The essential harmony in the classical equations of mathematical physics

M C Nucci[†] and P G L Leach[‡*]

[†]Dipartimento di Matematica e Informatica, Università di Perugia
06123 Perugia, Italia
[‡]GEODYSYC, Department of Mathematics, University of the Aegean
Karlovassi 83 200, Greece

Abstract

The possibility to transform any system of linear ordinary differential equations into a system of constant coefficient equations is demonstrated using Lie theory. Some examples relate the classical equations of mathematical physics to the simple harmonic oscillator. The rôles of the third order form of the Ermakov-Pinney equation and of Fleischen-von Weltunter systems are explained.

1 Introduction

Transformation methods to facilitate the solution of differential equations have a history just slightly shorter than that of the subject of differential equations. The development of transformation methods was, as for all methods of solving differential equations, something of an *ad hoc* affair for some two centuries. Then Sophus Lie, applying the geometric ideas of transformation groups which he developed [37, 38, 39], established a systematic approach by determining the point [41] and contact [40] symmetries of differential equations and the simplifying transformations which follow from the knowledge of the symmetries. Noether's treatment of the symmetries of the Action Integral [45] paved the way for generalised symmetries as a common tool. The final decades of the last millennium saw the introduction

*permanent address: School of Mathematical and Statistical Sciences, University of Natal, Durban 4041, Republic of South Africa

of nonlocal symmetries to provide some sense of completeness to the class of symmetries of differential equations. In the subfield of partial differential equations a number of specialised symmetries, such as potential [8, 9] and nonclassical [7, 32] were introduced, although the latter were shown to be equivalent to standard symmetries in terms of the method of determination [49].

In the Floquet theory of systems of periodic differential equations [5, 43] a corollary to the theorem on the Floquet representation of the solution of such a system establishes that for a system of periodic differential equations there exists a nonsingular periodic transformation of variables transforming the system into one of constant coefficients. Central to this demonstration is the existence of the monodromy matrix which arises from the periodic nature of the differential equation.

A question which arises naturally when one sees the results for periodic systems is whether some transformation having a similar effect exists for nonperiodic systems. Since systems are often derived from higher order equations, the question may be extended to the existence of a transformation which converts a nonautonomous scalar equation to an autonomous scalar equation.

For second order linear equations this is known to be the case for the time-dependent oscillator

$$q'' + \omega^2(t)q = 0 \Leftrightarrow H = \tfrac{1}{2}\left(p^2 + \omega^2 q^2\right), \quad p = q' \tag{1.1}$$

(the prime denotes differentiation with respect to the independent variable time, t) which has been the subject of numerous investigations. Lewis [33, 34, 35, 36] provided the modern impetus by showing, using Kruskal's asymptotic method [27], that there exists an invariant

$$I = \frac{1}{2\rho^2}\left[\left(\rho p - \rho' q\right)^2 + \frac{q^2}{\rho^2}\right] \tag{1.2}$$

for the Hamiltonian (1.1), where $\rho(t)$ is a solution of the auxiliary equation

$$\rho'' + \omega^2\rho = \frac{1}{\rho^3}, \tag{1.3}$$

commonly known as the Ermakov-Pinney equation after Ermakov who, as far as it is known, in 1880 was the first to treat the problem of the integration of (1.1) [16] and Pinney who provided the statement of the solution of (1.3) in 1950 [53]. Subsequently (1.2) was shown to be the representation, (1.1), in the original coordinates of the Hamiltonian

$$\bar{H} = \tfrac{1}{2}\left(P^2 + Q^2\right) \tag{1.4}$$

by means of a time-dependent linear canonical transformation coupled with a change of time variable [20, 29],

$$Q = \frac{q}{\rho}, \quad P = \rho p - \rho' q, \quad T = \int \rho^{-2}(t)\mathrm{d}t, \tag{1.5}$$

an instance of the generalised canonical transformation of Burgan *et al* [10]. Instead of the generalised canonical transformation, (1.5), one could transform the original nonautonomous equation to

$$\ddot{Q} + Q = 0 \tag{1.6}$$

by means of the Kummer-Liouville transformation [28, 42]

$$Q = \frac{q}{\rho}, \quad T = \int \rho^{-2}(t)\mathrm{d}t \tag{1.7}$$

(overdot means differentiation with respect to the 'new time', T) so that the solution of (1.1) follows from that of (1.7) as

$$q(t) = \rho(t)\left[A \sin T(t) + B \cos T(t)\right], \tag{1.8}$$

where A and B are the required two arbitrary constants of integration. Some examples of classes of nonautonomous higher order scalar equations have been shown to be rendered autonomous by transformations of the type (1.7) by Hill [22] who used a method based on group rather than algebraic considerations. As has been demonstrated by Aguirre and Krause [3, 4], generally the use of the finite transformations of the group is more cumbersome than the infinitesimal transformations associated with the corresponding algebra.

In this paper we address two major questions. The first is the possibility of the existence of a transformation based on the presence of symmetries which transforms a nonautonomous system of first order ordinary differential equations into an autonomous system, thereby complementing the results following from the Floquet theory of systems of periodic differential equations. The second question is to determine the algebraic conditions which allow the possibility of the transformation of a nonautonomous higher order scalar equation to an autonomous one of the same order. Given that the higher order equation can be written in the form of a system of first order linear equations, the connection between the two results becomes also a matter of interest.

An nth order linear scalar equation of maximal symmetry ($n+4$ Lie point symmetries) is shown always to be transformable to an autonomous equation of the same order which is what one might expect from studies of other aspects of nth order linear ordinary differential equations [44, 17]. In the deductions we find that there is an intermediate class of nonautonomous equations which can be transformed to autonomous form. Members of the class have $n+2$ Lie point symmetries, not the $n+4$ of the equations of maximal order, and we find that they have a specific structure which demarcates them from the generic nth order equations possessing $n+1$ symmetries. This explanation of the reason for the differing number of symmetries is one of the new results featured in this paper.

This second class of scalar nth order ordinary differential equations shows that the third order Ermakov-Pinney equation

$$\xi''' + 2\omega^2\xi' + 2\omega\omega'\xi = 0 \tag{1.9}$$

– formally obtained by differentiation of (1.3) with respect to time and setting $\xi = \rho^2$; but in fact of more fundamental importance in the study of symmetries of differential equations [44] – plays a determining role. We see that this equation arises not only with scalar equations of higher order but also in systems of first order equations. Its integral, *videlicet*

$$\xi\xi'' - \tfrac{1}{2}\xi'^2 + \omega^2\xi^2 = K, \tag{1.10}$$

has an interesting role in the transformation of nonautonomous second order ordinary differential equations via two-dimensional linear systems. We show, firstly through demonstration by example and then in general, that the classical equations of Mathematical Physics, *videlicet* those scalar linear second order equations which arise from separation of variables in partial differential equations arising in field theory and quantum mechanics, reduce to the first order system for the simple harmonic oscillator with the frequency determined by the constant of integration, K, of the relevant Ermakov-Pinney equation. This process of transformation and solution provides the potential of unexpected relations for many special functions.

2 Transformations of nonautonomous systems to autonomous systems

We consider the n-dimensional homogeneous system of nonautonomous first order equations

$$\mathbf{u}' = A(t)\mathbf{u} \Leftrightarrow u_i' = a_{ij}(t)u_j, \tag{2.1}$$

where $A(t)$ is an $n \times n$ matrix the elements of which, a_{ij}, can be nontrivial functions of time and summation over repeated indices is implied, as always in the sequel unless specifically excluded. (Hereinafter we suppress the explicit dependence (t) unless the context demands it.) Since (2.1) is a first order system, it possesses an infinite number of Lie point symmetries, which are equally generalised symmetries, a feature to be born in mind when making the transition from scalar nth order equation to n-dimensional first order system. Apart from any other Lie point symmetries (2.1) possesses the obvious symmetries of

$$\begin{aligned} G_H &= u_i\partial_{u_i} \\ G_{Si} &= b_{ij}(t)\partial_{u_j} \quad j = 1, n \\ G_{sl} &= s_0\partial_t + s_{ij}u_j\partial_{u_i}, \quad i, j = 1, n, \end{aligned} \tag{2.2}$$

where G_H and G_{Si} represent the homogeneity and n solution symmetries respectively and G_{sl} the type of symmetry of the form associated with the rescaling symmetry in the algebra $sl(2, R)$ characteristic of linear systems of maximal order [19, 25]. When the matrix $A(t)$ has some specific substructure, there are additional homogeneity symmetries. For example, if $A(t)$ is diagonal, there exist n homogeneity symmetries of the form $x_i\partial_{x_i}$ with no summation on i.

The most suitable candidate for the determination of a linear transformation to render (2.1) autonomous is one of the type G_{sl}. The first extension is

$$G_{sl}^{[1]} = s_0\partial_t + s_{ij}u_j\partial_{u_i} + \left(s_{ij}'u_j + s_{ij}u_j' - s_0'u_j'\right)\partial_{u_i'} \tag{2.3}$$

and the application of (2.3) to (2.1) with (2.1) taken into account gives

$$s_{kj}'u_j + s_{kl}a_{lj}u_j - s_0'a_{kj}u_j = s_0a_{kj}'u_j + a_{ki}s_{ij}u_j, \quad k = 1, n \tag{2.4}$$

so that each independent variable yields the equation

$$s_{kj}' = s_0a_{kj}' + s_0'a_{kj} - s_{kl}a_{lj} + a_{ki}s_{ij}, \quad k, j = 1, n \tag{2.5}$$

and the system of n^2 equations to be solved to determine the symmetry is

$$S' = (s_0 A)' + AS - SA. \tag{2.6}$$

The system (2.6) of n^2 independent equations contains n^2+1 dependent variables. Evidently one of these, logically/æsthetically s_0, plays the role of a parametric function and may well be set at zero (or 1 if so desired). In this we already note a contrast with the transformation (1.7) in which the introduction of new time is mandatory. We see below that the type of relation required in (1.7) is a consequence of the *ab initio* restriction of S to be zero in its upper triangle.

The canonical variables are found from the characteristics of G_{sl} once the solution of (2.6) is obtained – not necessarily a trivial task. In general the associated Lagrange's system is

$$\frac{\mathrm{d}t}{s_0} = \frac{\mathrm{d}u_1}{s_{1j}u_j} = \frac{\mathrm{d}u_2}{s_{2j}u_j} = \ldots = \frac{\mathrm{d}u_n}{s_{nj}u_j}, \tag{2.7}$$

ie

$$0 = \frac{s_0 \mathrm{d}u_i}{\mathrm{d}t} - s_{ij}u_j, \quad i = 1, n. \tag{2.8}$$

We take the solution to be

$$\mathbf{u}(t) = B^{-1}\mathbf{u}_0, \quad T = \int \frac{\mathrm{d}t}{s_0}, \tag{2.9}$$

where $\mathbf{u}_0$ is a constant vector and B is the solution of

$$s_0 B' + BS = 0 \quad \Leftrightarrow \quad \dot{B} + BS = 0, \tag{2.10}$$

so that the new variables are given by

$$\mathbf{w} = B\mathbf{u}, \quad T = \int \frac{\mathrm{d}t}{s_0}. \tag{2.11}$$

The system of equations in the new variables is

$$\begin{aligned} \dot{\mathbf{w}} &= s_0^{-1}\left(B\mathbf{u}' + B'\mathbf{u}\right) \\ &= s_0^{-1} B(A-S)B^{-1}\mathbf{w} \\ &= C\mathbf{w}, \end{aligned} \tag{2.12}$$

where overdot means differentiation with respect to the new time, T, and the constancy of C may be verified by differentiation with respect to T when(2.6), (2.9) and the (2.10) are taken into account.

The existence of these solutions of (2.6) and (2.8) is guaranteed provided the elements of A are continuous [23] [p 72]. Thus we have

Theorem 1: *The system of linear equations* $\mathbf{u}' = A(t)\mathbf{u}$, *where the elements,* $a_{ij}(t)$, *of* A *are continuous functions of the independent variable* t *may be transformed to the autonomous system* $\dot{\mathbf{w}} = C\mathbf{w}$ *by means of the Kummer-Liouville transformation*

$$\mathbf{w} = B\mathbf{u}, \quad T = \int \frac{\mathrm{d}t}{s_0}, \tag{2.13}$$

where B *is the matrix determined by the coefficient functions of the Lie point symmetry*

$$\Gamma = s_0\partial_t + s_{ij}u_j\partial_{u_i} \tag{2.14}$$

of the original system given by the solution of the system

$$S' = (s_0A)' + AS - SA. \tag{2.15}$$

Without loss of generality s_0 can be set at 0 or 1 in which case there is no need for a transformation of the independent variable. The result for periodic systems employing the monodromy matrix of Floquet theory is now extended to all systems using a particular class of the Lie point symmetries of the system. These are the generalised rescaling symmetries. (Conventionally a rescaling symmetry has the form $\alpha_0 t\partial_t + \alpha_{ij}u_i\partial_{u_j}$ with the α_{ij} constants. Here the coefficients have been extended to include functions of time. Even nonlocal generalised rescaling symmetries have found application [26, 52], although they may be local symmetries of an equivalent system of first order equations [48].)

We can now envisage the process of the solution of a nonautonomous equation as a series of processes which are summarised in the diagram contained in Figure 1.

The nth order nonautonomous equation is converted to what is commonly called the equivalent n-dimensional system of linear equations although in the language of symmetries one should always be sensitive to the nonlocal change of variables which means that the type of symmetries, say point, is not necessarily preserved. The generalised symmetries of the nth order equation are now all point symmetries of the first-order system. The generalised self-similar symmetries of the system are calculated and their canonical variables are used to construct the autonomous system corresponding to the nonautonomous system. The solution of the autonomous system is trivial and with these solutions the solution to the original nonautonomous nth order equation is obtained through the inverse transformation which, since the original transformation is linear, is easily accomplished.

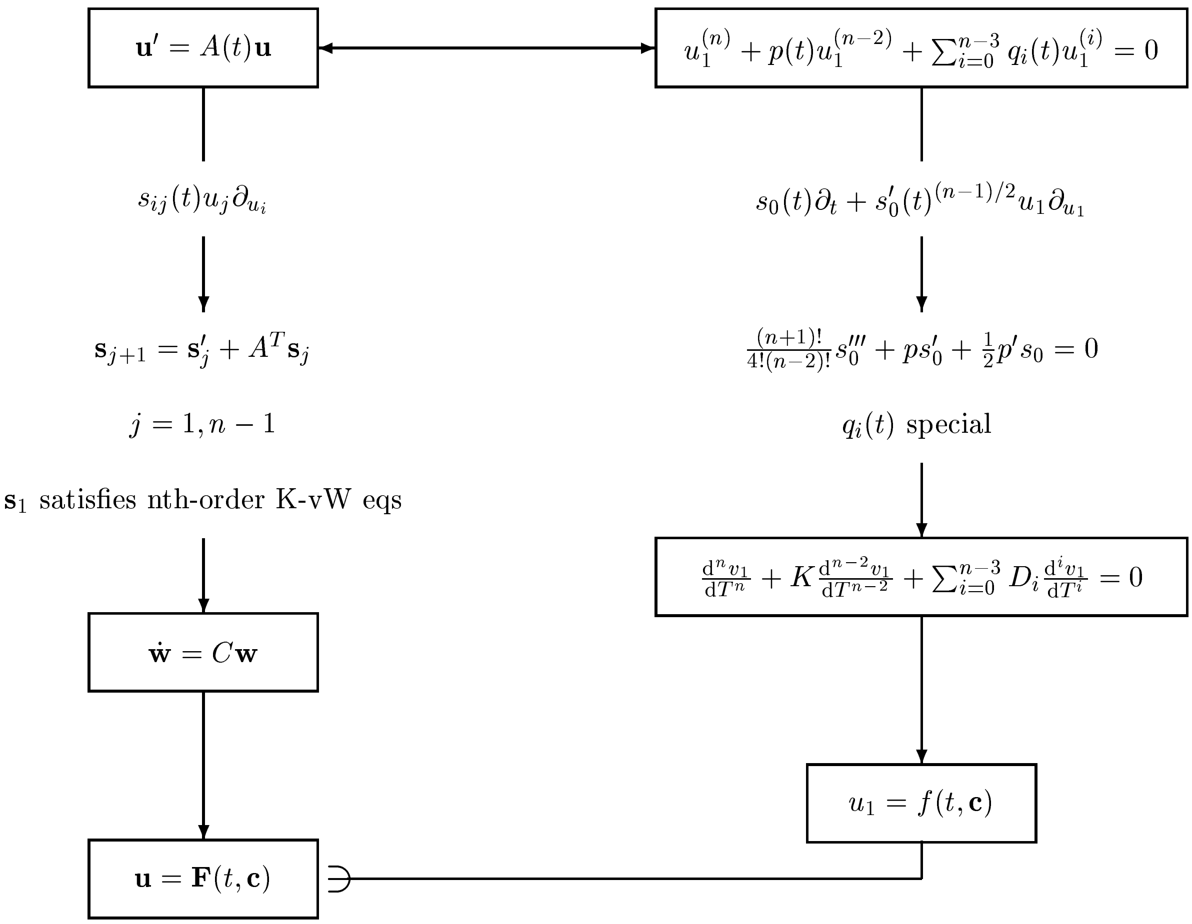

Figure 1:

Nth order equation.

3 Second order equations

We commence our considerations of second order equations with two examples. The first of these is Bessel's equation

$$x^2y'' + xy' + \left(x^2 - \mu^2\right) y = 0 \tag{3.1}$$

the solution set of which, Bessel's functions, is probably the most tabulated of all the higher transcendental functions. Bessel's functions arise from diverse sources apart from the differential equation (3.1), such as from the group representation of the Euclidean group, E_2, of translations and dilations in the plane [12]. Apart from preceding Bessel's systematic study of their properties by about a century, the use of particular Bessel's functions to solve specific problems is probably the first application of special functions to the solution of physical problems. The earliest known instance in Mechanics was in the investigation of transverse oscillations of a hanging heavy chain fixed only at its point of suspension by Daniel Bernoulli in 1732, but an earlier instance of application in the class of Riccati equations is reported in the correspondence of Leibniz (03:10:1702) [14]. (Colwell gives a date of 1703 without any details [13].) The work of Euler on what became known as elliptic functions was a few years later in the period 1756-1757. Curiously both areas were seriously established at about the same time [31, 24, 54, 6, 1, 18]. By the way, Hermite, who began his career with Jacobi's help, proposed a generalization of Floquet theory for double periodic functions in a series of papers on applications of elliptic functions [21].

We write (3.1) as the two-dimensional system

$$\left(\begin{array}{c} y \\ z \end{array}\right)' = \left(\begin{array}{cc} 0 & 1 \\ \dfrac{\mu^2}{x^2} - 1 & -\dfrac{1}{x} \end{array}\right)\left(\begin{array}{c} y \\ z \end{array}\right), \tag{3.2}$$

where now the prime denotes differentiation with respect to the independent variable, x, by the definition of the second variable $z = y'$. If we take $s_0 = 1$

in (2.6), we obtain the system

$$\begin{pmatrix} s_{11} \\ s_{12} \\ s_{21} \\ s_{22} \end{pmatrix}' = \begin{pmatrix} 0 & \left(1-\dfrac{\mu^2}{x^2}\right) & 1 & 0 \\ -1 & \dfrac{1}{x} & 0 & 1 \\ -\left(1-\dfrac{\mu^2}{x^2}\right) & 0 & \dfrac{1}{x} & \left(1-\dfrac{\mu^2}{x^2}\right) \\ 0 & -\left(1-\dfrac{\mu^2}{x^2}\right) & -1 & 0 \end{pmatrix} \begin{pmatrix} s_{11} \\ s_{12} \\ s_{21} \\ s_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2\dfrac{\mu^2}{x^3} \\ \dfrac{1}{x^2} \end{pmatrix} \tag{3.3}$$

from which we may write s_{11}, s_{21} and s_{22} in terms of s_{12} which satisfies the nonhomogeneous third order ordinary differential equation

$$s_{12}''' + \left(4 - \frac{4\mu^2 - 1}{x^2}\right) s_{12}' + \frac{4\mu^2 - 1}{x^3} s_{12} = \frac{4\mu^2 - x^2}{x^3}. \tag{3.4}$$

Equation (3.4) has three linearly independent solutions and so there are three independent transformations available to render (3.2) into autonomous form. However, the determination of the characteristics is a by no means trivial task and we employ a different stratagem. We set $s_{12} = 0$ and allow s_0 to remain unspecified. Now (2.6) with (3.2) gives

$$\begin{pmatrix} s_{11} \\ 0 \\ s_{21} \\ s_{22} \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -\left(1-\dfrac{\mu^2}{x^2}\right) & 0 & -\dfrac{1}{x} & \left(1-\dfrac{\mu^2}{x^2}\right) \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} s_{11} \\ 0 \\ s_{21} \\ s_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2\dfrac{\mu^2}{x^3} \\ \dfrac{1}{x^2} \end{pmatrix} s_0 + \begin{pmatrix} 0 \\ 1 \\ -\left(1-\dfrac{\mu^2}{x^2}\right) \\ -\dfrac{1}{x} \end{pmatrix} s_0' \tag{3.5}$$

and we find that the governing equation is that for s_0, *videlicet*

$$s_0''' + \left(4 + \frac{1-4\mu^2}{x^2}\right) s_0' - \frac{1-4\mu^2}{x^3} s_0 = 0, \tag{3.6}$$

which we recognise as the third-order Ermakov-Pinney equation, and

$$s_{11} = \tfrac{1}{2}\left(s_0' - \frac{s_0}{x}\right), \quad s_{21} = \tfrac{1}{2}\left(s_0' - \frac{s_0}{x}\right)', \quad s_{22} = -\tfrac{1}{2}\left(s_0' + \frac{s_0}{x}\right). \tag{3.7}$$

The canonical variables are found from the associated Lagrange's system

$$\frac{\mathrm{d}x}{s_0} = \frac{\mathrm{d}y}{\frac{1}{2}\left(s_0' - \frac{s_0}{x}\right) y} = \frac{\mathrm{d}z}{\frac{1}{2}\left(s_0' - \frac{s_0}{x}\right)' y - \frac{1}{2}\left(s_0' + \frac{s_0}{x}\right) z} \tag{3.8}$$

the solution of which is particularly simple due to the point nature of the symmetry. We find that

$$\begin{aligned} v_1 &= y\left(\frac{x}{s_0}\right)^{\frac{1}{2}} \\ v_2 &= (s_0 x)^{\frac{1}{2}} z - \left[\left(\frac{x}{s_0}\right)^{\frac{1}{2}}\right]' y \\ T &= \int \frac{\mathrm{d}x}{s_0(x)}. \end{aligned} \tag{3.9}$$

It is easy enough to show that

$$\frac{\mathrm{d}v_1}{\mathrm{d}T} = v_2 \tag{3.10}$$

and somewhat less easy to show that

$$\frac{\mathrm{d}v_2}{\mathrm{d}T} = -\tfrac{1}{2} v_1 \left[s_0 s_0'' - \tfrac{1}{2} s_0'^{\,2} + \tfrac{1}{2} s_0^2 \left(4 - \frac{4\mu^2 - 1}{x^2}\right)\right]. \tag{3.11}$$

The Ermakov-Pinney equation admits the integrating factor s_0 and we obtain

$$s_0 s_0'' - \tfrac{1}{2} s_0'^{\,2} + \tfrac{1}{2} s_0^2 \left(4 - \frac{4\mu^2 - 1}{x^2}\right) = 2\Omega^2 \tag{3.12}$$

so that the autonomous system is

$$\frac{\mathrm{d}}{\mathrm{d}T}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{3.13}$$

which is the easily solved equation for the simple harmonic oscillator provided $\Omega^2 > 0$. Under the standard change of variable $s_0 = \rho^2$, (3.12) takes the usual form for the second order Ermakov-Pinney equation

$$\rho'' + \left(1 - \frac{\mu^2 - \frac{1}{4}}{x^2}\right)\rho = \frac{\Omega^2}{\rho^3} \tag{3.14}$$

which has been given fruitful interpretation [15] as the radial component of the equation of motion

$$\mathbf{r}'' + \left(1 - \frac{\mu^2 - \frac{1}{4}}{x^2}\right)\mathbf{r} = 0, \tag{3.15}$$

where $\rho = |\mathbf{r}|$. The term on the right side of (3.14) is related to the conservation of the square of angular momentum and so one expects $\Omega^2 \geq 0$. (One notes that quantum mechanically this requirement is reduced to $\Omega^2 > -\frac{1}{4}$ to prevent 'collapse into the centre' [11], but that should not be the case here for which (3.15) is quite classical.)

Before we move to an interpretation of these results we consider a second example which is that of the quantal oscillator with Schrödinger equation

$$\left(-\tfrac{1}{2}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}x^2\right)\psi = i\frac{\partial\psi}{\partial t} \tag{3.16}$$

for which the spatial part of the wavefunction, ψ, satisfies

$$u_1'' + \left(\lambda - x^2\right)u_1 = 0 \tag{3.17}$$

in the usual process of separation of variables. For the two-dimensional system,

$$\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right)' = \left(\begin{array}{cc} 0 & 1 \\ x^2 - \lambda & 0 \end{array}\right)\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right), \tag{3.18}$$

we use a generalised self-similarity symmetry of the type (2.2) with $s_{12} = 0$ and $s_0 \neq 0$. We obtain

$$\Gamma = s_0\partial_x + \tfrac{1}{2}s_0'u_1\partial_{u_1} + \tfrac{1}{2}\left(s_0''u_1 - s_0'u_2\right)\partial_{u_2}, \tag{3.19}$$

where

$$s_0''' + 4\left(\lambda - x^2\right)s_0' - 4xs_0 = 0 \tag{3.20}$$

which is of Ermakov-Pinney type with the integrated form

$$s_0 s_0'' - \tfrac{1}{2} s_0'^{\,2} + 2\left(\lambda - x^2\right) s_0^2 = 2\Omega^2. \tag{3.21}$$

The canonical variables are

$$v_1 = \frac{u_1}{s_0^{1/2}}, \quad v_2 = s_2^{1/2} u_2 - \left(s_0^{1/2}\right)' u_1, \quad T = \int \frac{\mathrm{d}x}{s_0} \tag{3.22}$$

and the autonomous system is the simple harmonic oscillator (3.13).

If we again write $s_0 = \rho^2$, the solution of (3.13) gives for the solution of (3.17)

$$u_1 = \rho\left[A_1 \sin \Omega T + A_2 \cos \Omega T\right] \tag{3.23}$$

which is a very classical way of writing the solution of (3.17)! We note that the solution of (3.20) is (courtesy of Maple V)

$$s_0 = \frac{1}{x}\left\{C_1 W_W^2 + C_2 W_W W_M + C_3 W_M^2\right\}, \tag{3.24}$$

where $W_W\left(\frac{1}{4}\lambda, \frac{1}{4}, x^2\right)$ and $W_M\left(\frac{1}{4}\lambda, \frac{1}{4}, x^2\right)$ are Whittaker's W and M functions [2] [Ch 13, p. 504].

We observe that the transformation takes no note of the eigenvalues of the original equation (3.17) which, of course, are a consequence of imposed boundary conditions. We are now in a position to state

Theorem 2: *A second order ordinary differential equation is equivalent under a point transformation to the simple harmonic oscillator except in special circumstances.*

Proof: Without loss of generality the second order ordinary differential equation may be taken in the normal form

$$u_1'' + q(t) u_1 = 0 \tag{3.25}$$

(the prime denotes differentiation with respect to the independent variable time). The corresponding system of first-order equations

$$\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right)' = \left(\begin{array}{ccc} 0 & , & 1 \\ -q & , & 0 \end{array}\right)\left(\begin{array}{c} u_1 \\ u_2 \end{array}\right) \tag{3.26}$$

always has a Lie symmetry of the form

$$\Gamma = s_0 \partial_x + \tfrac{1}{2} s_0' u_1 \partial_{u_1} + \tfrac{1}{2}\left(s_0'' u_1 - s_0' u_2\right) \partial_{u_2}, \tag{3.27}$$

where s_0 is a solution of the third order Ermakov-Pinney equation

$$s_0''' + 4qs_0' + 2q's_0 = 0. \tag{3.28}$$

Under the transformation

$$v_1 = \frac{u_1}{s_0^{1/2}}, \quad v_2 = s_0^{1/2}u_2 - \left(s_0^{1/2}\right)' u_1, \quad T = \int \frac{\mathrm{d}x}{s_0} \tag{3.29}$$

(3.26) becomes the autonomous system

$$\frac{\mathrm{d}}{\mathrm{d}T}\left(\begin{array}{c} v_1 \\ v_2 \end{array}\right) = \left(\begin{array}{cc} 0 & 1 \\ -\Omega^2 & 0 \end{array}\right)\left(\begin{array}{c} v_1 \\ v_2 \end{array}\right), \tag{3.30}$$

where the constant Ω^2 is obtained from the integration of (3.28) once using the integrating factor s_0 and is given by

$$s_0s_0'' - \tfrac{1}{2}s_0'^2 + 2qs_0^2 = 2\Omega^2. \tag{3.31}$$

The solution of the original problem is given by

$$u_1 = \rho\left[A_1\sin\Omega T + A_2\cos\Omega T\right], \tag{3.32}$$

where $s_0 = \rho^2$.
Corollary: *The transformation to autonomous form can be achieved without a change in the independent variable by demanding the symmetry*

$$\Gamma = s_{ij}u_j\partial_{u_i}, \tag{3.33}$$

ie by making $s_0 = 0$.

The price to pay is a more complicated transformation of the dependent variables. The essential feature of either method of transformation is a nonconstant rescaling of the dependent variable. In the new coordinates, v_1 varies sinusoidally. The effect of the transformation can make u_1 vary quite erratically [30]. The schema for second order equations and two-dimensional systems is illustrated in Figure 2.

We note that we have achieved, albeit in a formal sense, the unity of all scalar second order order differential equations of maximal symmetry which is presaged by their possession of the common algebra of Lie point symmetries, $sl(2,R)$. There may be some possibility to derive usable relationships between various special functions. A curious point is that naturally the transformations considered here lead to a simple harmonic oscillator rather than the free particle equation which is the simplest representative of all scalar second order ordinary differential equations. This is a consequence of the general integral of the third order Ermakov-Pinney equation. The presence of the oscillator constant guarantees that there is no singularity in the rescaling due to an accidental zero in s_0. In the case of Bessel's equation it is amusing to note that one of the first applications [14] of Bessel's functions was in the solution of Kepler's equation

$$M = E - e\sin E, \tag{3.34}$$

where E is the eccentric anomaly, M the mean anomaly and e the eccentricity of an inverse square orbit. The Kepler Problem is also reducible to a simple harmonic oscillator by means of Lie symmetries [50].

$$
\begin{tikzcd}
\boxed{\mathbf{u}' = A(t)\mathbf{u}} \arrow[rr, leftrightarrow] \arrow[d, "s_{ij}(t)u_j\partial_{u_i}"] & & \boxed{u_1'' + q(t)u_1 = 0} \arrow[d, "s_0(t)\partial_t + \frac{1}{2}s_0'(t)u_1\partial_{u_1}"] \\
\begin{gathered} \mathbf{s}_2 = \mathbf{s}_1' + A^T\mathbf{s}_1 \\ \mathbf{s}_1 = (-s_2'/2, s_2) \\ s_2''' + 4qs_2' + 2q's_2 = 0 \end{gathered} \arrow[d] & & s_0''' + 4qs_0' + 2q's_0 = 0 \arrow[d] \\
\boxed{\dot{\mathbf{w}} = C\mathbf{w}} \arrow[d] & & \boxed{\frac{\mathrm{d}^2 v_1}{\mathrm{d}T^2} + Dv_1 = 0} \arrow[d] \\
\boxed{\mathbf{u} = \mathbf{F}(t, \mathbf{c})} \arrow[rr, leftrightarrow] & & \boxed{u_1 = F_1(t, \mathbf{c})}
\end{tikzcd}
$$

Figure 2:

Second order equation.

4 Equations of higher order

We have dwelt upon second order equations at some length for two reasons apart from the frequency of their occurrences in applications. In the first place linear second order ordinary differential equations are exceptional in that all have the same Lie algebra, $sl(3,R)$, of point symmetries. This is not the case with higher order linear equations for which the number of Lie point symmetries can be $n+1$, $n+2$ and $n+4$, where n is the order of the equation. Consequently one would expect some differences in the ability to find a transformation to convert a nonautonomous nth order ordinary differential equation to an autonomous nth order ordinary differential equation. The second point is that the happy facility one had for the second order linear equations in the choice of symmetry, *ie* putting $s_0 = 0$ and $s_2 \neq 0$ or vice versa, does not have the same impact. In both instances the symmetry selected for the linear system is a generalised symmetry of the nth order ordinary differential equation.

We consider firstly the third order equation written in normal form, *videlicet*

$$u_1''' + p(t)u_1' + q(t)u_1 = 0. \tag{4.1}$$

The corresponding system of first order equations is

$$\mathbf{u}' = A(t)\mathbf{u}, \tag{4.2}$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -q & -p & 0 \end{bmatrix}. \tag{4.3}$$

In the case that $s_0 = 0$, (2.6) leads to the system of determining equations for the symmetry, $s_{ij}u_j\partial_{u_i}$,

$$\begin{aligned} \mathbf{s}_1' &= -A^T\mathbf{s}_1 + \mathbf{s}_2 \\ \mathbf{s}_2' &= -A^T\mathbf{s}_2 + \mathbf{s}_3 \\ \mathbf{s}_3' &= -q\mathbf{s}_1 - p\mathbf{s}_2 - A^T\mathbf{s}_3, \end{aligned} \tag{4.4}$$

where the vectors $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$ consist of the elements s_{1j}, s_{2j} and s_{3j} in $s_{ij}u_j\partial_{u_i}$ of the symmetry. It is immediately apparent that the task of determining the s_{ij} is reducible to that of determining $\mathbf{s}_1$ from the system of three third order equations obtained by the elimination of $\mathbf{s}_2$ and $\mathbf{s}_3$ and then obtaining $\mathbf{s}_2$ and $\mathbf{s}_3$ recursively using (4.4a,b).

We note that this procedure works *mutatis mutandis* for higher order equations. For an nth order equation the three vectors $\mathbf{s}_i, i = 1, 3$ become n-vectors $\mathbf{s}_i, i = 1, n$, the 3×3 matrix A becomes $n \times n$ and the number of equations in (4.4) n with the addition of more of the type of (4.4a,b). For the nth order differential equation

$$u_1^{(n)} + pu_1^{(n-2)} + \sum_{i=0}^{n-3} q_i u_1^{(i)} = 0 \tag{4.5}$$

the bottom row of A is

$$-q_0, -q_1, \ldots, -q_{n-3}, -p, 0. \tag{4.6}$$

The inclusion of s_0 requires the addition of $(010)^T s_0'$, $(001)^T s_0'$ and $-\left[(qp0)^T s_0\right]'$ to (4.4a,b,c) respectively with suitable extensions for higher orders.

In determining the Lie point symmetries of (4.1) which have the form

$$\Gamma = s_0 \partial_t + s_1 u_1 \partial_{u_1} \tag{4.7}$$

we arrive at two equations for s_0, *videlicet*

$$\begin{aligned} 2s_0''' + 2ps_0' + p_0' &= 0 \\ s_0^{\mathrm{IV}} + ps_0'' + 3qs_0' + q's_0 &= 0. \end{aligned} \tag{4.8}$$

The form of (4.8a) is the familiar third order Ermakov-Pinney equation which determines s_0 given p. Differentiation of the former of (4.8) and substitution into the latter of (4.8) gives the compatibility condition

$$6qs_0' + 2q's_0 = 3p's_0' + p''s_0 \tag{4.9}$$

which, with the integrating factor s_0^2, is easily integrated to give

$$q = \tfrac{1}{2}p' - \frac{D}{s_0^3}. \tag{4.10}$$

Consequently for (4.1) to be transformable to an autonomous equation it must have the form

$$u_1''' + pu_1' + \left(\tfrac{1}{2}p' - \frac{D}{s_0^3}\right) u_1 = 0, \tag{4.11}$$

where s_0 is given by the solution of (4.8a). Alternatively one could write

$$
\begin{aligned}
& u_1''' - \frac{1}{s_0^2}\left[2s_0s_0'' - s_0'^{\,2} - K\right]u_1' \\
& -\frac{1}{s_0^3}\left[s_0^2s_0''' - 2s_0s_0's_0'' + s_0'^{\,3} + Ks_0' + D\right]u_1 = 0
\end{aligned} \tag{4.12}
$$

in which s_0 is now be to taken as an arbitrary differentiable function and K, which was the value of the integral $2s_0s_0'' - s_0'^{\,2} + ps_0^2$, is now some parameter. The symmetry is

$$\Gamma = s_0\partial_t + \dot{s}_0u_1\partial_{u_1} \tag{4.13}$$

for which the canonical variables are

$$T = \int\frac{\mathrm{d}t}{s_0} \quad \text{and} \quad v_1 = \frac{u_1}{s_0}. \tag{4.14}$$

and (4.11) ($\text{\ae}q$ (4.4)) becomes the constant coefficient equation

$$\dddot{v}_1 + K\dot{v}_1 + Dv_1 = 0. \tag{4.15}$$

(As usual overdot means differentiation with respect to new time.)

Here we note an interesting result. When $D = 0$, (4.1), and so (4.15), is self-adjoint and possesses seven Lie point symmetries, the maximum for a third order ordinary differential equation. When $D \neq 0$, the equation is not self-adjoint and has only five Lie point symmetries as it loses two of the elements of the $sl(2,R)$ subalgebra. Thus we have the curious situation that the number of Lie point symmetries, five or seven, does not affect the possibility to transform an equation of the form of (4.1) to autonomous form. The two additional symmetries in the maximal symmetry case are not necessary for this purpose. We conclude that (4.12) with $D \neq 0$ represents the most general form of a third order equation in normal form which can be rendered autonomous by a point transformation (as in (4.14)). We do observe, however, that there exist three possible transformations of the third order equation of maximal symmetry to render it autonomous since any one of the three solutions of (4.8a) can be used. In the case that $D \neq 0$ there is only one transformation and that is the one with the s_0 of the equation (4.12).

When we determine the generalised self-similar symmetries

$$\Gamma = s_{ij}u_j\partial_{u_i}, \tag{4.16}$$

in which s_0 is put at zero without loss of generality, of the linear system (4.4), we obtain three third order equations for s_{11}, s_{12} and s_{13}, *videlicet* the so-called Fleischen-van Weltunter (F-vW) system mentioned in the Abstract

$$\begin{aligned} s_{11}''' &= q''s_{13} + 3q's_{13}' + q's_{12} - s_{11}'p + 3s_{12}'q + 3s_{13}''q & (4.17) \\ s_{12}''' &= p''s_{13} + 3p's_{13}' + p's_{12} + 2q's_{13} - 3s_{11}'' + 2s_{12}'p \\ &\quad +3s_{13}''p + 3s_{13}'q & (4.18) \\ s_{13}''' &= 2p's_{13} - 3s_{11}' - 3s_{12}'' + 2s_{13}'p & (4.19) \end{aligned}$$

(using the interactive code of Nucci [46, 47]). These equations are compatible and so all third order ordinary differential equations can be rendered autonomous in this way. There is no contradiction in terms of the numbers of Lie point symmetries in the case of a (4.1) with only four Lie point symmetries as the transformation is generalised and so conservation of the number of point symmetries is not assured.

The same procedure applies to higher order equations. Only in the case of second order equations is there a difference due to the peculiar nature of the Lie point symmetries of scalar second order linear differential equations. One has a choice of applying a symmetry as in (4.7) to obtain a point transformation to be able to render autonomous equations having $n+4$ or $n+2$ Lie point symmetries or the generalised symmetry (4.16) to render autonomous all linear differential equations through the underlying linear system. As the pattern is established with the third order equations, we summarise the procedure in Figure 3.

We noted that our procedure enabled us to identify the general form of a linear third order ordinary differential equation with five Lie point symmetries ((4.11), or equivalently (4.12)). This same process can be applied to higher order linear equations almost as easily using the interactive REDUCE code of Nucci [46, 47]. As Hill [22] has already given the result, obtained by a different approach, for the class of fourth order equations, we give the result for the fifth order equation

$$u_1^{\mathrm{V}} + p(t)u_1''' + q_2(t)u_1'' + q_1(t)u_1' + q_0(t)u_1 = 0. \qquad (4.20)$$

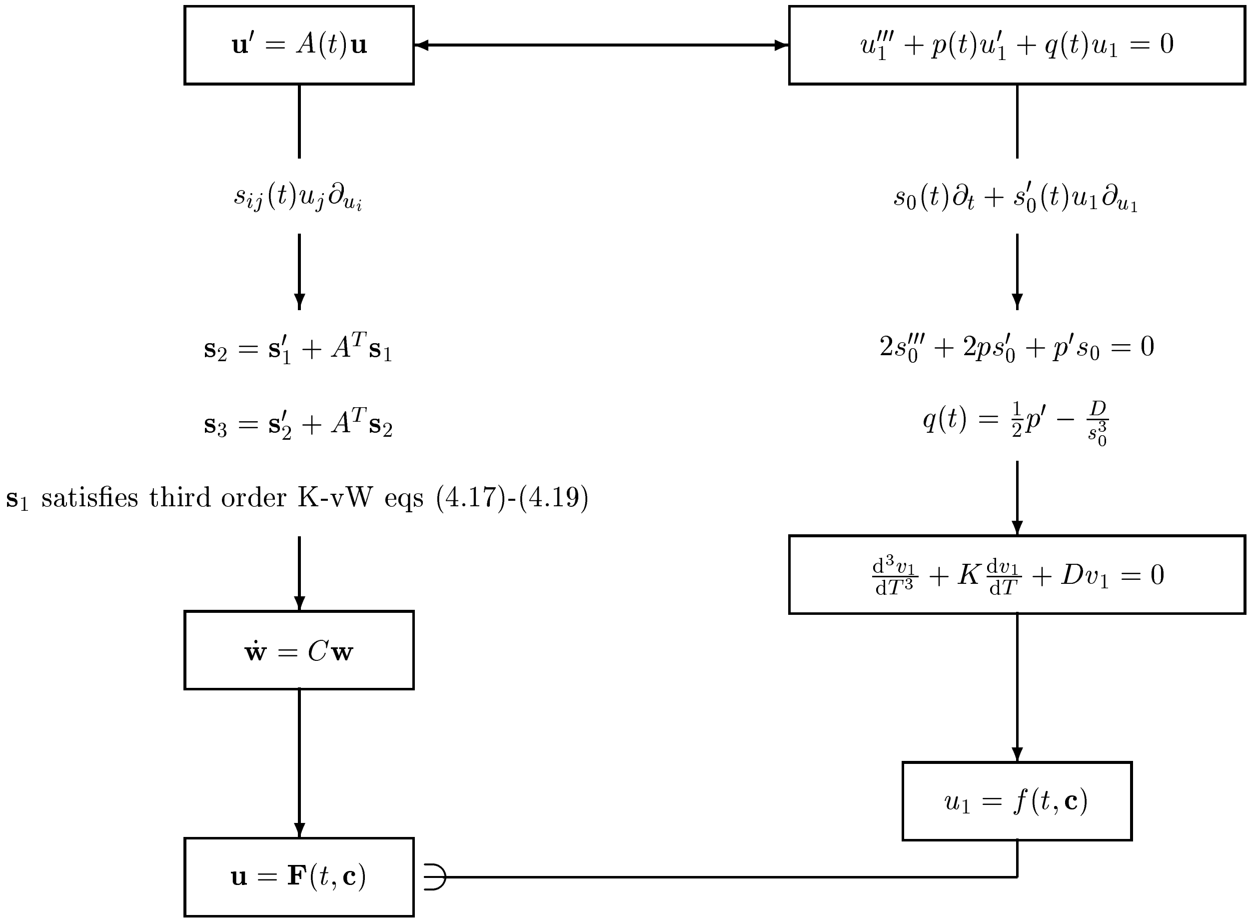

Figure 3:

Third order equation.

We find that the conditions on the functions $q_i(t), i = 0, 2$ for (4.20) to be rendered autonomous by this type of transformation are

$$\begin{aligned}
q_2 &= \frac{3}{2}p' - \frac{1}{2}\frac{D_2}{s_0^3} \\
q_1 &= \frac{9}{10}p'' + \frac{4}{25}p^2 + \frac{3}{2}\frac{s_0' D_2}{s_0^4} + \frac{D_1}{50 s_0^4} \\
q_0 &= \frac{1}{5}p''' + \frac{4}{25}pp' - \left(\frac{3}{2}\frac{s_0'^{\,2}}{s_0^5} - \frac{p}{10 s_0^3}\right) D_2 - \frac{s_0' D_1}{25 s_0^5} + \frac{D_0}{50 s_0^5} \qquad (4.21)
\end{aligned}$$

and the canonical variables for the transformation are found from the Lie point symmetries

$$\Gamma = s_0 \partial_t + 2 s_0' u_1 \partial_{u_1}, \qquad (4.22)$$

where s_0 is the solution of the Ermakov-Pinney equation

$$10 s_0''' + 2 p s_0' + p' s_0 = 0. \qquad (4.23)$$

The fifth order equations of maximal symmetry are obtained when all the arbitrary constants $D_i, i = 0, 2$, are zero, *cf* the results given in Mahomed and Leach [44]. If any of the constants, D_i, are nonzero, the number of point symmetries is reduced to seven. If any of the functions $q_i, i = 0, 2$, departs from the prescriptions of (4.21)-(4.23), the number of symmetries is reduced to six. When the conditions are satisfied, the autonomous form of (4.20) is

$$\frac{\mathrm{d}^5 v_1}{\mathrm{d}T^5} + K\frac{\mathrm{d}^3 v_1}{\mathrm{d}T^3} + D_2\frac{\mathrm{d}^2 v_1}{\mathrm{d}T^2} + \left(\frac{4}{25}K + D_1\right)\frac{\mathrm{d}v_1}{\mathrm{d}T} + D_0 v_1 = 0. \qquad (4.24)$$

The F-vW system for (4.20) is

$$\begin{aligned}
s_{11}^{V} = {} & -6p'' q_0 s_{15} - 4p' q_0' s_{15} - 15 p' s_{15}' q_0 - 3p' q_0 s_{14} + q_0^{IV} s_{15} \\
& + 5 q_0''' s_{15}' + q_0''' s_{14} + 5 q_0'' s_{14}' + 10 q_0'' s_{15}'' + q_0'' s_{13} + 5 q_0' s_{13}' \\
& + 10 q_0' s_{14}'' + 10 q_0' s_{15}''' - 2 q_0' s_{15}' p + q_0' s_{12} - 4 q_2' q_0 s_{15} \\
& - s_{11}''' p - s_{11}'' q_2 - s_{11}' q_1 + 5 s_{12}' q_0 + 10 s_{13}'' q_0 + 10 s_{14}''' q_0 \\
& - 2 s_{14}' p q_0 + 5 s_{15}^{IV} q_0 - 7 s_{15}'' p q_0 - 3 s_{15}' q_0 q_2 \qquad (4.25)
\end{aligned}$$

$$\begin{aligned}
s_{12}^{V} = {} & -6p'' q_1 s_{15} - 4p' q_1' s_{15} - 15 p' s_{15}' q_1 - 4p' q_0 s_{15} - 3p' q_1 s_{14} \\
& + 4 q_0''' s_{15} + 15 q_0'' s_{15}' + 3 q_0'' s_{14} + 10 q_0' s_{14}' + 20 q_0' s_{15}'' + 2 q_0' s_{13} \\
& + q_1^{IV} s_{15} + 5 q_1''' s_{15}' + q_1''' s_{14} + 5 q_1'' s_{14}' + 10 q_1'' s_{15}'' + q_1'' s_{13}
\end{aligned}$$

$$
\begin{aligned}
&+5q_1's_{13}' + 10q_1's_{14}'' + 10q_1's_{15}''' - 2q_1's_{15}'p + q_1's_{12} - 4q_2'q_1s_{15}\\
&-5s_{11}^{IV} - 3s_{11}''p - 2s_{11}'q_2 - s_{12}'''p - s_{12}''q_2 + 4s_{12}'q_1 + 10s_{13}''q_1\\
&+5s_{13}'q_0 + 10s_{14}'''q_1 + 10s_{14}''q_0 - 2s_{14}'pq_1 + 5s_{15}^{IV}q_1\\
&+10s_{15}'''q_0 - 7s_{15}''pq_1 - 2s_{15}'pq_0 - 3s_{15}'q_1q_2 \qquad (4.26)
\end{aligned}
$$

$$
\begin{aligned}
s_{13}^V = {} & -6p''q_2s_{15} - 4p'q_2's_{15} - 15p's_{15}'q_2 - 4p'q_1s_{15} - 3p'q_2s_{14}\\
&+6q_0''s_{15} + 15q_0's_{15}' + 3q_0's_{14} + 4q_1'''s_{15} + 15q_1''s_{15}' + 3q_1''s_{14}\\
&+10q_1's_{14}' + 20q_1's_{15}'' + 2q_1's_{13} + q_2^{IV}s_{15} + 5q_2'''s_{15}'\\
&+q_2'''s_{14} + 5q_2''s_{14}' + 10q_2''s_{15}'' + q_2''s_{13} + 5q_2's_{13}'\\
&+10q_2's_{14}'' + 10q_2's_{15}''' - 2q_2's_{15}'p - 4q_2'q_2s_{15} + q_2's_{12}\\
&-10s_{11}''' - 3s_{11}'p - 5s_{12}^{IV} - 3s_{12}''p + 3s_{12}'q_2\\
&-s_{13}'''p + 9s_{13}''q_2 + 4s_{13}'q_1 + 10s_{14}'''q_2 + 10s_{14}''q_1\\
&-2s_{14}'pq_2 + 5s_{14}'q_0 + 5s_{15}^{IV}q_2 + 10s_{15}'''q_1 - 7s_{15}''pq_2\\
&+10s_{15}''q_0 - 2s_{15}'pq_1 - 3s_{15}'q_2^2 \qquad (4.27)
\end{aligned}
$$

$$
\begin{aligned}
s_{14}^V = {} & p^{IV}s_{15} + 5p'''s_{15}' + p'''s_{14} + 5p''s_{14}' + 10p''s_{15}'' - 6p''ps_{15}\\
&+p''s_{13} - 4p'^2s_{15} + 5p's_{13}' + 10p's_{14}'' + 10p's_{15}''' - 17p's_{15}'p\\
&-3p'ps_{14} - 4p'q_2s_{15} + p's_{12} + 4q_0's_{15} + 6q_1''s_{15} + 15q_1's_{15}'\\
&+3q_1's_{14} + 4q_2'''s_{15} + 15q_2''s_{15}' + 3q_2''s_{14} + 10q_2's_{14}' + 20q_2's_{15}''\\
&-4q_2'ps_{15} + 2q_2's_{13} - 10s_{11}'' - 10s_{12}''' + 2s_{12}'p - 5s_{13}^{IV}\\
&+7s_{13}''p + 3s_{13}'q_2 + 9s_{14}'''p + 9s_{14}''q_2 - 2s_{14}'p^2 + 4s_{14}'q_1\\
&+5s_{15}^{IV}p + 10s_{15}'''q_2 - 7s_{15}''p^2 + 10s_{15}''q_1 - 5s_{15}'pq_2 + 5s_{15}'q_0 \qquad (4.28)
\end{aligned}
$$

$$
\begin{aligned}
s_{15}^V = {} & 4p'''s_{15} + 15p''s_{15}' + 3p''s_{14} + 10p's_{14}' + 20p's_{15}''\\
&-4p'ps_{15} + 2p's_{13} + 4q_1's_{15} + 6q_2''s_{15} + 15q_2's_{15}' + 3q_2's_{14}\\
&-5s_{11}' - 10s_{12}'' - 10s_{13}''' + 2s_{13}'p - 5s_{14}^{IV} + 7s_{14}''p + 3s_{14}'q_2\\
&+9s_{15}'''p + 9s_{15}''q_2 - 2s_{15}'p^2 + 4s_{15}'q_1. \qquad (4.29)
\end{aligned}
$$

The outstanding question is whether there is any feature distinguishing equations of $n+4$ and $n+2$ Lie point symmetries. In the case of the third order equation (4.1) the answer is to be found in its F-vW system (4.17)-(4.19). As a set of higher order equations, the F-vW system can be analysed for its Lie point symmetries. The maximum number of Lie point symmetries for a system of n mth order ordinary differential equations ($m \geq 3$) is $n^2 + mn + 3$ [51], comprising the three symmetries of $sl(2, R)$, mn solution symmetries and n^2 homogeneity symmetries. The structure of the general symmetry is

$$\Gamma = \xi \partial_t + \sum_{i=1}^{m} (\eta_i u_i + \zeta_i) \partial_{u_i}. \tag{4.30}$$

For (4.1) the number of Lie point symmetries of its F-vW system is either the maximal 21 or 18. The difference is determined by the relation (4.9) which arises in the simplification of the determining equations for the symmetries of the system.

The F-vW system (4.17)-(4.19) is the simplest example of that type as the size and order of the system increases with the order of the original equation. Thus for the fourth order equation in normal form

$$u_1^{\mathrm{IV}} + pu_1'' + q_1 u_1' + q_0 u_1 = 0 \tag{4.31}$$

its F-vW system comprises the four fourth order equations

$$\begin{aligned}
s_{11}^{IV} &= -3p'q_0 s_{14} + q_0''' s_{14} + 4q_0'' s_{14}' + q_0'' s_{13} + 4q_0' s_{13}' + 6q_0' s_{14}'' + q_0' s_{12} \\
&\quad - s_{11}'' p - s_{11}' q_1 + 4s_{12}' q_0 + 6s_{13}'' q_0 + 4s_{14}''' q_0 - 2s_{14}' p q_0 && (4.32) \\
s_{12}^{IV} &= -3p'q_1 s_{14} + q_1''' s_{14} + 4q_1'' s_{14}' + q_1'' s_{13} + 4q_1' s_{13}' + 6q_1' s_{14}'' \\
&\quad + q_1' s_{12} + 3q_0'' s_{14} + 8q_0' s_{14}' + 2q_0' s_{13} - 4s_{11}''' - 2s_{11}' p - s_{12}'' p \\
&\quad + 3s_{12}' q_1 + 6s_{13}'' q_1 + 4s_{13}' q_0 + 4s_{14}''' q_1 + 6s_{14}'' q_0 - 2s_{14}' p q_1 && (4.33) \\
s_{13}^{IV} &= p''' s_{14} + 4p'' s_{14}' + p'' s_{13} + 4p' s_{13}' + 6p' s_{14}'' - 3p' p s_{14} + p' s_{12} \\
&\quad + 3q_1'' s_{14} + 8q_1' s_{14}' + 2q_1' s_{13} + 3q_0' s_{14} - 6s_{11}'' - 4s_{12}''' + 2s_{12}' p \\
&\quad + 5s_{13}'' p + 3s_{13}' q_1 + +4s_{14}''' p + 6s_{14}'' q_1 - 2s_{14}' p^2 + 4s_{14}' q_0 && (4.34) \\
s_{14}^{IV} &= 3p'' s_{14} + 8p' s_{14}' + 2p' s_{13} + 3q_1' s_{14} - 4s_{11}' - 6s_{12}'' - 4s_{13}''' \\
&\quad + 2s_{13}' p + 5s_{14}'' p + 3s_{14}' q_1. && (4.35)
\end{aligned}$$

The maximum number of Lie point symmetries of this system is 31 and is achieved only if the condition that (4.31) have eight Lie point symmetries, *videlicet*

$$q_1 = p \quad \text{and} \quad q_0 = \frac{3}{10}p'' + \frac{9}{100}p^2 \tag{4.36}$$

is achieved. (The symmetry condition is stronger than the requirement that the equation be self-adjoint. The latter places no constraint on q_0.) If (4.31) can be rendered autonomous by a point transformation, *ie* q_1 and q_0 are such as to permit six Lie point symmetries in (4.31), the Lie point symmetries of the F-vW system (4.32)-(4.35) number 28.

We infer that this property holds for all F-vW systems arising from the analysis of linear ordinary differential equations of order $m(\geq 3)$ for the generalised self-similar symmetry.

5 Concluding remarks

The results reported in this paper were motivated by a corollary in Floquet theory which states that there exists a linear transformation from a first-order linear system with periodic coefficient matrix to one with a constant coefficient matrix. We have shown that this result is general and can be extended to scalar linear equations of higher order – equally linear systems of higher order equations – by means of the common technique of writing a higher order linear equation as a first-order system.

In the case of linear scalar higher order equations we saw that equations with both $n+4$ and $n+2$ Lie point symmetries could be rendered autonomous by a Kummer-Liouville transformation linear in the dependent variable. This is not possible for equations with only $n+1$ Lie point symmetries. The F-vW system which arises from the Lie point symmetries of these equations was shown to maintain the distinction in symmetry between the class of equations with $n+4$ Lie point symmetries and that with $n+2$ Lie point symmetries.

In this analysis the system of determining equations to be solved quickly becomes manually unhandable and the success of the analysis is dependent upon the use of a suitable code. Interpretation of the results also requires a knowledge of the Lie point symmetries and the algebras for systems of higher order linear differential equations, a subject which is yet to receive extensive attention.

Acknowledgements

PGLL expresses his deep appreciation of the hospitality of MCN and the provision of facilities by the Dipartimento di Matematica e Informatica, Università di Perugia, during the period in which this work was initiated, and of GEODYSYC, Department of Mathematics, University of the Aegean, during the period in which this work was finalised and acknowledges the support of the National Research Foundation of South Africa and the University of Natal.

References

[1] Abel NH (1826-27) Recherches sur les fonctions elliptiques *J für Math* **II** 101-196

[2] Abramowitz M & Stegun IA (1970) *Handbook of Mathematical Functions* (Dover, New York)

[3] Aguirre M & Krause J (1988) Finite point transformations and linearisation of $\ddot{x} = f(x)$ *J Phys A: Math Gen* **21** 2841-2845

[4] Aguirre M & Krause J (1988) $SL(3, R)$ as the group of symmetry transformations for all one-dimensional linear systems. II Realisations of the Lie algebra *J Math Phys* **29** 1746-1752

[5] Arscott FM (1964) *Periodic Differential Equations* (Pergamon, London)

[6] Bessel FW (1824) Untersuchung des Teils der planetarischen Störungen, welcher aus der Bewegung der Sonne entsteht *Abhandlungen der Berliner Akademie* (pub 1826) 1-52

[7] Bluman GW & Cole JD (1969) The general similarity solution of the heat equation *J Math Mech* **18** 1025-1042

[8] Bluman GW & Reid GJ (1988) New symmetries for ordinary differential equations *IMA J Appl Math* **40** 87-94

[9] Bluman GW & Kumei S (1989) *Symmetries and Differential Equations* (Springer-Verlag, New York)

[10] Burgan J-R, Feix MR, Fijalkow E, Gutierrez J & Munier A (1978) Utilization des groupes de transformation pour la resolution des équations aux derivées partielles in *Applied Inverse Problems* ed Sabatier PC (Springer-Verlag, Berlin)

[11] Camiz P, Geradi A, Marchioro C, Presutti E & Scacciatelli E (1971) Exact solution of a time-dependent quantal harmonic oscillator with a singular perturbation *J Math Phys* **12** 2040-2043

[12] Chukwumah GC (1991) Lie group theory of the Bessel function of the first kind of integral order *Int J Theoret Phys* **30** 1011-1031

[13] Colwell P (1992) Bessel functions and Kepler's equation *Am Math Monthly* **99** 45-48

[14] Dutka J (1995) On the early history of Bessel functions *Arch Hist Exact Sci* **49** 105-134

[15] Eliezer CJ & Grey A (1976) A note on the time-dependent harmonic oscillator *SIAM J Appl Math* **30** 463-468

[16] Ermakov V (1880) Second order differential equations. Conditions of complete integrability *Univ Izvestia Kiev Ser III* **9** 1-25 (trans Harin AO)

[17] Flessas GP, Govinder KS & Leach PGL (1997) Characterisation of the algebraic properties of first integrals of scalar ordinary differential equations *J Math Anal Appl* **212** 349-374

[18] Gauss CF (1876) *Werke III* (Königlichen Gesellschaft der Wissenschaften, Göttingen) 404-460

[19] Gorringe VM & Leach PGL (1988) Lie point symmetries for systems of second order ordinary differential equations *Quæst Math* **14** 277-289

[20] Günther NJ & Leach PGL (1977) Generalised invariants for the time-dependent harmonic oscillator *J Math Phys* **18** 572-576

[21] Hermite C (1877) Sur quelques applications des fonctions ellliptiques in *Oeuvres de Charles Hermite. Tome III* (Gauthier-Villars, Paris, 1908) 267-418

[22] Hill JM (1992) *Differential Equations and Group Methods for Scientists and Engineers* (CRC Press, Boca Raton)

[23] Ince EL (1927) *Ordinary Differential Equations* (Longmans, Green & Co, London)

[24] Jacobi KGJ (1829) *Fundamenta Nova Theoriæ Functionum Ellipticarum* (Fratrum Borntraeger, Regiomonti)

[25] Kara AH & Mahomed FM (2000) Relationship between symmetries and conservation laws *Int J Theoret Phys* **39** 23-40

[26] Krause J & (1984) On the complete symmetry group of the classical Kepler system *J Math Phys* **35** 5734-5748

[27] Kruskal MJ (1962) Asymptotic theory of Hamiltonian and other systems with all solutions nearly periodic *J Math Phys* **3** 806-828

[28] Kummer EE (1887) De generali quadam æquatione differentiali tertii ordinis *J für Math* **100** 1-9 (reprinted from the Programm des evangelischen Königl und Stadgymnaziums in Liegnitz for the year 1834)

[29] Leach PGL (1977) On the theory of time-dependent linear canonical transformations as applied to Hamiltonians of harmonic oscillator type *J Math Phys* **18** 1608-1611

[30] Leach PGL, Govinder KS, Brazier TI & Schei CS (1993) The ubiquitious time-dependent simple harmonic oscillator *S Afr J Sci* **89** 126-130

[31] Legendre A-M (1825) *Traité des Fonctions Elliptiques* (Gauthier, Paris)

[32] Levi D & Winternitz P (1989) Nonclassical symmetry reduction: example of the Boussinesq equation *J Phys A: Math Gen* **22** 2915-2924

[33] Lewis HR Jr (1967) Classical and quantum systems with time-dependent harmonic oscillator-type Hamiltonians *Phys Rev Lett* **18** 510-512

[34] Lewis HR Jr (1968) Motion of a time-dependent harmonic oscillator and of a charged particle in a time-dependent, axially symmetric, electromagnetic field *Phys Rev* **172** 1313-1315

[35] Lewis HR Jr (1968) Class of exact invariants for classical and quantum time-dependent harmonic oscillators *J Math Phys* **9** 1976-1986

[36] Lewis HR Jr & Riesenfeld WB (1969) An exact quantum theory of the time-dependent harmonic oscillator and of a charged particle in a time-dependent electromagnetic field *J Math Phys* **10** 1458-1473

[37] Lie S (1888) *Theorie der Transformationsgruppen. Erster Abschnitt* (Teubner,Leipzig)

[38] Lie S (1890) *Theorie der Transformationsgruppen. Zweiter Abschnitt* (Teubner, Leipzig)

[39] Lie S (1893) *Theorie der Transformationsgruppen. Dritter und Letzter Abschnitt* (Teubner, Leipzig)

[40] Lie S (1896) *Geometrie der Berührungstransformationen* (Teubner, Leipzig)

[41] Lie S (1912) *Vorlesungen über Differentialgleichungen* (Teubner, Leipzig)

[42] Liouville J (1837) Sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équatione différentielle du second ordre contenent un paramètre variable *J de Math* **II** 16-35

[43] MacLachlan NW (1947) *The Theory and Application of Mathieu Functions* (Clarendon, Oxford)

[44] Mahomed FM & Leach PGL (1990) Symmetry Lie algebras of nth order ordinary differential equations *J Math Anal Appl* **151** 80-107

[45] Noether E (1918) Invariante Variationsprobleme *Nach König Gesell Wissen Göttingen Math-phys Kl* 235-257

[46] Nucci MC (1990) Interactive REDUCE programs for calculating classical, non-classical and Lie-Bäcklund symmetries of differential equations *Preprint GT Math: 062090-051*

[47] Nucci MC (1996) Interactive REDUCE programs for calculating Lie point, non-classical, Lie-Bäcklund, and approximate symmetries of differential equations: manual and floppy disk *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 3: New Trends* ed N H Ibragimov (Boca Raton: CRC Press) 415-481.

[48] Nucci MC (1996) The complete Kepler group can be derived by Lie group analysis *J Math Phys* **37** 1772-1775

[49] Nucci MC (2000) Nonclassical symmetries as special solutions of heir-equations (preprint: nlin.SI/0011008)

[50] Nucci MC & Leach PGL (2001) The harmony in the Kepler and related problems *J Math Phys* **43** 746-764

[51] Nucci MC & Leach PGL (2001) Maximal algebras of Lie point symmetries for systems of ordinary differential equations (preprint: Dipartimento di Matematica e Informatica, Università di Perugia, 06123 Perugia, Italia)

[52] Pillay T & Leach PGL (1999) Generalized Laplace-Runge-Lenz vectors and nonlocal symmetries *S Afr J Sci* **95** 403-407

[53] Pinney E (1950) The nonlinear differential equation $y''(x) + p(x)y + cy^{-3} = 0$ *Proc Amer Math Soc* **1** 681

[54] Rosati L, Nucci MC & Mezzanotte F (2001) Italian edition and rereading of FUNDAMENTA NOVA by K.G.J. Jacobi 150 years since his death (February 18, 1851 - February 18, 2001) [in Italian] *Preprint RT n. 2001-4*