Generalised boundary conditions for the Aharonov-Bohm effect combined with a homogeneous magnetic field

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Abstract

The most general admissible boundary conditions are derived for an idealised Aharonov-Bohm flux intersecting the plane at the origin on the background of a homogeneous magnetic field. A standard technique based on self-adjoint extensions yields a four-parameter family of boundary conditions; other two parameters of the model are the Aharonov-Bohm flux and the homogeneous magnetic field. The generalised boundary conditions may be regarded as a combination of the Aharonov-Bohm effect with a point interaction. Spectral properties of the derived Hamiltonians are studied in detail.

1 Introduction

The purpose of this paper is to determine the most general admissible boundary conditions for the Aharonov-Bohm (AB) effect in the plane on the background of a homogeneous magnetic field, and also to investigate the basic properties of Hamiltonians obtained this way. The history of the effect goes four decades back and starts from the observation of Aharonov and Bohm [1] that the behavior of a charged quantum particle is influenced by a magnetic flux even if the field is zero in the region where the particle is localized. A particularly elegant treatment is possible in case of an idealized setup in which the AB flux is concentrated along a line perpendicularly intersecting the plane, conventionally at the origin [2].

The boundary conditions of the last mentioned paper are not the most general ones; the full family of such conditions giving the AB effect in the plane was derived in [7], and simultaneously also in [8]. These generalised boundary conditions may be interpreted as a combination of the AB effect with a point interaction supported, too, at the origin, although this is just one possible point of view. In any case they can be described and investigated by the technique of self-adjoint extensions which is in principle the same one as that used in the paper [6] in which two-dimensional point interactions were introduced.

A natural question is what happens if such a system is placed into a background homogeneous magnetic field. This problem attracted some attention recently, even with a controversy: the papers [3, 4, 5] consider the "pure" AB effect in this setting for the Pauli operator, i.e. a spin 1/2 particle. The last named property leads to specific behavior related to the Aharonov-Casher effect, which we will not discuss here.

Our aim here is different: we are going to consider a spinless particle with a point flux and a homogeneous background, and ask about the most general class of boundary conditions analogous to those of [7, 8]. The basic difference between the situations without and with a homogeneous magnetic field is that in the former case the spectrum is absolutely continuous and equal to the positive half-line possibly augmented with at most two negative eigenvalues (depending on the choice of boundary conditions) while in the latter case the spectrum is pure point and the point flux and interaction gives rise to eigenvalues in each gap between neighboring Landau levels. Our goal is to discuss these spectral properties in detail.

2 Formulation of the problem, preliminaries

We consider the symmetric operator

$$L = -(\nabla - A(\nabla))^2, \;\; \mathrm{Dom}(L) = C_0^\infty(\mathbb{R}^2 \setminus \{0\}),$$

where the vector potential A is a sum of two parts, $A = A_{\mathrm{hmf}} + A_{\mathrm{AB}}$, with the part A_{hmf} corresponding to the homogeneous magnetic field in the circular gauge,

$$A_{\mathrm{hmf}} = -\frac{\imath B}{2}(-x_2dx_1 + x_1dx_2),$$

and with the part A_{AB} corresponding to the idealised AB effect,

$$A_{\mathrm{AB}} = \frac{\imath\Phi}{2\pi r^2}(-x_2dx_1 + x_1dx_2), \;\; r^2 = x_1^2 + x_2^2.$$

Without loss of generality we may assume that $B > 0$. Further, we rescale the Aharonov-Bohm flux,

$$\alpha = -\frac{\Phi}{2\pi},$$

to have a variable which expresses the number of flux quanta and, as usual, we make use of the gauge symmetry allowing us to assume that $\alpha \in \,]0,1[\,$. Hence the case $\Phi \in 2\pi\mathbb{Z}$ is excluded since it is gauge equivalent to the vanishing AB flux. Our goal is to describe all the self-adjoint extensions of L as well as to investigate their basic properties.

It is straightforward to determine the adjoint operator L^*,

$$\begin{aligned}\psi \in \mathrm{Dom}(L^*) \quad \Longleftrightarrow \quad & \psi \in L^2(\mathbb{R}^2, \mathrm{d}^2x) \cap H^{2,2}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\}) \\ & \text{and } (\nabla - A(\nabla))^2\psi \in L^2(\mathbb{R}^2, \mathrm{d}^2x).\end{aligned}$$

Next we can employ the rotational symmetry when using the polar coordinates (r, θ) and decomposing the Hilbert space into the orthogonal sum of the eigenspaces of the angular momentum,

$$L^2(\mathbb{R}^2, \mathrm{d}^2x) = \sum_{m\in\mathbb{Z}}^{\oplus} L^2(\mathbb{R}_+, r\,\mathrm{d}r) \otimes \mathbb{C}\, e^{\imath m\theta}. \tag{1}$$

In the polar coordinates the operator L (and correspondingly L^*) takes the form

$$L = -\frac{1}{r}\,\partial_r r \partial_r + \frac{1}{r^2}\left(-\imath\partial_\theta + \alpha + \frac{Br^2}{2}\right)^2.$$

The operator L^* commutes on $\mathrm{Dom}(L^*)$ with the projectors P_m onto the eigenspaces of the angular momentum,

$$P_m\psi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \psi(r,\theta')\, e^{\imath m(\theta-\theta')}\,\mathrm{d}\theta',$$

and therefore L^* decomposes in correspondence with the orthogonal sum (1),

$$L^* = \sum_{m\in\mathbb{Z}}^{\oplus} (L^*)_m. \tag{2}$$

Thus we can reduce the problem and work in the sectors $\mathrm{Ran}\, P_m$, $m \in \mathbb{Z}$. For a given spectral parameter $\lambda \in \mathbb{C}$ we choose two independent solutions (except of particular values of λ) of the differential equation

$$\left(-\frac{1}{r}\,\partial_r r \partial_r + \frac{1}{r^2}\left(m + \alpha + \frac{Br^2}{2}\right)^2\right) g(r) = \lambda\, g(r), \tag{3}$$

namely

$$\begin{aligned} g_m^1(\lambda; r) &= r^{|m+\alpha|}\, F\left(\beta(m,\lambda), \gamma(m), \frac{Br^2}{2}\right) \exp\left(-\frac{Br^2}{4}\right), \\ g_m^2(\lambda; r) &= r^{|m+\alpha|}\, G\left(\beta(m,\lambda), \gamma(m), \frac{Br^2}{2}\right) \exp\left(-\frac{Br^2}{4}\right), \end{aligned} \tag{4}$$

where

$$\begin{aligned} \beta(m,\lambda) &= \frac{1}{2}\left(1+m+\alpha+|m+\alpha|-\frac{\lambda}{B}\right), \\ \gamma(m) &= 1+|m+\alpha|\,. \end{aligned} \tag{5}$$

Here F and G are confluent hypergeometric functions [9, Chp. 13],

$$F(\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{(\beta)_n\,z^n}{(\gamma)_n\,n!}\,,$$

and

$$G(\beta,\gamma,z)=\frac{\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)}\,F(\beta,\gamma,z)+\frac{\Gamma(\gamma-1)}{\Gamma(\beta)}\,z^{1-\gamma}F(\beta-\gamma+1,2-\gamma,z). \tag{6}$$

Notice that $F(\beta,\gamma,z)$ and $G(\beta,\gamma,z)$ are linearly dependent if and only if $\beta\in-\mathbb{Z}_+$. Moreover, $F(\beta,\gamma,z)$ is an entire function, particularly, it is regular at the origin while $G(\beta,\gamma,z)$ has a singularity there provided $\gamma>1$ and $\beta\notin-\mathbb{Z}_+$, and in that case it holds true that

$$\lim_{z\to 0_+} z^{\gamma-1}G(\beta,\gamma,z)=\frac{\Gamma(\gamma-1)}{\Gamma(\beta)}\,.$$

Thus in the case when $1<\gamma<2$ we have the asymptotic behaviour, as $z\to 0_+$,

$$G(\beta,\gamma,z)=\frac{\Gamma(\gamma-1)}{\Gamma(\beta)}\,z^{1-\gamma}+\frac{\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+1)}+O(z^{2-\gamma}). \tag{7}$$

We shall also need some information about the asymptotic behaviour at infinity. When $z\to+\infty$ it holds true that

$$F(\beta,\gamma,z)=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)}\,(-z)^{-\beta}\left(1+O\left(z^{-1}\right)\right)+\frac{\Gamma(\gamma)}{\Gamma(\beta)}\,e^z z^{\beta-\gamma}\left(1+O\left(z^{-1}\right)\right) \tag{8}$$

and

$$G(\beta,\gamma,z) = z^{-\beta}\left(1 + O\left(z^{-1}\right)\right).$$

3 The standard Aharonov-Bohm Hamiltonian

With the above preliminaries it is straightforward to solve the spectral problem for the standard AB Hamiltonian as we mentioned in the introduction. This means to solve the eigenvalue problem

$$L^*\psi = \lambda\psi$$

with the boundary condition

$$\lim_{r\to 0_+} \psi(r,\theta) = 0. \tag{9}$$

By virtue of the decomposition (2) the problem is reduced to the countable set of equations

$$(L^*)_m f = \lambda f,\ m \in \mathbb{Z},$$

and hence to the differential equations (3).

The solution $g_m^2(\lambda;r)$ of (3) is ruled out because it contradicts the condition (9) and the solution $g_m^1(\lambda;r)$ belongs to $L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ if and only if $\beta(m,\lambda) = -n$, with $n \in \mathbb{Z}_+$. Since it holds

$$F(-n, 1+\sigma, z) = \frac{n!\,\Gamma(\sigma+1)}{\Gamma(n+\sigma+1)}\, L_n^\sigma(z),\ n \in \mathbb{Z}_+,$$

we get a countable set of eigenvalues,

$$\lambda_{m,n} = B\,(m + \alpha + |m+\alpha| + 2n + 1),\ m \in \mathbb{Z},\ n \in \mathbb{Z}_+,$$

with the corresponding eigenfunctions

$$f_{m,n}(r,\theta) = C_{m,n}\, r^{|m+\alpha|}\, L_n^{|m+\alpha|}\left(\frac{Br^2}{2}\right) \exp\left(-\frac{Br^2}{4}\right) e^{\imath m\theta}$$

where

$$C_{m,n} = \left(\frac{B}{2}\right)^{\frac{1}{2}(|m+\alpha|+1)} \left(\frac{n!}{\pi\,\Gamma(n+|m+\alpha|+1)}\right)^{1/2}$$

are the normalisation constants.

As it is well known if we fix $m \in \mathbb{Z}$ then the functions $\{f_{m,n}(r,\theta)\}_{n=0}^{\infty}$ form an orthonormal basis in $L^2(\mathbb{R}_+, r\,\mathrm{d}r) \otimes \mathbb{C}\, e^{\imath m\theta}$ and so the complete set of eigenfunctions $\{f_{m,n}(r,\theta)\}_{m\in\mathbb{Z},\, n\in\mathbb{Z}_+}$ is an orthonormal basis in $L^2(\mathbb{R}_+, r\,\mathrm{d}r)\otimes L^2([\,0,2\pi\,],\mathrm{d}\theta)$. Since all the eigenvalues $\lambda_{m,n}$ are real we get this way a well defined self-adjoint operator which is an extension of L. We conventionally call it the standard AB Hamiltonian and denote it by H^{AB}. Thus the spectrum of H^{AB} is pure point and can be written as a union of two parts,

$$\sigma(H^{AB}) = \sigma_{pp}(H^{AB}) = \{B(2k+1);\ k\in\mathbb{Z}_+\} \cup \{B(2\alpha+2k+1);\ k\in\mathbb{Z}_+\}.$$

Notice that the eigenvalues belonging to the first part are nothing but the Landau levels. All the eigenvalues $B(2k+1)$ have infinite multiplicities while the multiplicity of the eigenvalue $B(2\alpha+2k+1)$ is finite and equals $k+1$.

A final short remark concerning the Hamiltonian H^{AB} is devoted to the Green function. Naturally, the Green function is expressible as an infinite series

$$G^{AB}(z; r_1, \theta_1, r_2, \theta_2) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} G_m^{AB}(z; r_1, r_2)\, e^{\imath m(\theta_1-\theta_2)}$$

where

$$
\begin{aligned}
G^{AB}_m(z;r_1,r_2) &= 2\left(\frac{B}{2}\right)^{|m+\alpha|+1}(r_1r_2)^{|m+\alpha|}\exp\left(-\frac{1}{4}\,B({r_1}^2+{r_2}^2)\right)\\
&\quad\times\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+|m+\alpha|+1)}\\
&\quad\quad\times\frac{L_n^{|m+\alpha|}(\frac{1}{2}B{r_1}^2)\,L_n^{|m+\alpha|}(\frac{1}{2}B{r_2}^2)}{B(m+\alpha+|m+\alpha|+2n+1)-z}\,.
\end{aligned}
$$

The radial parts can be rewritten with the aid of the standard construction of the Green function for ordinary differential operators of second order,

$$
\begin{aligned}
G^{AB}_m(z;r_1,r_2) &= \left(\frac{B}{2}\right)^{|m+\alpha|+1}(r_1r_2)^{|m+\alpha|}\exp\left(-\frac{1}{4}\,B({r_1}^2+{r_2}^2)\right)\\
&\quad\times\frac{\Gamma\big(-w(m,z)\big)}{\Gamma(|m+\alpha|+1)}\,F(-w(m,z),|m+\alpha|+1,r_<)\\
&\quad\times G(-w(m,z),|m+\alpha|+1,r_>)
\end{aligned}
$$

where

$$
w(m,z)=\frac{z}{2B}-\frac{1}{2}(m+\alpha+|m+\alpha|+1)
$$

and $r_<=\min(r_1,r_2)$, $r_>=\max(r_1,r_2)$. This amounts to the identity

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{n!}{\Gamma(n+\sigma+1)}\,\frac{L_n^\sigma(y_1)\,L_n^\sigma(y_2)}{n-w}\\
&\quad=\frac{\Gamma(-w)}{\Gamma(\sigma+1)}\,F(-w,\sigma+1,y_<)\,G(-w,\sigma+1,y_>).
\end{aligned}
$$

We do not expect that a simpler form for the Green function could be derived since the Hamiltonian H^{AB} enjoys only the rotational symmetry.

4 Self-adjoint extensions of L

Recalling what has been summarised in Section 2 it is easy to determine the deficiency indices. The solution $g_m^1(\pm\imath; r)$ diverges exponentially at infinity (cf. (8)) while $g_m^2(\pm\imath; r)$ behaves well at infinity but has a singularity at the origin of the order $r^{-|m+\alpha|}$. Thus $g_m^2(\pm\imath; r) \in L^2(\mathbb{R}_+, r\,\mathrm{d}r)$ if and only if $m = -1$ or $m = 0$. This means that the deficiency indices are $(2, 2)$. For a basis in the deficiency subspaces $\mathcal{N}_{\pm\imath}$ we can choose

$$\{f_{m,\pm}(r,\theta) = \frac{1}{\sqrt{2\pi}} N_m\, g_m^2(\pm\imath; r)\, e^{\imath m\theta};\ m = -1, 0\}.$$

Thus

$$\begin{aligned}
f_{-1,\pm}(r,\theta) &= \frac{1}{\sqrt{2\pi}} N_{-1}\, r^{1-\alpha}\, G\left(\frac{1}{2} \mp \frac{\imath}{2B}, 2-\alpha, \frac{Br^2}{2}\right) \exp\left(-\frac{Br^2}{4}\right) e^{-\imath\theta},\\
f_{0,\pm}(r,\theta) &= \frac{1}{\sqrt{2\pi}} N_0\, r^{\alpha}\, G\left(\frac{1}{2} + \alpha \mp \frac{\imath}{2B}, 1+\alpha, \frac{Br^2}{2}\right) \exp\left(-\frac{Br^2}{4}\right),
\end{aligned}$$

where N_{-1} and N_0 are normalisation constants making the basis orthonormal.

We shall need the explicit values of N_{-1} and N_0. Using the relation

$$W_{\upsilon,\tau}(z) = z^{\tau+\frac{1}{2}} e^{-z/2}\, G\left(\frac{1}{2} - \upsilon + \tau, 2\tau + 1, z\right)$$

where W is the Whittaker function we get

$$\begin{aligned}
N_m^{-2} &= \int_0^\infty |g_m^2(\pm\imath; r)|^2\, r\mathrm{d}r\\
&= \frac{1}{2}\left(\frac{2}{B}\right)^{|m+\alpha|+1} \int_0^\infty x^{-1} W_{\varrho,\sigma}(x)\, W_{\bar{\varrho},\sigma}(x)\,\mathrm{d}x
\end{aligned}$$

where

$$\varrho = \frac{1}{2}\left(-m - \alpha + \frac{\imath}{B}\right),\ \ \sigma = \frac{1}{2}\,|m+\alpha|.$$

Combining the identities [10, 2.19.24.6]

$$\int_0^\infty x^{-1} W_{\varrho,\sigma}(x)\, W_{\mu,\sigma}(x)\,\mathrm{d}x = \frac{\pi}{\sin(2\pi\sigma)} \times \left(-\frac{1}{\Gamma\left(\frac{1}{2}-\sigma-\mu\right)\,\Gamma\left(\frac{3}{2}+\sigma-\varrho\right)}\,{}_2F_1\left(\frac{1}{2}+\sigma-\mu, 1; \frac{3}{2}+\sigma-\varrho; 1\right) + \frac{1}{\Gamma\left(\frac{1}{2}+\sigma-\mu\right)\,\Gamma\left(\frac{3}{2}-\sigma-\varrho\right)}\,{}_2F_1\left(\frac{1}{2}-\sigma-\mu, 1; \frac{3}{2}-\sigma-\varrho; 1\right)\right)$$

and

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,{}_2F_1(a,b;a+b-c+1;1-z) + \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}\,(1-z)^{c-a-b}{}_2F_1(c-a,c-b;c-a-b+1;1-z)$$

we arrive at the relation

$$\int_0^\infty x^{-1} W_{\varrho,\sigma}(x)\, W_{\mu,\sigma}(x)\,\mathrm{d}x = \frac{\pi}{\sin(2\pi\sigma)(\mu-\varrho)} \times \left(-\frac{1}{\Gamma\left(\frac{1}{2}-\mu-\sigma\right)\,\Gamma\left(\frac{1}{2}-\varrho+\sigma\right)} + \frac{1}{\Gamma\left(\frac{1}{2}-\mu+\sigma\right)\,\Gamma\left(\frac{1}{2}-\varrho-\sigma\right)}\right).$$

Finally we get

$$\begin{aligned} N_{-1} &= \left(\frac{B}{2}\right)^{\frac{1}{2}(1-\alpha)} \sqrt{\frac{\sin(\pi\alpha)}{2\pi}} \left(\mathrm{Im}\,\frac{1}{\Gamma\left(-\frac{1}{2}+\alpha+\frac{\imath}{2B}\right)\,\Gamma\left(\frac{1}{2}-\frac{\imath}{2B}\right)}\right)^{-1/2}, \\ N_0 &= \left(\frac{B}{2}\right)^{\frac{1}{2}\alpha} \sqrt{\frac{\sin(\pi\alpha)}{2\pi}} \left(\mathrm{Im}\,\frac{1}{\Gamma\left(\frac{1}{2}+\frac{\imath}{2B}\right)\,\Gamma\left(\frac{1}{2}+\alpha-\frac{\imath}{2B}\right)}\right)^{-1/2}. \end{aligned}$$

Let us have a look at the asymptotic behaviour at the origin of the basis functions in the deficiency subspaces $\mathcal{N}_{\pm\imath}$. By (4) and (7) we have

$$\begin{aligned} g^2_{-1}(\pm\imath;r) &= a_{-1,\pm}\,r^{-1+\alpha} + b_{-1,\pm}\,r^{1-\alpha} + O(r^{1+\alpha}),\\ g^2_0(\pm\imath;r) &= a_{0,\pm}\,r^{-\alpha} + b_{0,\pm}\,r^{\alpha} + O(r^{2-\alpha}), \end{aligned} \tag{10}$$

where

$$\begin{aligned} a_{-1,\pm} &= \frac{\Gamma(1-\alpha)}{\Gamma\left(\frac{1}{2}\mp\frac{\imath}{2B}\right)}\left(\frac{B}{2}\right)^{-1+\alpha}, \quad & b_{-1,\pm} &= \frac{\Gamma(-1+\alpha)}{\Gamma\left(-\frac{1}{2}+\alpha\mp\frac{\imath}{2B}\right)},\\ a_{0,\pm} &= \frac{\Gamma(\alpha)}{\Gamma\left(\frac{1}{2}+\alpha\mp\frac{\imath}{2B}\right)}\left(\frac{B}{2}\right)^{-\alpha}, \quad & b_{0,\pm} &= \frac{\Gamma(-\alpha)}{\Gamma\left(\frac{1}{2}\mp\frac{\imath}{2B}\right)}. \end{aligned}$$

The coefficients $a_{m,\pm}$, $b_{m,\pm}$ are related to the normalisation constants N_m for it holds true that

$$\det M_{-1} = -\frac{\imath}{1-\alpha}\,(N_{-1})^{-2},\;\; \det M_0 = -\frac{\imath}{\alpha}\,(N_0)^{-2}. \tag{11}$$

where

$$M_m = \begin{pmatrix} a_{m,+} & b_{m,+}\\ a_{m,-} & b_{m,-} \end{pmatrix}.$$

Particularly, we shall need the fact that the matrices M_{-1} and M_0 are regular.

Let us now describe the closure of the operator L. In virtue of the decomposition (2) we have

$$\bar{L} = \sum_{m\in\mathbb{Z}}^{\oplus} \bar{L}_m$$

where $\bar{L}_m = (L^*)^*_m$. As it is well known, $\psi\in\mathrm{Dom}(L^*)$ belongs to $\mathrm{Dom}(\bar{L})$ if and only if $\langle\psi, L^*\varphi\rangle = \langle L^*\psi,\varphi\rangle$ for all $\varphi\in\mathcal{N}_\imath+\mathcal{N}_{-\imath}$. Thus $(L^*)_m = \bar{L}_m$ for $m\neq\{-1,0\}$, and if $m\in\{-1,0\}$ then $\varphi(r)\,e^{\imath m\theta}\in\mathrm{Dom}((L^*)_m)$ belongs to $\mathrm{Dom}(\bar{L}_m)$ if and only if

$$\lim_{r\to 0_+} r\,W(\overline{\varphi(r)}, g^2_m(\pm\imath,r)) = 0$$

where $W(f,g) = (\partial_r f)g - f\,\partial_r g$ is the Wronskian. Using the asymptotic behaviour (10) and the regularity of the matrix M_m we arrive at two conditions

$$\begin{aligned}\lim_{r\to 0_+}(-|m+\alpha|\,r^{-|m+\alpha|}\varphi(r) - r^{-|m+\alpha|+1}\partial_r\varphi(r)) = 0,\\ \lim_{r\to 0_+}(|m+\alpha|\,r^{|m+\alpha|}\varphi(r) - r^{|m+\alpha|+1}\partial_r\varphi(r)) = 0,\end{aligned}$$

which can be rewritten in the equivalent form,

$$\lim_{r\to 0_+} r^{-2|m+\alpha|+1}\partial_r(r^{|m+\alpha|}\varphi(r)) = 0,\quad \lim_{r\to 0_+} r^{|m+\alpha|}\varphi(r) = 0.$$

But since

$$r^{-|m+\alpha|}|\varphi(r)| \leq \frac{1}{2|m+\alpha|}\sup_{x\in\,]0,r[}|x^{-2|m+\alpha|+1}\partial_x(x^{|m+\alpha|}\varphi(x))|$$

we finally get a sufficient and necessary condition for $\varphi(r)\,e^{\imath m\theta} \in \mathrm{Dom}((L^*)_m)$ to belong to $\mathrm{Dom}(\bar{L})$, namely

$$\begin{aligned}&\lim_{r\to 0_+} r^{-1+\alpha}\varphi(r) = 0 \text{ and } \lim_{r\to 0_+} r^{\alpha}\varphi'(r) = 0 &&\text{if } m=-1,\\ &\lim_{r\to 0_+} r^{-\alpha}\varphi(r) = 0 \text{ and } \lim_{r\to 0_+} r^{-\alpha+1}\varphi'(r) = 0 &&\text{if } m=0.\end{aligned} \tag{12}$$

This shows that if $\psi \in \mathrm{Dom}(L^*) = \mathrm{Dom}(\bar{L}) + \mathcal{N}_\imath + \mathcal{N}_{-\imath}$ then

$$\begin{aligned}\psi(r,\theta) \;&=\; \left(\Phi_1^1(\psi)r^{-1+\alpha} + \Phi_2^1(\psi)r^{1-\alpha}\right)e^{-\imath\theta} + \Phi_1^2(\psi)r^{-\alpha} + \Phi_2^2(\psi)r^{\alpha}\\ &\quad + \text{ a regular part.}\end{aligned}$$

Let us formally introduce the functionals Φ_j^k on $\mathrm{Dom}(L^*)$,

$$\begin{aligned}
\Phi_1^{-1}(\psi) &= \lim_{r\to 0_+} r^{1-\alpha}\frac{1}{2\pi}\int_0^{2\pi}\psi(r,\theta)\,e^{\imath\theta}\mathrm{d}\theta,\\
\Phi_2^{-1}(\psi) &= \lim_{r\to 0_+} r^{-1+\alpha}\left(\frac{1}{2\pi}\int_0^{2\pi}\psi(r,\theta)\,e^{\imath\theta}\mathrm{d}\theta-\Phi_1^1(\psi)\,r^{-1+\alpha}\right),\\
\Phi_1^0(\psi) &= \lim_{r\to 0_+} r^{\alpha}\frac{1}{2\pi}\int_0^{2\pi}\psi(r,\theta)\mathrm{d}\theta,\\
\Phi_2^0(\psi) &= \lim_{r\to 0_+} r^{-\alpha}\left(\frac{1}{2\pi}\int_0^{2\pi}\psi(r,\theta)\mathrm{d}\theta-\Phi_1^2(\psi)\,r^{-\alpha}\right).
\end{aligned}$$

Notice that the upper index refers to the sector of angular momentum while the lower index refers to the order of the singularity. If $\psi\in\mathrm{Dom}(\bar{L})$ then according to (12) it actually holds $\Phi_j^k(\psi)=0$ for $j=1,2$, $k=-1,0$. On the other hand, if $\psi\in\mathcal{N}_\imath+\mathcal{N}_{-\imath}$ and $\Phi_j^k(\psi)=0$ for all indices $j=1,2$, $k=-1,0$, then $\psi=0$ (this is again guaranteed by the regularity of the matrices M_{-1} and M_0).

Let us introduce some more notation. It is convenient to arrange the functionals Φ_j^k into column vectors as follows,

$$\Phi_j(\psi)=\begin{pmatrix}\Phi_j^{-1}(\psi)\\ \Phi_j^0(\psi)\end{pmatrix},\ j=1,2.$$

Further, applying the functionals to the basis functions in $\mathcal{N}_\imath+\mathcal{N}_{-\imath}$ we obtain four 2×2 diagonal matrices. More precisely, set

$$(\Phi_{j,\pm})_{k\ell}=\sqrt{2\pi}\,\Phi_j^{k-2}(f_{\ell-2,\pm}),\ j,k,\ell=1,2.$$

Then

$$\Phi_{1,\pm}=\begin{pmatrix}N_{-1}a_{-1,\pm} & 0\\ 0 & N_0\,a_{0,\pm}\end{pmatrix},\ \Phi_{2,\pm}=\begin{pmatrix}N_{-1}b_{-1,\pm} & 0\\ 0 & N_0\,b_{0,\pm}\end{pmatrix}.$$

Now it is straightforward to give a formal definition of a self-adjoint extension H^U of the symmetric operator L determined by a unitary operator $U:\mathcal{N}_\imath\to\mathcal{N}_{-\imath}$. We identify U with a unitary 2×2 matrix via the choice of the orthonormal bases $\{f_{-1,\pm}, f_{0,\pm}\}$ in $\mathcal{N}_{\pm\imath}$.The self-adjoint operator H^U is unambiguously defined by the condition: $H^U \subset L^*$ and $\psi \in \mathrm{Dom}(L^*)$ belongs to $\mathrm{Dom}(H^U)$ if and only if

$$\begin{pmatrix} \Phi_1(\psi) \\ \Phi_2(\psi) \end{pmatrix} \in \mathrm{Ran} \begin{pmatrix} \Phi_{1,+} + \Phi_{1,-}U \\ \Phi_{2,+} + \Phi_{2,-}U \end{pmatrix}. \tag{13}$$

However condition (13) is rather inconvenient and we shall replace it in the next section by another one which is more suitable for practical purposes.

5 Boundary conditions

To turn (13) into a convenient requirement which would involve boundary conditions we shall need the following proposition. Set

$$D = \begin{pmatrix} 1-\alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

There is a one-to-one correspondence between unitary matrices $U \in U(2)$ and couples of matrices $X_1, X_2 \in \mathrm{Mat}(2,\mathbb{C})$ obeying

$$\mathrm{rank} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 2 \tag{14}$$

and

$$X_1^* D X_2 = X_2^* D X_1 \tag{15}$$

modulo the right action of the group of regular matrices $GL(2,\mathbb{C})$. The one-to-one correspondence is given by the equality

$$\mathrm{Ran} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathrm{Ran} \begin{pmatrix} \Phi_{1,+} + \Phi_{1,-}U \\ \Phi_{2,+} + \Phi_{2,-}U \end{pmatrix}.$$

Let us note that the equivalence class of a couple (X_1, X_2) modulo $GL(2,\mathbb{C})$

corresponds to a two-dimensional subspace in $\mathbb{C}^4$ and hence to a point in the Grassmann manifold $\mathbb{G}_2(\mathbb{C}^4)$. The complex dimension of $\mathbb{G}_2(\mathbb{C}^4)$ equals 4, i.e. $\dim_{\mathbb{R}} \mathbb{G}_2(\mathbb{C}^4) = 8$. The points of $\mathbb{G}_2(\mathbb{C}^4)$ obeying the ("real") condition (15) form a real 4-dimensional submanifold which is diffeomorphic, according to the proposition, to the unitary group $U(2)$.

To verify the proposition we first show that to any couple (X_1, X_2) with the properties (14), (15) there are related unique $Y \in GL(2,\mathbb{C})$ and $U \in U(2)$ such that

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} Y = \boldsymbol{J} \begin{pmatrix} I \\ U \end{pmatrix} \tag{16}$$

where we have set

$$\boldsymbol{J} = \begin{pmatrix} \Phi_{1,+} & \Phi_{1,-} \\ \Phi_{2,+} & \Phi_{2,-} \end{pmatrix} = \begin{pmatrix} N_{-1}a_{-1,+} & 0 & N_{-1}a_{-1,-} & 0 \\ 0 & N_0 a_{0,+} & 0 & N_0 a_{0,-} \\ N_{-1}b_{-1,+} & 0 & N_{-1}b_{-1,-} & 0 \\ 0 & N_0 b_{0,+} & 0 & N_0 b_{0,-} \end{pmatrix}.$$

Using (11) one easily finds that $\boldsymbol{J}$ is regular and

$$\boldsymbol{J}^{-1} = \imath \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \Phi_{2,-} & -\Phi_{1,-} \\ -\Phi_{2,+} & \Phi_{1,+} \end{pmatrix}.$$

Let us introduce another couple of matrices, $V_+, V_- \in \mathrm{Mat}(2,\mathbb{C})$, by the relation

$$\begin{pmatrix} V_- \\ V_+ \end{pmatrix} = \boldsymbol{J}^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

thus $V_\pm = \mp \imath D(\Phi_{2,\pm} X_1 - \Phi_{1,\pm} X_2)$. It follows that

$$V_\pm^* V_\pm = \begin{pmatrix} X_1^* & X_2^* \end{pmatrix} \begin{pmatrix} \Phi_{2,\pm}^* D^2 \Phi_{2,\pm} & -\Phi_{2,\pm}^* D^2 \Phi_{1,\pm} \\ -\Phi_{1,\pm}^* D^2 \Phi_{2,\pm} & \Phi_{1,\pm}^* D^2 \Phi_{1,\pm} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

and, consequently,

$$V_-^*V_- - V_+^*V_+ = \begin{pmatrix} X_1^* & X_2^* \end{pmatrix} \begin{pmatrix} 0 & -\imath D \\ \imath D & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \imath(X_2^*DX_1 - X_1^*DX_2)$$

for $\Phi_{j,\pm}$ and D commute (all of them are diagonal), $\Phi_{j,\pm}^* = \Phi_{j,\mp}$ and

$$-\Phi_{1,+}\Phi_{2,-} + \Phi_{1,-}\Phi_{2,+} = \imath D^{-1}$$

(cf. (11)). Owing to the property (15) we have

$$V_-^*V_- = V_+^*V_+ \tag{17}$$

which jointly with the property (14) implies that

$$\operatorname{Ker} V_- = \operatorname{Ker} V_+ = \operatorname{Ker} \begin{pmatrix} V_- \\ V_+ \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = 0.$$

The only possible choice of the matrices Y and U satisfying (16) is

$$Y = V_-^{-1},\ U = V_+V_-^{-1}.$$

The matrix U is actually unitary because of (17).

Conversely, we have to show that any couple of matrices X_1, X_2 related to a unitary matrix U according to the rule

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \boldsymbol{J} \begin{pmatrix} I \\ U \end{pmatrix}$$

obeys (14) and (15). Condition (14) is obvious since $\boldsymbol{J}$ is regular and condition (15) is again a matter of a direct computation. In more detail, since it holds

$$X_1^*DX_2 - X_2^*DX_1 = \begin{pmatrix} I & U^* \end{pmatrix} \boldsymbol{J}^* \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \boldsymbol{J} \begin{pmatrix} I \\ U \end{pmatrix}$$

it suffices to verify that

$$\boldsymbol{J}^* \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \boldsymbol{J} = \imath \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

This concludes the proof of the above proposition.

Using this correspondence one can relate to a couple $X_1, X_2 \in \mathrm{Mat}(2, \mathbb{C})$ obeying (14) and (15) a self-adjoint extension H determined by the condition

$$\psi \in \mathrm{Dom}(H) \Longleftrightarrow \begin{pmatrix} \Phi_1(\psi) \\ \Phi_2(\psi) \end{pmatrix} \in \mathrm{Ran} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}. \tag{18}$$

Two couples (X_1, X_2) and (X_1', X_2') determine the same self-adjoint extension if and only if there exists a regular matrix Y such that $(X_1', X_2') = (X_1 Y, X_2 Y)$. Moreover, all the self-adjoint extensions can be obtained in this way.

We shall restrict ourselves to an open dense subset in the space of all self-adjoint extensions by requiring the matrix X_2 to be regular. In that case we can set directly $X_2 = I$ and rename $X_1 = \Lambda$. Thus Λ is a 2×2 complex matrix satisfying

$$D\Lambda = \Lambda^* D. \tag{19}$$

The corresponding self-adjoint extension will be denoted H^Λ. The condition (18) simplifies in an obvious way. We conclude that $H^\Lambda \subset L^*$ and $\psi \in \mathrm{Dom}(L^*)$ belongs to $\mathrm{Dom}(H^\Lambda)$ if and only if

$$\Phi_1(\psi) = \Lambda \Phi_2(\psi), \tag{20}$$

and this is in fact the sought boundary condition.

Matrices Λ obeying (19) can be parametrised by four real parameters (or two real and one complex). We choose the parameterisation

$$\Lambda = \begin{pmatrix} u & \alpha \bar{w} \\ (1-\alpha) w & v \end{pmatrix}, \ u, v \in \mathbb{R}, \ w \in \mathbb{C}.$$

The relation between Λ and U reads

$$\Lambda = (\Phi_{1,+} + \Phi_{1,-}U)(\Phi_{2,+} + \Phi_{2,-}U)^{-1} \tag{21}$$

(provided the RHS makes sense).

The "most regular" among the boundary conditions is $\Phi_1(\psi) = 0$, i.e. the one determined by $\Lambda = 0$, and the corresponding self-adjoint extension is nothing but the standard Aharonov-Bohm Hamiltonian H^{AB} discussed in Section 3. According to (21) H^{AB} corresponds to the unitary matrix

$$U = -\Phi_{1,-}^{-1}\Phi_{1,+} = \operatorname{diag}\left\{ -\frac{\Gamma\left(\frac{1}{2} + \frac{\imath}{2B}\right)}{\Gamma\left(\frac{1}{2} - \frac{\imath}{2B}\right)}, -\frac{\Gamma\left(\frac{1}{2} + \alpha + \frac{\imath}{2B}\right)}{\Gamma\left(\frac{1}{2} + \alpha - \frac{\imath}{2B}\right)} \right\}.$$

6 The spectrum

Let us now proceed to the discussion of spectral properties of the described self-adjoint extensions. It is clear from what has been explained up to now that everything interesting is happening in the two critical sectors of the angular momentum labeled by $m = -1$ and $m = 0$. To state it more formally we decompose the Hilbert space into an orthogonal sum of the "stable" and "critical" parts,

$$\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_c$$

where

$$\mathcal{H}_s = \sum_{m\in\mathbb{Z}\setminus\{-1,0\}}^{\oplus} L^2(\mathbb{R}_+, r\,\mathrm{d}r) \otimes \mathbb{C}\,e^{\imath m\theta},\ \ \mathcal{H}_c = L^2(\mathbb{R}_+, r\,\mathrm{d}r) \otimes (\mathbb{C}\,e^{-\imath\theta} \oplus \mathbb{C}1).$$

A self-adjoint extension H^Λ decomposes correspondingly,

$$H^\Lambda = H^\Lambda|_{\mathcal{H}_s} \oplus H^\Lambda|_{\mathcal{H}_c}\,,$$

and we know that on $\mathcal{H}_s$ the operator H^Λ coincides with the standard AB Hamiltonian,

$$H^\Lambda|_{\mathcal{H}_s} = H^{AB}|_{\mathcal{H}_s}.$$

Thus

$$\sigma(H^\Lambda) = \sigma(H^{AB}|_{\mathcal{H}_s}) \cup \sigma(H^\Lambda|_{\mathcal{H}_c})$$

and, as explained in Section 3,

$$\sigma(H^{AB}|_{\mathcal{H}_s}) = \{B(2k+1);\ k \in \mathbb{Z}_+\} \cup \{B(2k+2\alpha+1);\ k \in \mathbb{N}\}$$

where the multiplicity of the eigenvalue $B(2k+1)$ is infinite while the multiplicity of the eigenvalue $B(2k+2\alpha+1)$ equals k. On the other hand,

$$\sigma(H^{AB}|_{\mathcal{H}_c}) = \{B(2k+1);\ k \in \mathbb{Z}_+\} \cup \{B(2k+2\alpha+1);\ k \in \mathbb{Z}_+\}$$

where all the eigenvalues are simple (the first set is a contribution of the sector $m=-1$ while the second one comes from the sector $m=0$). Since the deficiency indices are finite the Krein's formula jointly with Weyl Theorem [11, Theorem XIII.14] tells us that the essential spectrum $\sigma_{ess}(H^\Lambda|_{\mathcal{H}_c})$ is empty for any Λ. Thus the spectrum of $H^\Lambda|_{\mathcal{H}_c}$ is formed by eigenvalues which are at most finitely degenerated and have no finite accumulation points.

Let us derive the equation on eigenvalues for the restriction $H^\Lambda|_{\mathcal{H}_c}$. Let $\lambda \in \mathbb{R}$. In each of the sectors $m=-1,0$ there exists exactly one (up to a multiplicative constant) solution of the equation $(L^*)_m f = \lambda f$ which is L^2-integrable at infinity (with respect to the measure $r\,\mathrm{d}r$) and we may take for it the function $g^2_m(\lambda;r)\,e^{\imath m\theta}$ (cf. (4)). For a second linearly independent solution one may take $g^1_m(\lambda;r)\,e^{\imath m\theta}$ provided $\beta(m,\lambda) \notin -\mathbb{Z}_+$ (cf. (5)). If $\beta(m,\lambda) \in -\mathbb{Z}_+$ then a possible choice of a second linearly independent solution is

$$r^{|m+\alpha|}\, H\left(\beta(m,\lambda), \gamma(m), \frac{Br^2}{2}\right)\, \exp\left(-\frac{Br^2}{4}\right)$$

where

$$H(\beta,\gamma,z) = z^{1-\gamma} F(\beta-\gamma+1, 2-\gamma, z)$$

(cf. (6)).

Thus λ is an eigenvalue of $H^\Lambda|_{\mathcal{H}_c}$ if and only if there exists a vector $(\mu,\nu) \in \mathbb{C}^2 \setminus \{0\}$ such that the function

$$\psi_\lambda(r,\theta) = \mu\, g_{-1}^2(\lambda; r)\, e^{-\imath\theta} + \nu\, g_0^2(\lambda; r)$$

satisfies the boundary condition (20). Using again (4) and (7) one finds that

$$\Phi_1(\psi_\lambda) = \begin{pmatrix} a_{-1} & 0 \\ 0 & a_0 \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \ \Phi_2(\psi_\lambda) = \begin{pmatrix} b_{-1} & 0 \\ 0 & b_0 \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix},$$

where

$$\begin{aligned} a_{-1} &= \frac{\Gamma(1-\alpha)}{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2B}\right)} \left(\frac{B}{2}\right)^{-1+\alpha} \quad , \quad b_{-1} = \frac{\Gamma(-1+\alpha)}{\Gamma\left(-\frac{1}{2}+\alpha-\frac{\lambda}{2B}\right)}\,, \\ a_0 &= \frac{\Gamma(\alpha)}{\Gamma\left(\frac{1}{2}+\alpha-\frac{\lambda}{2B}\right)} \left(\frac{B}{2}\right)^{-\alpha} \quad , \quad b_0 = \frac{\Gamma(-\alpha)}{\Gamma\left(\frac{1}{2}-\frac{\lambda}{2B}\right)}\,. \end{aligned}$$

This immediately leads to the desired equation on eigenvalues which takes the form $\det \boldsymbol{A} = 0$ where

$$\boldsymbol{A} = \begin{pmatrix} a_{-1} & 0 \\ 0 & a_0 \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} - \Lambda \begin{pmatrix} b_{-1} & 0 \\ 0 & b_0 \end{pmatrix}.$$

After the substitution

$$z = \frac{1}{2} - \frac{\lambda}{2B}, \text{ i.e. } \lambda = B(1-2z),$$

we get

$$\frac{\Gamma(1-\alpha)\Gamma(\alpha)}{\Gamma(z)\Gamma(z+\alpha)}\,\frac{2}{B}-\frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(z+\alpha-1)\Gamma(z+\alpha)}\left(\frac{2}{B}\right)^{\alpha}u$$
$$-\frac{\Gamma(1-\alpha)\Gamma(-\alpha)}{\Gamma(z)^2}\left(\frac{2}{B}\right)^{1-\alpha}v$$
$$+\frac{\Gamma(\alpha-1)\Gamma(-\alpha)}{\Gamma(z)\Gamma(z+\alpha-1)}\left(uv-\alpha(1-\alpha)|w|^2\right)\;=\;0.$$

To simplify somewhat the form of the equation it is convenient to rescale the parameters as follows,

$$\xi=\left(\frac{B}{2}\right)^{1-\alpha}\frac{\Gamma(\alpha)}{\Gamma(2-\alpha)}\,u,\;\;\eta=\left(\frac{B}{2}\right)^{\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}\,v,\;\;\zeta=\sqrt{\frac{B}{2}}\,|w|. \qquad (22)$$

Finally we arrive at an equation depending on three real parameters ξ, η, ζ, namely

$$\frac{1}{\Gamma(z)\,\Gamma(z+\alpha)}+\frac{\xi}{\Gamma(z+\alpha-1)\,\Gamma(z+\alpha)}+\frac{\eta}{\Gamma(z)^2}+\frac{\xi\,\eta-\zeta^2}{\Gamma(z)\,\Gamma(z+\alpha-1)}=0. \qquad (23)$$

There is no chance to solve equation (23) explicitly apart of some particular cases. One of them, of course, corresponds to the standard AB Hamiltonian. This case is determined by the values of parameters $\xi=\eta=\zeta=0$ and the roots of (23) form the set $-\mathbb{Z}_+\cup(-\alpha-\mathbb{Z}_+)$. Consider also the case when when $\xi=\eta=0$ and $\zeta\neq 0$ with the set of roots equal to $-\mathbb{Z}_+\cup(-\alpha-\mathbb{Z}_+)\cup\{1-\alpha+\zeta^{-2}\}$. Comparing the latter case to the former one we see that there is one additional root, namely $1-\alpha+\zeta^{-2}$, which escapes to infinity when $\zeta\to 0$.

In the last particular case one can also consider the limit $\zeta\to\infty$. More generally, suppose that $\det\Lambda\neq 0$, i.e. $\xi\eta-\zeta^2\neq 0$, replace Λ with $t\,\Lambda$ in (20) and take the limit $t\to\infty$. The limiting boundary condition reads

$$\Phi_2(\psi)=0$$

and the corresponding self-adjoint extension which we shall call H^{∞} is one of those omitted when we restricted ourselves to an open dense subset in the space of all self-adjoint extensions (regarded as a a 4-dimensional real manifold). Equation (23) reduces in this limit to the equation

$$\frac{1}{\Gamma(z)\,\Gamma(z+\alpha-1)}=0 \tag{24}$$

with the set of roots $-\mathbb{Z}_{+}\cup(1-\alpha-\mathbb{Z}_{+})$.

Another case when equation (23) simplifies though it is not solvable explicitly is $\zeta=0$. This is easy to understand since if $\zeta=0$ then the matrix Λ is diagonal and the two critical sectors of angular momentum do not interfere. This is reflected in the fact that the equation (23) splits into two independent equations,

$$\frac{1}{\Gamma(z)}+\frac{\xi}{\Gamma(z+\alpha-1)}=0,\ \frac{1}{\Gamma(z+\alpha)}+\frac{\eta}{\Gamma(z)}=0.$$

Let us shortly discuss the dependence of roots of equation (23) on the parameters ξ,η,ζ. Since the derivative of the LHS of (23) with respect to z and with the values of parameters $(\xi,\eta,\zeta)=(0,0,0)$ equals

$$\frac{(-1)^{m}m!}{\Gamma(-m+\alpha)}\neq 0 \text{ for } z=-m, \text{ and } \frac{(-1)^{m}m!}{\Gamma(-m-\alpha)}\neq 0 \text{ for } z=-m-\alpha,$$

where $m\in\mathbb{Z}_{+}$, the standard Implicit Function Theorem (analytic case) is sufficient to conclude that the roots are analytic functions in ξ,η,ζ at least in some neighbourhood of the origin (depending in general on the root). Let us denote by $z_{1,m}(\xi,\eta,\zeta)$ and $z_{2,m}(\xi,\eta,\zeta)$ the roots of (23) regarded as analytic functions in ξ,η,ζ and such that $z_{1,m}(0,0,0)=-m$ and $z_{2,m}(0,0,0)=-\alpha-m$, with $m\in\mathbb{Z}_{+}$. A straightforward computation results in the following power series truncated at degree 4.

Set

$$
\begin{aligned}
h_m^0(z) &= \sum_{j=1}^{m} \frac{1}{j} - \gamma - \psi(z), \\
h_m^1(z) &= \frac{\pi^2}{6} + \sum_{j=1}^{m} \frac{1}{j^2} - \psi'(z), \\
h_m^2(z) &= -2\,\boldsymbol{\zeta}(3) + 2\sum_{j=1}^{m} \frac{1}{j^3} - \psi''(z),
\end{aligned}
$$

where γ is the Euler constant, $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and $\boldsymbol{\zeta}$ is the zeta function. Then

$$
\begin{aligned}
z_{1,m}(\xi,\eta,\zeta) &= -m + \frac{(-1)^{m+1}}{m!\,\Gamma(-1-m+\alpha)}\,\xi + \frac{h_m^0(-1-m+\alpha)}{(m!)^2\,\Gamma(-1-m+\alpha)^2}\,\xi^2 \\
&\quad + \frac{(-1)^{m+1}\left(3\,h_m^0(-1-m+\alpha)^2 + h_m^1(-1-m+\alpha)\right)}{2\,(m!)^3\,\Gamma\left(-1-m+\alpha\right)^3}\,\xi^3 \\
&\quad + \frac{(-1)^m\,\left(1+m-\alpha\right)}{m!\,\Gamma(-1-m+\alpha)}\,\xi\,\zeta^2 \\
&\quad + \frac{1}{6\,(m!)^4\,\Gamma(-1-m+\alpha)^4}\,\Big(4\,h_m^0(-1-m+\alpha) \\
&\quad \times \big(4\,h_m^0(-1-m+\alpha)^2 + 3\,h_m^2(-1-m+\alpha)\big) \\
&\quad + h_m^2(-1-m+\alpha)\Big)\,\xi^4 \\
&\quad + \frac{3 - 2\,(1+m-\alpha)\,h_m^0(-m+\alpha)}{(m!)^2\,\Gamma(-1-m+\alpha)^2}\,\xi^2\zeta^2 + \cdots,
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
z_{2,m}(\xi,\eta,\zeta) \;=\;& -\alpha-m+\frac{(-1)^{m+1}}{m!\,\Gamma(-m-\alpha)}\,\eta+\frac{h_m^0(-m-\alpha)}{(m!)^2\,\Gamma(-m-\alpha)^2}\,\eta^2\\
&+\frac{(-1)^{m+1}\,(3\,h_m^0(-m-\alpha)^2+h_m^1(-m-\alpha))}{2\,(m!)^3\,\Gamma\,(-m-\alpha)^3}\,\eta^3\\
&+\frac{(-1)^m\;(m+1)}{m!\,\Gamma(-m-\alpha)}\,\eta\,\zeta^2\\
&+\frac{1}{6\,(m!)^4\,\Gamma(-m-\alpha)^4}\,\big(4\,h_m^0(-m-\alpha)\\
&\times\big(4\,h_m^0(-m-\alpha)^2+3\,h_m^2(-m-\alpha)\big)\\
&+h_m^2(-m-\alpha)\big)\,\eta^4\\
&+\frac{1-2\,(m+1)\,h_m^0(-m-\alpha)}{(m!)^2\,\Gamma(-m-\alpha)^2}\,\eta^2\zeta^2+\cdots\,.
\end{aligned}
\tag{26}
$$

A similar analysis can be carried out to get the asymptotic behaviour of roots for ξ,η,ζ large. To this end assume that $\xi\eta-\zeta^2\neq 0$ and set

$$
\xi'=\frac{\xi}{\xi\eta-\zeta^2}\,,\;\;\eta'=\frac{\eta}{\xi\eta-\zeta^2}\,,\;\;\zeta'=\frac{\zeta}{\xi\eta-\zeta^2}\,.
$$

Notice that $\xi'\eta'-\zeta'^2=(\xi\eta-\zeta^2)^{-1}$. Equation (23) becomes

$$
\frac{\xi'\eta'-\zeta'^2}{\Gamma(z)\,\Gamma(z+\alpha)}+\frac{\xi'}{\Gamma(z+\alpha-1)\,\Gamma(z+\alpha)}+\frac{\eta'}{\Gamma(z)^2}+\frac{1}{\Gamma(z)\,\Gamma(z+\alpha-1)}=0.
\tag{27}
$$

Roots of (27) are analytic functions in ξ',η',ζ' at least in some neighbourhood of the origin. Again, it would be possible to compute the beginning of the corresponding power series and to derive formulae similar to those of (25), (26) but we avoid doing it here explicitly.

Instead we prefer to plot two graphs in order to give a reader some impression about how the eigenvalues may depend on the parameters, i.e. on the boundary conditions. In each graph we choose a line in the parameter space, $\{(\xi t,\eta t,\zeta t)\in\mathbb{R}^3;\ t\in\mathbb{R}\}$, and we depict the dependence on t of several first eigenvalues for the corresponding self-adjoint extension restricted to $\mathcal{H}_c$ (see (22) for the substitution). In the both graphs we have set $\alpha=0.3$ and $B=1$.

Probably the most complete general information which is available about solutions of equation (23) might be a localisation of roots of this equation with respect to a suitable splitting of the real line into intervals. Let us choose the splitting into intervals with boundary points coinciding with the roots of equation (24). To get the localisation let us rewrite equation (23), equivalently provided $z \neq -\mathbb{Z}_+ \cup (1-\alpha-\mathbb{Z}_+)$, as follows

$$\left(\frac{\Gamma(z-1+\alpha)}{\Gamma(z)}+\xi\right)\left(\frac{\Gamma(z)}{\Gamma(z+\alpha)}+\eta\right)=\zeta^2. \tag{28}$$

Put

$$F_\alpha(z)=\frac{\Gamma(z-1+\alpha)}{\Gamma(z)}$$

so that equation (28) can be rewritten as

$$\left(F_\alpha(z)+\xi\right)\left(F_{1-\alpha}(z+\alpha)+\eta\right)=\zeta^2. \tag{29}$$

It is easy to carry out some basic analysis of the function $F_\alpha(z)$. We have $F_\alpha{}'(z)=F_\alpha(z)\left(\psi(z-1+\alpha)-\psi(z)\right)$. One observes that $F_\alpha(z)>0$ for $z\in\,]1-\alpha,+\infty[\,\cup\left(\bigcup_{m\in\mathbb{Z}_+}\,]-\alpha-m,-m[\,\right)$, and $F_\alpha(z)<0$ for $z\in\bigcup_{m\in\mathbb{Z}_+}\,]-m,1-\alpha-m[\,$, and in any case $F_\alpha{}'(z)<0$. In the former case this follows from the fact that $\psi(z)$ is strictly increasing on each of the intervals $]0,+\infty[$ and $]-m-1,-m[$, with $m\in\mathbb{Z}_+$. In the latter case this is a consequence of the identity

$$\psi(z-1+\alpha)-\psi(z)=\frac{\pi\,\sin(\pi\alpha)}{\sin(\pi z)\,\sin(\pi(z+\alpha))}+\int_0^\infty\frac{e^{-(1-z)\,t}\left(1-e^{-(1-\alpha)\,t}\right)}{1-e^{-t}}\,\mathrm{d}t.$$

Moreover,

$$\lim_{z\to+\infty}F_\alpha(z)=0,\quad \lim_{z\to(1-\alpha-m)\pm}F_\alpha(z)=\pm\infty \text{ and } F_\alpha(-m)=0 \text{ for } m\in\mathbb{Z}_+.$$

This also implies that $F_{1-\alpha}(z+\alpha)>0$ for $z\in\,]0,+\infty[\,\cup\left(\bigcup_{m\in\mathbb{Z}_+}\,]-1-m,-\alpha-m[\,\right)$

and $F_{1-\alpha}(z) < 0$ for $z \in \bigcup_{m\in\mathbb{Z}_+} \,]-\alpha-m, -m[$, in any case $F_{1-\alpha}{}'(z+\alpha) < 0$, and

$$\lim_{z\to+\infty} F_{1-\alpha}(z+\alpha) = 0, \quad \lim_{z\to-m\pm} F_{1-\alpha}(z+\alpha) = \pm\infty,$$
$$\text{and } F_{1-\alpha}(-\alpha-m) = 0 \text{ for } m \in \mathbb{Z}_+.$$

With the knowledge of these basic properties of the function $F_\alpha(z)$ it is a matter of an elementary analysis to determine the number of roots of equation (29) in each of the intervals $]1-\alpha, +\infty[$, $]-m, 1-\alpha-m[$ and $]-\alpha-m, -m[$, with $m \in \mathbb{Z}_+$. The result is summarised in the following tables.

interval $]1-\alpha, +\infty[$			
conditions			number of roots
$\xi \geq 0$	$\eta \geq 0$	$\zeta^2 > \xi\eta$	1
$\xi \geq 0$	$\eta \geq 0$	$\zeta^2 \leq \xi\eta$	0
$\xi \geq 0$	$-\Gamma(1-\alpha) < \eta < 0$	no condition	1
$\xi \geq 0$	$\eta \leq -\Gamma(1-\alpha)$	no condition	0
$\xi < 0$	$\eta \geq 0$	no condition	1
$\xi < 0$	$-\Gamma(1-\alpha) < \eta < 0$	$\zeta^2 \geq \xi\eta$	1
$\xi < 0$	$-\Gamma(1-\alpha) < \eta < 0$	$\zeta^2 < \xi\eta$	2
$\xi < 0$	$\eta \leq -\Gamma(1-\alpha)$	$\zeta^2 \geq \xi\eta$	0
$\xi < 0$	$\eta \leq -\Gamma(1-\alpha)$	$\zeta^2 < \xi\eta$	1

interval $]0, 1-\alpha[$		
conditions		number of roots
$\xi \leq 0$	$\eta \geq -\Gamma(1-\alpha)$	0
$\xi \leq 0$	$\eta < -\Gamma(1-\alpha)$	1
$\xi > 0$	$\eta \geq -\Gamma(1-\alpha)$	1
$\xi > 0$	$\eta < -\Gamma(1-\alpha)$	2

intervals $]-\alpha-m,-m[\,,\ m\in\mathbb{Z}_+$		
conditions		number of roots
$\xi\geq 0$	$\eta\leq 0$	0
$\xi\geq 0$	$\eta>0$	1
$\xi<0$	$\eta\leq 0$	1
$\xi<0$	$\eta>0$	2

intervals $]-1-m,-\alpha-m[\,,\ m\in\mathbb{Z}_+$		
conditions		number of roots
$\xi\leq 0$	$\eta\geq 0$	0
$\xi\leq 0$	$\eta<0$	1
$\xi>0$	$\eta\geq 0$	1
$\xi>0$	$\eta<0$	2

This is to be completed with the simple observation that $1-\alpha$ is a root of (23) if and only if $\eta=-\Gamma(1-\alpha)$, and $-m$, with $m\in\mathbb{Z}_+$, is a root if and only if $\xi=0$, and finally $-\alpha-m$, with $m\in\mathbb{Z}_+$, is a root if and only if $\eta=0$.

Let us note that this localisation is in agreement with a general result according to which if A and B are two self-adjoint extensions of the same symmetric operator with finite deficiency indices (d,d) then any interval $J\subset\mathbb{R}$ not intersecting the spectrum of A contains at most d eigenvalues of the operator B (including multiplicities) and no other part of the spectrum of B [12, §8.3]. Thus in our example if J is an open interval whose boundary points are either two subsequent eigenvalues of H^∞ or the lowest eigenvalue of H^∞ and $-\infty$ then any self-adjoint extension H^Λ has at most two eigenvalues in J.

7 Concluding remarks

The above discussion does not exhaust all questions related to the system under consideration. One may ask, for instance, how the state of such a particle evolves under an adiabatic change of parameters. In particular, since the model exhibits eigenvalue crossings, one may expect that there are parameter loops exhibiting a nontrivial Berry phase. Another question concerns the physical meaning of our idealized model. More specifically, one is interested in which sense the model Hamiltonian can be approximated by those with smeared flux and a regular interaction. We leave these problems to a future publication.

ask the authors or see quant-ph/0111050

Figure 1: The Hamiltonian is determined by the boundary conditions corresponding to the parameters $(\xi, \eta, \zeta) = (0.95\,t, 0.25\,t, 0.25\,t)$, $\alpha = 0.3$, $B = 1$.

ask the authors or see quant-ph/0111050

Figure 2: The Hamiltonian is determined by the boundary conditions corresponding to the parameters $(\xi, \eta, \zeta) = (0.95\,t, -0.25\,t, 0)$, $\alpha = 0.3$, $B = 1$.

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