Phase Space Tunneling in Multistate Scattering

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Abstract

We prove exponentially small estimates on the off-diagonal terms of the scattering matrix associated to two-state semiclassical Schrödinger Hamiltonians. Our method is based on phase space tunneling estimates and

a splitting of the operator by means of Toeplitz-type phase space cut-off operators.

1 Introduction

During the last few years, various techniques and new tools have been introduced for the study of microlocal tunneling (i.e. tunneling in phase space): see e.g. [Ma2, Na1, Na2, Sj]. These techniques have permitted in various situations to obtain very accurate estimates on exponentially small quantities attached to microlocal tunneling effects, such as the semiclassical widths of resonances ([Ba, Ma2, Na3]), the adiabatic transition probabilities ([Ju, Ma3, MaNe]), the effect of magnetic fields ([MaSo, Na4]) and the off-diagonal coefficients of the scattering operator of multistate systems ([BeMa, Na2]).

Here we plan to generalize [Na2] to multidimensional two-state scattering systems (improving at the same time the result of [BeMa] by eliminating the regularizing weights appearing there). More precisely, we consider a two-state semiclassical Schrödinger Hamiltonian of the type $P(h) = -h^2\Delta_x + V(x) + hR(x;hD_x)$ on $\mathcal{H} = L^2(I\!R^n) \oplus L^2(I\!R^n)$, where V is a 2×2 matrix of multiplication operators with a gap between its two eigenvalues (and therefore $V(x)$ can actually be assumed to be diagonal without loss of generality). Under additional assumptions of decay at infinity, one can define the scattering matrix

$$S(\lambda) = \begin{pmatrix} S_{1,1}(\lambda) & S_{1,2}(\lambda) \\ S_{2,1}(\lambda) & S_{2,2}(\lambda) \end{pmatrix}$$

where e.g. λ is greater than the limit at infinity of the largest eigenvalue of $V(x)$. Then, depending on the regularity of V, we give estimates on $S_{1,2}(\lambda)$ and $S_{2,1}(\lambda)$ in the case where λ is a non-trapping energy. In particular, if V is C^∞ we show that (see Theorem 3.7):

$$\|S_{1,2}(\lambda)\| + \|S_{2,1}(\lambda)\| = \mathcal{O}(h^\infty)$$

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while if V admits a holomorphic extension in a complex strip $\Gamma = \{x \in \mathbb{C}^n \; ; \; |\operatorname{Im} x| < \gamma\}$ (for some $\gamma > 0$), then (see Theorem 4.5):

$$\|S_{1,2}(\lambda)\| + \|S_{2,1}(\lambda)\| = \mathcal{O}(e^{-\tau/h})$$

where the exponential rate $\tau > 0$ is related to the behavior in the complex phase space of the characteristic set of P (or more precisely of the complex extensions of the two connected components of the real characteristic set of P). Let us observe that related results are obtained in [NeSo] by using different techniques involving almost invariant subspaces of P.

Our approach is very similar in spirit to the one usually adopted for studying interactions between 2 potential wells (see e.g. [HeSj, Si]), and already used for the one-dimensional scattering case in [Na2]: it consists in splitting both the space and the operator in two parts (via identification operators in the same spirit as in [BCD]), in such a way that each part corresponds to a problem with a connected real

characteristic set. Then the interaction between these two connected sets appears as very small off-diagonal coefficients in the representation of P obtained in this way. However, since we want to keep some analyticity (when required), we have to be careful in cutting-off the operator, and this problem is solved by using Toeplitz operators of the type $T^*\chi T$ where T is a global FBI transform (see e.g. [Sj]) and χ is a cut-off function on the phase space. Such operators have the nice property of satisfying microlocal weight exponential a priori estimates (similar to those of [Ma1, Na1]), although their natural symbol χ is not analytic. Moreover, since this technique permits to cut the phase-space in a way essentially arbitrary, we believe it can be useful in many other problems involving microlocal tunneling.

In Section 2, we prepare basic notations and introduce our assumptions on $P(h)$. In Section 3, we consider the case when the symbol is C^∞-smooth, and introduce scattering theoretical machinery, which will be also used in the next section.In Section 4, we study the case when the symbol is analytic, and prove our main result, namely the exponential decay of the off-diagonal terms of the scattering matrix. We discuss the calculus of cut-off operators in phase space in terms of Toeplitz operators in Appendix.

2 Notations and Assumptions

We consider the two-state Schrödinger Hamiltonian

$$P(h) = -h^2\Delta_x + V(x) + hR(x; hD_x) \tag{2.1}$$

on $\mathcal{H} = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. Here

$$V(x) := \begin{pmatrix} V_1(x) & 0 \\ 0 & V_2(x) \end{pmatrix}$$

is a 2×2 diagonal symmetric matrix and

$$R(x; hD_x) = \sum_{|\alpha| \le 1} c_\alpha(x)(hD_x)^\alpha$$

is a differential operator of order 1. Moreover, $V(x)$ and $c_\alpha(x)$ are real-valued smooth functions on $\mathbb{R}^n$ and there exist $\rho > 1$ and a real symmetric matrix

$$V_\infty := \begin{pmatrix} V_1^\infty & 0 \\ 0 & V_2^\infty \end{pmatrix}$$

such that for all multi-index β:

$$|\partial_x^\beta (V(x) - V_\infty)| + \sum_{|\alpha|\leq 1} |\partial_x^\beta c_\alpha(x)| = \mathcal{O}(\langle x\rangle^{-\rho-|\beta|}) \tag{2.2}$$

uniformly on $\mathbb{R}^n$. Setting

$$P_0 = \begin{pmatrix} P_1^0 & 0 \\ 0 & P_2^0 \end{pmatrix} := -h^2\Delta_x + V_\infty$$

we can prove as in the scalar case that the wave operators:

$$W_\pm(P, P_0) \quad = \quad \mathrm{s}-\lim_{t\to\pm\infty} e^{itP}e^{-itP_0}$$

exist and are complete. Hence, we can define the scattering operator

$$S = S(P, P_0) =: W_+(P, P_0)^* W_-(P, P_0)$$

which is a unitary operator on $\mathcal{H}$.
In the following, we work near an energy level $E > \mathrm{Max}\{V_1^\infty, V_2^\infty\}$ and we assume that E is non-trapping, that is, denoting by $q_j(x,\xi) = \xi^2 + V_j(x)$ $(j = 1, 2)$, we assume that for any $(x,\xi) \in \mathbb{R}^{2n}$ such that $q_j(x,\xi) = E$ one has:

$$|\exp tH_{q_j}(x,\xi)| \to \infty \quad \text{as } |t| \to \infty \tag{2.3}$$

for $j = 1, 2$. Here H_{q_j} is the Hamiltonian flow generated by q_j.
Let I be a small neighborhood of E. Since S commutes with P_0 then, for $\lambda \in I$, one can define the scattering matrix

$$S(\lambda) = \begin{pmatrix} S_{1,1}(\lambda) & S_{1,2}(\lambda) \\ S_{2,1}(\lambda) & S_{2,2}(\lambda) \end{pmatrix} \in \mathcal{L}(L^2(S^{n-1}) \oplus L^2(S^{n-1}))$$

such that, for any $\phi \in \mathcal{H}$ and $\lambda \in I$

$$F_0(\lambda) S\phi = S(\lambda) F_0(\lambda)\phi,$$

where $F_0(\lambda)$ stands for the spectral representation of P_0. We denote

$$\Sigma_1(E) \quad = \quad \{(x,\xi) \in \mathbb{R}^{2n}\ ;\ \xi^2 + V_1(x) - E = 0\}, \tag{2.4}$$

$$\Sigma_2(E) \quad = \quad \{(x,\xi) \in \mathbb{R}^{2n}\ ;\ \xi^2 + V_2(x) - E = 0\} \tag{2.5}$$

and we assume

$$\mathrm{dist}(\Sigma_1(E), \Sigma_2(E)) > 0. \tag{2.6}$$

In particular $V_1^\infty \neq V_2^\infty$, and $V_1(x)$ and $V_2(x)$ are never equal on $\mathcal{A}_E := \{x\ ;\ V_1(x) \leq E\ ,\ V_2(x) \leq E\}$. In fact, let us prove:

Lemma 2.1 *Assume (2.2), (2.3) and $\Sigma_1(E)\cap\Sigma_2(E)=\emptyset$. Then the difference $V_2(x)-V_1(x)$ keeps a constant sign on $\mathcal{A}_E$ for all $E>\mathrm{Max}\{V_1^\infty, V_2^\infty\}$.*

Proof - For $j=1,2$ denote $M_j:=\{x\in I\!R^n\ ;\ V_j(x)\leq E\}$. Then by (2.2) M_j contains a neighborhood of infinity, and by (2.3) it has no bounded connected component. Moreover V_2-V_1 never vanishes on M_j, otherwise the point where it does would belong to $M_1\cap M_2=\mathcal{A}_E$. As a consequence, since $V_2-V_1\to V_2^\infty-V_1^\infty\neq 0$ at infinity, it keeps a constant sign on each M_j and thus also on A_E. $\diamond$

Now, assuming, e.g., that $V_2^\infty>V_1^\infty$, Lemma 2.1 implies that $V_2>V_1$ on $\mathcal{A}_E$, and Assumption (2.6) is actually equivalent to:

$$\inf_{x\in\mathcal{A}_E}(V_2(x)-V_1(x))\ >\ 0. \tag{2.7}$$

Then, we also set

$$\begin{aligned}\Sigma_j^+(E) &= \{(x,\xi)\in I\!R^{2n}\ ;\ \xi^2+V_j(x)-E>0\}, &(2.8)\\ \Sigma_j^-(E) &= \{(x,\xi)\in I\!R^{2n}\ ;\ \xi^2+V_j(x)-E<0\}, &(2.9)\\ \Sigma(E) &= \Sigma_1^-(E)\cap\Sigma_2^+(E). &(2.10)\end{aligned}$$

3 Estimates in the Smooth Case

Let us define two functions $\chi^+,\chi^-\in C^\infty$ by:

$$\chi^+(x,\xi)=\chi_0^+(\xi^2+V_2-E),$$

$$\chi^-(x,\xi)=\chi_0^-(\xi^2+V_1-E),$$

where $\chi_0^\pm\in C^\infty(I\!R\ ;\ [0,1])$ is such that

$$\begin{aligned}&\chi_0^+(s)=1\quad\text{if}\ \ s\geq 2\delta,\\ &\chi_0^+(s)=0\quad\text{if}\ \ s\leq\delta,\\ &\chi_0^-(s)=\chi_0^+(-s),\end{aligned}$$

where $\delta>0$ is fixed small enough (and will possibly be shrinked a finite number of times in the sequels).

Given a real $n\times n$ symmetric positive definite matrix A, we denote $q_A(x)=\langle Ax,x\rangle$ and we consider the following global FBI transform:

$$T_Au(x,\xi)=2^{-n/2}(\pi h)^{-3n/4}(\mathrm{det}A)^{1/4}\int e^{i(x-y)\xi/h-q_A(x-y)/2h}u(y)\,dy$$

which is an isometry from $L^2(I\!R^n)$ to $L^2(I\!R^{2n})$ (see e.g. [Ma1]). Then we set:

$$\begin{aligned}P^+ &= P+T_A^*(1-\chi^+)(L-p)T_A &(3.1)\\ P^- &= P+T_A^*(1-\chi^-)(L-p)T_A &(3.2)\end{aligned}$$

where $p = \xi^2 + V$ is the principal symbol of P and $L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$ with $L_1, L_2 \in I\!R$, $L_1 < E < L_2$. Let us observe that if a is a symbol then, denoting Op_h^W the usual semiclassical Weyl quantization of symbols, we have:

$$T_A^* a T_A = \mathrm{Op}_h^W(b) \tag{3.3}$$

with

$$b(x,\xi) = \frac{1}{(2\pi h)^n} \int e^{-q_A(x-y)/2h - q_{A^{-1}}(\xi-\eta)/2h} a(y,\eta) dy d\eta \sim \sum_{j=0}^{\infty} \frac{h^j}{4^j j!} \Delta_A^j a(x,\xi) \tag{3.4}$$

where $\Delta_A := \langle A^{-1}\partial_x, \partial_x\rangle + \langle A\partial_\xi, \partial_\xi\rangle$ (this can be seen by a direct computation). In particular, $P^\pm$ is a pseudodifferential operator and if $p^\pm = \chi^\pm p + (1-\chi^\pm)L$ denotes its principal symbol we see:

$$\det(p^+(x,\xi) - E) = 0 \qquad \text{iff} \quad \xi^2 + V_1(x) = E \tag{3.5}$$

$$\det(p^-(x,\xi) - E) = 0 \qquad \text{iff} \quad \xi^2 + V_2(x) = E. \tag{3.6}$$

Now the main idea of our proof consists in comparing P with the operator

$$Q = P^+ \oplus P^- \tag{3.7}$$

acting on $\mathcal{H}\oplus\mathcal{H}$. For this purpose we introduce two additional functions $j^+, j^- \in C^\infty(I\!R^{2n})$ such that $\mathrm{Supp}\, j^+ \subset \{\xi^2 + V_2(x) - E \geq 3\delta\}$, $j^+ = 1$ on $\{\xi^2 + V_2(x) - E \geq 4\delta\}$, $\mathrm{Supp}\, j^- \subset \{\xi^2 + V_1(x) - E \leq -3\delta\}$, $j^- = 1$ on $\{\xi^2 + V_1(x) - E \leq -4\delta\}$. We can also assume that $j^\pm$ does not depend on x for $|x|$ large enough, and we observe that $\mathrm{Supp}\,(1-\chi^\pm) \cap \mathrm{Supp}\, j^\pm = \emptyset$.

We also set $\hat{J}^\pm(x, hD_x) = T_A^* j^\pm T_A$ and we define

$$J : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$$

$$J\phi = (\hat{J}^+\phi) \oplus (\hat{J}^-\phi)$$

which will play the role of an identification operator (observe that J preserves the H^2-Sobolev regularity). Now if I is a sufficiently small interval containing E and $f \in C_0^\infty(I)$, then

$$\mathrm{Supp}\,(f(p^\pm)) \subset\subset \{j^\pm = 1\}.$$

Therefore, it follows from the functional calculus of pseudodifferential
operators (see e.g. [DiSj, Ma1, Ro]) that:

$$f(P^\pm)\hat{J}^\pm = f(P^\pm) + R^\pm \tag{3.8}$$

with

$$R^\pm \in OPS(h^\infty \langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty}) \tag{3.9}$$

where $OPS(h^\infty \langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty})$ denotes the space of semiclassical pseudodifferential operators with symbol bounded (together with all its derivatives) by $\mathcal{O}(h^N\langle x\rangle^{-N}\langle\xi\rangle^{-N})$ for all $N > 0$. More generally, for any given function $m \in C^\infty(I\!R^{2n}; I\!R_+)$, we denote $OPS(m)$ the space of semiclassical pseudodifferential operators with (possibly h-dependent) symbol $a \in C^\infty(I\!R^{2n})$ satisfying:

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) = \mathcal{O}(m(x,\xi)\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}) \tag{3.10}$$

uniformly with respect to $(x,\xi) \in I\!\!R^{2n}$ and $h>0$ small enough (the corresponding space of these symbols will be denoted by $S_{2n}(m)$). Then we have the following result:

Lemma 3.1 *Let $F \in OPS(\langle x\rangle^k \langle \xi\rangle^\ell)$ with $k,\ell \in I\!\!R$ arbitrary. Then*

$$T_A^*(1-\chi^\pm)(L-p)T_A F\hat{J}^\pm \in OPS(h^\infty \langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty}).$$

Proof: Applying the property (3.3)-(3.4) to $a=j^\pm$ and to $a=(1-\chi^\pm)(L-p)$ (which have disjoint supports), the result becomes an easy consequence of standard pseudodifferential calculus with symbols in the classes defined by (3.10). $\diamond$

In the sequels, we also set

$$M_1 = QJ - JP_0, \qquad M_2 = PJ^* - J^*Q \tag{3.11}$$

and we let I be any small interval around E such that the assumptions (2.3) and (2.6) remain valid for any energy in I. Moreover, in all the sequels δ is supposed to be sufficiently small in order that (3.5) and (3.6)
are also valid for any energy in I.

Now, with Q defined in (3.7), we first study the scattering for the pair (Q,P_0) and we show:

Proposition 3.2 *The wave operators*

$$W_\pm^1 E_I(P_0) = W_\pm(Q,P_0;J)E_I(P_0) := \mathrm{s}-\lim_{t\to\pm\infty} e^{itQ}Je^{-itP_0}E_I(P_0)$$

exist and are complete (i.e $\mathrm{Ran}W_\pm^1 E_I(P_0) = E_I(Q)(\mathcal{H})$ where, for any selfadjoint operator A, $E_I(A)$ denotes the spectral projection of A on I). Moreover, they are partial isometries with initial space $E_I(P_0)$, that is:

$$\|W_\pm^1 E_I(P_0)\phi\| = \|E_I(P_0)\phi\|$$

Proof: Following [Na2], we first show that

$$M_1 = M^+ \oplus M^- \in OPS(\langle x\rangle^{-\rho}\langle\xi\rangle^2)$$

Here we have

$$M^\pm = P^\pm \hat{J}^\pm - \hat{J}^\pm P_0 = P\hat{J}^\pm - \hat{J}^\pm P_0 + R_1$$

with $R_1 = (P^\pm - P)\hat{J}^\pm = T_A^*(1-\chi^\pm)(L-p)\hat{J}^\pm$. Hence by Lemma 3.1, $R_1 \in OPS(h^\infty\langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty})$. On the other hand, since $j^\pm = j_0^\pm(\xi)$ does not depend on x for $|(x,\xi)|$ large enough, we have:

$$P\hat{J}^\pm - \hat{J}^\pm P_0 = PJ_0^\pm(hD_x) - J_0^\pm(hD_x)P_0 + R_2$$

with $R_2 \in OPS(\langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty})$ and thus

$$P\hat{J}^\pm - \hat{J}^\pm P_0 = (V-V^\infty)J_0^\pm(hD_x) + hR(x,hD_x)J_0^\pm(hD_x) + R_2 \in OPS(\langle x\rangle^{-\rho}\langle\xi\rangle^2)$$

This implies that

$$M^{\pm} \in OPS(h^2\langle x\rangle^{-\rho}\langle\xi\rangle^2)$$

and the existence of the wave operators follows by the standard Cook-Kuroda method. Their asymptotic completeness is a consequence of the limiting absorption principle for P, which in turns follows from the existence of a global escape function for ξ^2+V_1 and ξ^2+V_2 at energy E and from Mourre estimates as in [GeMa]. $\diamond$

Now we can define the following scattering operator for the pair (Q,P_0):

$$S_1E_I(P_0) = S(Q,P_0;J)E_I(P_0) = (W_+(Q,P_0;J)E_I(P_0))^*(W_-(Q,P_0;J)E_I(P_0))$$

and we have:

Proposition 3.3

$$S_1E_I(P_0) =: \begin{pmatrix} S_1^1 & 0 \\ 0 & S_2^1 \end{pmatrix} E_I(P_0)$$

Proof: For any $\varphi_j \in \mathcal{S}(I\!R^n)$ such that $0 \notin \operatorname{Supp}\hat{\varphi}_j$, one has

$$\langle W_-^1 f(P_0)\begin{pmatrix}\varphi_1\\0\end{pmatrix}, W_+^1 f(P_0)\begin{pmatrix}0\\\varphi_2\end{pmatrix}\rangle = \lim_{t\to-\infty}\lim_{s\to+\infty}\langle A(t),B(s)\rangle$$

with:

$$\begin{aligned} A(t) &= e^{itP^+}\begin{pmatrix}\hat{J}^+ f(P_1^0)e^{-itP_1^0}\varphi_1\\0\end{pmatrix} \oplus e^{itP^-}\begin{pmatrix}\hat{J}^- f(P_1^0)e^{-itP_1^0}\varphi_1\\0\end{pmatrix}\\ B(s) &= e^{isP^+}\begin{pmatrix}0\\\hat{J}^+ f(P_2^0)e^{-isP_2^0}\varphi_2\end{pmatrix} \oplus e^{isP^-}\begin{pmatrix}0\\\hat{J}^- f(P_2^0)e^{-isP_2^0}\varphi_2\end{pmatrix}. \end{aligned}$$

Now, since for N large enough one has

$$\operatorname{Supp} j^+ \cap \{|x|\geq N\} \cap \operatorname{Supp} f(\xi^2+V_2^\infty) = \emptyset$$

we see that

$$\|\hat{J}^+ f(P_2^0)e^{-isP_2^0}\varphi_2\| \to 0 \quad (s\to-\infty)$$

and in the same way:

$$\|\hat{J}^- f(P_1^0)e^{-itP_1^0}\varphi_1\| \to 0 \quad (t\to+\infty).$$

As a consequence:

$$\langle W_-^1 f(P_0)\begin{pmatrix}\varphi_1\\0\end{pmatrix}, W_+^1 f(P_0)\begin{pmatrix}0\\\varphi_2\end{pmatrix}\rangle = 0.$$

$\diamond$

It remains to study the scattering for the pair (P,Q). We have:

Proposition 3.4 *The wave operators*

$$W_\pm^2 E_I(Q) = W_\pm(P,Q;J^*)E_I := \mathrm{s}-\lim_{t\to\pm\infty} e^{itP}J^*e^{-itQ}E_I^{ac}(Q)$$

exist and are complete i.e $\mathrm{Ran}W_\pm^2 E_I(Q) = E_I(P)(\mathcal{H})$.

Proof: By (3.11) we have

$$M_2(\varphi_1 \oplus \varphi_2) = M_2^+\varphi_1 + M_2^-\varphi_2$$

where

$$M_2^\pm = [P, \hat{J}^\pm] + R^\pm \tag{3.12}$$

with

$$R^\pm = \hat{J}^\pm T_A^*(1-\chi^\pm)(p-L)T_A$$

Hence, $R^\pm \in OPS(h^\infty\langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty})$ and thus

$$M_2 \in OPS(\langle x\rangle^{-\rho}\langle\xi\rangle).$$

Then the Cook-Kuroda method gives the existence of $W_\pm^2 E_I(Q)$. For the asymptotic completeness, we first observe:

$$W_\pm^2 E_I(Q)W_\pm^1 E_I(P_0) = \mathrm{s}-\lim_{t\to\pm\infty} e^{itP}J^*Je^{-itP_0}E_I(P_0)$$

$$andthesymbolof$$

J^*J is given by $j^+(x,\xi)^2 + j^-(x,\xi)^2 = 1 + k(x,\xi)$ where k is supported in $\Sigma(E)$. In particular, if I has been chosen small enough then $\mathrm{Supp}\, k \cap \bigcup_{\lambda\in I}\{\det(p-\lambda) = 0\} = \emptyset$. As a consequence, $(1-J^*J)E_I(P_0)$ is a compact operator and since $e^{-itP_0}\varphi \to 0$ weakly as $|t|\to\infty$ for any $\varphi\in\mathcal{H}$ we get:

$$W_\pm^2 E_I(Q)W_\pm^1 E_I(P_0) = \mathrm{s}-\lim_{t\to\pm\infty} e^{itP}e^{-itP_0}E_I(P_0)$$

$= \mathrm{W}_\pm(P,P_0)E_I(P_0) andthereforetheasymptoticcompletenessof \mathrm{W}_\pm^2 E_I(Q)$ follows from the one of $W_\pm(P,P_0)E_I(P_0)$. $\diamond$

Let us denote by $S_2 = S(P,Q,J^*) = (W_+^2)^*W_-^2$ the scattering operator and by $S_2(\lambda)$ the scattering matrix associated to the pair of operators (P,Q). If $F_0(\lambda)$ is the spectral representation of P_0, we set

$$F_1(\lambda) = F_0(\lambda)(W_+^1)^*$$

for $\lambda\in I$. Then, denoting $\mathcal{H}^\alpha = L^{2,\alpha}(I\!R^n)\oplus L^{2,\alpha}(I\!R^n)$ with

$L^{2,\alpha}(I\!R^n) := \langle x\rangle^{-\alpha}L^2(I\!R^n)$, we see that $F_1(\lambda)$ is a spectral representation of Q on $\mathrm{Ran}E_I(Q)$. Moreover $F_1(\lambda)$ is bounded from $\mathcal{H}^\alpha\oplus\mathcal{H}^\alpha$ to $[L^2(S^{n-1})\oplus L^2(S^{n-1})]^2$ for any $\alpha > 1/2$ and its operator norm is bounded by $c\,h^{-2}$ uniformly for $\lambda\in I$ (this can be seen as in [JeNa] by using a stationary representation of W_1^+ and a semiclassical resolvent estimate for Q, see also [Is]). Then we can apply the representation formula for the scattering matrix to obtain:

$$S_2(\lambda) = 1 - 2\pi i F_1(\lambda)[M_2^*J^* - M_2^*(P-\lambda-i0)^{-1}M_2]F_1(\lambda)^* \tag{3.13}$$

and in order to estimate it we first prove:

Lemma 3.5 *If I is a sufficiently small neighborhood of E and $f \in C_0^\infty(I)$ then $M_2 f(Q) \in OPS(h^\infty \langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty})$ and, in particular,*

$$\|\langle x\rangle^N M_2 f(Q)\langle x\rangle^N\| \leq Ch^N$$

for any $N \geq 0$.

Proof: Since

$$\mathrm{Supp}\,(f(p^\pm)) \cap \mathrm{Supp}\,(\nabla j_\pm) = \emptyset$$

one can easily see (e.g. by using (3.3) and (3.4)) that:

$$[P, \hat{J}_\pm]f(P^\pm) \in OPS(h^\infty\langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty})$$

and thus by (3.12):

$$M_2 f(Q) \in OPS(h^\infty\langle x\rangle^{-\infty}\langle\xi\rangle^{-\infty})$$

$\diamond$

Now the key result of this section is:

Proposition 3.6 *If I is a sufficiently small neighborhood of E, then for any $N \geq 0$ there exists $C_N > 0$ such that:*

$$\|S_2(\lambda) - I\|_{\mathcal{L}(L^2(S^{n-1}\oplus S^{n-1}))} \leq Ch^N$$

uniformly for $h > 0$ small enough.

Proof: We have

$$\begin{aligned}
&(S_2(\lambda) - I)f(\lambda)^2 \\
&\quad = \quad 2\pi i F_1(\lambda)[f(Q)M_2^* J^* f(Q) - f(Q)M_2^*(P - \lambda - i0)^{-1} M_2 f(Q)]F_1(\lambda)^* \\
&\quad = \quad 2\pi i F_1(\lambda)[(M_2 f(Q))^* J^* f(Q) - (M_2 f(Q))^*(P - \lambda - i0)^{-1} M_2 f(Q)]F_1(\lambda)^*
\end{aligned}$$

and since E is non-trapping, we have the resolvent estimates

$$\|\langle x\rangle^{-\rho/2}(P - \lambda \pm i0)^{-1}\langle x\rangle^{-\rho/2}\| = \mathcal{O}(h^{-1})$$

for $\lambda \in I$. Therefore, the result follows by using Lemma 3.5. $\diamond$

Now we can state and prove the main result of this section:

Theorem 3.7 *Under assumptions (2.2), (2.3) and (2.6), let I be a sufficiently small interval around E. Then, for any $N \geq 0$ there exists $C_N > 0$ such that:*

$$\|S_{1,2}(\lambda)\|_{\mathcal{L}(L^2(S^{n-1}))} + \|S_{2,1}(\lambda)\|_{\mathcal{L}(L^2(S^{n-1}))} \leq Ch^N$$

uniformly for $h > 0$ small enough and $\lambda \in I$.

Proof: We have

$$\begin{aligned}
SE_I(P_0) &= (W_+ E_I(P_0))^* W_- E_I(P_0) \\
&= (W_+^1 E_I(P_0))^*(W_+^2 E_I(Q))^*(W_-^2 E_I(Q))(W_-^1 E_I(P_0)) \\
&= (W_+^1 E_I(P_0))^*(S_2 E_I(Q))(W_-^1 E_I(P_0)) \\
&= (W_+^1 E_I(P_0))^*(S_2 - 1)E_I(Q)(W_-^1 E_I(P_0)) + S_1 E_I(P_0).
\end{aligned}$$

By Proposition 3.6, the first term is $\mathcal{O}(h^N)$ and, by Proposition 3.3, the off-diagonal term of S_1 are 0. $\diamond$

4 Analytic Case: Exponential Decay

Here we add the following assumption on V and the c_α's:

$$\begin{array}{l} V(x) \text{ and } c_\alpha(x)\ (|\alpha|\leq 1) \text{ admit holomorphic extensions in the} \\ \text{complex strip } \Gamma = \{x \in \mathbb{C}^n\ ;\ |\operatorname{Im} x| < \gamma\} \text{ for some } \gamma > 0, \\ \text{and (2.2) holds uniformly on } \Gamma. \end{array} \tag{4.1}$$

For any given connected open set $\Omega \subset \mathbb{R}^{2n}$, let us consider a function $\psi \in C_b^\infty(\mathbb{R}^{2n})$ real valued satisfying:

$$|\nabla_\xi \psi(x,\xi)| \leq \gamma \tag{4.2}$$

and

$$\operatorname{Supp} \nabla\psi \cap \Omega = \emptyset \tag{4.3}$$

In particular, ψ is constant on Ω, and, denoting by ψ_0
this constant value, we assume that there exists $\nu > 0$ such that

$$|\psi(x,\xi) - \psi_0| \leq \frac{1-\nu}{4} \mathrm{d}_A((x,\xi),\Omega) \tag{4.4}$$

for any $(x,\xi) \in \mathbb{R}^{2n}$, where d_A denotes the distance on $\mathbb{R}^{2n}$ associated with the metric:

$$Q_A(x,\xi) = \langle Ax, x\rangle + \langle A^{-1}\xi, \xi\rangle.$$

In the following we denote by $\mathcal{M}_\Omega$ the set of such functions ψ which satisfy (4.2), (4.3) and (4.4).
In relation with microlocal weight estimates (see appendix) we also set

$$p_{\psi,A}(x,\xi) := p(x - A^{-1/2}\partial_A\psi, \xi + iA^{1/2}\partial_A\psi)$$

where $\partial_A := A^{-1/2}\partial_x + iA^{1/2}\partial_\xi$, and we denote $\mathcal{A}_A$ the set of pairs of functions $(\phi_1,\phi_2) \in \left(\mathcal{M}_{\Sigma_1^+(E)} \cap \mathcal{M}_{\Sigma_2^-(E)}\right)^2$ such that:

$$\begin{array}{l} \operatorname{Supp}\phi_1 \subset \Sigma_1^-(E)\ ;\ \operatorname{Supp}\phi_2 \subset \Sigma_2^+(E)\ ; \\ \operatorname{Supp}\nabla\phi_1 \cup \operatorname{Supp}\nabla\phi_2 \subset \Sigma(E)\ ; \\ |\det(p_{\phi_1,A}(x,\xi) - E)| \cdot |\det(p_{\phi_2,A}(x,\xi) - E)| > 0 \text{ for all } (x,\xi) \in \Sigma(E). \end{array}$$

In particular ϕ_1 and ϕ_2 are constant on $\Sigma_2^-(E)$ and on $\Sigma_1^+(E)$ respectively, and we set:

$$\tau_A = \sup_{(\phi_1,\phi_2)\in\mathcal{A}_A} \operatorname{Min}\left\{ \phi_1\big|_{\Sigma_2^-(E)}, \phi_2\big|_{\Sigma_1^+(E)} \right\}.$$

One can easily see that $\tau_A > 0$: indeed, a possible choice for (ϕ_1,ϕ_2) consists in taking $\phi_j = f_j(\xi^2 + W(x) - E)$ with $V_1 < W < V_2$, $\operatorname{Supp} f_1 \subset [-\varepsilon, +\infty)$ with $\varepsilon > 0$ small enough, $f_1 > 0$ constant on $[\varepsilon, +\infty)$, $f_1' > 0$ small enough on $(-\varepsilon,\varepsilon)$, and $f_2(s) = f_1(-s)$ for all $s \in \mathbb{R}$ (with this choice one has $\phi_1\big|_{\Sigma_2^-(E)} = \phi_2\big|_{\Sigma_1^+(E)} = f_1(\varepsilon) > 0$).

Finally we set

$$\tau_0 = \sup_A \tau_A$$

where A runs over the set of all $n \times n$ real symmetric positive definite matrices. Then for any $\varepsilon_1 > 0$ there exist A and $(\phi_1, \phi_2) \in \mathcal{A}_A$ such that, denoting

$$\tau_1 := \phi_1\big|_{\Sigma_2^-(E)} \;\; ; \;\; \tau_2 := \phi_2\big|_{\Sigma_1^+(E)}$$

we have:

$$\tau_1 \geq \tau_0 - \varepsilon_1 \text{ and } \tau_2 \geq \tau_0 - \varepsilon_1.$$

Moreover, since $\nabla\phi_1$ and $\nabla\phi_2$ are supported in $\Sigma(E)$, if we take the value of δ (used in the constructions of the previous section) sufficiently small, we see that:

$$\begin{aligned}
&\mathrm{Supp}\,(\nabla\phi_1) \cap \mathrm{Supp}\,(\nabla j^{\pm}) = \mathrm{Supp}\,(\phi_1) \cap \mathrm{Supp}\,(1 - j^-) = \emptyset \\
&\mathrm{Supp}\,(\nabla\phi_2) \cap \mathrm{Supp}\,(\nabla j^{\pm}) = \mathrm{Supp}\,(\phi_2) \cap \mathrm{Supp}\,(1 - j^+) = \emptyset
\end{aligned}$$

and for some open connected neighborhood $\Omega^{\pm}$ of $\mathrm{Supp}\,(1-j^{\pm})$:

$$\phi_j \in \mathcal{M}_{\Omega^+} \cap \mathcal{M}_{\Omega^-} \;\; (j = 1, 2).$$

(In particular, the ϕ_j's satisfy the conditions of the appendix with $\chi = 1 - j^{\pm}$ and with $\chi = 1 - \chi^{\pm}$.) Moreover, by construction $|\det(p_{\phi_j,A} - E)| > 0$ on $\Sigma(E)$. Then we have:

Proposition 4.1 *If $u \in \mathcal{H}^{-\rho/2}$ is a generalized eigenfunction of P^+ with eigenvalue $\lambda \in I$, then for any $N \geq 0$:*

$$\|\langle(x,\xi)\rangle^N \chi^- T_A u\| = \mathcal{O}(h^N)\|\langle x\rangle^{-\rho/2} u\|.$$

Proof: It is an immediate consequence of the pseudodifferential calculus with symbols in the classes defined by (3.10): just write $u = f(P^+)u$ with f supported in an arbitrarily small neighborhood of λ, and observe that in this case χ^- and the symbol of $f(P^+)$ have disjoint supports. ⋄

Proposition 4.2 *Let m be as in (A.4). Then there exist two positive constants c_m and C_m such that for all $v \in \mathcal{S}(I\!R^n)$ one has:*

$$\|m e^{\phi_1/h} T_A (P^+ - \lambda) v\| \geq c_m \|m\langle\xi\rangle^2 e^{\phi_1/h} T_A v\| - C_m \|m\langle\xi\rangle^2 T_A v\|.$$

Proof: Using Corollary A.2 of the appendix (with $\chi = 1 - \chi^+$) and the fact that $\phi_1 = 0$ on $\mathrm{Supp}\,(1 - \chi_-)$, we can write:

$$\begin{aligned}
&\|m e^{\phi_1/h} T_A (P^+ - \lambda) v\| \\
&= \|m\left((p_{\phi_1,A} - \lambda)\chi^+ + (L - \lambda)(1 - \chi^+)\right) e^{\phi_1/h} T_A v\| \\
&\qquad\qquad\qquad\qquad + \mathcal{O}(\sqrt{h})\|m\langle\xi\rangle^2 e^{\phi_1/h} T_A v\| \\
&= \|m\left((p_{\phi_1,A} - \lambda)\chi^+ + (L - \lambda)(1 - \chi^+)\right) \chi^- e^{\phi_1/h} T_A v\| + \mathcal{O}(\|m\langle\xi\rangle^2 T_A v\|) \\
&\qquad\qquad\qquad\qquad + \mathcal{O}(\sqrt{h})\|m\langle\xi\rangle^2 e^{\phi_1/h} T_A v\| \\
&\geq (c + \mathcal{O}(\sqrt{h}))\|m\langle\xi\rangle^2 \chi^- e^{\phi_1/h} T_A v\| + \mathcal{O}(\|m\langle\xi\rangle^2 T_A v\|) \\
&\geq c_1 \|m\langle\xi\rangle^2 e^{\phi_1/h} T_A v\| + \mathcal{O}(\|m\langle\xi\rangle^2 T_A v\|)
\end{aligned}$$

uniformly for $h > 0$ small enough. $\diamond$

Of course, by a density argument Proposition 4.2 can be extended to any $v \in \mathcal{S}'(I\!R^n)$ such that $m\langle\xi\rangle^2 Tv \in L^2(I\!R^{2n})$.

Now, writing

$$\mathcal{F}_1(\lambda) =: \mathcal{F}_1^+(\lambda) \oplus \mathcal{F}_1^-(\lambda)$$

for the spectral representation of $Q = P^+ \oplus P^-$, we have:

Proposition 4.3 *For any $N \geq 0$ and any $\varepsilon > 0$ there exists $C = C(I, N, \varepsilon) > 0$ such that for all $\lambda \in I$:*

$$\|\langle(x,\xi)\rangle^N \chi^- e^{\phi_1/h} T_A \mathcal{F}_1^+(\lambda)^*\| \leq Ce^{\varepsilon/h}.$$

Proof: Let $\varphi \in (C_0^\infty(I\!R^n))^2$ be arbitrary. Then, applying Proposition 4.2 to $v = \mathcal{F}_1^+(\lambda)^*\varphi \in \mathcal{H}^{-\rho/2}$ and with $m = \langle x\rangle^{-\rho/2}\langle\xi\rangle^2$ one gets (since $P^+v = \lambda v$):

$$\|\langle x\rangle^{-\rho/2} e^{\phi_1/h} T_A \mathcal{F}_1^+(\lambda)^*\varphi\| \leq C\|\langle x\rangle^{-\rho/2} T_A \mathcal{F}_1(\lambda)^*\varphi\|$$

and thus, since $\langle x\rangle^{-\rho/2}\mathcal{F}_1^+(\lambda)^*$ and $\langle x\rangle^{-\rho/2} T_A \langle x\rangle^{\rho/2}$ are uniformly bounded operators:

$$\|\langle x\rangle^{-\rho/2} e^{\phi_1/h} T_A \mathcal{F}_1^+(\lambda)^*\| = \mathcal{O}(1). \tag{4.5}$$

In particular $\langle x\rangle^{-\rho/2} e^{\phi_1/h}\chi^- T_A \mathcal{F}_1^+(\lambda)^*$ is a uniformly bounded operator. On the other hand, we have by Proposition 4.1:

$$\|\langle(x,\xi)\rangle^{tN} e^{\phi_1/h}\chi^- T_A \mathcal{F}_1^+(\lambda)^*\| = \mathcal{O}(h^N e^{F/h})$$

with $F = \mathrm{Sup}\,\phi_1$. By interpolation with (4.5) we obtain for any $t \in [0,1]$ and any $N > 0$:

$$\|\langle(x,\xi)\rangle^{tN} \langle x\rangle^{-(1-t)\rho/2} e^{\phi_1/h}\chi^- T_A \mathcal{F}_1^+(\lambda)^*\| \leq C' h^{tN} e^{tF/h}$$

with a new positive constant C'. Since t can be taken arbitrarily small, the result follows. $\diamond$

In the same way, we obtain:

$$\|\langle(x,\xi)\rangle^N \chi^+ e^{\phi_2/h} T_A \mathcal{F}_1^-(\lambda)^*\| \leq Ce^{\varepsilon/h} \tag{4.6}$$

for any $N > 0$ and $\varepsilon > 0$.

Now we can state and prove the key result of this section:

Proposition 4.4 *If I is a sufficiently small neighborhood of E, then for any $\varepsilon > 0$, $N \geq 0$ and $\lambda \in I$, there exists $C > 0$ such that:*

$$\|\langle(x,\xi)\rangle^N M_2 \mathcal{F}_1(\lambda)^*\| \leq Ce^{-(\tau-\varepsilon)/h}$$

where $\tau := \mathrm{Min}\{\tau_1, \tau_2\}$.

Proof: For $\varphi = (\varphi_1, \varphi_2) \in \mathcal{H} \oplus \mathcal{H}$ we have by (3.11):

$$M_2\varphi = (P\hat{J}^+ - \hat{J}^+P^+)\varphi_1 + (P\hat{J}^- - \hat{J}^-P^-)\varphi_2 =: M_2^+\varphi_1 + M_2^-\varphi_2$$

and we can decompose M_2^+ as:

$$M_2^+ = P(\hat{J}^+ - 1) - (\hat{J}^+ - 1)P^+ + T_A^*(p - L)(1 - \chi^+)T_A. \tag{4.7}$$

Now, observing that $\mathrm{Supp}\,(1 - \chi^+) \subset \mathrm{Supp}\,(1 - j^+) \subset \{\chi^- = 1\} \subset \{\phi_1 = \tau_1\}$, we obtain directly from Proposition 4.3:

$$\|\langle (x,\xi)\rangle^N (1 - \chi^+)T_A\mathcal{F}_1^+(\lambda)^*\| + \|\langle (x,\xi)\rangle^N (1 - j^+)T_A\mathcal{F}_1^+(\lambda)^*\| \le Ce^{-(\tau_1 - \varepsilon)/h}$$

and since also $P^+\mathcal{F}_1^+(\lambda)^* = \lambda\mathcal{F}_1^+(\lambda)^*$, we can see immediately on (4.7) that:

$$\|\langle (x,\xi)\rangle^N M_2^+\mathcal{F}_1^+(\lambda)^*\| = \mathcal{O}(e^{-(\tau_1 - \varepsilon)/h}).$$

We get in the same way by using (4.3):

$$\|\langle (x,\xi)\rangle^N M_2^-\mathcal{F}_1^-(\lambda)^*\| = \mathcal{O}(e^{-(\tau_2 - \varepsilon)/h}).$$

and the result follows. ◇

By using (3.13), we can finally deduce the main result of this section:

Theorem 4.5 *Assume (2.2), (2.3), (2.6) and (4.1). Then, for any $\varepsilon > 0$ there exist an interval I around E and a constant $C > 0$ such that for any $\lambda \in I$ one has:*

$$\|S_{1,2}(\lambda)\|_{\mathcal{L}(L^2(S^{n-1}))} + \|S_{2,1}(\lambda)\|_{\mathcal{L}(L^2(S^{n-1}))} \le Ce^{-(\tau_0 - \varepsilon)/h})$$

uniformly with respect to $h > 0$ small enough.

Remark 4.6 *In particular, if we assume that*

$$E_2 - E_1 > 0$$

with

$$E_1 = \sup(V_1(x))\ ,\ E_2 = \inf(V_2(x))$$

then we can give a geometric meaning to τ_0. Let us define

$$\kappa(\xi) = \sup\{\kappa \in [0, \gamma[\ ;\ |\det(p(x - iy, \xi) - E)| > 0, \forall (x, y) \in I\!R^{2n} \text{ such that } |y| < \kappa\}$$

for $E - E_2 < \xi^2 < E - E_1$, and $\kappa(\xi) = 0$ elsewhere. We also denote by $d(\xi, \xi')$ the Agmon distance associated to κ, i.e. the pseudodistance on $I\!R^n$ associated to the metric $\kappa(\xi)d\xi^2$. Then we can take

$$\tau_0 = d(\Sigma_1^0, \Sigma_2^0)$$

where

$$\Sigma_j^0 = \{\xi \in I\!R^n\ ;\ \xi^2 + E_j = E\}, \quad j = 1, 2.$$

Proof: Just take $A = \mu I$ with $\mu > 0$ small enough. ◇

Appendix: Cut-off Operators

Let $p = p(x,\xi) \in C^\infty(I\!R^n)$ be such that for all $\alpha \in I\!N^{2n}$ one has

$$\partial^\alpha p = \mathcal{O}(m_0(x,\xi)) \tag{A.1}$$

uniformly in $I\!R^n$, where $m_0(x,\xi) = \langle x\rangle^k \langle \xi\rangle^\ell$ for some $k, \ell \in I\!R$.

We associate to p its semiclassical Weyl-quantization $P = \mathrm{Op}_h^W(p)$ and our purpose is to cut-off P in some special way and to obtain (under an additional analyticity assumption) exponential microlocal estimates for the resulting operator. Such estimates are well known for pseudodifferential operators with analytic symbols (see [Ma1, Na1]) but are known to be false for general pseudodifferential operators with C^∞ symbol. Therefore one has to be very careful in the way of cutting-off P, and the main idea will consist in using Toeplitz-type cut-off operators.

We consider $\chi \in C_b^\infty(I\!R^{2n})$ (the space of smooth functions on $I\!R^{2n}$ which are bounded together with all their derivatives) and we denote Ω_1, ..., Ω_N the different connected components of $\mathrm{Supp}\,\chi$ (χ will be our cut-off function). We also consider $\psi \in C_b^\infty(I\!R^{2n})$ real-valued such that:

$$\mathrm{Supp}\,\nabla\psi \cap \mathrm{Supp}\,\chi = \emptyset. \tag{A.2}$$

In particular ψ is constant on every Ω_j ($j = 1, ..., N$), and we denote ψ_j its constant value on Ω_j. For a given $n \times n$ symmetric positive definite matrix A, we associate the positive definite quadratic form on $I\!R^{2n}$ defined by:

$$Q_A(x,\xi) := \langle Ax, x\rangle + \langle A^{-1}\xi, \xi\rangle$$

and we denote $d_A(X,Y) = \sqrt{Q_A(X-Y)}$ the corresponding distance on $I\!R^{2n}$. We assume there exists some $\nu > 0$ such that:

$$|\psi(X) - \psi_j| \le \frac{1-\nu}{4} d_A(X, \Omega_j)^2 \tag{A.3}$$

for all $X \in I\!R^{2n}$ and for $j = 1, ..., N$.

In order to construct our cut-off operator, we use the global FBI transform T_A defined by:

$$T_A u(x,\xi) = 2^{-\frac{n}{2}} (\pi h)^{-\frac{3n}{4}} (\det A)^{\frac{1}{4}} \int e^{i(x-y)\xi/h - q_A(x-y)/2h} u(y)\, dy$$

where $q_A(x) := \langle Ax, x\rangle$. Then T_A is an isometry from $L^2(I\!R^n)$ to $L^2(I\!R^{2n})$ (see e.g. [Ma1]). In all the sequels we also consider a positive function m on $I\!R^{2n}$ of the form:

$$m(X) = \Pi_{j=1}^k \langle X_j' \rangle^{\ell_j} \tag{A.4}$$

where the X_j''s denote some (arbitrary) components of X and $\ell_j \in I\!R$. Then we show:

Theorem A.1 *Under assumptions (A.1), (A.2) and (A.3) there exists $C > 0$ such that for all $u \in C_0^\infty(I\!R^n)$:*

$$\| m e^{\psi/h} T_A T_A^* p \chi T_A u - m p \chi e^{\psi/h} T_A u \|_{L^2(I\!R^{2n})} \le C\sqrt{h} \| m m_0 e^{\psi/h} T_A u \|_{L^2(I\!R^{2n})}$$

uniformly with respect to h small enough.

Proof - Using that $T_A^* T_A = 1$, we see that:

$$\begin{aligned} me^{\psi/h}T_A T_A^* p\chi T_A u - mp\chi e^{\psi/h}T_A u &= me^{\psi/h}(T_A T_A^* p\chi - p\chi T_A T_A^*)T_A u \\ &= Rmm_0 e^{\psi/h}T_A u \end{aligned}$$

with

$$R := me^{\psi/h}(T_A T_A^* p\chi - p\chi T_A T_A^*)e^{-\psi/h}m^{-1}m_0^{-1}$$

and it remains to show that $\|R\| = \mathcal{O}(\sqrt{h})$. Denoting $K_R(X,Y)$ its distributional kernel, we have:

$$K_R(X,Y) = \frac{m(X)}{m(Y)} K_{T_A T_A^*}(X,Y) e^{(\psi(X)-\psi(Y))/h} \frac{p(Y)\chi(Y) - p(X)\chi(X)}{m_0(Y)} \quad \text{(A.5)}$$

where $K_{T_A T_A^*}(X,Y)$ is the distributional kernel of $T_A T_A^*$. A direct computation gives:

$$K_{T_A T_A^*}(X,Y) = (2\pi h)^{-n} e^{i\theta(X,Y)/h - Q_A(X-Y)/4h} \quad \text{(A.6)}$$

with $\theta((x,\xi),(y,\eta)) := (x-y)(\xi+\eta)/2$. In particular, we see on (A.5) that $K_R(X,Y) = 0$ if X and Y do not belong to $\cup_{j=1}^N \Omega_j$. Assume first that $Y \in \Omega_j$ for some j. then it follows from the assumptions that

$$|\psi(X) - \psi(Y)| = |\psi(X) - \psi_j| \leq \frac{1-\nu}{4} d_A(x, \Omega_j)^2 \leq \frac{1-\nu}{4} Q_A(X-Y)$$

Thus, in this case we deduce from (A.5)-(A.6) that

$$\begin{aligned} |K_R(X,Y)| &\leq (2\pi h)^{-n} \frac{m(X)}{m(Y)} e^{-\nu Q_A(X-Y)/4h} \frac{|p(Y)\chi(Y) - p(X)\chi(X)|}{m_0(Y)} \\ &\leq Ch^{-n} \frac{m(X)[m_0(X) + m_0(Y)]}{m(Y)m_0(Y)} e^{-\nu Q_A(X-Y)/4h} |X-Y| \quad \text{(A.7)} \end{aligned}$$

where we have used the fact that the derivatives of $p\chi$ on the segment $[X,Y] \subset \mathbb{R}^{2n}$ are bounded by $\mathcal{O}(m_0(X) + m_0(Y))$. Since the same estimate holds in the case $X \in \Omega_j$, the result follows from the Schur lemma and the fact that $m(X)/m(Y)$ and $m_0(X)/m_0(Y)$ remain uniformly bounded on $\{|X-Y| \leq \frac{1}{4}(|X| + |Y|)\}$. $\diamond$

Now we assume moreover that $p = p(x,\xi)$ is analytic in the complex strip

$$S = \{(x,\xi) \in \mathbb{C}^{2n}\ ;\ |\mathrm{Im}\,(x,\xi)| < a\} \quad \text{(A.8)}$$

for some positive a, and that the estimates (A.1) hold uniformly in S. We also assume that

$$|\nabla\psi(x,\xi)| < a \quad \text{(A.9)}$$

for all $(x,\xi) \in \mathbb{R}^{2n}$. Then, applying to P the microlocal weight exponential estimates of [Ma1, Na1], we immediately get the following corollary:

Corollary A.2 *Under assumptions (A.2), (A.3), (A.8) and (A.9), there exists $C > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$:*

$$\|me^{\psi/h}T_A(P - T_A^* p\chi T_A)u - m(1-\chi)p_{\psi,A}e^{\psi/h}T_A u\| \leq C\sqrt{h}\|mm_0 e^{\psi/h}T_A u\|_{L^2(\mathbb{R}^{2n})}$$

uniformly with respect to h small enough, with

$$p_{\psi,A} := p(x - A^{-1/2}\partial_A\psi, \xi + iA^{1/2}\partial_A\psi).$$

Proof: Just observe that $p\chi = \chi p_{\psi,A}$ because of (A.2). $\diamond$

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