Stochastic variational principle on riemannian manifolds

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Most of the conceptual appeal of Stochastic Mechanics [4] disappears as soon as one realizes that it may be ultimatively seen as a corollary of Fukushima's Theorem on symmetric Dirichlet forms (generalized by Albeverio, Ma and Röckner for non-symmetric Dirichlet forms) applied in the **ground-state representation** (sometimes called the **Darboux representation**) [5], [7], [8],[6].

Given an Hilbert space $\mathcal{H}$ and a positive, symmetric, bilinear form on $\mathcal{H}$ $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \rightarrow \mathbb{R}$ (a **symmetric form** from here and beyond) defined on a dense subset $D(\mathcal{E}) \times D(\mathcal{E})$ of $\mathcal{H} \times \mathcal{H}$ (that I will denote as $(D(\mathcal{E}), \mathcal{E})$) let us introduce the symmetric form on $\mathcal{H}$:

$$\mathcal{E}_1 : D(\mathcal{E}) \times D(\mathcal{E}) \rightarrow \mathbb{R} \tag{1}$$

$$\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + (f,g) \tag{2}$$

Clearly $(D(\mathcal{E}), \mathcal{E}_1)$ is a **pre-Hilbert space**.

We will say that the form $(D(\mathcal{E}), \mathcal{E})$ is **closed** if $(D(\mathcal{E}), \mathcal{E}_1)$ is also an **Hilbert space**, i.e. if it is complete in the norm $\| \cdot \|_1 := \sqrt{\mathcal{E}_1(\cdot, \cdot)}$.

We will say that $(D(\mathcal{E}), \mathcal{E})$ is **closable** if it has a closed extension.

In this case we define the **closure** of $\mathcal{E}$ as the smallest closed extension $(D(\bar{\mathcal{E}}), \bar{\mathcal{E}})$ of $(D(\mathcal{E}), \mathcal{E})$.

The closure condition guarantees the existence of a whole plethora of bijective correspondences:

1. there is a one-to-one correspondence between **closed symmetric forms** and **positive self-adjoint operators** on $\mathcal{H}$ according to which the operator $(D(H), H)$ corresponds the closed symmetric form:

$$(D(E) := D(\sqrt{H}) \, , \, E(f,g) := (\sqrt{H} \, f \, , \, \sqrt{H} \, g) \tag{3}$$

2. there is a one-to-one correspondence between **positive self-adjoint operators** and **strongly continuous contraction semigroups** on $\mathcal{H}$ according to which the operator $(D(H), H)$ corresponds to the strongly-continuous contraction semigroup $(T_t := e^{-tH})_{t \geq 0}$

3. there is a one-to-one correspondence between **strongly continuous contraction semigroups** and **strongly continuous contraction resolvents** according to which the strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ corresponds to the strongly-continuous contraction-resolvent $(G_\alpha)_{\alpha \geq 0}$:

$$G_\alpha f := \int_0^\infty e^{-\alpha s} \, T_s f \, ds \ \ \forall f \in \mathcal{H} \tag{4}$$

The full power of this chain of one-to-one correspondences arises, anyway, when one consider **Dirichlet forms**:

in fact, for Dirichlet forms, one can add a link to the chain of one-to-one correspondences associating to a Dirichlet form the Markovian stochastic process having as transition-probability-function the integral kernel of the associated strongly-continuous contraction semigroup.

Supponing that $\mathcal{H} = L^2(M,\mu)$ for a proper Borel measure space $(M\,,\,\mathcal{B}\,,\,\mu)$ will say that the **closed symmetric form**: $(D(\mathcal{E}),\mathcal{E})$ is a **Dirichlet form** iff:

$$\mathcal{E}(f^\sharp,f^\sharp) \;\leq\; \mathcal{E}(f,f) \;\;\forall f \in D(\mathcal{E}) \tag{5}$$

where $f^\sharp \;:=\; \min(1,\max(f,0))$.

The **Dirichlet-condition** of a symmetric form corresponds to specular conditions on the other links of the chain of one-to-one correspondences above introduced:

namely one has that:

$$(D(\mathcal{E}),\mathcal{E}) \text{ is Dirichlet } \;\Leftrightarrow\; (D(H),H) \text{ is Dirichlet } \;\Leftrightarrow\; (T_t)_{t\geq 0} \text{ is markovian} \tag{6}$$

where a self-adjoint operator $(D(A),A)$ is said **Dirichlet** iff:

$$(Af\,,\,\max(0,f-1)) \;\geq\; 0 \;\;\forall f \in D(A) \tag{7}$$

while a strongly-continuous contraction semigroup is said **markovian** iff:

$$0 \leq f \;\leq\; 1 \;\Rightarrow\; 0 \leq T_t f \;\leq\; 1 \;\;\forall f \in \mathcal{H} \tag{8}$$

Let us now assume that $(M,\mathcal{B})$ is Hausdorff and that the measure μ is $\sigma - finite$.

We will say that a symmetric form $\mathcal{E}$ is **properly associated** with a Markov process $(X_t)_{t\geq 0}$ on M if $E.[f(X_t)]$ is an $\mathcal{E}-$quasi-continuous μ-version of $(e^{-tH}f)(\cdot)$ where $\mathcal{E}$-quasi-continuous means continuous modulo $\mathcal{E}$- exceptional sets, i.e. modulo set $S \subset M$:

$$\mathcal{E} - cap(S) \;=\; 0 \tag{9}$$

where the **$\mathcal{E}$-capacity** of S is defined as:

$$\mathcal{E} - cap(S) \;:=\; \inf\{\mathcal{E}_1(f,f) \;:\; f \in D(\mathcal{E})\,,\; f \geq 1\,,\mu - a.e.\; on\; S\,\} \tag{10}$$

if S is open, and:

$$\mathcal{E} - cap(S) \;:=\; \inf\{\mathcal{E} - cap(U) \;: U\; open\; U \supset S\} \tag{11}$$

if S is arbitrary.

We will say that a Dirichlet form $(D(\mathcal{E}),\mathcal{E})$ is **regular** if $C_0(M)\bigcap D(\mathcal{E})$ is dense both in $(C_0(M)\,,\,\|\cdot\|_\infty)$ and in $(D(\mathcal{E})\,,\,\|\cdot\|_1)$, where $C_0(M)$ denotes the set of compact-supported functions on M, $\|\cdot\|_\infty$ denotes the usual uniform norm, while $\|\cdot\|_1 := \sqrt{\mathcal{E}_1(\cdot,\cdot)}$.

The fundamental Fukushima Theorem grants that if a Dirichlet form is regular than it is properly associated to a Markov process (X_t).

Let us now pass to analyze concretely the more important application of this general mathematical framework, starting from the definition of the brownian motion on (M , g):

the starting point is the symmetric form:

$$(D(E) := C_0^\infty(M) \ , \ E(f,g) \ := < df, dg >) \tag{12}$$

where $C_0^\infty(M)$ denotes the set of compact-supported smooth functions on M, while $< \cdot, \cdot >$ is the **g - metric induced scalar product** on the space $\Gamma(T^{(0,n)}M)$ of the sections of the (0,n)-tensor bundle (i.e. the space of the n-differential forms on M):

$$< \omega, \nu > := \int_M \omega \wedge \star \nu \ \ \omega, \nu \in \Gamma(T^{(0,n)}M) \, , \, n \in \mathbb{N} \tag{13}$$

where $\star$ is the Hodge duality operator on (M,g).

The closure $(D(\bar{\mathcal{E}}), \bar{\mathcal{E}})$ is a Dirichlet form, usually called the **heat Dirichlet form on (M , g)** whose associated positive self-adjoint operator is precisely the Laplace-Beltrami operator defined on the 2^{th} Sobolev space on M:

$$(-\Delta_g \ := \ d\, d^\dagger \ + \ d^\dagger \, d \ , \ H^2(M)) \tag{14}$$

The **brownian motion on (M,g)** is then defined precisely as the Markov process on (M,g) properly associated to the **heat Dirichlet form on (M , g)**.

Let us now modify a bit the scenario introducing the Hilbert space:

$$\mathcal{H} \ := \ L^2(M, d\mu_g) \tag{15}$$

and associating to a generic **wave-function** $\psi \in \mathcal{H}$ the symmetric form:

$$(D(E_\psi) := C_0^\infty(M) \ , \ E_\psi(f,g) \ := \int_M |\psi|^2 df \ \wedge \star dg) \tag{16}$$

on the Hilbert space:

$$\mathcal{H}_\psi \ := \ L^2(M, |\psi|^2 \, d\mu_g) \tag{17}$$

Since its closure $(D(\bar{\mathcal{E}_\psi}), \bar{\mathcal{E}_\psi})$ is a Dirichlet form, the Fukushima's Theorem allows to associate to the **wave-function** ψ the **markovian stochastic process** x_ψ over (M , g) properly associated to $(D(\bar{\mathcal{E}_\psi}), \bar{\mathcal{E}_\psi})$.

For example, in the particular case in which (M,g) is the one-dimensional euclidean space $(\mathbb{R}, \ \delta)$ while:

$$\psi(y) \ := \ \frac{1}{\pi^{\frac{1}{4}}} \, e^{-\frac{y^2}{2}} \tag{18}$$

is the ground-state wave-function of the harmonic-oscillator's hamiltonian, x_ψ is the Ornstein-Uhlenbeck stochastic process over $(\mathbb{R}, \ \delta)$.

Returning to the general case let us denote by $\hat{H}_{\psi}$ the positive self-adjoint operator over $\mathcal{H}_{\psi}$ associated to the Dirichlet form $(D(\bar{\mathcal{E}_{\psi}}), \bar{\mathcal{E}_{\psi}})$.

One has that ψ is the ground-state eigenfunction of $\hat{H}_{\psi}$, correponding to the zero eigenvalue.

It easily follows that ψ is the solution of the variational problem:

$$\delta S_1(\psi) \;=\; 0 \qquad (19)$$

$$\delta^2 S_1(\psi) \;>\; 0 \qquad (20)$$

where $S_1 \;:\; \mathcal{H} \mapsto \mathbb{R}$ is the functional:

$$S_1(\psi) \;:=\; \frac{< \psi|\hat{H}_{\psi}|\psi >}{< \psi|\psi >} \qquad (21)$$

Denoted by $\mathcal{M}(M, g)$ the set of all the markovian stochastic processes over (M , g) let us introduce the functional $S_2 \;:\; \mathcal{H} \mapsto \mathbb{R}$ defined as:

$$S_2(x_{\psi}) \;:=\; S_1(\psi) \qquad (22)$$

One has then that, obviously, x_{ψ} is the solution of the stochastic action principle:

$$\delta S_2(x_{\psi}) \;=\; 0 \qquad (23)$$

$$\delta^2 S_2(x_{\psi}) \;>\; 0 \qquad (24)$$

Let us now observe that such a stochastic variational principle is different and incompatible with Guerra's way of formulating the stochastic action principle on manifolds [3], [1],[2] as we will know show:

let us introduce the following one-parameter family of wave functions belonging to the Hilbert space $\mathcal{H} \;:=\; L^2(M, d\mu_g)$:

$$\psi_p(q) \;:=\; N(p) e^{\frac{(d_g(p,q))^2}{2}} \quad p \in M \qquad (25)$$

where $d_g(p, q)$ is the geodesic distance among the points p and q of (M,g), while N(p) is a normalization factor such that:

$$||\psi_p||^2_{\mathcal{H}} \;=\; 1 \;\; \forall p \in M \qquad (26)$$

Let us now observe that in the particular case in which (M, g) is the one-dimensional euclidean space $(\mathbb{R},\, \delta)$ and $p := 0$ the wave-function of eq.25 reduces to the one-dimensional harmonic-oscillator's ground-state's wave-function of eq.18 and conseguentially, the markovian process x_{ψ} reduces to the one-dimensional Ornstein-Uhlenbeck process.

This doesn't occurs, instead in the Guerra's approach to the same situation whose starting point is the ill-defined construction of a stochastic action functional that is:

- expressed in terms of local coordinates but not invariant under change of local coordinates

- dependent among the totally arbitrary choice of an affine connection that is not fixed to be the Levi-Civita connection of (M ,g) and whose variation is, in a totally inconsistent way, claimed to correspond to a sort of gauge transformation
- depends critically upon an ad-hoc regularization procedure, resulting in the more possibly arbitary renormalized stochastic action

The absolute mathematical inconsistence of such an approach is definitively proved by the own fact that, in the one-dimensional euclidean limit, the solution of the stochastic variational equation it gives rise to is not the Ornstein-Uhlenbeck process.

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