A Bound on Binding Energies and Mass Renormalization in Models of Quantum Electrodynamics

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Abstract

We study three well known models of matter coupled to the ultraviolet cutoff, quantized radiation field and to the Coulomb potential of arbitrarily many nuclei. Two are nonrelativistic: the first uses the kinetic energy $(p + eA(x))^2$ and the second uses the Pauli-Fierz energy $(p + eA(x))^2 + e\sigma \cdot B(x)$. The third, no-pair model, is relativistic and replaces the kinetic energy with the Dirac operator $D(A)$, but restricted to its positive spectral subspace, which is the "electron subspace". In each case we are able to give an upper bound to the *binding* energy – as distinct from the less difficult ground state energy. This implies, for the first time we believe, an estimate, albeit a crude one, of the mass renormalization in these theories.

Dedicated to two masters of mathematical physics, David Ruelle and Yakov Sinai, on the occasion of their 65th birthdays

1 Introduction

There has been a great deal of recent work dedicated to the construction of a theory of ordinary bulk matter interacting with the quantized radiation field. In such theories the number of electrons, N, is usually held fixed (i.e., pair production is not allowed) and these N particles interact with each other and with K fixed nuclei via the ordinary electrostatic Coulomb potential – in the Coulomb gauge. (The nuclei are fixed because they are, relative to the electrons, infinitely massive.) The electrons also interact with the magnetic vector potential A which is *quantized* and which has the well known quantized field energy. It is essential, however, to have an ultraviolet cutoff Λ in the

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interaction of electrons and A field, for otherwise almost everything in the theory becomes infinite. This is not an enormous drawback since we are interested only in the low energy physics of atoms, molecules and photons.

Attention has been paid mostly to the stability of matter, namely that the ground state energy is bounded below by $C(N+K)$, where C is some universal constant (depending on the parameters of the theory but not on N and K) [2, 5, 6, 10, 11, 13]. Almost no attention has been paid to the estimation of the atomic binding energy, i.e., to the difference between the ground state energies with and without the Coulomb potential.

In this paper we shall consider two nonrelativistic theories and one relativistic theory. In the absence of quantized fields, the ground state energy (or 'self energy') of a free electron (i.e., without other electrons and nuclei) is zero. In the presence of other electrons, but without nuclei, the energy is still zero because the electrons can move infinitely far apart. Therefore, in the presence of nuclei, the ground state energy is equal to the binding energy when there is no quantization.

The situation changes dramatically when the A field is quantized. The self-energy of a free electron (i.e., the ground state energy without Coulomb potentials but with the quantized A field) is large if Λ is not too small. If the fine structure constant $\alpha = e^2/\hbar c$ is not too large (e.g., $1/137$) and if the nuclear charges are not too large then the change in the ground state energy is not very large. Thus, the binding energy is the difference of two large quantities and its calculation is like "looking for a needle in a haystack".

All three models use 'minimal coupling', i.e., p is replaced by $p + eA(x)/c$ in the kinetic energy. The first has no explicit spin interaction with the magnetic field while the second – the Pauli-Fierz model – has a $\sqrt{\alpha}\,\sigma \cdot B(x)$ term. The latter is much more delicate than the former (e.g., it requires bounds on α and Z, as well as the presence of a field energy, for stability, while the former needs no such restrictions), and our results in the second case are not as good as in the first. Still, they are meaningful. The third theory uses the Dirac operator and is relativistic.

Clearly, it is essential to have a decent grasp of the binding energy, which is the truly physical quantity, in order to be able to proceed with a non-perturbative renormalization program. It is useful to recall that the binding energy for one electron and one nucleus of charge Z should be $\frac{1}{2}m_{\text{phys}}c^2\alpha^2 Z^2$, where m_{phys} is the physical, renormalized electron mass. In the case of hydrogen, $Z = 1$, this is a comparatively tiny energy. (There is no charge renormalization in a theory,without pair production, but there is one in a theory with pair production, in which case α here must be replaced by the renormalized α.)

Our purpose here is to find an upper bound to the binding energy in the two non-relativistic QED theories that have been extensively studied in the literature, and we believe it is the first time that such a rigorous estimate has been made. Note that the binding energy (14) is always positive, by definition, and thus an upper bound corresponds to a lower bound on the energy of the fully interacting system relative to the energy of free electrons.

If we equate the binding energy (which depends on m) to the physical binding energy (which depends on $m_{\rm phys}$) we obtain a (upper) bound to the amount of renormalization that is needed (see (23) and (26) for the first model). We believe these are the first rigorous renormalization estimates of their kind in a (admittedly simple-minded) quantized theory of electrons and photons.

Let us note an interesting feature of our results about the mass. There are several ways to define the renormalized mass. The usual one is to look at the ground state energy of a free electron with *fixed* total momentum (electron + field) and to define $1/2m_{\rm phys}$ to be the coeefficient of p^2 in the energy at $p = 0$. Another way is to set the binding energy equal to the physical binding energy, as done here.

• The latter definition has the property that for every value of Λ and of $m_{\rm phys}$ there is a value of m that gives equality. (Our bound is unique, but the true answer is, conceivably, not unique.)

• The former, usual definition very likely yields a solution for m only if Λ and $1/m_{\rm phys}$ are sufficiently small.

We cannot prove this last statement but it agrees with the prediction of perturbation theory and with classical electromagnetism. It also agrees with Van Kampen's exact solution of Kramer's "dipole approximation" model [4], which was the model that gave impetus to the renormalization program in QED. Instead of $(p+A(x))^2$ as in our first model, one takes $(p+A(0))^2$. In such a model the energy of N free electrons goes as $C\Lambda^{3/2}\sqrt{N}$ instead of $C\Lambda^{3/2}N$, as in our model (see (21)), but this is not the most significant point. For $N = 1$ one can compute the energy as a function of momentum p and finds that, with the first definition, $m_{\rm phys} = C'm + \alpha\Lambda$. Consequently, there is no solution for m if Λ or $1/m_{\rm phys}$ are large.

The first Hamiltonian H_N we consider is given by

$$\frac{1}{\hbar c}H_N = \mathcal{T} + \alpha V_c + H_f, \tag{1}$$

where $\mathcal{T}$ is proportional to the kinetic energy of N electrons

$$\mathcal{T} = \sum_{j=1}^{N} T_j \tag{2}$$

with

$$T_j = \frac{\hbar}{2mc}\left(p_j + \sqrt{\alpha}A(x_j)\right)^2 \ , \tag{3}$$

with $p_j = i\nabla_{x_j}$, and where m is the (unrenormalized) mass of the electron. The quantized, ultraviolet cutoff electromagnetic vector potential is

$$A(x) = \frac{1}{2\pi}\sum_{\lambda=1}^{2}\int_{|k|\le\Lambda}\frac{\varepsilon_\lambda(k)}{\sqrt{\omega(k)}}\left(a_\lambda(k)e^{ik\cdot x} + a_\lambda^*(k)e^{-ik\cdot x}\right)dk \ , \tag{4}$$

where Λ is the ultraviolet cutoff on the wave-numbers $|k|$. The operators a_λ, a_λ^* satisfy the usual commutation relations

$$[a_\lambda(k), a_\nu^*(q)] = \delta(k-q)\delta_{\lambda,\nu} \ , \quad [a_\lambda(k), a_\nu(q)] = 0, \quad \text{etc} \tag{5}$$

and the vectors $\varepsilon_\lambda(k)$ are the two possible orthonormal polarization vectors perpendicular to k and to each other. The field energy is

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \omega(k) a_\lambda^*(k) a_\lambda(k) dk \ ; \tag{6}$$

the physical choice of ω is $\omega(k) = |k|$, but our Theorems 2.1 and 3.1 are not restricted to this choice. No infrared cutoff is needed.

Finally, there is the Coulomb potential. There are K nuclei with positive charges $eZ_1, ..., eZ_K$ and with fixed locations $R_1, ..., R_K$ in $\mathbb{R}^3$. In this model the nuclei will preferentially locate themselves at those R_j that minimize the total energy, but these special locations are irrelevant for our theorem.

$$V_c = -\sum_{i=1}^{N}\sum_{k=1}^{K} Z_k \frac{1}{|x_i - R_k|} + \sum_{1\le i<j\le N} \frac{1}{|x_i - x_j|} + \sum_{1\le k<l\le K} Z_k Z_l \frac{1}{|R_k - R_l|} \ . \tag{7}$$

The Hilbert space is $\mathcal{H}_N = \wedge_{i=1}^N L^2(R^3; C^q) \bigotimes \mathcal{F}$, where $\wedge$ denotes the antisymmetric tensor product (Fermi statistics), $\mathcal{F}$ is the photon Fock space, q is the number of spin states for each electron ($q = 2$ in nature). The Hamiltonian (1) is bounded below by $C(N + K)$ (even without the aid of H_f); this was first stated in [1, p.857] and [3, p.2], who noted that the stability proof in [12] with $A = 0$ extends to the $A \neq 0$ case by virtue of a diamagnetic inequality. In [1] a remark of J. Fröhlich is presented, that this applies equally well to the quantized A field (4) because $[A(x), A(y)] = 0$ for all x, y. We use and discuss this fact again in (17) below.

The Pauli-Fierz Hamiltonian, $H_N^{\rm PF}$, which is treated in Section 3, is

$$H_N^{\rm PF} = H_N + \frac{\hbar^2}{2m}\sqrt{\alpha}\sum_{j=1}^{N} \sigma_j \cdot B(x_j) \qquad \text{with} \qquad B(x) = \operatorname{curl} A(x) \ . \tag{8}$$

The Pauli matrices σ_j are the spin-$\frac{1}{2}$ operators for particle j.

The third model is the relativistic no-pair model treated in [6], whose Hamiltonian is

$$\frac{1}{\hbar c} H_N^{\rm rel} = P^+ \left\{ \sum_{i=1}^{N} D_i(A) + \alpha V_c + H_f \right\} P^+ \ . \tag{9}$$

Here, $D(A)$ is the Dirac operator

$$D(A) := \vec{\alpha} \cdot (-i\nabla + \sqrt{\alpha} A(x)) + \frac{mc}{\hbar}\beta \ , \tag{10}$$

and P^+ is the projector onto the positive spectral subspace of $D(A)$ for all of the N electrons. Since the N Dirac operators commute with each other, this definition of P^+ as a projector makes sense. In other words, we start with the usual Hilbert space $\mathcal{H}_N$ and then replace it by the smaller physical Hilbert space $\mathcal{H}_N^{\rm phys} = P^+\mathcal{H}_N$. In $\mathcal{H}_N^{\rm phys}$ it is impossible to separate the L^2 spaces from the Fock space.

2 Binding Energy Bound (simple version)

In this section we analyze the binding energy for H_N.

We define $Z = \max\{Z_1, ..., Z_K\} \geq 0$ and set

$$\kappa = \frac{\pi}{2} Z + 2.2159 q^{1/3} Z^{2/3} + 1.0307 q^{1/3}. \tag{11}$$

We also define the (positive) free-electron ground state energy

$$E_0(N) = \inf \mathrm{spec} H_N(0) > 0 \tag{12}$$

(where $H_N(0)$ is the Hamiltonian (1) without the V_c term), the total ground state energy

$$E(N) = \inf \mathrm{spec} H_N \leq E_0(N) \ , \tag{13}$$

and the (positive) binding energy

$$\Delta E(N) = E_0(N) - E(N) \ . \tag{14}$$

THEOREM 2.1 (Binding energy for H_N). *Assume one of two cases:*

$$E_0(N) \begin{cases} \leq \frac{1}{2}(\kappa\alpha)^2 mc^2 N, & \text{Case A} \\ \geq \frac{1}{2}(\kappa\alpha)^2 mc^2 N, & \text{Case B.} \end{cases} \tag{15}$$

Then the binding energy per electron satisfies

$$\frac{\Delta E(N)}{N} \leq \begin{cases} (\kappa\alpha)^2 mc^2 & \text{Case A} \\ \kappa\alpha\sqrt{2mc^2}\sqrt{E_0(N)/N} & \text{Case B.} \end{cases} \tag{16}$$

Proof. We use the known result for the stability of "relativistic" matter in the form given in [7, eqs. 2.9 and 5.2] (which improves some of the results in [13]).

$$V_c \geq -\kappa \sum_{j=1}^{N} |p_j + \sqrt{\alpha} A(x_j)| \tag{17}$$

for any vector field $A(x)$. (Note that although (17) was proved for ordinary, non-quantized A fields, we are allowed to use it for our operator-valued field (4) since the commutator $[A(x), A(y)] = 0$ for all x, y, and hence there is a representation in which $A(x)$ is an ordinary vector field. Of course, $A(x)$ does not commute with H_f but that is immaterial.)

By Schwarz's inequality

$$V_c \geq -\kappa\sqrt{2mc/\hbar}\sqrt{N}\sqrt{\mathcal{T}} \ . \tag{18}$$

Since $H_f \geq 0$ and since the square root is operator monotone,

$$\frac{1}{\hbar c} H_N \geq \mathcal{T} + H_f - \kappa\alpha\sqrt{2mc/\hbar}\sqrt{N}\sqrt{\mathcal{T}} \geq \mathcal{T} + H_f - \kappa\alpha\sqrt{2mc/\hbar}\sqrt{N}\sqrt{\mathcal{T} + H_f} \ . \tag{19}$$

The final step is to note that the function $f(x) = x - \kappa\alpha\sqrt{2mc/\hbar}\sqrt{N}\sqrt{x}$ has its minimum at $x = \kappa^2\alpha^2 mcN/2\hbar$. Therefore, when $x \le \kappa^2\alpha^2 mcN/2\hbar$ we can say that $f(x)$ is not less than the minimum of $f(x)$, which is $-\frac{1}{2}(\kappa\alpha)^2 mcN/\hbar$. This is Case A of (15). Therefore $\Delta E \le E_0(N) + \frac{1}{2}(\kappa\alpha)^2 mc^2 N \le (\kappa\alpha)^2 mc^2 N$, which is Case A of (16). On the other hand, if $x \ge \kappa^2\alpha^2 mcN/2\hbar$ we can say that $f(x)$ is monotone increasing in x. Since $x = \mathcal{T} + H_f \ge E_0(N)/\hbar c \ge \kappa^2\alpha^2 mcN/2\hbar$, the infimum of the spectrum of the right side of (19) is given by $f(E_0(N)/\hbar c)$, which is Case B of (16). □

To apply this Theorem 2.1 we must have a decent estimate of $E_0(N)$. Let us consider the physical case $\omega(k) = |k|$ and let us define (with $\lambda_C =$ Compton wavelength)

$$\mathcal{R} = \frac{\hbar c\Lambda}{mc^2} = \Lambda\lambda_C \ , \tag{20}$$

which is the ratio of the cutoff photon energy to the self energy that an electron would have in a relativistic theory. A bound on $E_0(N)$ in this case is provided in [8] where a proof is announced and outlined that for *fermions* there are constants C_1, C_2 (depending on q) such that (for large Λ)

$$C_1\, mc^2\alpha^{1/2}\mathcal{R}^{3/2}N \le E_0(N) \le C_2\, mc^2\alpha^{2/7}\mathcal{R}^{12/7}N \tag{21}$$

The exact exponent is still not known but we lean towards 12/7. In any case, it differs from the perturbation theory value 2. Fermions are most important here because one can show [8] that $C_3\, mc^2\alpha^{1/2}\mathcal{R}^{3/2}N^{1/2} < E_0(N) < C_4\, mc^2\alpha^{2/7}\mathcal{R}^{12/7}N^{5/7}$ for bosons, and this would be useless for our purposes. Unfortunately, the bounds in (21) do not imply that $E_0(N)$ is strictly linear in N, as one would hope. We also note that (21) holds even if the Coulomb repulsion among the electrons is omitted.

If $\mathcal{R}$ is small then we are in Case A. This will surely be so if $\kappa^2\alpha^{12/7} \ge 2C_2\,\mathcal{R}^{12/7}$. Let us note that $\kappa \le 5.67Z$ for $Z \ge 1$ and $q = 2$. Then,

$$\frac{\Delta E(N)}{N} \le (32.2)\alpha^2 Z^2 mc^2 \qquad \text{Case A} \ , \tag{22}$$

which compares not unfavorably with the hydrogenic value $e_{\text{hydrogenic}}(Z) = Z^2\alpha^2 m_{\text{phys}}c^2/2$, where m_{phys} is the physical electron mass. As Λ increases the bare mass m should decrease. If we set $\Delta E(N)/N = e_{\text{hydrogenic}}(Z)$, inequality (22) tells us that the required bare mass cannot be too small, namely

$$m \ge m_{\text{phys}}/64.4 \qquad \text{Case A} \ . \tag{23}$$

We turn now to Case B, which surely holds if $\kappa^2\alpha^{3/2} \le 2C_1\mathcal{R}^{3/2}$. With the help of (21) we can conclude that

$$\frac{\Delta E(N)}{N} \le \sqrt{2C_2}\ \kappa\alpha^{8/7}\mathcal{R}^{6/7}mc^2 \qquad \text{Case B} \ . \tag{24}$$

To understand Case B further, let us use $\kappa \leq 5.67Z$ and note that (24) becomes

$$\frac{\Delta E(N)}{N e_{\text{hydrogenic}}(Z)} \leq \frac{5.67}{Z} \frac{C_2^{1/2}}{C_1^{4/7}} 2^{13/14} \mu^{8/7} \frac{m}{m_{\text{phys}}} \qquad \text{if} \qquad 5.67Z \leq \mu \ , \tag{25}$$

with $\mu = \alpha^{-3/4}\sqrt{2C_1}\,\mathcal{R}^{3/4}$.

To satisfy the condition in (25) for all $Z \leq 92$ we can take $\mu = 5.67 \cdot 92 = 522$ or $\mathcal{R} = (522)^{4/3}(2C_1)^{-2/3}\alpha = 30.7(2C_1)^{-2/3}$ (with $\alpha = 1/137$)). Assuming that we choose C_1 to be not too large (which can always be done since C_1 refers to a lower bound in (21)), this allows for a sizeable value of the cutoff Λ (see (20)). Now let us set the left side of (25) equal to 1, in order to make contact with experiment. We then find (since $Z \geq 1$) that

$$m \geq m_{\text{phys}}\, C_1^{4/7} C_2^{-1/2}/13,800 \qquad \text{All Cases} \ . \tag{26}$$

3 Binding Energy Bound (Pauli-Fierz version)

In this section we analyze the binding energy for H_N^{PF} in (8). The ground state energy and binding energy are defined as before in (12), (13), (14), but with H_N^{PF}, and we do not encumber the notation with a superscript PF. As far as constants are concerned, the following theorem is not the best possible one, but it is presented this way for simplicity. In particular, we do not have to assume that α is bounded — as we do in the hypothesis of Theorem 3.1.

Some constants have to be defined. The maximum nuclear charge Z is defined as before and we then define $\Omega = \max\{Z,\ 20.6\}$. (Note, for later use, that $20.6 = 64.5/\pi$.) We also define $\xi = (0.060)(8\pi)(3/4)^{-3/2} = 2.322$ and

$$\Gamma = 9.65 \left(\frac{\alpha K}{N}\right)^{1/4} \Lambda \ . \tag{27}$$

We also define the operator $\hbar c \mathcal{T}'$ to be the total Pauli-Fierz kinetic energy, namely,

$$\mathcal{T}' = \sum_{j=1}^{N} T_j^{PF} = \mathcal{T} + \frac{\hbar}{2mc}\sqrt{\alpha} \sum_{j=1}^{N} \sigma_j \cdot B(x_j) \geq 0 \ , \tag{28}$$

where

$$T_j^{PF} = \frac{\hbar}{2mc}\left\{ \left(p_j + \sqrt{\alpha}A(x_j)\right)^2 + \sqrt{\alpha}\,\sigma_j \cdot B(x_j) \right\} = \frac{\hbar}{2mc}\left\{\sigma_j \cdot \left(p_j + \sqrt{\alpha}A(x_j)\right)\right\}^2 \ . \tag{29}$$

THEOREM 3.1 (Binding energy for H_N^{PF}). *Assume that $2\pi\Omega\xi\alpha^2 < 1$ and assume one of two cases:*

$$E_0(N) \begin{cases} \leq 2(1 - 2\pi\Omega\xi\alpha^2)^{-2}(\pi\Omega\alpha)^2\, mc^2 N, & \text{Case A} \\ \geq 2(1 - 2\pi\Omega\xi\alpha^2)^{-2}(\pi\Omega\alpha)^2\, mc^2 N, & \text{Case B.} \end{cases} \tag{30}$$

Then the binding energy per electron satisfies

$$\frac{\Delta E(N)}{N} \leq \begin{cases} 2(\pi\Omega\alpha)^2(1-2\pi\Omega\xi\alpha^2)^{-2}(2-2\pi\Omega\xi\alpha^2)\, mc^2 + 2\pi\Omega\alpha\Gamma\hbar c & \text{Case A} \\ 2\pi\Omega\xi\alpha^2\, E_0(N)/N + 2\pi\Omega\alpha\sqrt{2mc^2}\sqrt{E_0(N)/N} + 2\pi\Omega\alpha\Gamma\hbar c & \text{Case B.} \end{cases} \tag{31}$$

Proof. The strategy is the same as in Theorem 2.1. An analogue of (17) is provided by [6].

$$\frac{1}{2\pi\Omega}V_c \geq -\sum_{j=1}^{N} |\sigma_j \cdot \left(p_j + \sqrt{\alpha}A(x_j)\right)| - \xi\alpha H_f - \Gamma N \ , \tag{32}$$

The derivation of (32) from [6] will be explained at the end of the proof. For the moment let us continue with the proof of the theorem. The analogue of (18) is then

$$\frac{1}{2\pi\Omega}V_c \geq -\sqrt{2mc/\hbar}\sqrt{N}\sqrt{\mathcal{T}'} - \xi\alpha H_f - \Gamma N \ . \tag{33}$$

Consequently, since $\mathcal{T}' \geq 0$,

$$\frac{1}{\hbar c}H_N \geq \left(\mathcal{T}' + H_f\right)(1-2\pi\Omega\xi\alpha^2) - 2\pi\Omega\alpha\sqrt{2mc/\hbar}\,\sqrt{N}\sqrt{\mathcal{T}' + H_f} - 2\pi\Omega\alpha\Gamma N \ . \tag{34}$$

The rest follows as in the proof of Theorem 2.1.

It remains to show how (32) arises from ref. [6]. We consider a Hamiltonian, $\widehat{H}_N$, similar to H''_N in [6], but with some auxiliary parameters.

$$\widehat{H}_N = \sum_{j=1}^{N} |\sigma_j \cdot \left(p_j + \sqrt{\alpha}A(x_j)\right)| + \alpha' V_c + \gamma H_f \ , \tag{35}$$

with $\alpha',\ \gamma > 0$. Note that $\widehat{H}_N$ has the dimension of length^{-1} and not an energy. Note also that the α appearing in $p_j + \sqrt{\alpha}A(x_j)$ is the true α and not α'.

We shall prove the analogue of Theorem 2.2 of [6] with $\widehat{H}_N$ in place of H''_N and with an appropriate substitute for the lower bound stated in Theorem 2.2. Theorem 3.1 of [6] is unchanged, and we turn to Sect. 4. We take $\varepsilon = 0$ (it can be taken to be zero even if $m \neq 0$). All the equations in this section remain true if we replace $\kappa\alpha$ by $\kappa\alpha'$ and $\kappa^2\alpha^2$ by $\kappa^2(\alpha')^2$. Thus, in the lower bound for the operator H_2 in Sect. 4 of [6], there is α in the numerator and α' in the denominator.

The next step is to use the inequality in Example 1 of Appendix B of [6] to bound $\int B^2$ appearing in the lower bound for H_2, but we have to remember that we have γH_f and not H_f. Thus, the analogous conditions on the parameters are $\kappa \leq \pi\Omega,\ \kappa\alpha' < 1$ and $\gamma \geq 8\pi(0.060)\alpha\left(1-\kappa^2(\alpha')^2\right)^{-3/2}$.

We make the choices

$$\kappa = \pi\Omega, \qquad \alpha' = (2\pi\Omega)^{-1}, \qquad \gamma = \xi\alpha \ . \tag{36}$$

The lower bound to $\widehat{H}_N$ is as in [6]. We find that the corresponding number C_2 is bounded by

$$C_2^4 = \frac{N}{K}\left[\frac{6 + (\alpha'/2)(\sqrt{2Z}+2.3)^2}{(27/2\pi)\gamma}\right] < \frac{N}{K}\left[\frac{13\pi}{27\gamma}\right] \ , \tag{37}$$

since $(\sqrt{2Z}+2.3)^2/2\pi\Omega<1$.

The analogous lower bound for the operator $\widehat{H}_N$, is then

$$\frac{1}{N}\widehat{H}_N > -\frac{K}{N}\left[\frac{18\Lambda\gamma}{\pi}\right]C_2^3 > -\left(\frac{\alpha K}{N}\right)^{1/4}\Lambda\nu \tag{38}$$

with $\nu=(\xi/\pi)^{1/4}(2/3)(39)^{3/4}=9.65$.

This completes the derivation of (32) from [6]. □

Using Theorem 3.1 we could proceed to derive more explicit bounds for the binding energy and $m_{\rm phys}$ – as in the discussion after 2.1, but we leave this task to the interested reader. The only needed information is the analogue of (21) for the Pauli-Fierz operator. As announced in [8]

$$C_1 mc^2\frac{\alpha}{1+\alpha^{2/3}}\mathcal{R}N \le E_0(N) \le C_2 mc^2\alpha^{1/2}\mathcal{R}^{3/2}N\ . \tag{39}$$

4 Binding Energy Bound (relativistic version)

We consider the Hamiltonian $H_N^{\rm rel}$ in (9). Our results here are very crude and we state them mainly to point out that realistic results on the binding energy could be obtained if one were able to improve the estimates of various constants. In the present situation we do not have any bounds on $E_0(N)$ (other than the simple one $E_0(N)>mc^2$, but we expect something like $E_0(N)/N\sim mc^2+(\text{const.})\hbar c\Lambda$ for not too small Λ.

We set $\mathcal{T}''=P^+\sum_{i=1}^N|D_i(A)|P^+$. With $P^+D(A)P^+$ in place of $|\sigma_j\cdot(p_j+\sqrt{\alpha}A(x_j))|$, (32) is valid for $P^+V_cP^+$ — in the same way that the inequalities of Theorems 2.1 and 2.2 of [6] are valid with the same constants. We find that

$$\begin{aligned}\frac{1}{\hbar c}H_N^{\rm rel} &\ge -2\pi\Omega\alpha\mathcal{T}''-2\pi\Omega\xi\alpha^2H_f-2\pi\Omega\alpha\Gamma N+\mathcal{T}''+H_f\\ &=(1-2\pi\Omega\alpha)\,\mathcal{T}''+\left(1-2\pi\Omega\xi\alpha^2\right)H_f-2\pi\Omega\alpha\Gamma N\\ &\ge\left(1-2\pi\Omega\xi\alpha^2\right)\left(\mathcal{T}''+H_f\right)-2\pi\Omega\alpha\Gamma N\end{aligned} \tag{40}$$

since $\xi\alpha<1$. Therefore,

$$\frac{\Delta E(N)}{N}\le 2\pi\Omega\alpha\frac{E_0(N)}{N}+2\pi\Omega\alpha\Gamma\hbar c\ . \tag{41}$$

We note that — apart from the unnaturally large constant — the binding energy appears to be bounded by α times the self-energy. We also note that (40) and (41) can be improved a little by using the free parameter $0\le\varepsilon<1$ that appears in Sect. 4 of [6]; we have taken $\varepsilon=0$ here, as we did in Sect. 3.

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