On persistence of invariant tori and a theorem by Nekhoroshev

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Summary. We give a proof of a theorem by N.N. Nekhoroshev concerning Hamiltonian systems with n degrees of freedom and s integrals of motion in involution, where $1 \leq s \leq n$. Such a theorem ensures persistence of s-dimensional invariant tori under suitable nondegeneracy conditions generalizing Poincaré's condition on the Floquet multipliers. We also deal in detail with perturbations of systems having reducible tori: in this case persistence can be ensured by a nonresonance condition expressed in terms of linear combinations of determinants involving the frequencies of the motion on the torus and the frequencies of small oscillations about the torus.

Introduction.

A Hamiltonian system in n degrees of freedom having n independent integrals of motion in involution is integrable [1]. When the system has a number s of independent integrals of motion greater than one, but smaller than the number of degrees of freedom, i.e. $1 < s < n$, we say the system is "partially integrable".

In the early nineties Nekhoroshev stated, under suitable nondegeneracy conditions, an interesting result on the existence of s-parameter families of tori in partially integrable Hamiltonian systems with n degrees of freedom ($1 < s < n$) [9]; this constitutes a bridge between the Poincaré-Lyapunov theorem (case of periodic trajectories, $s = 1$) and the Liouville-Arnold one (complete integrability, $s = n$).

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Unfortunately Nekhoroshev never published a proof of this theorem. Here we want to provide a proof of it; in addition we reformulate the nondegeneracy condition in an equivalent but simpler way.

We also extend the theorem to the perturbative framework and apply it to the study of persistence of reducible lower dimensional tori; in this context the nondegeneracy condition can be stated in purely algebraic terms. Finally we consider as concrete applications a three-body and a four-body problems. Our approach and result is applied elsewhere [3] to the study of quasiperiodic breathers in Hamiltonian lattices.

We recall that, roughly speaking (the precise statement will be recalled in sect. 1), Nekhoroshev's theorem states that if one has s integrals of motion in involution and an invariant torus $\Lambda \simeq \mathbf{T}^s$, then (under suitable nondegeneracy conditions) there is a $2s$–dimensional symplectic submanifold N, with $\Lambda \subset N$, which is fibered by invariant tori $\Lambda_\beta \simeq \mathbf{T}^s$; moreover it is possible to build action angle coordinates on N. The main problem lies in identifying the relevant nondegeneracy condition. We find that it can be expressed in terms of Floquet multipliers of periodic orbits of suitable Hamiltonian vector fields. In the case of elliptic tori the condition can be reformulated in a purely algebraic form.

The **plan of the paper** is as follows. In section 1 we recall the precise statement of Nekhoroshev's theorem, and provide an extension of it to the perturbative frame. Section 2 is devoted to a detailed proof of the main theorem, based on a lemma which provides "good" coordinates. In section 3 the theory is applied to the case of *reducible tori* (whose definition is recalled there); in such a case the nondegeneracy condition reduces to a condition on determinants of s-dimensional matrices, see eq.(3.19). The final section 4 is devoted to concrete applications, namely a three-body problem and a four-body problem.

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1 Statement of results

Let (M, Ω) be a symplectic manifold (with symplectic form Ω) of dimension $\dim(M) = 2n$, differentiable of class C^r ($r \geq 2$). Let $\mathbf{F} := \{F_1, ..., F_s\}$ be s real functions on M, also differentiable of class C^r; we assume they are independent and in involution; the latter condition means that $\{F_i, F_j\} = 0$, $i, j \in \{1, ..., s\}$.

We can then consider the Hamiltonian vector fields X_i generated by the functions F_i, namely defined by $X_i \lrcorner \Omega = \mathrm{d}F_i$; the conditions on the F_i imply that $[X_i, X_j] = 0$ and that all of the X_i admit the F_j's as first integrals.

We will *assume* that there is a compact and connected manifold $\Lambda \subset M$ of dimension s, differentiable of class C^r, which is invariant under all the X_i, and such that the X_i are linearly independent at all points $m \in \Lambda$. Obviously this Λ will be a submanifold of $\mathbf{F}^{-1}(\beta_0)$ for some $\beta_0 \in R^s$.

Note that the condition of independence of the X_i (and of the F_i) on Λ is equivalent to the requirement that

$$\eta := \mathrm{d}F_1 \wedge ... \wedge \mathrm{d}F_s \neq 0 \quad \text{on } \Lambda \ ;$$

since the F_i are smooth (i.e. C^1) functions, the relation $\eta \neq 0$ holds in a neighbourhood of Λ.

It is actually immediate to see that, being s-dimensional, connected, compact and invariant under the s commuting vector fields X_i, the C^r manifold Λ is necessarily a torus: $\Lambda = \mathbf{T}^s$; see e.g. the proof of the Liouville-Arnold theorem in [1]. Thus, under our assumptions the system has an invariant s-torus.

It is well known that for $s = 1$ (i.e. $\Lambda \approx S^1$ is a periodic orbit), if the Floquet multiplier 1 has multiplicity two, then Λ is part of a one-parameter local family of such orbits (Poincaré-Lyapunov theorem). We want to show that a similar property holds for arbitrary s, $1 \leq s \leq n$; the main problem lies in identifying the appropriate nondegeneracy condition.

In order to state the theorem we need to fix some notation.

First of all, using the same notation as in [1], we will denote by $g_i^t(.)$ the flow of the vector field X_i, and, for $\tau \equiv (\tau_1, \tau_2, ..., \tau_s) \in \mathcal{T} \subseteq \Re^s$, we define

$$g^\tau(p) = g_1^{\tau_1} \circ g_2^{\tau_2} \circ ... \circ g_s^{\tau_s}(p) \ . \tag{1.1}$$

Consider the first homotopy group $\pi_1(\Lambda) = \mathbf{Z}^s$ of Λ, and fix $\alpha = (\alpha_1, ..., \alpha_s) \in \pi_1(\Lambda)$; then there exist real constants $(c_1, ..., c_s) \equiv \mathbf{c}$ such that the flow of the vector field

$$X_\alpha \ := \ \sum_{i=1}^{s} \ c_i \, X_i \tag{1.2}$$

with initial data on Λ is periodic of period one (if $\alpha \not\equiv 0$) and has closed trajectories in the homotopy class α. Such periodic orbits are given by $g^{t\mathbf{c}}(m)$, $t \in [0, 1]$, $m \in \Lambda$.

We recall that for a fixed $m \in \Lambda$, the Floquet multipliers of the periodic orbit $g^{t\mathbf{c}}(m)$ are the eigenvalues of the Jacobian matrix $[dg^{\mathbf{c}}](m)$ (where the

differentiation is with respect to the variable m). It turns out (as discussed in lemma 2.9 below) that the Floquet multipliers of the orbit $g^{t\mathbf{c}}(m)$ do not depend on the point $m \in \Lambda$, but only on the vector field X_α. Moreover, due to the symmetries of the problem (see the proof of lemma 2.10 below), 1 is a Floquet multiplier of such a periodic orbit with multiplicity at least $2s$.

Theorem 1.3 (Nekhoroshev). *Let (M, Ω) be a $2n$-dimensional symplectic manifold, differentiable of class C^r. Let $\mathbf{F} := \{F_1, ..., F_s\}$, $1 \leq s \leq n$, be s real functions on M, differentiable of class C^r, and denote by X_i the corresponding Hamiltonian vector fields. Assume that:*
(i) *There exists a C^r compact and connected s-dimensional manifold $\Lambda \subset M$ invariant under all the X_i;*
(ii) *The functions F_i are independent and in involution (namely $\mathrm{d}F_1 \wedge ... \wedge \mathrm{d}F_s \neq 0$ and $\{F_i, F_j\} = 0$ $\forall i, j \in \{1, ..., s\}$) in a neighbourhood W of Λ;*
(iii) *There exists $\alpha \in \pi_1(\Lambda)$ such that the periodic orbits of X_α (as defined in 1.2) on Λ have Floquet multiplier 1 with multiplicity exactly equal to $2s$.*

Then:
(1) *In a neighbourhood $U \subseteq W$ of Λ in M, there is a symplectic submanifold $N \subset U$ of dimension $2s$ which is fibered over a domain $B \subset R^s$, having as fibers C^r-differentiable tori $\Lambda_\beta = N \cap \mathbf{F}^{-1}(\beta) \simeq \mathbf{T}^s$ (with $\beta \in B$), C^r-diffeomorphic to Λ and invariant under the X_i.*
(2) *There exist global symplectic action angle coordinates $(I_1, ..., I_s, \varphi_1, ..., \varphi_s)$ in N, such that the functions $F_i\big|_N$ are independent of the angles φ, i.e. $F_i\big|_N = F_i\big|_N(I_1, ..., I_s)$, with $i = 1, ..., s$.*

Remark 1. By (2) one has that, in terms of the coordinates (I, φ), the restriction of X_i to N has the expression

$$X_i\big|_N = \sum_j \frac{\partial F_i}{\partial I_j}(I)\, \frac{\partial}{\partial \varphi_j} \ ,$$

and therefore the flows of the vector fields X_i $i = 1, ..., s$ on the tori Λ_β are quasiperiodic or periodic depending on the resonance relation fulfilled by the frequencies $\omega_j^{(i)} := (\partial F_i / \partial I_j)$.

Remark 2. The theorem applies in particular to the case of a hamiltonian system with a compact symmetry group acting properly and freely, and which admits an Ad^* equivariant momentum map. To fix ideas suppose that F_1 is the Hamiltonian of the system and $F_2, ..., F_s$ are the integrals in involution related to the momentum map. In this case the invariant torus Λ is just a relative periodic orbit (namely a periodic orbit of the system obtained by the Marsden–Weinstein reduction procedure). Thus in this case the statement (1) is very close to the Poincaré continuation theorem for relative periodic orbits. We remark however that the assumptions of theorem 1.3 are given in terms of objects living in the original phase space, while the assumptions for the Poincaré

continuation theorem are naturally formulated in the reduced phase space and it is not obvious how to lift them to the original (non reduced) phase space. ⊙

Remark 3. Consider again the situation of the previous remark. If the symmetry group acts linearly, the solutions of the equations of motion of the Hamiltonian vector fields of $F_2, ..., F_s$ are automatically periodic also for initial data outside Λ. We stress that this is not assumed here. ⊙

Remark 4. A related result, based on topological arguments, has been recently proved by J.P.Ortega [10], who extended the Weinstein–Moser theorem in order to find a lower bound to the number of relative periodic orbits (i.e. invariant tori of the original system) close to relative equilibria. ⊙

It follows immediately from the proof of theorem 1.3 (given in the next section) that this result and its proof can be extended to a perturbative setting. In order to state this extension, consider a perturbation of the set of functions $F_1, ..., F_s$, namely, for ε in a neighbourhood E of zero in $\mathbf{R}$, define

$$F_i^\varepsilon \ := \ F_i \ + \ \varepsilon \, G_i^\varepsilon \ ,$$

where the G_i^ε are C^r functions in W (introduced above), and denote by X_i^ε the associated hamiltonian vector fields; now $X_i \equiv X_i^\varepsilon\big|_{\varepsilon=0} = X_i^0$ and X_α is the vector field defined in (1.2).

Theorem 1.4 *Let (M, Ω) be a $2n$-dimensional symplectic manifold. Let $\mathbf{F}_\varepsilon := \{F_1^\varepsilon, ..., F_s^\varepsilon\}$ be s real functions on $M \times E$, differentiable of class C^r jointly in the M variables and in ε; and denote by X_i^ε the corresponding Hamiltonian vector fields.*

Assume that:
(i) *there exists a C^r compact and connected s-dimensional manifold $\Lambda \subset M$ invariant under all the $X_i \equiv X_i^\varepsilon\big|_{\varepsilon=0}$;*
(ii) *the functions F_i^ε are independent and in involution – namely $\mathrm{d}F_1^\varepsilon \wedge ... \wedge \mathrm{d}F_s^\varepsilon \neq 0$ and $\{F_i^\varepsilon, F_j^\varepsilon\} = 0 \ \forall i, j \in \{1, ..., s\}$ – in a neighbourhood W of Λ, for all $\varepsilon \in E$;*
(iii) *there exists $\alpha \in \pi_1(\Lambda)$ such that the Floquet multiplier 1 of the periodic orbits of X_α^0 on Λ has multiplicity $2s$.*

Then there exists $\varepsilon_* > 0$, such that, for all $\varepsilon \in E_0$, $E_0 := (-\varepsilon_*, \varepsilon_*)$, the following holds true:
(1) *In a neighbourhood $U \subseteq W$ of Λ in M, there is a family of symplectic submanifolds $N^\varepsilon \subset U$, of dimension $2s$, which are fibered over a domain $B \subset R^s$ having as fibers C^r-differentiable tori $\Lambda_\beta^\varepsilon = N^\varepsilon \cap \mathbf{F}_\varepsilon^{-1}(\beta) \simeq \mathbf{T}^s$ (with $\beta \in B$). For each $\varepsilon \in E_0$, the tori $\Lambda_\beta^\varepsilon$ are C^r-diffeomorphic to Λ and invariant under the X_i^ε; the tori $\Lambda_\beta^\varepsilon$ and the manifolds N^ε depend in a C^r way jointly on $(\varepsilon, \beta) \in E_0 \times B$.*
(2) *There exist global symplectic action angle coordinates $(I_1^\varepsilon, ..., I_s^\varepsilon; \varphi_1^\varepsilon, ..., \varphi_s^\varepsilon)$ in N, such that $F_i^\varepsilon\big|_{N^\varepsilon} = F_i^\varepsilon\big|_{N^\varepsilon}(I_1^\varepsilon, ..., I_s^\varepsilon)$, with $i = 1, ..., s$. The coordinates $(I^\varepsilon; \varphi^\varepsilon)$ depend in a C^r way on $\varepsilon \in E$, and so do the functions $F_i^\varepsilon\big|_{N^\varepsilon}$.*

Remark 5. This result allows to continue invariant tori of a given system ($\varepsilon = 0$) to invariant tori of its perturbations ($\varepsilon \in E_0$). $\odot$

Remark 6. In the same situation one could try to apply KAM theory for lower dimensional tori [5, 6, 11, 4] (note that this would require a stronger nondegeneracy condition), and this would ensure persistence of a Cantor family of invariant tori; on the other hand we ensure here existence of a *continuous* family of invariant tori. In particular we ensure also persistence of the *resonant* tori. Obviously this is due to the fact that the systems admit some integrals of motions independent of the Hamiltonian, and thus our situation is exceptional. $\odot$

Remark 7. When studying infinite dimensional systems, one meets cases where a continuous spectrum arises, and this makes KAM theory non applicable at all. On the other hand theorem 1.4 extends immediately to some infinite dimensional situations of this kind; in particular it is used in [3] to construct quasiperiodic breathers in infinite lattices. $\odot$

In the perturbative case (i.e. in the case of theorem 1.4) the statement (2) is particularly useful in that it allows to characterize the dynamics on the invariant tori of the perturbed system. In particular one has the following

Corollary 1.5 *In the same hypotheses of theorem 1.4, assume also that for some $\kappa \in \{1, ..., s\}$, and for some $m \in \Lambda$ one has*

$$\det \left(\frac{\partial^2 F_\kappa^0}{\partial I_j \partial I_k} \right) (m) \neq 0 \ , \tag{1.6}$$

and fix ε small enough; then there exists a neighbourhood B_0 of β_0 in $\Re^s$ such that for almost all $\beta \in B_0$ the flow of X_κ^ε on $\Lambda_\beta^\varepsilon$ is quasiperiodic with s frequencies independent over the rationals.

Proof. By smooth dependence of action angle variables and of F_κ^ε on ε, eq.1.6 holds also for ε small but different from zero. It follows that the map from the actions to the frequencies is a local isomorphism. $\triangle$

2 Proofs

In this section we will always *assume that all the hypotheses of theorem 1.3 hold*, without stating this explicitly in each lemma.

In order to prove the two theorems stated in the previous section, the main point is to introduce suitable coordinates in a neighbourhood U of an arbitrary point of Λ. In order to do that we introduce, for any m in Λ a manifold Σ_m of codimension s passing through m. By the tubular neighbourhood theorem [8], the manifolds Σ_m can be chosen in such a way to define a foliation in U.

Lemma 2.7 *Let m be an arbitrary point in Λ. There exists a neighbourhood $\mathcal{V}_m \subset W \subset M$ of m and a coordinate map*

$$\mathcal{V}_m \ni p \mapsto (\beta, \tau, y) \in \mathbf{R}^s \times \mathbf{R}^s \times \mathbf{R}^{2(n-s)}$$

with the following properties (where we identify a point with its coordinates):
(i) $F_i(\beta, \tau, y) = \beta_i$,
(ii) $m = (\beta_0, 0, 0)$
(iii) $(\partial/\partial\tau_i) = X_i$, $\forall i = 1, ..., s$
(iv) $p \in \Sigma_m \iff p = (\beta, 0, y)$
(v) the coordinate map depends smoothly on m

Proof. First one has that $\mathbf{F}$ restricted to Σ_m is a submersion and therefore one can introduce coordinates (β, y) in Σ_m defined in an open set of $\mathbf{R}^s \times \mathbf{R}^{2(n-s)}$ with $\mathbf{F}(\beta, y) = \beta$ and such that the coordinates of $m \in \Sigma_m$ are $(\beta_0, 0)$.

For (τ, z) in some subset of $\mathbf{R}^s \times \Sigma_m$, consider the map $(\tau, z) \mapsto g^\tau(z)$. Using the coordinates (β, y) in Σ_m this can be expressed as a map $(\tau, \beta, y) \mapsto g^\tau(\beta, y)$. It is easily verified that its differential at m is an isomorphism and therefore, by the implicit function theorem, the (β, τ, y) are a local coordinate system at m, which by construction (and by commutativity of the flows g_i) have the properties stated in the lemma. $\triangle$

The coordinates constructed in this lemma will be called *adapted coordinates based at the point m*

Let us now fix a nontrivial homotopy class $\alpha \in \pi_1(\Lambda)$ and, as in the previous section, consider a vector field $X_\alpha = \sum_i c_i X_i$ such that its trajectories are periodic of period 1 on Λ and belong to the homotopy class α. For ease of writing, we denote by Φ the time 1 flow of X_α, i.e. $\Phi(p) = g^{\mathbf{c}}(p)$ where $\mathbf{c} := (c_1, ..., c_s)$.

Consider now a point $m \in \Lambda$ and introduce adapted coordinates based at m; we write $(\widehat{\beta}, \widehat{\tau}, \widehat{y}) := \Phi(\beta, \tau, y)$. Remark that since $\Phi(\beta_0, 0, 0) = (\beta_0, 0, 0) \equiv m$, by smooth dependence of solutions on initial data there exists a neighbourhood $\widetilde{\mathcal{V}}_m$ of m which is mapped under Φ in the domain $\mathcal{V}_m$ of definition of the above coordinates. We restrict Φ to such a neighbourhood.

Lemma 2.8 *In the adapted coordinates, the map $\Phi : (\beta, \tau, y) \to (\widehat{\beta}, \widehat{\tau}, \widehat{y})$ is described by*

$$\widehat{\beta}(\beta, \tau, y) = \beta \ ; \ \widehat{\tau}(\beta, \tau, y) = \tau + \tau_0(\beta, y) \ ,$$

with τ_0 a suitable function. Moreover,

$$\partial\widehat{y}/\partial\tau = 0 \ .$$

Proof. The first equality is a trivial consequence of the way the coordinates are defined. To prove the other two equalities fix $\mu \in \Re^s$ small and consider

$$\widehat{\beta}(\beta, \tau + \mu, y) \ ; \ \widehat{\tau}(\beta, \tau + \mu, y) \ ; \ \widehat{y}(\beta, \tau + \mu, y)$$

by definition these are the coordinates of the point

$$\Phi(\beta,\tau+\mu,y)=\Phi\left(g^{\mu}(\beta,\tau,y)\right)=g^{\mu}\left(\Phi\left(\beta,\tau,y\right)\right)=(\beta,\hat{\tau}(\beta,\tau,y)+\mu,\widehat{y}(\beta,\tau,y))\ .$$

From this

$$\widehat{\tau}(\beta,\tau+\mu,y)=\widehat{\tau}(\beta,\tau,y)+\mu\ ,\quad \widehat{y}(\beta,\tau+\mu,y)=\widehat{y}(\beta,\tau,y)$$

which shows that $\widehat{y}$ is independent of τ. To conclude the proof just put $\tau_0(\beta,y):=\hat{\tau}(\beta,0,y)$. $\triangle$

It is useful to remark that the Floquet multipliers of the periodic orbits of X_α do not depend on the initial point of the orbit in Λ.

Lemma 2.9 *Let m and m_1 be two points of Λ; then the Floquet multipliers of $g^{t\mathbf{c}}(m)$ and of $g^{t\mathbf{c}}(m_1)$ coincide.*

Proof. Remark that there exists τ_1 such that $m_1=g^{\tau_1}(m)$. Then, by the commutation of flows we have $g^{\mathbf{c}}=g^{-\tau_1}\circ g^{\mathbf{c}}\circ g^{\tau_1}$. Taking the differential (with respect to the space variable) of this equality at the point m we get

$$dg^{\mathbf{c}}(m)=dg^{-\tau_1}(m_1)dg^{\mathbf{c}}(m_1)dg^{\tau_1}(m)\ ,$$

which gives the relation between $dg^{\mathbf{c}}(m)$ and $dg^{\mathbf{c}}(m_1)$; the eigenvalues of the former are the Floquet multipliers at m, and the eigenvalues of the latter are the Floquet multipliers at m_1. This shows that $dg^{\mathbf{c}}(m)$ and $dg^{\mathbf{c}}(m_1)$ are conjugated and therefore have the same eigenvalues. $\triangle$

Lemma 2.10 *Fix $m\in\Lambda$ and introduce adapted coordinates based at m, consider the map $\Phi:(\beta,\tau,y)\to(\widehat{\beta},\widehat{\tau},\widehat{y})$; then there exists β_* independent of m such that*

$$\widehat{y}(\beta,\rho_m(\beta))=\rho_m(\beta)\ .$$

defines a unique smooth map $\rho_m(\beta)$ for all β with $|\beta-\beta_0|<\beta_*$.

Proof. In order to be guaranteed that we can solve the equation $\widehat{y}(\beta,y)=y$ by means of the implicit function theorem, we show now that our assumption (iii) on the Floquet multipliers (see theorems 1.3, 1.4) implies that 1 is not an eigenvalue of the Jacobian of the map $y\mapsto\hat{y}$ at $(\beta_0,0,0)$.

To prove this fact remark that by lemma 2.8 the Jacobian matrix J of Φ at m takes the block form

$$J=\begin{pmatrix}\dfrac{\partial\hat{\beta}}{\partial\beta} & \dfrac{\partial\hat{\tau}}{\partial\beta} & \dfrac{\partial\hat{y}}{\partial\beta}\\ \dfrac{\partial\hat{\beta}}{\partial\tau} & \dfrac{\partial\hat{\tau}}{\partial\tau} & \dfrac{\partial\hat{y}}{\partial\tau}\\ \dfrac{\partial\hat{\beta}}{\partial y} & \dfrac{\partial\hat{\tau}}{\partial y} & \dfrac{\partial\hat{y}}{\partial y}\end{pmatrix}=\begin{pmatrix}1 & \dfrac{\partial\tau_0}{\partial\beta} & \dfrac{\partial\hat{y}}{\partial\beta}\\ 0 & 1 & 0\\ 0 & \dfrac{\partial\tau_0}{\partial y} & \dfrac{\partial\hat{y}}{\partial y}\end{pmatrix}\ . \tag{2.11}$$

It follows that the secular equation for J takes the form (with I the identity matrix)

$$\det(\lambda I - J) \;=\; (\lambda - 1)^{2s} \;\det\left(\lambda I - \frac{\partial \hat{y}}{\partial y}\right) \;=\; 0$$

which gives the relation between the Floquet multipliers λ and the eigenvalues of the Jacobian matrix of the map $y \mapsto \widehat{y}$. As J has, by assumption, *exactly* $2s$ Floquet multipliers equal to 1, it follows that 1 is not an eigenvalue of $(\partial\widehat{y}/\partial y)$. Note that this also shows that we have always a multiplicity at least $2s$ for the eigenvalue 1

We can thus apply the implicit function theorem, which ensures existence uniqueness and smoothness of the map ρ_m. We can choose β_* independent of m: indeed by compactness of Λ the C^r norm of $\widehat{y}$ can be bounded uniformly with respect to m and the eigenvalues of $(\partial\widehat{y}/\partial y)$ are uniformly bounded away from 1, see lemma 2.9. $\triangle$

Define now the map $\sigma_\beta : \Lambda \to U$ as $\sigma_\beta(m) \equiv (\beta, 0, \rho_m(\beta))$ where we used adapted coordinates based at m. Remark that since $\sigma_\beta(m) \in \Sigma_m$, one has

$$\sigma_\beta(m) \neq \sigma_\beta(m') \quad \text{if} \quad m \neq m'.$$

Define

$$\Lambda_\beta := \sigma_\beta(\Lambda) \;,$$

and remark that this set is in one to one correspondence with Λ. We are going to prove that actually Λ_β is a smooth manifold, and that the above correspondence is a diffeomorphism. To this end, having fixed m and β with $|\beta - \beta_0| < \beta_*$, define

$$\mathcal{M}_m^\beta := \left\{ p \in \widetilde{\mathcal{V}}_m \;:\; p = (\beta, \tau, y) \;,\; y = \rho_m(\beta) \right\}$$

where we use adapted coordinates based at m. Then $\sigma_\beta(m) \in \mathcal{M}_m^\beta$, and moreover $\mathcal{M}_m^\beta$ is an s–dimensional smooth submanifold of M.

Lemma 2.12 *One has $\mathcal{M}_m^\beta \subset \Lambda_\beta$.*

Proof. Let $p \in \mathcal{M}_m^\beta$, $p \neq \sigma_\beta(m)$, then there exists a unique m_1 such that $p \in \Sigma_{m_1}$. Consider $\widehat{p} := \Phi(p)$. Using adapted coordinates based at m, if $p = (\beta, \tau, y)$ then, by definition of $\mathcal{M}_m^\beta$, the coordinates of $\hat{p}$ are $(\beta, \widehat{\tau}, y)$. From this it follows $g^{-(\widehat{\tau}-\tau)}(\widehat{p}) = p$. Introduce adapted coordinates (β, τ_1, y_1) based at the point m_1; one has $p = (\beta, 0, y_1)$, and denote

$$\widehat{p} = (\beta, \widehat{\tau}_1, \widehat{y}_1) \;.$$

Applying $g^{-(\widehat{\tau}-\tau)}$ we obtain

$$(\beta, \widehat{\tau}_1 - (\widehat{\tau} - \tau), \widehat{y}_1) = g^{-(\widehat{\tau}-\tau)}(\widehat{p}) = p = (\beta, 0, y_1)$$

from which in particular one has $y_1 = \widehat{y}_1(\beta, y_1)$. By uniqueness of the solution to $y_1 = \widehat{y}(\beta, y_1)$, it follows $y_1 = \rho_{m_1}(\beta)$ and therefore $p = \sigma_\beta(m_1)$. $\triangle$

Remark 8. The sets $\mathcal{M}_m^\beta$ are a covering of Λ_β: indeed $\mathcal{M}_m^\beta$ contains at least $\sigma_\beta(m)$.

Lemma 2.13 *Λ_β is a smooth compact manifold, C^r-diffeomorphic to Λ.*

Proof. First remark that the map

$$\psi_m^\beta : \mathcal{U}_m \to \mathcal{M}_m^\beta \quad , \quad \psi_m^\beta(\beta_0, \tau, 0) = (\beta, \tau, \rho_m(\beta))$$

(where we used adapted coordinates based at m) is smooth from a neighbourhood $\mathcal{U}_m$ of m in Λ to a neighbourhood of $\sigma_\beta(m)$ in Λ_β. Use this map to introduce the τ as local coordinates in Λ_β. Then the maps ψ_m^β yield a C^r atlas of Λ_β; the transition functions between the coordinate system τ on $\mathcal{M}_m^\beta$ and the coordinate system τ_1 on $\mathcal{M}_{m_1}^\beta$ are given simply by the functions $\tau_1(\beta, \tau, \rho_m(\beta))$, which are C^r-smooth. It follows that Λ_β is a C^r manifold; it is also C^r diffeomorphic (by the map σ_β) to Λ, and therefore is compact. $\triangle$

Remark 9. By construction the fields X_i are tangent to each of the $\mathcal{M}_m^\beta$, and therefore the manifold Λ_β is invariant under the flows of all the X_i.

Lemma 2.14 *The C^r manifold $\widetilde{N}$ obtained as the union of Λ_β for $|\beta - \beta_0| < \beta_*$ is symplectic and fibered in isotropic tori $\mathbf{T}^s$.*

Proof. It is clear that $\widetilde{N}$ is C^r and that it is fibered in tori $\mathbf{T}^s \equiv \Lambda_\beta$. The tori Λ_β are integral manifolds of the Hamiltonian vector fields generated by the functions F_i, and lie in common level manifolds of the F_i; thus we can choose a basis of variables τ_i along Λ_β such that $\Omega(\partial/\partial F_i, \partial/\partial \tau_j) = \delta_{ij}$. This shows at once that the Λ_β are isotropic and that the restriction $\Omega_{\widetilde{N}}$ of the symplectic form Ω defined on M to the submanifold $\widetilde{N} \subset M$ is non-degenerate. As $\mathrm{d}\Omega = 0$ implies $\mathrm{d}\Omega_{\widetilde{N}} = 0$, the proof is complete. $\triangle$

Corollary 2.15 *There exist a $2s$ dimensional symplectic submanifold N, $\Lambda \subset N \subset \widetilde{N}$, and global action angle coordinates on N.*

Proof. The restriction of the system to $\widetilde{N}$ is integrable in the Arnold Liouville sense. Thus the standard construction of action angle coordinates for integrable systems applies [1] and allows to construct action angle coordinates in a neighbourhood N of Λ in $\widetilde{N}$. $\triangle$

This concludes the proof of theorem 1.3. We leave to the reader to check that the same proof, with minor modifications, also applies to theorem 1.4.

3 Reducible tori

We want to apply the result proved above to the case of reducible tori. It is clear that only the Floquet multipliers lying on the unit circle in $\mathbf{C}$ can break the nondegeneracy condition required by theorems 1.3 and 1.4. Thus the case

of elliptic tori contains all the difficulties met in the application of Nekhoroshev theorem (and of its perturbative extension), so we will limit to consider this. Actually, due to a result by Kuksin [7], elliptic tori are always reducible under the hypotheses assumed in this paper.

Consider a Hamiltonian system with hamiltonian function H, and assume that it has an invariant s-dimensional torus Λ. It is known [2] that if $s = 1$ and the periodic orbit has Floquet multipliers with modulus 1, then in a neighbourhood of the periodic orbit there exists a system of canonical coordinates $(I, \psi, p, q) \in \mathbf{R} \times T^1 \times \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$, such that the Hamiltonian takes the form

$$\omega_1 I \ + \ \sum_{j=1}^{n-1} \nu_j \frac{p_j^2 + q_j^2}{2} + \texttt{h.o.t.} \ .$$

Here and below h.o.t. denotes higher order terms, i.e. terms which are at least quadratic in (I, p, q) if they depend on I, and terms which are independent of I and at least cubic in p, q. Therefore the periodic orbit is just $I = p = q = 0$ and $\psi(t) = \omega_1 t + \psi_0$.

In the case of invariant tori of dimension $s \geq 2$, the existence of such a system of coordinates is not guaranteed in generic systems (see the discussion in [6]); if there exist variables in which the Hamiltonian H takes the form (here and below $r := n - s$)

$$H \ = \ \sum_{j=1}^{s} \omega_j \, I_j \ + \ \sum_{j=1}^{r} \nu_j \frac{p_j^2 + q_j^2}{2} \ + \ \texttt{h.o.t.}$$

then the torus is said to be *reducible* for H. We have denoted by ω the frequencies along the invariant torus, and by ν the transversal ones.

Here we want to apply Nekhoroshev's theorem to the case where the invariant torus Λ is reducible for each of the Hamiltonians $F_1, ..., F_s$ in the same system of coordinates. The main point is that in this situation the assumption on the Floquet multipliers takes a very simple and explicit form.

Remark 10. Kuksin proved that under the assumptions considered in this paper, such coordinates exist [7] $\odot$

Since the discussion on the Floquet multipliers involves only the linearized dynamics around the torus, and thus only the quadratic part Φ_i of the expansion of F_i around Λ, and the linearization Y_i of the hamiltonian vector fields X_i associated to the F_i at Λ (these also satisfy $Y_i \lrcorner \Omega = \mathrm{d}\Phi_i$), from now on we neglect the higher order terms, and consider just Φ_i and Y_i. The results we obtain hold, of course, for the complete hamiltonian system and not just for the linearized one.

Thus our assumption of reducibility of Λ for all the F_i means that there exists a system of canonical coordinates (I, ψ, p, q), with $\psi \in \mathbf{T}^s$ and (I, p, q)

defined in a subset of $\Re^s \times \Re^{n-s} \times \Re^{n-s}$, in which the functions Φ_i take the form

$$\Phi_i \;=\; \sum_{j=1}^{s} \omega_j^{(i)}\, I_j \;+\; \sum_{j=1}^{r} \nu_j^{(i)} \left(\frac{p_j^2 + q_j^2}{2} \right) \;.$$

We write $\partial_j := (\partial/\partial\psi_j)$; we also introduce, for ease of notation,

$$\begin{aligned} J_j &:= \; (p_j^2 + q_j^2)/2 \\ \partial_j &:= \; p_j(\partial/\partial q_j) \;-\; q_j(\partial/\partial p_j) \end{aligned} \qquad\qquad j = s+1, ..., n \;.$$

We also define, again for ease of notation, matrices A and B with elements given by

$$A_{ij} \;:=\; \omega_j^{(i)} \;\;;\;\; B_{ij} = \nu_j^{(i)} \;.$$

Hence A is a $(s \times s)$ matrix built with the frequencies of motion in the invariant torus Λ, and B is a $(s \times r)$ matrix built with the frequencies of small oscillations in the transversal directions to the invariant torus.

In this way we have

$$\Phi_i \;=\; \sum_{k=1}^{s} A_{ik}\, I_k \;+\; \sum_{k=1}^{r} B_{ik}\, J_k$$

and the corresponding linear vector field Y_i takes the form

$$Y_i \;=\; \sum_{j=1}^{s} A_{ij}\, \partial_j \;+\; \sum_{j=1}^{r} B_{ij}\, \partial_{j+s} \qquad (i = 1, ..., s) \tag{3.16}$$

We will denote by $\Omega(k; j)$ the matrix obtained from A by substituting its k-th column with the j-th column of B. For ease of notation, we also write $P := (A^T)^{-1}$. We will also denote by $|M|$ the determinant of a matrix M.

Theorem 3.17 *Let F_i, Φ_i, Y_i and Λ be as above in this section. Let X_α be the vector field $X_\alpha = \sum_i c_i Y_i$ having periodic orbits γ_α on Λ with period one and in the homotopy class $\alpha \in \mathbf{Z}^s$. Then:*
(1) *The periodic orbit γ_α has $2s$ Floquet multipliers equal to 1, and $2r$ "transversal" Floquet multipliers given by $\exp[\pm 2\pi i Q_j(\alpha)]$ $(j = 1, ..., r)$ with*

$$Q_j(\alpha) \;=\; \sum_{i,k=1}^{s} B_{ji}^T P_{ik} \alpha_k \;.$$

(2) *The condition that the multiplicity of the Floquet multiplier 1 does not exceed $2s$ is equivalent to $Q_j(\alpha) \notin \mathbf{Z}$.*
(3) *The nondegeneracy assumption (iii) of theorem 1.3 holds if and only if there exists $\alpha \equiv (\alpha_1, \alpha_2, ..., \alpha_s) \in \mathbf{Z}^s$ such that*

$$Q_j(\alpha) \;\neq\; m \quad \forall m \in \mathbf{Z} \;,\; \forall j = 1, ..., r \;. \tag{3.18}$$

(4) *Condition 3.18 is equivalent to*

$$\sum_{k=1}^{s} \alpha_k \, |\Omega(k;j)| \;\neq\; m \; |A| \qquad \forall m \in \mathbf{Z} \;, \quad \forall j = 1, ..., r \;. \tag{3.19}$$

Corollary. *Under the assumption 3.18, or equivalently 3.19, the considered invariant torus is part of an s parameter family of invariant tori and such a family persists under perturbation.*

The corollary follows immediately from theorem 1.3.

We would like to stress that both the conditions given in the theorem can be checked once we know the matrices A and B; however condition 3.18 requires to consider only one matrix $B^T(A^T)^{-1}$ but also to perform the inversion of the $(s \times s)$ matrix A, while condition 3.19 requires to consider $s \cdot (n-s)$ matrices $\Omega(k,j)$, but does not require to consider any inversion of matrices. Thus it can be more convenient to use one or the other of them depending on the problem at hand; however condition 3.19 can always be explicitly checked and requires only simple linear algebra. As far as we know, condition 3.19 has not been considered before.

Remark 11. In the case $s = 1$ condition 3.19 reduces to

$$\exists \alpha \in \mathbf{Z} \;:\; \alpha\omega_k \neq m\omega_1 \;,\; \forall m \in \mathbf{Z} \;,\; \forall k = 2, ..., n \;, \tag{3.20}$$

which is easily seen to be equivalent to the standard condition under which the Lyapunov center theorem holds, namely that $\omega_k/\omega_1 \notin \mathbf{Z}$, $\forall k = 2, ..., n$. Indeed, if $\omega_k/\omega_1 \in \mathbf{Z}$ then, for any choice of $\alpha \in \mathbf{Z}$ one has that $\alpha\omega_k/\omega_1 \in \mathbf{Z}$, and therefore 3.20 is also violated. Conversely, if $\omega_k/\omega_1 \notin \mathbf{Z}$ then simply choose $\alpha = 1$ and 3.20 also holds. $\odot$

Proof of theorem 3.17. It is obvious that (1) $\Rightarrow$ (2) $\Rightarrow$ (3); so we have only to prove (1) and (4).

Fix $\alpha \in \pi_1(\Lambda)$; first of all we want to determine the vector field X_α, or, more precisely, its linearization Y_α about the torus. To this end we remark that the projection Z_c of a general vector field $Y_c = \sum_{i=1}^{s} c_i Y_i$ to invariant tori is just the projection onto $\{\partial_1, ..., \partial_s\}$. Hence, see 3.16,

$$Z_c \;=\; \sum_{i=1}^{s} \sum_{k=1}^{s} \; c_i \, A_{ik} \, \partial_k \;.$$

When we require that this has closed orbits with period 1 and winding number α_i around the cycles of Λ, we are requiring

$$Z_c \;=\; \sum_{k=1}^{s} \; (2\pi) \, \alpha_k \, \partial_k \;.$$

Thus we are asking $\sum_i c_i A_{ik} = 2\pi\alpha_k$, i.e. $A^T\mathbf{c} = 2\pi\alpha$, and we get (recall $P := (A^T)^{-1}$)

$$c_i = \sum_{k=1}^{s} (2\pi)\, P_{ij}\, \alpha_j,$$

which explicitly defines $Y_\alpha := \sum_{j=1}^{s} c_i Y_i$.

In order to compute the transversal Floquet multipliers we have to study the dynamics of Y_α in the transversal direction. In particular, recalling again 3.16, the flow generated by Y_α in the planes p_j, q_j $(j = 1, ..., r)$ take the form

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix} \mapsto \begin{pmatrix} p_j \cos(2\pi\, Q_j(\alpha)\, t) - q_j \sin(2\pi\, Q_j(\alpha)\, t) \\ q_j \cos(2\pi\, Q_j(\alpha)\, t) + q_j \sin(2\pi\, Q_j(\alpha)\, t) \end{pmatrix} ,$$

with

$$Q_j(\alpha) := \sum_i c_i \nu_{ij} := \sum_i (B^T)_{ji} c_i = \sum_{i,k=1}^{s} B^T_{ji}\, P_{ik}\, \alpha_k.$$

This shows that the transversal Floquet exponents are indeed the $Q_j(\alpha)$, as claimed in the statement, and proves (1). As remarked above, (2) and (3) are then obvious.

It remains to prove (4), i.e. to reformulate the condition $Q_j(\alpha) \notin \mathbf{Z}$ in a convenient way. Let us consider the matrix

$$M := B^T\, P \equiv B^T (A^T)^{-1}$$

and the matrices $\Omega(k, j)$ introduced above. We have to show that the elements m_{kj} of M can be written as a ratio of determinants, $m_{kj} = |\Omega(k, j)|/|A|$.

Let us introduce a useful notation: given a matrix M of elements m_{ij}, we denote by $\widetilde{M}$ the matrix of its algebraic complements; this means that the element $(\widetilde{m})_{ij}$ is the algebraic complement of m_{ij} in M. We recall that $\sum_j m_{ij}(\widetilde{m})_{kj} = \delta_{ik}|M|$, that $(\widetilde{M^T}) = \left(\widetilde{M}\right)^T$, and that by Cramer's theorem M^{-1} is obtained as $(M^{-1})_{ij} = |M|^{-1} \left(\widetilde{M}\right)^T$.

For ease of writing, we will think of k and j as fixed and write simply R for $[\Omega(k, j)]^T$. By elementary linear algebra, as recalled above, $|R| = \sum_\ell R_{i\ell}(\widetilde{R})_{i\ell}$, for any $i = 1, ..., s$; we consider $i = k$ and compute the determinant in this way. By definition, however, $R_{k\ell} = (B^T)_{j\ell}$. As for $(\widetilde{R})_{k\ell}$, we note that R differs from A^T only on the row k; thus the k-th rows of $\widetilde{R}$ and of $\widetilde{A^T}$ are identical, i.e. $\widetilde{R}_{k\ell} = \widetilde{A^T}_{k\ell} = \widetilde{A}_{\ell k}$.

In this way we obtain that

$$|R| = \sum_\ell R_{k\ell}\, (\widetilde{R})_{k\ell} = \sum_\ell (B^T)_{j\ell}\, \widetilde{A}_{\ell k} = \sum_\ell (B^T)_{j\ell}\, [(A^T)^{-1}]_{\ell k}\, (|A|) .$$

Using $|R^T| = |R|$ and writing again in full $R^T \equiv \Omega(k, j)$, we have shown that

$$M_{jk} := (B^T P)_{jk} \equiv (B^T (A^T)^{-1})_{jk} = \frac{|\Omega(k, j)|}{|A|} ,$$

hence that $Q_j(\alpha) = \sum_k \alpha_k(|\Omega(k,j)|/|A|)$; this completes the proof of (4) and thus of the theorem. $\triangle$

4 Applications

4.1 A three body problem

Consider the system composed by a heavy star and two light planets lying in a fixed plane. For simplicity we consider the case where the star is fixed at some point and the two planets move in the plane under the action of the gravitational force due to the star and of the gravitational force due to the other planet. If the interaction between the two planet is neglected then it is clear that there exist solutions in which both planet move on circular orbits around the star at distances given respectively by R_1 and R_2; we will show that, provided the masses of the two planets are small enough and a suitable nonresonance condition is fulfilled, such solutions persist in the coupled case (where the interaction is also considered).

Denote by M the mass of the star, by εm_1 and εm_2 the masses of the two planets. Take the position of the star as the origin of the coordinates and introduce polar coordinates $r_1, \theta_1,\ r_2, \theta_2$ for the position of the two planets. Then the dynamics of the system is described by the Hamiltonian

$$H^\varepsilon = \frac{P_1^2}{2m_1} + \frac{\pi_1^2}{2m_1 r_1^2} - \frac{GMm_1}{r_1} + \frac{P_2^2}{2m_2} + \frac{\pi_2^2}{2m_2 r_2^2} - \frac{GMm_2}{r_2} - \varepsilon K_1 \qquad (4.21)$$

where P_i is conjugated to r_i, π_i is conjugated to θ_i, and the function G_1 is given by

$$K_1 := \frac{Gm_1m_2}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}} \ ;$$

G is the gravitational constant. This Hamiltonian system has an additional integral of motion given by the total angular momentum

$$F_2 := \pi_1 + \pi_2 \ .$$

We will apply theorem 3.17 with $F_1^\varepsilon := H^\varepsilon$ and F_2 given by the total angular momentum. In order to identify Λ, fix two positive numbers R_1 and R_2 and define

$$L_1 := m_1\sqrt{GMR_1}\ , \quad L_2 := m_2\sqrt{GMR_2}\ ;$$

then we have

$$\Lambda \ := \ \left\{r_i = R_i,\ P_i = 0,\ \pi_i = L_i,\ \theta_i \in S^1\right\} \ .$$

Denote by

$$\omega_i := \frac{L_i}{m_i R_i^2} \equiv \sqrt{\frac{GM}{R_i^3}}\ , \quad i = 1, 2$$

the frequencies of revolution of the two planets, and introduce new coordinates I_1, I_2 (conjugated to the angles θ_1, θ_2) and (p_1, q_1), (p_2, q_2) by

$$I_i := \pi_i - L_i \ , \quad p_i := \frac{P_i}{\sqrt{m_i \omega_i}} \ , \quad q_i := (r_i - R_i)\sqrt{m_i \omega_i} \ , \quad i = 1, 2$$

then a straightforward computation shows that the two function F_1^0 and F_2 take the form

$$F_1^0 = \omega_1 I_1 + \omega_2 I_2 + \omega_1 \frac{p_1^2 + q_1^2}{2} + \omega_2 \frac{p_2^2 + q_2^2}{2} + \mathtt{h.o.t.} \equiv \Phi_1 + \mathtt{h.o.t.}$$

$$F_2 = I_1 + I_2 \equiv \Phi_2 \ ,$$

which have a form suitable for the application of theorem 3.17.

Theorem 4.22 *Assume that the frequencies fulfill the nonresonance condition*

$$\frac{\omega_1}{\omega_1 - \omega_2} \notin \mathbf{Z} \tag{4.23}$$

then, provided ε is small enough the conclusions of theorem 1.4 hold.

In particular there exists a family of two dimensional invariant tori which can be parametrized by the radii R_1, R_2 which continue in the coupled case the above solutions. Remark that since in the unperturbed case the frequencies of revolution are in one to one correspondence with the parameters R_1, R_2, the same holds for the coupled case, and in particular on most of the tori the motion will be quasiperiodic with two nonresonant frequencies, while on the remaining tori the motion will be periodic.

Proof. We have just to check that 4.23 is equivalent to the nonresonance condition 3.19. We have

$$A = \begin{pmatrix} \omega_1 & \omega_2 \\ 1 & 1 \end{pmatrix} \ , \quad B = \begin{pmatrix} \omega_1 & \omega_2 \\ 0 & 0 \end{pmatrix}$$

from which

$$\begin{aligned} \Omega(1;1) &= \begin{pmatrix} \omega_1 & \omega_2 \\ 0 & 1 \end{pmatrix} \ , \quad \Omega(1;2) = \begin{pmatrix} \omega_2 & \omega_2 \\ 0 & 1 \end{pmatrix} \\ \Omega(2;1) &= \begin{pmatrix} \omega_1 & \omega_1 \\ 1 & 0 \end{pmatrix} \ , \quad \Omega(2;2) = \begin{pmatrix} \omega_1 & \omega_2 \\ 1 & 0 \end{pmatrix} \ . \end{aligned}$$

Computing the determinants and inserting these in 3.19, the nonresonance condition takes the form

$$\exists (\alpha_1, \alpha_2) \in \mathbf{Z}^2 \ : \ \forall \, (m_1, m_2) \in \mathbf{Z}^2 \ , \ \begin{cases} \omega_1(\alpha_1 - \alpha_2) \neq m(\omega_1 - \omega_2) \\ \omega_2(\alpha_1 - \alpha_2) \neq m(\omega_1 - \omega_2) \end{cases}$$

and this is easily seen to be equivalent to 4.23. $\triangle$

Remark 12. It is possible to generalize theorem 4.22 to the case of unperturbed elliptic (rather than circular) orbits, but the computations would be more involved. ⊙

The result of theorem 4.22 can be obtained also by first passing to the reduced system obtained by reducing the angular momentum and then applying standard Poincaré theory. In particular the reduction of the angular momentum is elementary in the case of the system 4.21: just perform the change of variable $J_1 := I_1 + I_2$, $J_2 = I_2$. However there are cases in which the reduction of the angular momentum is not elementary and however Nekhoroshev's theory works very well. In the next subsection we present an example of this kind.

4.2 A four body problem

Consider three particles lying in a plane attracted by a fourth particle lying at a fixed point of the plane (that we choose as the origin of the coordinates) through a spherically symmetric force. The particles also interact by a small spherically symmetric potential. The system is described by

$$F_1^\varepsilon \equiv H := \sum_{k=1,2,3} \left(\frac{\|\mathbf{p}_k\|^2}{2} + V(\|\mathbf{q}_k\|) \right) + \varepsilon \sum_{1 \le i < j \le 3} W(\|\mathbf{q}_i - \mathbf{q}_j\|) \ , \qquad (4.24)$$

where $\mathbf{p}_i = (p_{i,1}, p_{i,2}) \in \Re^2$ are canonically conjugated to $\mathbf{q}_i = (q_{i,1}, q_{i,2}) \in \Re^2$ and the norm is the euclidean norm. In this subsection, the index i will always take the values $i = 1, 2$. The functions W and V are C^3; we assume that V has a zero of second order at the origin. Moreover, we assume that

$$V(r) = \frac{1}{2}\omega_0^2 r^2 + \mathcal{O}(r^3) \ ,$$

with some positive ω_0.

Due to the rotation invariance of the hamiltonian there is an integral of motion which is the total angular momentum, namely

$$F_2 := \sum_{k=1,2,3} (p_{k,1} q_{k,2} - p_{k,2} q_{k,1}) \ .$$

We consider the unperturbed solution in which the first two particles rotate on circular orbits and the third one is at rest at the origin. We will prove that under a suitable nonresonance condition this solution persist in the coupled case. To describe Λ introduce polar coordinates (r_i, θ_i) for the first two particles, and the corresponding momenta (P_i, π_i). Fix L_i and assume that the equations

$$-\frac{L_i^2}{r^3} + V(r)' = 0 \ , \qquad (i = 1, 2)$$

have unique solutions R_1, R_2 respectively, which are the radii of the circular orbits with angular momenta L_i. We put

$$\Lambda := \left\{ r_i = R_i,\ P_i = 0,\ \pi_i = L_i,\ \theta_i \in S^1 \ ,\ \mathbf{p}_3 = \mathbf{q}_3 = 0 \right\} \ .$$

Denote

$$I_i := \pi_i - L_i \ , \quad , \quad r'_i := (r_i - R_i) \ ,$$

then a straightforward computation shows that the two function F_1^0 and F_2 take the form

$$\begin{aligned} F_1^0 \ &= \ \omega_1 I_1 + \omega_2 I_2 + (1/2) \left[\ (P_1^2 + \omega_{r1} r_1'^2) + \left(P_2^2 + \omega_{r2} r_2'^2\right) \right. \\ &\quad \left. + \left(p_{3,1}^2 + \omega_0^2 q_{3,1}^2\right) + \left(p_{3,2}^2 + \omega_0^2 q_{3,2}^2\right) \ \right] \ + \ \texttt{h.o.t.} \\ &\equiv \ \Phi_1 \ + \ \texttt{h.o.t.} \end{aligned}$$

with suitable $\omega_1, \omega_2, \omega_{r1}, \omega_{r2}$, and

$$F_2 = I_1 + I_2 + (p_{3,1} q_{3,2} - p_{3,2} q_{3,1}) \ .$$

In order to apply our theorem we have still to put F_2 in the standard form. This is done via the change of coordinates

$$p_3 := \frac{1}{\sqrt{2}} \left(\frac{p_{3,1}}{\sqrt{\omega_0}} + q_{3,2} \sqrt{\omega_0} \right) \ , \quad q_3 = \frac{1}{\sqrt{2}} \left(\frac{p_{3,2}}{\sqrt{\omega_0}} - q_{3,1} \sqrt{\omega_0} \right)$$

$$p_4 := \frac{1}{\sqrt{2}} \left(\frac{p_{3,2}}{\sqrt{\omega_0}} + q_{3,1} \sqrt{\omega_0} \right) \ , \quad q_4 = \frac{1}{\sqrt{2}} \left(\frac{p_{3,1}}{\sqrt{\omega_0}} - q_{3,2} \sqrt{\omega_0} \right) \ .$$

Rescaling the radial variables by $p_i = P_i / \sqrt{\omega_{ri}}$, $q_i = r'_i \sqrt{\omega_{ri}}$, $i = 1, 2$, one obtains

$$\begin{aligned} \Phi_1 \ &= \ \omega_1 I_1 + \omega_2 I_2 \\ &\quad + (1/2) \left[\omega_{r1} \left(p_1^2 + q_1^2\right) + \omega_{r2} \left(p_2^2 + q_2^2\right) + \omega_0 \left(p_3^2 + q_3^2\right) + \omega_0 \left(p_4^2 + q_4^2\right) \right] \\ F_2 \ &= \ I_1 + I_2 + (1/2) \left[\left(p_3^2 + q_3^2\right) - \left(p_4^2 + q_4^2\right) \right] \ . \end{aligned}$$

Theorem 4.25 *Assume that the frequencies fulfill the nonresonance conditions*

$$\frac{\omega_{r1}}{\omega_1 - \omega_2} \notin \mathbf{Z} \ , \quad \frac{\omega_{r2}}{\omega_1 - \omega_2} \notin \mathbf{Z} \ , \quad \frac{\omega_0 - \omega_2}{\omega_1 - \omega_2} \notin \mathbf{Z} \ , \quad \frac{\omega_0 + \omega_2}{\omega_1 - \omega_2} \notin \mathbf{Z} \ ;$$

then, provided ε is small enough, the conclusions of theorem 1.4 hold.

Proof. The frequency matrices are given by

$$A := \begin{pmatrix} \omega_1 & \omega_2 \\ 1 & 1 \end{pmatrix} \qquad B := \begin{pmatrix} \omega_{r1} & \omega_{r2} & \omega_0 & \omega_0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

So the needed nonresonance conditions is that there exist $(\alpha_1, \alpha_2) \in \mathbf{Z}$ such that

$$\begin{aligned} & m(\omega_1 - \omega_2) \neq (\alpha_1 - \alpha_2)\omega_{r1} \\ & m(\omega_1 - \omega_2) \neq (\alpha_1 - \alpha_2)\omega_{r2} \\ & m(\omega_1 - \omega_2) \neq \alpha_1(\omega_0 \omega_2) + \alpha_2(\omega_{r1} - \omega_0) \\ & m(\omega_1 - \omega_2) \neq \alpha_1(\omega_0 + \omega_2) - \alpha_2(\omega_1 + \omega_0) \end{aligned}$$

for all $m \in \mathbf{Z}$; this is equivalent to the one we stated. $\triangle$

Note that in this case it is not elementary to introduce the total angular momentum as a canonical coordinate belonging to a system which has no singularities close to Λ. Nevertheless Nekhoroshev's theory works without any difficulty.

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