Dynamics of traffic jams: order and chaos

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Abstract

By means of a novel variational approach we study ergodic properties of a model of a multi lane traffic flow, considered as a (deterministic) wandering of interacting particles on an infinite lattice. For a class of initial configurations of particles (roughly speaking satisfying the Law of Large Numbers) the complete description of their limit (in time) behavior is obtained, as well as estimates of the transient period. In this period the main object of interest is the dynamics of 'traffic jams', which is rigorously defined and studied. It is shown that the dynamical system under consideration is chaotic in a sense that its topological entropy (calculated explicitly) is positive. Statistical quantities describing limit configurations are obtained as well.

Keywords: traffic flow, dynamical system, variational principle, topological entropy.

AMS Subject Classification: Primary 37B99; Secondary 37B15, 37B40, 37A60, 60K.

1 Introduction

Despite a self evident practical importance of the analysis of traffic flows and a relatively long history of attempts of their scientific treatment (going back to the fifties) only recently (in the end of nineties) reasonable mathematical models of traffic flows and method to study them were obtained. Previous attempts were based on ideas borrowed from such classical fields of physics as mechanics and hydrodynamics. Not going into details of a qualitative and quantitative comparison of the hydrodynamic type models with practice (which one can find, for example, in a recent review [19] and references therein), we consider the following practical observation. It turns out that going by foot in a slowly moving crowd it is faster to go against the "flow" than in the same direction as other people go. A mathematical model describing this effect in the case of the one lane traffic was introduced in [2]. A standard probabilistic model of a diffusion of particles against/along the flow clearly contradicts to this observation, which very likely indicates a very special (nonrandom) intrinsic structure of the flow in this case. The main aim of the present paper is to study how this structure emerges from arbitrary (random) initial configurations in a simple model of the multi lane traffic flow.

A recent progress in the analysis of traffic flows was due to the introduction of discrete (in time and in space) cellular automata models of the one lane traffic flow in [14, 13] and later studied by many authors (see [4] for review and further references). Various approaches starting

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from the mean-field approximation [11] to combinatorial techniques and statistical mechanics methods [10] were used in their analysis. All these models were based on the idea to describe the dynamics in terms of deterministic or random cellular automata (see results about stochastic models in [6, 7, 8, 9]) and to a large extent were studied by means of numerical simulation (especially because of low computational cost of the numerical realization of cellular automata rules, which made it possible to realize large-scale real-time simulations of urban traffic [20]).

Roughly speaking the one lane road in these models is associated to a finite one-dimensional integer lattice of size N with periodic boundary conditions and each position on the lattice is either occupied by a particle (represented a vehicle), or empty. On the next time step each particle remains on its place if the next position is occupied, and moves forward by one place otherwise. In [14, 10, 4] it was shown (mainly numerically and by some physical argument) that limit (as time going to infinity) behavior of the dynamical system under configuration depends only on the density of particles in the initial configuration. This result was generalized for the case of the dynamics on the infinite lattice and proved mathematically in [2], where a novel variational approach was introduced.

Despite various generalizations the one lane restriction of these models was crucial, for example, in an attempt to study a multi lane model in [2], satisfying standard traffic rules, no mathematically interesting phenomena were found. Only two years ago in [15, 16] a first non-trivial multi lane generalization was introduced for the case of a motion on a finite lattice with periodic boundary conditions, based on a so called ultradiscrete limit of the well known Burgers equation.

In the present paper is we study ergodic properties of this model. As we shall show this analysis boils down to the study of the dynamics of 'traffic jams' (see the rigorous definition in Section 4), which mainly depends of the density of particles in the initial configuration.

One of the main quantity of interest in traffic models – the average velocity of cars and its dependence on the density of cars ρ (called the fundamental diagram) is typically studied in the steady state. From our results it follows that in the multi lane model that we consider the average velocity in the steady state is equal to $\max\{1, K/\rho - 1\}$, which immediately reminds the similar result known in the one lane case.

The paper is organized as follows. In Section 2 we describe the model in detail and introduce the basic notation including important notions of dual configurations and maps. In Section 3 we introduce the space of regular (statistically defined) configurations, show that this space is invariant with respect to dynamics and formulate the main result of the paper – Theorem 3.1. Qualitatively this result means that in a steady state any configuration either consists of free (moving independently) particles, or this property holds for all empty places on the lattice. In terms of mentioned above variational principle this can be formulated that the total number of free particles between two fixed ones can only grow in time. The proof of this result in the next Section 4 is based on the detailed analysis of the dynamics of traffic jams. It is worth notice that in distinction to the one lane case the formal description of the traffic jam is rather nontrivial and some individual particles in it can move still representing obstacles to the motion of other particles. Section 5 is dedicated to the proof of chaoticity of this model: we explicitly calculate its topological entropy and show that it is strictly positive. In the last section 6 we derive statistical quantities describing typical limit configurations.

We tried to define rigorously all important objects that we consider in the text, however of course we were unable to introduce all standard mathematical definitions. The reader can find exact definitions and further references related to dynamical systems (especially acting on discrete phase spaces), for example, in [1, 12].

2 Multi lane traffic flow models: dynamics in space of configurations

The model corresponds to the highway traffic flow on a road with K lanes. Let $\mathbf{X}_0^K := \{0,1,\dots,K\}^{\mathbb{Z}^1}$ be an infinite lattice, positions on which we call (lattice) sites. For a sequence $X \in \mathbf{X}_0^K$ and $x \in \mathbb{Z}^1$ by $X(x)$ we denote the x-th element of this sequence. Consider a map $T : \mathbf{X}_0^K \to \mathbf{X}_0^K$, defined by the relation

$$TX(x) = X(x) + \min\{X(x-1), K - X(x)\} - \min\{X(x), K - X(x+1)\}. \qquad (2.1)$$

Remark. In [15, 16] the above map were introduced for the case of the finite lattice with periodic boundary conditions. Observe that a finite lattice of arbitrary size $N < \infty$ with periodic boundary conditions is a particular case of $\mathbb{Z}^1$ lattice considered in our paper restricted to only N-periodic configurations. The paper [16] claims an estimate of the transient period in the $2N$-periodic case as N. Since the construction in this paper sensitively depends on the length of the period it cannot be extended to the case of unbounded lattices with general nonperiodic initial configurations that we consider. It is of interest that $2N+1$ periodic initial configurations lead (as we shall show) to much worse estimate of the transient period $2N$.

The above map can be described in a different way in terms of configurations of particles. Let us introduce several definitions.

A collection of particles Ξ on the lattice $\mathbb{Z}^1$ we shall call *ordered* if there is a function (called index function) $I : \Xi \to \mathbb{Z}^1$ such that for any two particles $\xi, \xi' \in \Xi$, $\xi \neq \xi'$ the corresponding indices satisfy the inequality $I(\xi) \neq I(\xi')$ and if these particles are located at sites $|\xi| < |\xi'|$ on the lattice then $I(\xi) < I(\xi')$, where $|\xi|$ stands for the location of the particle ξ on the lattice.

To a configuration $X \in \mathbf{X}_0^K$ we associate an ordered collection (finite or infinite) of particles on the lattice $\mathbb{Z}^1$ containing not more than K particles at each site, such that $X(x)$ for $x \in \mathbb{Z}^1$ means the number of particles located at the site x. Then the set of all possible ordered configurations of particles containing not more than K particles at each site forms the phase space $\mathbf{X}_0^K := \{0,1,\dots,K\}^{\mathbb{Z}^1}$ of the system under configuration.

For a given positive integer K and a given configuration $X \in \mathbf{X}_0^K$ the action of the map T can be described as follows. For each site $x \in \mathbb{Z}^1$ (independently from other sites) we move $\min\{X(x), K - X(x+1)\}$ particles with the largest indices from the site x to the site $x+1$.

Lemma 2.1 *The order function I is preserved under the action of the map T.*

Proof. Straightforward. ∎

For a given configuration X, associated to the collection of particles Ξ, for each particle $\xi \in \Xi$ we introduce the notion of velocity $v(\xi)$ which is equal to 1 if the particle moves after the application of the map T or 0 otherwise. Accordingly we shall say that the particle ξ is *free* if $v(\xi) > 0$ and *jammed* otherwise. Summing up velocities of individual particles we obtain moments (the total velocity of particles at a given site) of lattice sites in the configuration X:

$$v(X,x) := \sum_{|\xi|=x} v(\xi).$$

An immediate calculation shows that

Lemma 2.2 $v(X,x) = \min\{X(x), X^*(x+1)\}$.

¿From the point of view of the description in terms of individual particles we introduce the notion of the dual configuration $X^*(x) := K - X(x)$ for any $x \in \mathbb{Z}^1$, which describes empty places in the original configurations of particles X. Therefore to describe the dynamics of empty places we consider the *dual map* $T^* : \mathbf{X}_0^K \to \mathbf{X}_0^K = (\mathbf{X}_0^K)^*$ whose action is defined by the relation $T^*X = (TX^*)^*$ and can be written as follows.

Lemma 2.3 *For $X \in \mathbf{X}_0^K$ we have*

$$T^*X(x) = X(x) - \min\{X(x), X^*(x-1)\} + \min\{X(x+1), X^*(x)\}.$$

Observe that the dynamics of empty places is exactly the same as the dynamics of particles, but occurs in the opposite direction. Obviously both the above formula and the relation (2.1) describe the mass conservation rule: the number of particles at a given site in the new configuration is equal to the number of particles at the same site in the original configuration minus the number of particles leaving it and plus the number of particles coming to this site.

By a *jammed cluster* (of particles) we shall mean a locally maximal group of consecutive sites on the lattice containing at least one jammed particle at each site. Accordingly the jammed cluster in the dual configuration defines the cluster of free empty places in the original configuration. The locally maximal property means that any enlarging of the considered group breaks the definition, i.e. both immediate neighboring sites to the cluster do not contain jammed particles.

Consider two subspaces of the space of configurations $\mathbf{X}_0^K$ which shall play an important role in our analysis. The first of them is the space of configurations of free particles:

$$\mathrm{Free}(K) := \{X \in \mathbf{X}_0^K : \ v(\xi) = 1 \ \forall \xi \in X\},$$

and the second one is the space of (space) n-periodic configurations:

$$\mathrm{Per}(n, K) := \{X \in \mathbf{X}_0^K : \ X(x) = X(x+n) \ \forall x \in \mathbb{Z}^1.\}$$

A trivial calculation show that both of these spaces are invariant with respect to the dynamics.

Lemma 2.4 $T : \mathrm{Free}(K) \to \mathrm{Free}(K)$, $T : \mathrm{Per}(n, K) \to \mathrm{Per}(n, K)$ *for any* $n, K \in \mathbb{Z}^1_+$.

Proof. Observe that the restriction of the map T to the space of configurations of free particles is equivalent to the shift operator in this space, from where the first statement follows immediately. To prove the second statement notice that according to the formula (dynamics)

$$TX(x) = X(x) + \min\{X(x-1), K - X(x)\} - \min\{X(x), K - X(x+1)\}$$

$$= X(x+n) + \min\{X(x-1+n), K - X(x+n)\} - \min\{X(x+n), K - X(x+1+n)\} = TX(x+n)$$

due to the n-periodicity of the configuration $X \in \mathrm{Per}(n, K)$. ∎

Denote by $X = \langle \alpha \rangle$ the n-periodic configuration $X \in \mathrm{Per}(n, K)$ consisting of the periodically repeating word $\alpha = a_1 a_2 \ldots a_n$ with $a_i \in \{0, 1, \ldots, K\}$ and such that $X(1) = a_1$, for example $X = \langle 1234 \rangle = \ldots 123412341234 \ldots$.

It is worth notice that despite the statement of the previous Lemma the minimal period of the configuration may not be preserved under the dynamics. Indeed, consider a 4-periodic configuration $\langle 1100 \rangle$ and observe that for $K = 1$ we have $T\langle 1100 \rangle = \langle 1010 \rangle \in \mathrm{Per}(2, 1)$.

To deal with more general and still statistically homogeneous configurations in the next section we introduce a more interesting subset of configurations – regular configurations for which as we shall show the statistical description makes sense.

3 Space of regular configurations

For a configuration $X \in \mathbf{X}_0^K$ we define the notion of a subconfiguration $X_k^n := \{X(k), X(k+1), \ldots, X(n)\}$, i.e. a collection of entries of X between the pair of given indices $k < n$, and introduce the corresponding density and the average velocity:

$$\rho(X_k^n) := \frac{m(X_k^n)}{n-k+1}, \qquad \mathrm{V}(X_k^n) := \frac{1}{m(X_k^{n-1})} \sum_{x=k}^{n-1} v(X,x),$$

where $m(X_k^n) := \sum_{x=k}^n X(x)$ stays for the number of particles in the subconfiguration X_k^n. [1]

By the *density* and the *average velocity* (of particles) of a entire configuration $X \in \mathbf{X}_0^K$ we mean the following limits (if they are well defined):

$$\rho(X) := \lim_{n\to\infty} \rho(X_{-n}^n), \qquad \mathrm{V}(X) := \lim_{n\to\infty} \mathrm{V}(X_{-n}^n),$$

otherwise one can consider the corresponding upper and lower limits, which we denote by $\rho_\pm(X)$ and $\mathrm{V}_\pm(X)$.

Notice that both these quantities are well defined in the case of space periodic configurations (belonging to $\mathrm{Per}(K)$) in distinction even to the simplest case when a configuration X consists of free particles (i.e. belongs to $\mathrm{Free}(K)$). Thus in the general case important statistical quantities $\rho(X), \mathrm{V}(X)$ may be not well defined. To be able to deal with the space of configurations satisfying a reasonable statistical description we introduce the following space of configurations.

We shall say that a configuration X satisfies the *regularity* assumption (or simply *regular*) if there exists a number $\rho \in [0, K]$ and a strictly monotone function $\varphi(n) \to 0$ as $n \to \infty$ (which we call rate function), such that for any $n \in \mathbb{Z}^1$, $N \in \mathbb{Z}_+^1$ and any subconfiguration X_{n+1}^{n+N} of length N the number of particles in this subconfiguration $m(X_{n+1}^{n+N})$ satisfies the inequality

$$\left| \frac{m(X_{n+1}^{n+N})}{N} - \rho \right| \le \varphi(N). \tag{3.1}$$

It is clear that at for a configuration X satisfying the regularity assumption the density of particles $\rho(X)$ is well defined and is equal to the value ρ in the formulation of the assumption. The space of configurations from X_0^K satisfying the regular assumption with the density ρ and the rate function φ we shall denote by $\mathrm{Reg}(\rho, \varphi, K)$.

The main result of the paper formulated below describes the restriction of the dynamics to the space of regular configurations and will be proven in the rest of this section and the next one.

Theorem 3.1 *Let the initial configuration $X \in$ $Reg(\rho, \varphi)$ with $\rho \neq K/2$. Then after a finite number of iterations $t \le t_c = t_c(\rho, \varphi) := \frac{1}{4}(\varphi^{-1}(\frac{K}{2} - \rho) + 1)^2$ for the configuration $T^t X$ the average velocity of particles becomes well defined and is equal to* $\min\{1, \frac{K}{\rho} - 1\}$. *Moreover for any $t \ge t_c$ we have $T^t X \in \mathrm{Free}(K)$ if $\rho < K/2$ and $(T^t X)^* \in \mathrm{Free}(K)$ if $\rho > K/2$.*

To analyze properties of regular configurations we introduce the binary relation 'domination', which we denote by $\vdash$, on the set of configurations $\mathbf{X}_0^K$ as follows: $X \vdash Y$ if and only if for any $n \in \mathbb{Z}, N \in \mathbb{Z}^+$ there exists a pair $n_-, n_+ \in \mathbb{Z}$ such that

$$m(X_{n_-+1}^{n_-+N}) \le m(Y_{n+1}^{n+N}) \le m(X_{n_++1}^{n_++N}).$$

[1] Observe that in the definition of the average velocity we consider only particles from sites till $n-1$. This is related to the fact that velocities of particles in the site n is not defined by the subconfiguration X_k^n.

Lemma 3.1 *The relation $\vdash$ is an order relation, i.e. it is reflexive and transitive, but this relation is not symmetric.*

Proof. The proof of the reflexivity, i.e. that $X \vdash X$ for any $X \in \mathbf{X}_0^K$ is straightforward. To prove the second statement consider a pair of configurations $X \vdash Y \vdash Z$. By definition for any $n, k \in \mathbb{Z}$, $N \in \mathbb{Z}^+$ we have

$$m(X_{n_-+1}^{n_-+N}) \leq m(Y_{n+1}^{n+N}) \leq m(X_{n_++1}^{n_++N}),$$

$$m(Y_{n'_-+1}^{n'_-+N}) \leq m(Z_{k+1}^{k+N}) \leq m(Y_{n'_++1}^{n'_++N}).$$

Thus for any k, N there exists $n_-, n_+, n'_-, n'_+ \in \mathbb{Z}$ such that

$$m(X_{n_-+1}^{n_-+N}) \leq m(Y_{n'_-+1}^{n+N}) \leq m(Z_{k+1}^{k+N}) \leq m(Y_{n'_++1}^{n'_++N}) \leq m(X_{n_++1}^{n_++N}).$$

Therefore $X \vdash Z$. It remains to prove the absence of symmetry, i.e. that there exists a pair of configurations $X \vdash Y$ such that the relation $Y \vdash X$ does not hold. Let $X(1) = 1$, while $X(x) = 0$ for all $x \neq 1$, and let $Y(x) = 0$ for all $x \in \mathbb{Z}$. Then clearly $X \vdash Y$ but the opposite relation does not hold. ∎

Lemma 3.2 *Let $X \vdash Y$ and let $X \in \mathrm{Reg}(\rho, \varphi, K)$. Then $Y \in \mathrm{Reg}(\rho, \varphi, K)$ as well.*

Proof. According to the definition for any n, N there exists a pair n_-, n_+ such that

$$m(X_{n_-+1}^{n_-+N}) \leq m(Y_{n+1}^{n+N}) \leq m(X_{n_++1}^{n_++N}).$$

Thus

$$-\varphi(N) \leq \frac{m(X_{n_-+1}^{n_-+N})}{N} - \rho \leq \frac{m(Y_{n+1}^{n+N})}{N} - \rho \leq \frac{m(X_{n_++1}^{n_-+N})}{N} - \rho \leq \varphi(N),$$

which yields the desired statement. ∎

Now we are ready to prove that the set of regular configurations is invariant under the dynamics.

Lemma 3.3 $T : \mathrm{Reg}(\rho, \varphi, K) \to \mathrm{Reg}(\rho, \varphi, K)$ *for any triple* (ρ, φ, K).

Proof. Let a configuration $X \in \mathrm{Reg}(\rho, \varphi, K)$. We need to show that the configuration TX also satisfies the same assumption. To do it we shall prove that $X \vdash TX$, from where by Lemma 3.2 we shall get the desired statement. Fix arbitrary integers $n \in \mathbb{Z}$ and $N \in \mathbb{Z}^+$ and consider the subconfiguration $(TX)_{n+1}^{n+N}$. The number of particles in this subconfiguration differs from the number of particles in the subconfiguration X_{n+1}^{n+N} by the number of particles P_- coming from the site n to the site $n+1$ and the number of particles P_+ coming from the site $n+N$ to the site $n+N+1$, i.e.

$$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + P_- - P_+.$$

There might be four possible situations:

(a) $X(n) + X(n+1) \leq K$ and $X(n+N) + X(n+N+1) \leq K$. Then $P_- = X(n)$, $P_+ = X(n+N)$, and thus

$$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + X(n) - X(n+N) = m(X_n^{n+N-1}).$$

(b) $X(n) + X(n+1) \le K$ and $X(n+N) + X(n+N+1) > K$. Then $P_- = X(n)$, $P_+ = K - X(n+N+1)$, and

$$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + X(n) + K - X(n+N+1)$$

$$= m(X_n^{n+N-1}) + X(n+N) + X(n+N+1) - K > m(X_n^{n+N-1}).$$

On the other hand,

$$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + X(n) + K - X(n+N+1)$$

$$= m(X_{n+2}^{n+N+1}) + X(n) + X(n+1) - K \le m(X_{n+2}^{n+N+1}).$$

(c) $X(n) + X(n+1) > K$ and $X(n+N) + X(n+N+1) \le K$. Then $P_- = K - X(n+1)$, $P_+ = X(n+N)$, and

$$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + K - X(n+1) - X(n+N)$$

$$= m(X_n^{n+N-1}) - X(n) + X(n+N) + K - X(n+1) - X(n+N) > m(X_n^{n+N-1}).$$

On the other hand,

$$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + K - X(n+1) - X(n+N)$$

$$= m(X_{n+2}^{n+N+1}) - X(n+N+1) + K - X(n+N) \le m(X_{n+2}^{n+N+1}).$$

(d) $X(n) + X(n+1) > K$ and $X(n+N) + X(n+N+1) > K$. Then $P_- = K - X(n+1)$, $P_+ = K - X(n+N+1)$ and

$$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + K - X(n+1) - K + X(n+N+1) = m(X_{n+2}^{n+N+1}).$$

Therefore in all four possible cases we have found subconfigurations in X approximating (by the number of particles) those in TX from both hands, which yields the statement of Lemma. ■

Lemma 3.4 $(\mathrm{Reg}(\rho, \varphi, K))^* = \mathrm{Reg}(K - \rho, \varphi, K)$.

Proof. Let $X \in \mathrm{Reg}(\rho, \varphi, K)$. Then for any $n \in \mathbb{Z}^1, N \in \mathbb{Z}_+^1$ we have

$$m((X^*)_{n+1}^{n+N}) = m((\langle K \rangle - X)_{n+1}^{n+N}) = K \cdot N - m(X_{n+1}^{n+N}).$$

Therefore

$$\left|\rho((X^*)_{n+1}^{n+N}) - (K - \rho)\right| = \left|\frac{m((X^*)_{n+1}^{n+N})}{N} - (K - \rho)\right|$$

$$= \left|\frac{m(X_{n+1}^{n+N})}{N} - \rho\right| \le \varphi(N).$$

■

Consider now the connection between spaces of periodic configurations and regular ones. Clearly, for any configuration $X \in \mathrm{Per}(n, K)$ the notion of density $\rho(X)$ is well defined and $\rho(X) = m(X_1^n)/n$. To specify the density we denote by $\mathrm{Per}_\rho(n, K)$ the set of configurations from $\mathrm{Per}(n, K)$ having the same density ρ.

Lemma 3.5 *For any ρ, n, K we have $T : \mathrm{Per}_\rho(n, K) \to \mathrm{Per}_\rho(n, K)$ and $\mathrm{Per}_\rho(n, K) \subset \mathrm{Reg}(\rho,\ \rho(1-\frac{\rho}{K})\frac{n}{N},\ K)$.*

Proof. For a given configuration $X \in \mathrm{Per}(n, K)$ denote $\rho := \rho(X) = m(X_1^n)/n$. The first statement immediately follows from Lemma 2.4 and the fact that the number of particles on the period of the configuration cannot change under dynamics. Now each positive integer N can be represented as $N = kn + l$, where $k \in \{0, 1, \ldots\}$, $l \in 0, 1, \ldots, n-1$. For any $l \leq n - \frac{\rho}{K}n$ the number of particles in the subconfiguration X_{x+1}^{x+N} can be estimated from below as

$$m(X_{x+1}^{x+N}) \geq \rho kn.$$

Therefore

$$\rho - \frac{m(X_{x+1}^{x+N})}{N} \leq \rho - \frac{\rho kn}{kn+l} = \frac{\rho l}{N} \leq \rho(1 - \frac{\rho}{K})\frac{n}{N} =: \varphi(N).$$

Otherwise, if $l > n - \frac{\rho}{K}n$ we have

$$m(X_{x+1}^{x+N}) \geq \rho kn + K(l - n + \frac{\rho}{K}n) = \rho kn + Kl - Kn + \rho n.$$

Thus

$$\begin{aligned} \rho - \frac{m(X_{x+1}^{x+N})}{N} &\leq \rho - \frac{\rho kn + Kl - Kn + \rho n}{kn+l} \\ &= \frac{1}{N}(n-l)(K-\rho) < \frac{1}{N}\frac{\rho n}{K}(K-\rho) = \varphi(N). \end{aligned}$$

Now we shall use estimates for the number of particles in the subconfiguration X_{x+1}^{x+N} from above. If $l \leq \frac{\rho}{K}n$ then

$$m(X_{x+1}^{x+N}) \leq \rho kn + Kl$$

and

$$\frac{m(X_{x+1}^{x+N})}{N} - \rho \leq \frac{\rho kn + Kl}{kn+l} - \rho = \frac{1}{N}(K-\rho)l \leq \frac{1}{N}(K-\rho)\frac{\rho}{K}n = \varphi(N).$$

Otherwise, if $l > \frac{\rho}{K}n$ then

$$m(X_{x+1}^{x+N}) \leq \rho kn + \rho n$$

and

$$\begin{aligned} \frac{m(X_{x+1}^{x+N})}{N} - \rho &\leq \frac{\rho kn + \rho n}{kn+l} - \rho = \frac{1}{N}\rho(n-l) \\ &< \frac{1}{N}\rho(n - \frac{\rho}{K}n) = \frac{1}{N}\rho n(1 - \frac{\rho}{K}) = \varphi(N). \end{aligned}$$

∎

One can easily check that for any (space) periodic configuration the notion of the average velocity is well defined. It is of interest that for more general class of regular configurations this is not the case even for $K = 1$. Denote $a = 1100$ and $b = 1010$ and consider the configuration X constructed as follows:

$$\ldots\ bbbbaaaa\ bbaa\ ba\ ab\ aabb\ aaaabbbb\ \ldots, \tag{3.2}$$

i.e. $X_1^8 = ab$, $X_{-7}^0 = ba$, $X_9^{24} = aabb$, $X_{-23}^{-8} = bbaa$, etc. Notice that in each subsequent series the number of consequent elements $aa\ldots a$ and $bb\ldots b$ doubles.

Lemma 3.6 *The configuration X defined as (3.2) is regular ($X \in \mathrm{Reg}(1/2, 1/N, K)$), but the average velocity is not well defined.*

Proof. Observe that $m(X_{i+1}^{i+4k}) = 2k$ for any i,k, while $2k \le m(X_{i+1}^{i+4k+j}) \le 2k+2$ for any $j \in \{1,2,3\}$. Therefore the configuration (3.2) is regular with the density 1/2 and the rate function $\varphi(N) = 1/N$.

Let us calculate now the average velocity on various subconfigurations. First consider a subconfiguration starting from the 1st element and containing the full series $aa\ldots bb$, i.e. $X_1^{2(2^{k+1}-1)}$. This subconfiguration for any k contains the same number of elements a and b and hence

$$\mathrm{V}(X_1^{2(2^{k+1}-1)}) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1 = \frac{3}{4}.$$

Similarly due to the symmetry of the configuration X we have $\mathrm{V}(X^0_{-2(2^{k+1}-1)+1}) = 3/4$ and therefore $|\mathrm{V}(X_{-2^{k+2}}^{2^{k+2}}) - 3/4| < 5\cdot 2^{-(k+3)}$. Thus

$$\mathrm{V}(X_{-2^{k+2}}^{2^{k+2}}) \to 3/4 \quad \text{as} \quad k \to \infty.$$

Another type of subconfigurations that we consider differs from the previous one by the fact that it contains an additional (full) series of elements $a\ldots a$ in the end, i.e. $X_1^{2(2^{k+1}-1)+2^{k+1}} = X_1^{3\cdot 2^{k+1}-2}$.

$$\mathrm{V}(X_1^{3\cdot 2^{k+1}-2}) = \frac{\frac{1}{2}\cdot 2^{k+1} + \frac{3}{4}\cdot 2(2^{k+1}-1)}{2^{k+1} + 2(2^{k+1}-1)} = \frac{2\cdot 2^{k+1} - \frac{3}{2}}{3\cdot 2^{k+1} - 2} \to \frac{2}{3}$$

as $k \to \infty$. Therefore using again the symmetry of the configuration X we have

$$\mathrm{V}(X_{-3\cdot 2^{k+1}}^{3\cdot 2^{k+1}}) \to 2/3 \quad \text{as} \quad k \to \infty,$$

and thus different subsequences of k lead to different average velocities. ∎

4 Traffic jams and simple properties of the dynamics

Recall that in a configuration X sites between x' and x'' belong to the *jammed cluster* if for any integer $x \in [x', x'']$ the inequality $X(x) + X(x+1) > K$ holds true. Similarly consecutive sites for which this inequality does not hold belong to a free cluster. The site x is called *free* if $X(x) + X(x+1) \le K$.

Lemma 4.1 *For each configuration $X \in \mathbf{X}_0^K$ and for each site $x \in \mathbb{Z}^1$ we have*

$$\min\{X(x-1), X(x), X(x+1)\} \le (TX)(x) \le \max\{X(x-1), X(x), X(x+1)\}$$

and thus for any $t \in \mathbb{Z}^1_+$

$$\min_x\{X(x)\} \le \min_x\{(T^tX)(x)\} \le \max_x\{(T^tX)(x)\} \le \max_x\{X(x)\}.$$

Moreover, the upper and lower limits may be not preserved under dynamics:
- $\exists X \in \mathbf{X}_0^K$ *such that* $\max_x\{(TX)(x)\} < \max_x\{X(x)\}$*;*
- $\exists X \in \mathbf{X}_0^K$ *such that* $\min_x\{(TX)(x)\} > \min_x\{X(x)\}$*;*

Proof. Fix a configuration $X \in \mathbf{X}_0^K$ and a site $x \in \mathbb{Z}^1$. Then denoting by $P(x) := \min\{X(x-1), K - X(x)\}$ the number of particles moving to the site x (from the site $x-1$) under the action of the map T we get

$$(TX)(x) = X(x) + P(x) - P(x+1).$$

Consider all 4 possibilities:
(a) $X(x-1) \le K - X(x)$ and $X(x) \le K - X(x+1)$. Thus

$$(TX)(x) = X(x) + X(x-1) - X(x) = X(x-1).$$

(b) $X(x-1) > K - X(x)$ and $X(x) \le K - X(x+1)$. Thus

$$(TX)(x) = X(x) + (K - X(x)) - X(x) = K - X(x) < X(x-1).$$

On the other hand, in this case

$$(TX)(x) = K - X(x) \ge X(x+1).$$

(c) $X(x-1) \le K - X(x)$ and $X(x) > K - X(x+1)$. Thus

$$(TX)(x) = X(x) + X(x-1) - (K - X(x+1)) = X(x-1) - (K - X(x)) + X(x+1) \le X(x+1).$$

On the other hand, in this case

$$(TX)(x) = X(x-1) + X(x) - (K - X(x+1)) > X(x-1) + X(x) - X(x) = X(x-1).$$

(d) $X(x-1) > K - X(x)$ and $X(x) > K - X(x+1)$. Thus

$$(TX)(x) = X(x) + (K - X(x)) - (K - X(x+1)) = X(x+1).$$

Thus the first statement of Lemma holds true in all situations.

It remains to construct examples of configurations satisfying the last two statements of Lemma. Let $K = 2$. Then $T : \langle 221022 \rangle \to \langle 211122 \rangle$ and $T : \langle 002100 \rangle \to \langle 001110 \rangle$. In the first example the minimal value 0 becomes 1, while in the second example the maximal value 2 becomes 1 under the action of the dynamics. ∎

Introduce a map marking global maxima of a configuration $M : \mathbf{X}_0^K \to \mathbf{X}_0^1$ as follows: $MX(x) := 1$ if $X(x) = \max_y\{X(y)\}$ and $MX(x) := 0$ otherwise. We define also arithmetic operations with configurations $X, Y \in \mathbf{X}_0^K$:

$$(X+Y)(x) := \min\{X(x) + Y(x),\ K\}, \qquad (X-Y)(x) := \max\{X(x) - Y(x),\ 0\}.$$

Using this notation we can formulate the following decomposition result.

Lemma 4.2 *For a given $X \in \mathbf{X}_1^K$ if $\forall x \in \mathbb{Z}_+^1$ such that $MX(x) = 1$ holds $X(x-1) + X(x) \le K$ and $X(x) + X(x+1) \le K$ then*

$$TX = T(X - MX) + T(MX),$$

otherwise

$$TX = T_{|_{K-1}}(X - MX) + T_{|_1} MX,$$

where $T_{|_{K-1}}$ means the restriction of the map T to $\mathbf{X}_0^{K-1}$. On the other hand, there exists a configuration $X \in \mathbf{X}_0^K$ such that

$$(T^t X)(x) \ne (T^t_{|_{K-1}} X)(x).$$

even if $\max_x\{X(x)\} < K$.

Proof. The statement about the decomposition follows immediately from the definition of the dynamics while the following example demonstrate the second statement: $T_{|_3} : \langle 1221 \rangle \to \langle 1212 \rangle$, $T_{|_2} : \langle 1221 \rangle \to \langle 2211 \rangle$. ∎

These results demonstrate rather counter intuitive properties of the considered model of traffic flows. For example, from Lemma 4.1 it follows that if for a given initial configuration one traffic lane is not occupied (along the entire lattice), then these property holds for any moment of time. So it looks that the dynamics will not change if the road will be made narrower by one lane. However this is completely wrong, which was demontstrated in the second statement of Lemma 4.2.

Another example gives the following seemingly evident (but wrong) decomposition, which one would expect instead of the more complex decomposition described in Lemma 4.2. Assume that for a configuration X we have $X(x) > 1$ for all $x \in \mathbb{Z}^1$. Then it looks reasonable that the dynamics of the configuration, restricted to the lanes $2, 3, \ldots, K$ should be the same as in the original one, i.e.

$$T_{|_K} X = T_{|_{K-1}}(X - \langle 1 \rangle) + T_{|_K}(\langle 1 \rangle).$$

The following example of a periodic configuration shows that this is not the case:

$$T_{|_2}\langle 1221 \rangle = \langle 2211 \rangle, \qquad T_{|_1}\langle 0110 \rangle + T_{|_2}\langle 1111 \rangle = \langle 0101 \rangle + \langle 1111 \rangle = \langle 1212 \rangle.$$

Lemma 4.3 *Let X_{k+1}^{k+n} be a jammed cluster of length n in the configuration X. Then*

$$(TX)(x) = X(x+1) \quad \forall x \in \{k+2, \ldots, k+n\}, \tag{4.1}$$

$$(TX)(k+n+1) = K - X(k+n+1), \tag{4.2}$$

$$(TX)(k+1) = X(k) + X(k+1) + X(k+2) - K \tag{4.3}$$

and if the site $k-1$ does not belong to another jammed cluster, then

$$(TX)(k-1) = X(k-2), \quad (TX)(k) = X(k-1), \tag{4.4}$$

otherwise

$$(TX)(k-1) = X(k), \quad (TX)(k) = K - X(k). \tag{4.5}$$

Proof. First let us show that $TX(x) = X(x+1)$ for all $x \in \{k+2, \ldots, k+n\}$. Observe that by the definition of a jammed cluster we have

$$X(x-1) + X(x) > K, \quad X(x) + X(x+1) > K.$$

Thus

$$X(x-1) > K - X(x), \quad X(x) > K - X(x+1).$$

Therefore after the application of the map T exactly $K - X(x)$ particles comes to the site x from the site $x-1$, while $K - X(x+1)$ particles leaves it. Therefore

$$TX(x) = K - X(x) + X(x) - (K - X(x+1)) = X(x+1),$$

which proves the equality (4.1). Observe that this equality makes sense only if $n \geq 2$.

The site $k+n+1$ is the first free site after the jammed cluster. Therefore all particles from it moves to the site $k+n+2$ under the action of the map T, while exactly $K - X(k+n+1)$ particles moves to the site $k+n+1$ from the last site of the considered jammed cluster. This gives the equality (4.2). Notice that from this inequality we get that

$$(TX)(k+n) + (TX)(k+n+1) = K,$$

i.e. the site $k+n$ does not belong to a jammed cluster in the configuration TX.

Clearly the site k cannot belong to another jammed cluster, otherwise the site $k+1$ would not be the first site of the considered jammed cluster.

By definition under the action of the map T all particles from the site k moves to the site $k+1$, from where exactly $X(k+2)-K$ particles moves to the site $k+2$. Thus we get the equality (4.3).

Consider now the case when the site $k-1$ does not belong to another jammed cluster, i.e. $X(k-1)+X(k) \le K$. This immediately gives the formulae (4.4) for the number of particles in the sites $k-1$ and k.

If $n=1$ then

$$(TX)(k+1)+(TX)(k+2) = X(k)+X(k+1)+X(k+2)-K+K-X(k+2) = X(k)+X(k+1) \le K.$$

If $n \ge 2$ then

$$(TX)(k+i)+(TX)(k+i+1) = X(k+i-1)+X(k+i)$$

for all $i \in \{1, \dots, n-2\}$. Thus

$$(TX)(k+n-1)+(TX)(k+n) = X(k+n)+X(k+n+1) > K,$$

$$(TX)(k+n)+(TX)(k+n+1) = X(k+n+1)+K-X(k+n+1) = K.$$

Therefore in both cases the site which was the last one in the jammed cluster X_{k+1}^{k+n+1} becomes, and if $n \ge 2$ the site $k+n-1$ turns out to be the last site in the jammed cluster in TX.

Summarizing, in the case when the site $k-1$ is free we get the following representation for TX in the neighborhood of the considered jammed cluster:

```
    ...   k-1     k        k+1       k+2     k+3    ...  k+n-1    k+n
k+n+1      ...
 X= ...  X(k-1)  X(k)   [X(k+1)    X(k+2)  X(k+3) ... X(k+n-1) X(k+n)]
X(k+n+1)    ...
TX= ...  X(k-2)  X(k-1) (TX)(k+1)  X(k+3)  X(k+4) ... X(k+n)]  X(k+n+1)
K-X(k+n+1) ...
```

By square brackets we denote the boundaries of jammed clusters. Observe that the first site of the new cluster might be either k or $k+1$.

In the alternative case when the site $k-1$ is the last site of the previous jammed cluster, the representation differs only at sites $k-1$ and k:

```
    ...   k-1     k        k+1       k+2     k+3    ...  k+n-1    k+n
k+n+1      ...
 X= ...  X(k-1)]  X(k)   [X(k+1)   X(k+2)  X(k+3) ... X(k+n-1) X(k+n)]
X(k+n+1)    ...
TX= ...  X(k)    [K-X(k) (TX)(k+1) X(k+3)  X(k+4) ... X(k+n)]  X(k+n+1)
K-X(k+n+1) ...
```

Indeed, applying the first statement of Lemma (which has been already proven) to the previous cluster, we get that $(TX)(k-1) = X(k)$. On the other hand, the number of particles moving from the site $k-1$ to the site k is equal to $K-X(k)$, while all the particles that were at site k move to the site $k+1$ (since the site k does not belong to a jammed cluster).

$$(TX)(k)+(TX)(k+1) = K-X(k)+X(k)+X(k+1)+X(k+2)-K$$

$$= X(k+1)+X(k+2) > K,$$

therefore the new jammed cluster has the same length and is located at the sites from k to $k+n-1$.

■

Corollary 4.4 *For each jammed cluster $(TX)_{k+1}^{k+n+1}$ of length $n > 1$ we have $(TX)(k+n+2) + (TX)(k+n+3) = K$.*

Proof. Immediately follows from the equality (4.2). ■

This implies that the distance between two consecitive jammed clusters is at least 2.

Lemma 4.5 *Let X_{k+1}^{k+n} be a jammed cluster of length n in the configuration X. Then neither its length, nor the number of particles in it cannot increase under dynamics. Moreover, if $X(k-1) + X(k) < K$ then number of particles in the jammed cluster decreases at least by $K - (X(k-1) + X(k)) > 0$ after the application of the map.*

Proof. Consider first the case when the site $k-1$ does not belong to another jammed cluster, i.e.

$$X(k-1) + X(k) \leq K, \qquad X(k) + X(k+1) \leq K.$$

Clearly, in this case the site $k-1$ cannot be the first site of the jammed cluster in the configuration TX. Therefore in the worst case the jammed cluster is located at sites from k to $k+n-1$, i.e. its length is at least not larger than of the considered one. Applying Lemma 4.3 we can estimate from above the difference between the number of particles in the new cluster and the old one as follows:

$$\begin{aligned} &[(TX)(k) + (TX)(k+1)\} - \{X(k+1) + X(k+2)] \\ &= X(k-1) + X(k) + X(k+1) + X(k+2) - K - X(k+1) - X(k+2) \\ &= X(k-1) + X(k) - K \leq 0. \end{aligned}$$

Hence the number of particles in this case cannot increase, and moreover this number decreases if $X(k-1) + X(k) < K$.

It remains to consider the case when the considered jammed cluster is immediately preceded by another jammed cluster. Again by Lemma 4.3

$$(TX)(k-1) + (TX)(k) = X(k) + K - X(k) = K.$$

Hence the site $k-1$ does not belong to the jammed cluster. On the other hand,

$$\begin{aligned} (TX)(k) + (TX)(k+1) &= K - X(k) + X(k) + X(k+1) + X(k+2) - K \\ &= X(k+1) + X(k+2) > K, \end{aligned}$$

since the site $k+1$ belongs to the jammed cluster. Thus the site $k+1$ is the first site of the jammed cluster in the configuration TX, which lies in sites from k to $k+n$, i.e. its length is exactly the same as of the old one. Applying the same trick as above to calculate the difference between the number of particles in the new cluster and the old one we get:

$$\begin{aligned} &\{(TX)(k) + (TX)(k+1)\} - \{X(k+1) + X(k+2)\} \\ &= K - X(k) + X(k) + X(k+1) + X(k+2) - K - X(k+1) - X(k+2) = 0. \end{aligned}$$

Therefore even the number of particles in the jammed cluster is preserved in this case. ■

Lemma 4.6 *Let $n \in \mathbb{Z}_+^1$, $\rho(X_{k+1}^{k+2n+1}) \leq K/2$, and let in the subconfiguration X_{k+1}^{k+2n+1} there is at least one jammed site. Then there is an integer $i \in \{1, \dots, 2n-1\}$ such that $X(k-i) + X(k+i+1) < K$.*

t	X	v(X)	X	v(X)
0	<21200>	3/5	<22300>	4/7
1	<12020>	4/5	<13030>	6/7
2	<10202>	4/5	<10303>	6/7
3	<11021>	4/5	<21031>	6/7
4	<11111>	1	<12112>	1
	(a)		(b)	

Figure 1: Two examples of the dynamics of n-periodic configurations: (a) $K = 2$, $n = 5$, $\rho = 1 = K/2$, (b) $K = 3$, $n = 5$, $\rho = 7/5 < K/2 = 3/2$.

Proof. Assume that this statement does not hold true. Then for any two consecutive sites x and $x+1$ in this subconfiguration we have $X(x) + X(x+1) \geq K$. On the other hand, for the jammed site y we get $X(x) + X(x+1) \geq K + 1$. Thus

$$\sum_{x=1}^{2n} X(k+x) \geq n+1,$$

which contradicts to the fact that the density is less or equal to $1/2$. ∎

These results yield the following property: For any given subconfiguration the number of particles in any jammed cluster completely contained in this subconfiguration is a nonincreasing function of time and achieves its lowest possible level under dynamics.

Lemma 4.7 *Let $X \in \mathrm{Reg}(\rho, \varphi, K)$ with the density $\rho < K/2$. Then after at most $t_c = t_c(\rho, \varphi) = \frac{1}{4}(\varphi^{-1}(\frac{K}{2} - \rho) + 1)^2$ iterations all particles in $T^t X$ for $t \geq t_c$ will become free.*

Proof. According to the definition of regular configurations $M(n)$ – the maximal number of particles in subconfigurations of length n of the configuration X for each $n \in \mathbb{Z}^1_+$ satisfies the inequality

$$\frac{M(n)}{n} \leq \rho + \varphi(n).$$

Thus for any $n > N_c := \varphi^{-1}(\frac{K}{2} - \rho)$ it follows that $M(n) < n/(2K)$. By Lemma 4.6 in each subconfiguration of length n there is a pair of consequent sites whose total number of particles Q is strictly less than K. Consider the dynamics of this pair of sites. According to our previous results, while the site ahead of them is free these two sites will simply move one position forward. In the opposite case, when the next site is the first site of some jammed cluster by Lemma 4.5 the number of particles in this cluster will decrease by $K - Q > 0$. On the other hand, free particles and jammed clusters move in opposite directions each with the velocity 1. Thus the maximum time between the consecutive meetings of a jammed cluster and a pair of consequent sites with the total number of particles less than K does not exceed $n/2$. Let n be the smallest integer larger or equal to N_c. Since after each such meeting the number of particles in the corresponding jammed cluster decreases at least by 1 and since the number of particles in this cluster is less or equal to $M(n)$ we get the following upper estimate of the transient period:

$$t_c \leq \frac{M(n) \cdot n}{2} \leq \frac{n^2}{4K} < \frac{1}{4}(\varphi^{-1}(K/2 - \rho) + 1)^2.$$

∎

Proof of Theorem 3.1. After the preparation made in Lemmas 3.2–4.7 we are able to finish the proof of our main result. Indeed, in the case of a regular configuration $X \in \mathrm{Reg}(\rho, \varphi, K)$ with the density $\rho < K/2$ by Lemma 4.7 for any integer $t \geq t_c$ the configuration $T^t X$ consists of only free particles. In the opposite case, when $\rho > K/2$, we consider the dual configuration $X^* \in \mathrm{Reg}(K-\rho, \varphi, K)$ (by Lemma 3.4) and the since the action of the dual map is equivalent to the main one but proceeds in the opposite direction we get that $T^t X^* \in \mathrm{Free}(K)$.

It remains to prove the statement about the average velocity of the configuration $T^t X$ for each $t \geq t_c$. Again we start from the case of low density $\rho < K/2$. Since the configuration $T^t X$ consists of free particles, velocity of each particle is equal to 1. Thus

$$V((T^t X)_{-n}^{n}) \equiv 1 \quad \forall n \in \mathbb{Z}_+^1,$$

which both shows that the average velocity s well defined and that $\mathrm{V}(T^t X) = 1$. Now if $\rho > K/2$ the dual configuration to the configuration $Y := T^t X$ again consists of free particles. Hence

$$V(Y_{-n}^{n}) = \frac{1}{m(Y_{-n}^{n-1})} \sum_{x=-n}^{n-1} v(Y, x) = \frac{m((Y^*)_{-n}^{n-1})}{2nK - m((Y^*)_{-n}^{n-1})} \to \frac{K}{\rho} - 1$$

as $n \to \infty$ since the density of the configuration Y^* is equal to $K - \rho$. ■

Observe that in the proof of Theorem 3.1 we actually derived an estimate of the length of the transient period as $t \leq t_c = t_c(\rho, \varphi) := \frac{1}{4}(\varphi^{-1}(\frac{K}{2} - \rho) + 1)^2$ which goes to infinity as $\rho \to 1/2$. This is the reason why Theorem 3.1 does not cover the boundary case $\rho = 1/2$, which we discuss below.

Theorem 4.1 *Let the initial configuration $X \in \mathrm{Reg}(\frac{K}{2}, \varphi, K)$ and let $x'(t) < x''(t)$ be positions of two fixed arbitrary particles at the arbitrary moment t. Then the average velocity of the subconfiguration $X_{x'(t)}^{x''(t)}$ converges to 1 as $t \to \infty$.*

Proof. Denoting $\rho := \frac{K}{2}$ and choosing a positive integer M we consider a configuration ${}^{-M}X$ obtained from the configuration $X \in \mathrm{Reg}(\rho, \varphi, K)$ by the following operation: for each integer k we remove from the configuration X the closest from behind particle to the position kM. For a given positive integer M any integer N may be represented as $N = kM + l$ with $l \in \{-M, -M+1, \ldots, M-1, M\}$ and $k \in \mathbb{Z}^1$. Then

$$m({}^{-M}X_{n+1}^{n+kM+l}) = m(X_{n+1}^{n+kM+l}) - k$$

and thus

$$\left| \frac{m({}^{-M}X_{n+1}^{n+N})}{N} - (\rho - \frac{1}{M}) \right| = \left| \frac{m(X_{n+1}^{n+N})}{N} - \rho - (\frac{k}{N} - \frac{1}{M}) \right|.$$

On the other hand,

$$\left| \frac{k}{N} - \frac{1}{M} \right| = \left| \frac{k}{kM + l} - \frac{1}{M} \right| = \frac{l}{(kM + l)M} < \frac{1}{N}.$$

Therefore ${}^{-M}X \in \mathrm{Reg}(\rho - \frac{1}{M},\ \varphi + \frac{1}{N},\ K)$ and according to Theorem 3.1 after a finite number of iterations t_c the average velocity of the configuration $T^{t_c}({}^{-M}X)$ becomes equal to 1 (since all the particles in this configurations are free).

t	X	v(X)	X	v(X)
0	<0414232>	9/14	<1204440>	7/15
1	<0142313>	10/14	<0124404>	9/15
2	<3123131>	13/14	<4034040>	12/15
3	<1321313>	13/14	<0430404>	12/15
4	<3222131>	13/14	<4313040>	12/15
5	<2222213>	13/14	<3131304>	12/15
6	<2222222>	1	<1313133>	13/15
7	<2222222>	1	<3131331>	13/15
8	<2222222>	1	<1313313>	13/15
9	<2222222>	1	<3133131>	13/15
10	<2222222>	1	<1331313>	13/15
11	<2222222>	1	<3313131>	13/15
12	<2222222>	1	<3131313>	13/15
	(a)		(b)	

Figure 2: Long transient periods ($t_c = n-1$) of n-periodic configurations with $K = 4$ and $n = 7$: (a) $\rho = 2 = K/2$, (b) $\rho = 15/7 > K/2$.

Making an opposite operation, namely inserting a particle to the configuration X to the closest from behind to kM empty position for each integer k, we obtain another regular configuration ${}^{+M}X \in \mathrm{Reg}(\rho+\frac{1}{M}, \varphi+\frac{1}{N}, K)$. Again by Theorem 3.1 after a finite number of iterations the average velocity of this configuration becomes equal to

$$\frac{K}{\rho - \frac{1}{M}} - 1 = 1 + \frac{4}{KM-2} \to 1 \quad \text{as} \quad M \to \infty.$$

Thus both (arbitrary close as $M \to \infty$) approximations ${}^{\pm M}X$ to the configuration X have after a finite number of iterations (depending on M) the average velocity deviating from 1 by $\mathcal{O}(1/M)$. It remains to show that the average velocity of a subconfiguration of the configuration X can be estimated from above and from below by those from above approximations. Let X and Y be two configurations such that $X(x) \le Y(x)$ for all x and let $x'(t) < x''(t)$ be positions of two fixed particles in the configuration X at the arbitrary moment t. Denote by $y'(t) < y''(t)$ positions of the same particles in the configuration Y. Then

$$\mathrm{V}(X_{x'(t)}^{x''(t)}) \ge \mathrm{V}(X_{x'(t)}^{x''(t)})$$

for any moment of time t. Indeed, additional particles in the configuration Y present only obstacles to the motion of other particles, thus making the average velocity slower (or at least not faster). ∎

In the case of (space) n-periodic configurations numerical examples below demonstrate a much better estimate of the transient period: $t_c \le n-1$, but it is rather unclear if it is possible to generalize this result for more general regular configurations.

Moreover, it turns out that even the upper (attainable) estimate of the length of the transient period for an initial configuration from $\mathrm{Per}_\rho(n, K)$ is not monotonous on the length of the period n and heavily depends on its parity. The above example demonstrate the estimate $t_c \le n-1$ for odd values of n. Now following mainly ideas proposed in [15] we shall show that this estimate is rather different for even values of the period.

Lemma 4.8 *Let $X \in \mathrm{Per}_\rho(2n, K)$ for some $n \ge 1$. Then the length of the transient period $t_c \le n$.*

Proof. We introduce an operator G mapping the space of configurations X_0^K into the space of two-side sequences of real numbers defined as follows:

$$GX(x) := \sum_{i=0}^{x-1} X(i) - (x-1)K/2,$$

where we set $\sum_{i=0}^{-j} = \sum_{i=-j}^{0}$ for any positive integer j.

One can easily show that for any $x \in \mathbb{Z}^1$ we have

$$X(x) = GX(x+1) - GX(x) + K/2,$$

and if additionally $X \in \mathrm{Per}_\rho(2n, K)$, then

$$GX(x+2n) - GX(x) = \sum_{i=x}^{x+2n-1} (X(i) - K/2) = 2nK(\rho - 1/2) \tag{4.6}$$

and for any $x \in \mathbb{Z}^1$

$$G(TX)(x) = \max\{GX(x-1), GX(x) - K/2, GX(x+1)\}. \tag{4.7}$$

This yields that

$$G(T^tX)(x) = \max\{\max\{GX(x-t), GX(x-t+2), \ldots, GX(x+t)\},$$

$$\max\{GX(x-t+1), GX(x-t+3), \ldots, GX(x+t-1)\} \quad - K/2\}.$$

As usual we consider three possibilities.

(a) Assume first that $\rho < K/2$. Then from the equality (4.6) we get that $GX(x+2n) < GX(x)$ for each x. Then for $t \ge n$ we obtain

$$G(T^tX)(x) = \max\{\max\{GX(x-t), GX(x-t+2), \ldots\},$$

$$\max\{GX(x-t+1), GX(x-t+3), \ldots\} \quad - K/2\}.$$

Thus $G(T^{t+1}X)(x) = G(T^tX)(x-1)$ and $T^{t+1}X(x) = T^t(x-1)$. Substituting these equalities into (4.7) we obtain

$$0 = \max\{0,\ G(T^tX)(x) - G(T^tX)(x-1) - K/2,\ G(T^tX)(x+1) - G(T^tX)(x-1)\}$$

$$= \max\{0,\ T^tX(x-1) - K,\ T^tX(x-1) + T^tX(x) - K\},$$

from where $T^tX(x) \le K - T^tX(x+1)$ for any t, x. Therefore for $t \ge n$ all particles in the configuration T^tX are free.

(b) $\rho = K/2$. Since in this case $GX(x+2n) = GX(x)$ for all x we get that for $t \ge n$

$$G(T^tX)(x) = \max\{\max\{GX(2), GX(4), \ldots, GX(2n)\},$$

$$\max\{GX(1), GX(3), \ldots, GX(2n-1)\} - K/2\}$$

if $x - t$ is even and

$$G(T^tX)(x) = \max\{\max\{GX(1), GX(3), \ldots, GX(2n-1)\},$$

$$\max\{GX(2), GX(4), \ldots, GX(2n)\} - K/2\}$$

otherwise. Therefore $G(T^{t+1}X)(x) = G(T^t X)(x \pm 1)$ and thus $T^{t+1}X(x) = T^t X(x \pm 1)$ for all $x \in \mathbb{Z}^1$, which yields that $T^t X \in \mathrm{Free}(K)$.

(c) $\rho > K/2$. This case follows from the argument applied in the case (a), since we can consider the dual configuration X^*, for which the density of particles is less than $K/2$. ∎

Lemma 4.9 *Let $X \in \mathrm{Per}_\rho(2n+1, K)$ for some $n \geq 1$. Then the length of the transient period $t_c \leq 2n+1$.*

Proof. Any configuration $X \in \mathrm{Per}_\rho(2n+1, K)$ belongs also to $\mathrm{Per}_\rho(4n+2, K)$. On the other hand the number $4n+2$ is even and thus by Lemma 4.8 we get the desired estimate of the transient period. ∎

It is of interest that in the space periodic case we can give a more detailed information about the dynamics in time.

Proposition 4.10 *For each configuration $X \in \mathrm{Per}(n, K)$ and any integer $t \geq t_c = n$ the sequence $\{T^t X(x)\}_t$ is n-periodic on t for each $x \in \mathbb{Z}^1$.*

Proof. This result follows from the fact that for each t the configuration $T^t X$ is n-periodic in space and by Lemmas 4.8 and 4.9 the length of the transient period $t_c \leq n$. Thus for $t \geq t_c$ the configuration $T^t X$ consists either of free particles (if $\rho(X) \leq K/2$) or its dual satisfies this property. Therefore $T^{t+n}X(x) = T^t X(x)$ for any $x \in \mathbb{Z}^1$. Notice that this period in time might be not minimal (which can be as small as 2). ∎

Observe that this construction heavily depends on the periodic space structure of the configurations, which rules out the generalization for a more general situation.

5 On the chaoticity of the dynamics

In the previous sections it was shown that for sufficiently large time the dynamics occurs either in $\mathrm{Free}(K)$ or in $(\mathrm{Free}(K))^*$, i.e. the corresponding dual configurations belong to this space. Therefore to study asymptotic (as time goes to ∞) properties of the dynamics we consider its restriction to the space of configurations of free particles (which contains the union of all attractors of the map T restricted to the set of regular configurations with the density less than $K/2$). The following result shows that this map is chaotic in the sense that its topological entropy (see definitions in [12]) is positive.

Theorem 5.1 $h_{\mathrm{top}}(T, \mathrm{Free}(K)) = \ln\left(\frac{2(K+1)}{\pi} + \frac{1}{\pi} + \frac{R(K)}{(K+1)^2}\right) > 0$ *for any $K \in \mathbb{Z}^1_+$. The remainder term $R(K)$ above satisfies the inequality $|R(K)| \leq 2$.*

Proof. All the particles in configurations on the largest (i.e. containing all others) attractor are free, i.e. $X(x) + X(x+1) \leq K$. Thus the action of the map is equivalent to the right shift map with the upper triangular transition matrix (i.e. all elements in the first line are 1, all but the last are 1's in the 2nd line, etc.). It is well known (see, for example, [12]) that the logarithm of the largest eigenvalue of this matrix gives the topological entropy of the right shift map and thus the topological entropy $h_{\mathrm{top}}(T)$ of the traffic flow. Therefore the representation of the largest eigenvalue of the transition matrix which we shall give below finishes the proof. ∎

To simplify the notation we denote $N := K+1$. Let $A(N) = (a_{ij})$ be the $N \times N$ left triangular matrix, i.e. $a_{ij} = 1$ for all $i+j \le N+1$ and $a_{ij} = 0$ otherwise. This is a nonnegative symmetric matrix, therefore its spectrum belongs to the real line and its largest eigenvalue $\lambda_{\max}(A(N))$ is positive.

Theorem 5.2 $\lambda_{\max}(A(N)) = \frac{2}{\pi}N + \frac{1}{\pi} + \frac{R}{N^2}$, *where the remainder term* $|R| = |R(N)| \le 2$.

Proof. For an integrable function $f \in \mathbf{L}^2$ consider the operator

$$Lf(x) := \int_0^{1-x} f(s)\, ds.$$

According to [3] eigenvalues of the operator L ordered by their moduli are equal to

$$\lambda_k := \frac{(-1)^{k+1}}{(k-1/2)\pi},$$

while $e(x) := \cos(\frac{\pi}{2}x)$ is the eigenfunction corresponding to the leading eigenvalue λ_1. For simplicity we shall use the notation $A := A(N)$, $\lambda := \lambda_1 = \frac{2}{\Pi}$, $e_k := e(k/N)$, and $\varepsilon = \frac{1}{N}$. Now since the function $e(x)$ is analytical, decreases monotonically, and its second derivative satisfies the inequality $|\frac{d^2e(x)}{dx^2}| < \frac{\pi^2}{4}$, it follows that for each $k = 1, 2, \ldots, N$ we have

$$(Ae)_k = \int_0^{1-k/N} e(s)\, ds + \frac{1}{2N}\sum_{i=0}^{N-k}\left(e(\frac{i}{N}) - e(\frac{i+1}{N})\right) + \frac{R_1}{N^2}, \tag{5.1}$$

where the remainder term $|R_1| = |R_1(N)| \le \frac{\pi^2}{4\cdot 6} < 1$. Thus, introducing the operator $Gf(x) := f(0) - f(1-x)$, we rewrite the last equality as

$$(Ae)_k = Le(k/N) + \frac{\varepsilon}{2}Ge(k/N) + R_1\varepsilon^2.$$

On this step our aim is to show that there exists a function $g \in \mathbf{C}^1[0,1]$ orthogonal to e (which means that $\int g \cdot e = 0$) such that the following relation holds true:

$$(L + \frac{\varepsilon}{2}G)(e + \frac{\varepsilon}{2}g) = (\frac{2}{\pi} + \varepsilon\frac{1}{\pi})(e + \frac{\varepsilon}{2}g) + R_3\varepsilon^2, \tag{5.2}$$

where again $|R_3| \le 1$.

Since the operator L is symmetric the orthogonal complement to the function e is invariant with respect to L. Thus for some constant α and a bounded function $h \in \mathbf{C}^1$ independent on ε and orthogonal to e we have

$$Ge = \alpha e + h.$$

Let us calculate the constant α. Since h is orthogonal to e, then multiplying the both hands of the previous equality by e and integrating (observe that $\int_0^1 e \cdot h = 0$) we get

$$\int_0^1 e(x) \cdot (e(0) - e(1-x))\, dx = \alpha \int_0^1 e^2(x)\, dx.$$

On the other hand,

$$\int_0^1 \cos^2(\frac{\pi}{2}x)\, dx = \frac{1}{2}, \qquad \int_0^1 \cos(\frac{\pi}{2}x)\, dx = \frac{2}{\pi},$$

$$\int_0^1 \cos(\frac{\pi}{2}x)\ \cos(\frac{\pi}{2}(1-x))\ dx = \frac{1}{2}\int_0^1 \sin(\frac{\pi}{x}) = \frac{1}{\pi}.$$

Thus

$$(\frac{2}{\pi} - \frac{1}{\pi}) = \frac{1}{2}\alpha,$$

from where $\alpha = \frac{2}{\pi}$.

Therefore for any function $g \in \mathbf{C}^0$ we have

$$(L + \frac{\varepsilon}{2}G)(e + \frac{\varepsilon}{2}g) = Le + \frac{\varepsilon}{2}Lg + \frac{\varepsilon}{2}Ge + \frac{\varepsilon^2}{4}Gg = \left(\frac{2}{\pi} + \frac{\varepsilon}{2}\frac{2}{\pi}\right)\cdot e + \frac{\varepsilon}{2}(h + Lg) + \frac{\varepsilon^2}{4}Gg.$$

Comparing this relation with the equality (5.2) we come to the conclusion that $h + Lg = \frac{2}{\pi}g$ or $g := (L - \frac{2}{\pi})h$. Notice that the right hand side of the last expression makes sense since h is orthogonal to e. Thus

$$g(x) = (L - \frac{2}{\pi})^{-1}\left(e(0) - e(1-x) - \frac{1}{\pi}e(x)\right).$$

¿From the first two leading eigenvalues, we deduce that the norm of the operator $L - \frac{2}{\pi}$ restricted to the orthogonal complement to the leading eigenfunction can be estimated from above by $\frac{2}{\pi} - \frac{2}{3\pi} = \frac{4}{3\pi}$. Therefore

$$|g| \le \frac{4}{3\pi}(1 - \frac{1}{\pi}), \qquad \frac{1}{4}|Gg| \le \frac{3\pi(1 + \frac{1}{\pi})}{4\cdot 4} < 1.$$

Combining above estimates we deduce that there exist two vectors $v, \xi \in \mathbb{R}^N$ such that

$$Av = (\frac{2}{\pi}N + \frac{1}{\pi})v + \xi, \qquad |\xi| \le \frac{2}{N^2}|v|,$$

which yields the statement of Theorem by the following a'posteriori matrix perturbation argument [17]. Let the equality $Av = \mu v + \xi$ be satisfied for a symmetric matrix A, two vectors $v, \xi \in \mathbb{R}^n$ with $||\xi|| \le \varepsilon ||v||$ and a scalar μ. Then the closest to μ eigenvalue λ of the matrix A satisfies the inequality $|\lambda - \mu| \le \varepsilon$.

Indeed, for $\mu = \lambda$ the inequality becomes trivial, while otherwise

$$||v|| \le ||(A - \mu I)^{-1}|| \cdot ||(A - \mu I)v|| = \frac{1}{|\mu - \lambda|}\cdot ||\xi||.$$

Thus $|\mu - \lambda| \le \frac{||\xi||}{||v||} \le \varepsilon$. ∎

As we already mentioned this result corresponds to the steady states of our model for the case of initial regular configurations with 'low' traffic. Notice that in the opposite case of 'high' traffic $\rho > K/2$ each jammed configuration is in one-to-one correspondence with its dual one, for which $\rho < K/2$. Therefore the statement similar to Theorem 5.1 holds in this case as well, moreover

$$h_{\rm top}(T, \mathrm{Free}(K) \cup (\mathrm{Free}(K))^*) = h_{\rm top}(T, \mathrm{Free}(K)).$$

Observe that for any positive integer K the topological entropy for the map T is strictly positive, which yields the chaoticity of the map.

6 Statistics of typical configurations

In this section we shall derive statistical information about typical configurations of particles. For configurations $X \in \mathrm{Free}(K)$ let us denote by $S(n,K)$ the total number of different subconfigurations X_1^n of length $n \in \mathbb{Z}_+^1$.

Lemma 6.1 $S(n,K) = \lambda_{\max}^n(A(K+1)) + o(\lambda_{\max}^n(A(K+1)))$, *where* $A(N)$ *is* $N \times N$ *left triangular matrix.*

Proof. Denote by $S_i(n,K)$ the number of subconfigurations of length n consisting of only free particles and starting with the symbol $i \in \{0,1,\dots,K\}$. Then we have the following recurrence relation:

$$S_i(n+1,K) = \sum_{j=0}^{K-i} S_j(n,K).$$

The number of particles in each site of a configuration X may vary from 0 to K, i.e. it may admit $N := K+1$ different values, and the only additional relation that should be satisfied is

$$X(x) + X(x+1) \le K \quad \forall x \in \{1,2,\dots,n\}.$$

Therefore these configurations are completely described by $N \times N$ left triangular transition matrix $A = A(N)$. Thus we get $S(n,K) = \sum_{i=0}^{K} S_i(n,K)$, from where and from Theorem 5.2 the statement of Lemma follows. ∎

We shall say that a subconfiguration is *blocking (non-blocking)* if it contains (not contains) the symbol K. Then the number of non-blocking subconfigurations of length n is equal to $S(n,K-1)$. The fraction of blocking subconfigurations of length n is equal to

$$\frac{S(n,N) - S(n,N-1)}{S(n,N)} = 1 - \frac{S(n,N-1)}{S(n,N)}.$$

Applying the asymptotic representation for the leading eigenvalue of the matrix $A(N)$ we get the following estimate:

$$S(n,N) = \left(\frac{2}{\pi}N + \frac{1}{\pi} + o(\frac{1}{N})\right)^n.$$

Therefore

$$\frac{S(n,N-1)}{S(n,N)} = \left(\frac{\frac{2}{\pi}(N-1) + \frac{1}{\pi} + o(\frac{1}{N})}{\frac{2}{\pi}N + \frac{1}{\pi} + o(\frac{1}{N})}\right)^n = \left(1 - \frac{1}{N} + o(\frac{1}{N})\right)^n.$$

To derive a more deep information about the statistics and to be able to deal with periodic configurations one can use an approach based on Markov chain approximations.

For for a given integer $i \in \{0,1,\dots,K\}$ and a configuration $X \in \mathrm{Free}(K)$ denote by $\bar{\pi}_i(X_1^n)$ the fraction of sites $x \in \{1,2,\dots,n\}$ where $X(x) = i$, i.e.

$$\bar{\pi}_i(X_1^n) := \frac{1}{n}\ \#\{x \in \{1,\dots,n\}: \quad X(x) = i\},$$

while by $\bar{\pi}_i^{(n)}$ we denote the average of these fractions over all possible different subconfigurations of free particles of length n:

$$\bar{\pi}_i^{(n)} := \frac{\sum_{X \in \mathrm{Free}(X)} \bar{\pi}_i(X_1^n)}{S(n,K)}.$$

Lemma 6.2 $\bar{\pi}_i^{(n)} \to \frac{K+1-i}{K+1}\frac{2}{K+2}$ *as* $n \to \infty$.

Proof. Continuing the same argument as in the proof of Lemma 6.1 we see that each configuration can be considered as a realization of a Markov chain with $N = K+1$ states numbered as $0, 2, \ldots, K$ and the following transition probabilities:

$$p_{ij} := \begin{cases} \mathcal{P}\{X(x+1) = j \mid X(x) = i\} = \frac{1}{K-i+1} & \text{if } j \le K-i \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the N-th power P^N of the transition matrix $P = (p_{ij})$ is strictly positive and thus the Markov chain is ergodic. Denote by π_i, $i \in \{0, \ldots, K\}$ its stationary probabilities, i.e. the probability to have $X(x) = i$. Then these quantities should satisfy the following system of equalities:

$$\pi_i = \sum_{j=0}^{K-i} \frac{\pi_j}{K-j+1},$$

from where

$$\pi_i - \pi_{i+1} = \frac{\pi_{(K-i)}}{i+1}$$

for all $i = 0, 1, \ldots, K$. Solving the last system of difference equations we get

$$\pi_i = \frac{K+1-i}{K+1}\pi_0,$$

and eventually (since they sum up to 1) we come to

$$\pi_i = \frac{K+1-i}{K+1}\frac{2}{K+2}.$$

■

To study statistics of n-periodic configurations of free particles one should take into account that there is an additional constraint: $X(1) + X(n) \le N$. Denote by $\tilde{S}(n, N)$ the total number of subconfigurations of length n consisting of free particles and satisfying this constraint. Therefore the fraction of nonadmissible n-periodic configurations (which do not satisfy above constraint)

$$\sum_{i=0}^{K} \pi_i \sum_{j=K-i}^{K} \pi_j = \sum_{i=0}^{K} \pi_i \left(1 - \sum_{j=0}^{K-i} \pi_j\right)$$

is asymptotically (for large K) equal to

$$\int_0^1 \pi(x) \int_{1-x}^1 \pi(y)\ dy\ dx = \frac{1}{6},$$

where $\pi(x) := 2(1-x)$ corresponds to the limit (as $K \to \infty$) distribution.

Using these statistics one can easily obtain all correlation functions and to study large deviations. For example, we get that the fraction of blocking configurations in Free(K) is equal to $\pi_K = 2/((K+1)(K+2))$, which in terms of traffic estimates how often the road with K lanes is completely blocked by moving cars.

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