi/2} |\cos v|^{\frac{2n}{\eta_{2n-1}(y)}} dv. \end{aligned}$$

For $|v| < \pi/2$ we have that $|\cos v| \leq \exp(-cv^2/2)$, so that

$$\begin{aligned} I_1 &\leq 2b_n \int \exp(-c\frac{2nv^2}{\eta_{2n-1}(y)})dv \\ &\leq cb_n \frac{\eta_{2n-1}(y)^{1/2}}{n^{1/2}} \int \exp(-v^2/2)\frac{dv}{\sqrt{2\pi}} \\ &\leq c\frac{d_{n,2}^{1/2}}{d_{n,3}} \frac{n^{\delta_4}}{n^{1/2}}, \end{aligned}$$

and finally the overall probability is majorised by $\mathcal{O}(d_{n,3}(I_1 + I_2)) = \mathcal{O}(n^{-1/4})$. □

Corollary 4.7

$$\sum_{n\in\mathbb{N}} p_{n,2} < \infty.$$

Proof: Recalling that for the standard random walk $\mathbb{P}(Y_{2n} = 0) = \mathcal{O}(n^{-1/2})$ and from the estimates obtained in 4.5 and 4.6, we have

$$\begin{aligned} p_{n,2} &= \mathbb{P}(X_{2n} = 0; Y_{2n} = 0; A_n \setminus B_n) \\ &= \mathbb{E}(\mathbb{E}\left(\mathbb{1}_{Y_{2n}=0}\left[\mathbb{E}(\mathbb{1}_{A_n\setminus B_n}\mathbb{P}(X_{2n} = 0|\mathcal{F}\vee\mathcal{G})|\mathcal{F})\right]\right)) \\ &= \mathcal{O}(n^{-1/2}n^{-1/4}\sqrt{\frac{\ln n}{n}}) \\ &= \mathcal{O}(n^{-5/4}\ln n), \end{aligned}$$

proving thus the summability of $p_{n,2}$. □

Theorem 4.8 *For almost all realisations of the random environment ϵ, the random walk on the lattice $\mathbb{O}_\epsilon$ is transient.*

Proof: The transience is a simple consequence the previous propositions. As a matter of fact $p_n = p_{n,1} + p_{n,2} + p_{n,3}$ is summable because the partial probabilities $p_{n,i}$, for $i = 1, 2, 3$ are all shown to be summable. □

4.3 Strong law of large numbers for the random walk on $\mathbb{O}_\epsilon$

Proposition 4.9 *Denote $m_1 = \mathbb{E}\xi_1$. Then conditionally on $\mathcal{F}\vee\mathcal{G}$, the ratio $\frac{X_n - m_1\Delta_n}{n}$ converges in probability to 0, i.e. for all $\lambda > 0$,*

$$\lim_{n\to\infty} \mathbb{P}(|\frac{X_n - m_1\Delta_n}{n}| \geq \lambda|\mathcal{F}\vee\mathcal{G}) = 0.$$

Proof: Since $X_n = \sum_y \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$, it follows that $\mathbb{E}(X_n|\mathcal{F}\vee\mathcal{G}) = m_1(N_+ - N_-) = m_1\Delta_n$. Letting $m_2 = \mathbb{E}(\xi_1^2)$ and $s^2 = m_2 - m_1^2$, we have by developing

$$\begin{aligned} X_n^2 &= \sum_y \sum_{i=1}^{\eta_{n-1}(y)} (\xi_i^{(y)})^2 + \sum_y \sum_{i_1=1}^{\eta_{n-1}(y)} \sum_{i_2=1}^{\eta_{n-1}(y)} \xi_{i_1}^{(y)}\xi_{i_2}^{(y)} \\ &\quad + \sum_{y_1}\sum_{y_2\neq y_1} \epsilon_{y_1}\epsilon_{y_2} \sum_{i_1=1}^{\eta_{n-1}(y_1)} \sum_{i_2=1}^{\eta_{n-1}(y_2)} \xi_{i_1}^{(y_1)}\xi_{i_2}^{(y_2)}, \end{aligned}$$

that $\mathbb{E}(X_n^2|\mathcal{F}\vee\mathcal{G}) = s^2 n + m_1^2\Delta_n^2$. Consequently, $\mathbb{E}((X_n - m_1\Delta_n)^2|\mathcal{F}\vee\mathcal{G}) = s^2 n$ and the result follows by a straightforward application of Chebychev's inequality. □

Proposition 4.10 *We have*

$$\lim_{n\to\infty} \frac{\Delta_n}{n} = 0 \ \textit{almost surely.}$$

Proof: Since $\Delta_n = \sum_y \epsilon_y \eta_{n-1}(y)$, using the symmetry of the random variables ϵ_y, it follows immediately that $\mathbb{E}(\Delta_n|\mathcal{F}) = 0$. For some positive integer r, compute $\mathbb{E}(\Delta_n^{2r}|\mathcal{F})$. Using again the symmetry of the random variables ϵ_y, only terms containing even powers of ϵ_y will survive. Among these terms, the dominant one for large n is the term $\sum_{y_1} \eta_{n-1}(y_1)^2 \ldots \sum_{y_r} \eta_{n-1}(y_r)^2$ and each sum appearing is estimated by

$$\begin{aligned} \sum_y \eta_{n-1}(y)^2 &\leq \max_y \eta_{n-1}(y) \sum_y \eta_{n-1}(y) \\ &= \max_y \eta_{n-1}(y) n \\ &\leq d_{n,2} n 1\!\!1_{A_{n,2}} + n^2 1\!\!1_{A_{n,2}^c}. \end{aligned}$$

It follows that

$$\mathbb{E}(\Delta_n^{2r}|\mathcal{F}) \leq n^r d_{n,3}^r 1\!\!1_{A_{n,2}} + n^{2r} 1\!\!1_{A_{n,2}^c}$$

and consequently, using the estimate $\mathbb{P}(A_{n,2}^c) \leq \exp(-n^{\delta_2})$, obtained in the proof of proposition 4.1, we get

$$\mathbb{E}(\Delta_n^{2r}) \leq n^{3r/2 + r\delta_2} + n^{2r}\exp(-n^{\delta_2}),$$

so that, for $r \geq 3$,

$$\sum_{n\in\mathbb{N}} \mathbb{E}(\frac{\Delta_n^{2r}}{n^{2r}}) < \infty.$$

The result follows by straightforward application of Borel-Cantelli lemma. □

Remark: Using similar arguments (choosing r sufficiently large), it can easily be shown that for every $\beta > 3/4$,

$$\lim_{n\to\infty} \frac{\Delta_n}{n^\beta} = 0 \ \text{almost surely.}$$

Theorem 4.11 *The embedded random walk (X_n) on the $\mathbb{O}_\epsilon$ lattice has almost surely zero speed, i.e.*

$$\lim_{n\to\infty} \frac{X_n}{n} = 0, \quad \textit{almost surely.}$$

Proof: It is enough to show that for some positive integer r,

$$\sum_{n\in\mathbb{N}} \mathbb{E}(\frac{(X_n - m_1\Delta_n)^{2r}}{n^{2r}}) < \infty$$

since then the almost sure convergence to 0 of X_n/n will follow from the almost sure convergence of Δ_n/n to 0. Following exactly the same scheme as in the previous proposition, start by developing $\mathbb{E}(X_n^{2r}|\mathcal{F} \vee \mathcal{H})$. The symmetry of the random variables ϵ_y over which we integrate, guarantees that only terms with even powers of each random variable ϵ_y will remain. Perform now the integration over the random variables ξ. As it was the case in the proof of the previous proposition, the expectation $\mathbb{E}(X_n^{2r}|\mathcal{F})$ is estimated by $C(\sum_y \eta_{n-1}(y)^2)^r$ and the individual sum over y is again estimated as previously. Thus choosing $s \geq 3$ the result follows from Borel-Cantelli lemma. $\square$

5 Conclusion, open problems, and further developments

It is shown that random walks on oriented lattices exhibit novel and interesting features and arise in many situations in topological field theories. It was quite surprising for us to discover that so little was previously known about this kind of random walk, in sharp contrast with random walks on undirected lattices for which there are several hundreds of papers and also excellent books (like [10, 11] not to mention but the two more complete ones.)

For the alternate lattice $\mathbb{L}$, several different proofs for the recurrence can be given; we chose to present here the most elementary one. The main point is that the lattice is still periodic in the vertical direction. So group arguments can be used to prove the recurrence instead of our proof. Another possibility is to regroup even and odd ordinates to a new effective lattice that it is not any more oriented. This approach has been used in [4] for Manhattan lattices. It is easy to see that instead of taking period 2 in the vertical direction we use another arbitrary periodicity, *i.e.* take horizontal strips of width l such that all the horizontal lines inside a given strip are going in the same direction but the directions alternate for every strip to the following one, the lattice is fundamentally the same as $\mathbb{L}$ so that the random walk is still recurrent. The overall asymptotic behaviour of all these random walks is quite reminiscent of the random walk on unoriented ordinary $\mathbb{Z}^2$ lattice.

The lattice $\mathbb{H}$ is quite different. Here the random walk is transient. It can be shown that it has still a zero speed but a non zero angular speed, so that asymptotically it has infinite winding number around the origin. The study of the detailed behaviour of this walk is postponed in a subsequent publication.

The lattice $\mathbb{O}_\epsilon$ has a very rich structure. The reason for which this lattice is transient is the presence and the size of fluctuations. It should be interesting to ask whether other random environments modify this characteristic. Another interesting question is whether the random walk on this lattice verifies a functional limit theorem possibly with unconventional normalisation. All these questions are under investigation.

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