LOW-TEMPERATURE LOCALIZATION FOR CONTINUOUS SPIN-SYSTEMS

Jacob Schach Møller[1]
Université Paris-sud
Département de Mathématique
91405 Orsay CEDEX
email: moeller.jacob@ math.u-psud.fr

ABSTRACT. We study the free energy of continuous spin-systems on $\mathbb{Z}^d$, in the framework of Laplace integrals and transfer operators. Under a weak coupling condition, we show that the free energy in the low-temperature limit is determined, up to an exponentially small error, by the restriction to a neighbourhood of global minima of the energy. We have results for some single- and double-well problems.

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1. Introduction

This paper is mainly concerned with tying up some loose ends left over from a series of papers by Helffer [He1], [He2], [He3] and [He4], and Sjöstrand [Sj3]. These works were concerned with obtaining full low-temperature expansions of the free-energy per spin, for a class of $\mathbb{R}$-valued spin-systems on the cubic lattice $\mathbb{Z}^d$ (their results extend, with some extra work, to $\mathbb{R}^n$-valued spin-systems). Two approaches were used. The direct approach consists of a stationary phase analysis of the partition function defined as a Laplace integral. Another approach is via the transfer operator formalism, where the partition function is replaced by the highest eigenvalue of the associated transfer operator, which is a positive trace-class operator. The stationary phase analysis is then replaced by semiclassical analysis.

Let Γ be a countable collection of finite sets Λ. We assume

(Γ1) Γ is equipped with a partial ordering $\prec$ and has no maximal elements, that is, for any Λ there exists Λ' with $\Lambda \prec \Lambda'$.

(Γ2) $\forall \Lambda \in \Gamma$ there exists a map $\Lambda \ni \lambda \to n_\Lambda(\lambda) \in \mathbb{N}$ such that $n_0 := \sup_{\Lambda\in\Gamma} \max_{\lambda\in\Lambda} n_\Lambda(\lambda) < \infty$.

The configuration space over $\Lambda \in \Gamma$ is the Cartesian product of the single-spin configuration spaces over Λ, that is:

$$\mathbb{R}^\Lambda := \times_{\lambda\in\Lambda} \mathbb{R}^{n_\Lambda(\lambda)}.$$

Let $\{\Lambda_n\}_{n\in\mathbb{N}} \subset \Gamma$. We say $\Lambda_n \to \infty$ if for all m there exists $N > m$ such that $\Lambda_m \prec \Lambda_n$, $n \geq N$. (Such sequences exist by (Γ1).) Let $\{f_\Lambda\}_{\Lambda\in\Gamma} \subset \mathbb{R}$. We say the thermodynamic limit $\lim_{\Lambda\to\infty} f_\Lambda$ exists if for any sequence $\Lambda_n \to \infty$, the limit $\lim_{n\to\infty} f_{\Lambda_n}$ exists and is independent of the choice of sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$.

For a family of potentials $V^{(\Lambda)} : \mathbb{R}^\Lambda \to \mathbb{R}$, $\Lambda \in \Gamma$, and $\beta > 0$ we define the partition function as a Laplace integral

$$Z_\Lambda(\beta) := \int_{\mathbb{R}^\Lambda} e^{-\beta V^{(\Lambda)}(x)}\, d^\Lambda x. \tag{1.1}$$

Here $d^\Lambda x$ is the Lebesgue measure on $\mathbb{R}^\Lambda$. For a family of potentials $V_{\mathrm{tr}}^{(\Lambda)} : \mathbb{R}^\Lambda \to \mathbb{R}$, interactions $I^{(\Lambda)} : \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$ and coupling constant $J \in \mathbb{R}$ we define a family of transfer operators $K^{(\Lambda)}(\beta)$, $\beta > 0$, by their integral kernels

$$K^{(\Lambda)}(x, y; \beta) := e^{-\beta V_{\mathrm{tr}}^{(\Lambda)}(x)} e^{-\beta J I^{(\Lambda)}(x,y)} e^{-\beta V_{\mathrm{tr}}^{(\Lambda)}(y)}. \tag{1.2}$$

The potentials $V^{(\Lambda)}, V_{\mathrm{tr}}^{(\Lambda)}$, the interaction $I^{(\Lambda)}$ and the coupling constant J satisfies suitable assumptions (including a growth condition on $V^{(\Lambda)}$ and $V_{\mathrm{tr}}^{(\Lambda)}$ at infinity). The $K^{(\Lambda)}$'s are positive trace-class operators on $L^2(\mathbb{R}^\Lambda)$. Here $\beta > 0$ is the inverse temperature and the low-temperature limit is the limit $\beta \to \infty$. The coupling constant J will be assumed small in norm (compared with $V_{\mathrm{tr}}^{(\Lambda)}$ and $I^{(\Lambda)}$).

By the free energy per degree of freedom in the two cases we understand

$$F_\Lambda(\beta) := \frac{\ln Z_\Lambda(\beta)}{|\Lambda|} \tag{1.3}$$

and

$$f_\Lambda(\beta) := \frac{\ln \lambda^{(\Lambda)}(\beta)}{|\Lambda|}, \tag{1.4}$$

where $\lambda^{(\Lambda)}(\beta)$ is the highest eigenvalue of $K^{(\Lambda)}(\beta)$.

The analysis by Helffer and Sjöstrand where done in small l^∞ neighbourhoods of critical points of the potential and it remains to give a good estimate of the error made, when one throws away the part of the configuration space that is away from critical points. Such localization estimates is the main focus of this paper. We note that the problem here is to obtain errors that are uniform in $\Lambda \in \Gamma$. Our error estimates will be of the form $e^{-a\beta}$, for some $a > 0$, uniformly in $\Lambda \in \Gamma$. We will always require a weak coupling condition. In [He1] a localization result was presented for the Laplace integral under the requirement that $0 < C_1 I \leq \nabla^2 V^{(\Lambda)}(x) \leq C_2 I < \infty$ uniformly in $x \in \mathbb{R}^\Lambda$ and $\Lambda \in \Gamma$ (in the case $n_\Lambda(\lambda) = 1$, $\mathbb{R}$-valued spins). In particular this excludes local minima other than a unique global one and it also excludes $x^4 + x^2$ type self-energies. In [M] a localization result was obtained for transfer operators in fixed dimension (i.e. without uniformity in Λ), using an IMS localization technique. No localization results with uniform control in Λ exist for the transfer operator.

In [BJS], [Sj4] and [BM], the so-called Witten Laplacian approach is used to obtain low-temperature asymptotics of truncated two-point correlation functions for $\mathbb{R}$-valued spin-systems on $\mathbb{Z}^d$. The analysis in those papers is not local although a localization result related to ours is used in [BM].

Another point which will be discussed is the step that relates the study of the two free energies above, in the case of translation invariant $\mathbb{R}^n$-valued spin-systems on $\mathbb{Z}^d$, $d \geq 1$, with nearest neighbour interaction. Here we take $\Gamma = \{\Lambda(\underline{L})\}_{\underline{L} \in \mathbb{N}^d}$ where

$$\Lambda(\underline{L}) := \times_{i=1}^d [-L_i, L_i], \quad \text{for} \quad \underline{L} \in \mathbb{N}^d.$$

We take $n_\Lambda(i) = n$, for all $i \in \Lambda$, and the partial ordering on Γ is induced by the partial ordering on $\mathbb{N}^d$: $\underline{L} \prec \underline{L}' \Leftrightarrow L_k < L'_k, 1 \leq k \leq d$. (It is not important that the boxes are centered at 0). In this case the thermodynamic limit is the infinite volume limit of boxes in the sense of van Hove, see [R] or [T]. The associated transfer operator is defined on the lattice $\mathbb{Z}^{d-1}$ and it was proved in [He4] that the two free energies has the same thermodynamic limit provided the limit is taken essentially along the diagonal $L_k \simeq L_{k'}$. Here we extend this result to $\Lambda(\underline{L}) \to \infty$ in the sense of van Hove and we also show that the infinite volume free energy is independent of boundary conditions. See Proposition 3.2.

The fact that one can express the free energy in terms of the highest eigenvalue of an associated transfer operator is the continuous spin analogue of the transfer matrix formalism for the Ising model, introduced by Kramers and Wannier in [KW]. See also [HK] and [K]. See [F] for the use of transfer operators in the study of self-coupled scalar fields, via a lattice approximation of the corresponding Euclidian field theory.

Our main results are Theorem 2.6 (localization for the Laplace integral in the single-well case), Theorem 4.4 (localization for the transfer operator in the single-well case) and Theorem 5.2 (localization for the Laplace integral for a class of examples with a symmetric double-well).

A central tool in this paper is a covering of $\mathbb{R}^m$ introduced by Sjöstrand in [Sj2]. It was used to treat a localization problem for the first eigenvalue of a Schrödinger operator $H = -h^2\Delta + V$ on $L^2(\mathbb{R}^m)$, with uniform control in the dimension m. The result obtained for H does not involve any uniform convexity assumption on V. This covering was later applied in [He1] to Laplace integrals (where the result mentioned above was obtained). In the papers [HeSj1] and [HeSj2] the authors study a Schrödinger operator with a family of potentials coming from a mean-field model introduced by Kac, see [K]. The mean-field parameter plays the role of a semiclassical constant. Above the critical temperature the potential is convex and below the critical temperature it is a double-well. The covering is used to study both the first eigenvalue and the spectral gap between the first and the second eigenvalue (above and below the critical temperature). In the papers mentioned above the covering was designed to handle what amounts to $\mathbb{R}$-valued spins. Here we refine the covering to include $\mathbb{R}^n$-valued spins.

We write for $x \in \mathbb{R}^\Lambda$, $\Lambda \in \Gamma$,

$$x = \{x_\lambda\}_{\lambda\in\Lambda}, \quad x_\lambda \in \mathbb{R}^{n_\Lambda(\lambda)}.$$

Let $1 \le p \le \infty$. We write $|x|_p$ for the p-norm on $\mathbb{R}^\Lambda$. ($|x|_p = (\sum_{\lambda\in\Lambda}\sum_{1\le j\le n_\Lambda(\lambda)} |x_{\lambda,j}|^p)^{1/p}$, $1 \le p < \infty$ and $|x|_\infty = \max_{\lambda\in\Lambda}\max_{1\le j\le n_\Lambda(\lambda)}\{|x_{\lambda,j}|\}$.) We use the notation $\|A\|_{p,q}$ for the uniform norm of a linear operator A as a map from $\mathbb{R}^\Lambda$ equipped with the p-norm, to $\mathbb{R}^\Lambda$ equipped with the q-norm. ($\|A\|_{p,q} = \sup_{t\in\mathbb{R}^\Lambda, |t|_p\le 1} |At|_q$).

We are primarily motivated by the following class of potentials associated with the cubic lattice $\mathbb{Z}^d$. Lattices and single spin configuration spaces other than $\mathbb{Z}^d$ and $\mathbb{R}^n$ could be considered as well.

Example 1.1 ($\mathbb{R}^n$-valued spins on $\mathbb{Z}^d$). *Let $\Lambda = \Lambda(\underline{L})$, $\underline{L} \in \mathbb{N}^d$. The potential $V = V^{(\Lambda)}$ is of the form*

$$V(x) = \sum_{i\in\Lambda} v_i(x_i) + JW(x).$$

The interaction term W is

$$W(x) = \frac{1}{2}\sum_{i\neq j} \omega_{i,j}(x_i, x_j),$$

where the precise interpretation of the sum constitutes a choice of boundary condition, which we will make precise below. For $i \in \mathbb{Z}^d$ we write $\mathcal{M}_i = v_i^{-1}(\{0\})$ and $\mathcal{M}_i(R) = \{x \in \mathbb{R}^n : \exists z \in \mathcal{M}_i$ s.t. $|x - z|_2 \le R\}$. We assume there exist $R_0 > 0$, $a_1 > 0$ and $\tau > 0$ such that

(1) *For $i \neq j \in \mathbb{Z}^d$ we have $v_i \in C^3(\mathbb{R}^n)$ and $\omega_{i,j} \in C^3(\mathbb{R}^n_x \times \mathbb{R}^n_y)$. We assume $v_i \ge 0$.*
(2) *$\mathcal{M}_i \neq \emptyset$ and there exists $C > 0$ such that for all $i \in \mathbb{Z}^d$ and $z \in \mathcal{M}_i$ we have $\nabla^2 v_i(z) \le CI$.*
(3) *$\sup_{i\in\mathbb{Z}^d}\sup_{x\in\mathcal{M}_i(R_0)} |\partial^\alpha v_i(x)| < \infty$, for $2 \le |\alpha| \le 3$.*
(4) *$\omega_{i,j}(0,0) = 0$ and $\nabla_x\omega_{i,j}(0,0) = \nabla_y\omega_{i,j}(0,0) = 0$.*
(5) *We assume the two bounds*

$$\sup_{i\in\mathbb{Z}^d}\sum_{j(\neq i)} \frac{1}{2}\Big(\sup_{x,y\in\mathbb{R}^n} |\nabla_x^k\nabla_y^l\omega_{i,j}(x,y)|_{2,2} + \sup_{x,y\in\mathbb{R}^n} |\nabla_x^k\nabla_y^l\omega_{j,i}(x,y)|_{2,2}\Big) \le a_1$$

$$\sup_{i\in\mathbb{Z}^d}\sum_{j(\neq i)} \sup_{x,y\in\mathbb{R}^n} |\partial_x^\alpha\partial_y^\beta\omega_{i,j}(x,y)| + \sup_{x,y\in\mathbb{R}^n} |\partial_x^\alpha\partial_y^\beta\omega_{j,i}(x,y)| < \infty,$$

for $k + l = 2$ and $|\alpha + \beta|_1 = 3$.
(6) *$|J|a_1 < \tau$ and for all $i \in \mathbb{Z}^d$ we have $\liminf_{|x|_2\to\infty} |x|_2^{-2} v_i(x) > \tau$.*

Remarks:

1) We have phrased the example using only two-body interactions. One could generalize this to include many-body interactions as well. That is, interactions of the form

$$\{\omega_X \in C^3(\mathbb{R}^X)\}_{X\subset\mathbb{Z}^d, 2\le|X|<\infty}.$$

2) We will in applications only consider the single-well case, $\mathcal{M}_i = \{0\}$, and the symmetric double-well case, $|\mathcal{M}_i| = 2$. Our methods though, should extend beyond these two examples. Note that $\nabla v_i(z) = 0$, $z \in \mathcal{M}_i$, is implied by (1).

3) We impose no upper bounds on the self-energies v_i outside a neighbourhood of the critical points. Exponential growth is for example included.

4) One should view (4) and (5) in the light of Taylor's formula. In particular we get the estimate

$$\begin{aligned}\frac{1}{2}\Big|\sum_{\substack{i\neq j\\ i,j\in\Lambda}}\omega_{i,j}(x_i,x_j)\Big| \;&\le\; \frac{1}{2}\int_0^1(1-t)\sum_{\substack{i\neq j\\ i,j\in\Lambda}}\{|x_i|_2^2\left(|\nabla_x^2\omega_{i,j}(tx_i,tx_j)|_{2,2}\;+\;|\nabla_x\nabla_y\omega_{i,j}(tx_i,tx_j)|_{2,2}\right)\\ &\quad+\;|x_j|_2^2\left(|\nabla_y^2\omega_{i,j}(tx_i,tx_j)|_{2,2}\;+\;|\nabla_x\nabla_y\omega_{i,j}(tx_i,tx_j)|_{2,2}\right)\}\,dt\\ &\le\; a_1|x|_2^2. \end{aligned}\tag{1.5}$$

Boundary conditions will enter as a linear correction, see Definition 1.2 below. (Note that $\alpha,\beta\in\mathbb{N}^n$ are multiindicies.)

5) Condition (6) is a condition at infinity, which together with (1.5) ensures that the potential has (quadratic) growth and (1.1) is well-defined. In the case $J\ge 0$ and $\omega_{i,j}\ge 0$ one can replace this condition by $\liminf_{|x|_2\to\infty}|x|_2^{-1}v_i(x)>\tau$. We also note that in this case weak coupling is only needed for our localization results and it will depend on the potentials behavior at critical points, not on its behavior at infinity. For simplicity we omit treatment of this case.

6) When specifying the one- and double-well problems, we give precise lower bounds in the whole space. Note that the self-energies are allowed to have regions stretching out to infinity, where $\nabla^2V(x)$ has negative eigenvalues. This is the case for $v_i(x)=x^2+\frac{1}{2}x\sin x$ $(n=1)$. A more standard example is $v_i(x)=|x|_2^4\pm\alpha|x|_2^2+C_\pm(\alpha)$, with interaction $\omega_{i,j}(x,y)=J_{i,j}|x-y|^2$ or (equivalently) $\omega_{i,j}(x,y)=-J_{i,j}x\cdot y$.

We introduce four types of boundary conditions to be employed in the interaction term of Example 1.1.

Definition 1.2. *We introduce*

- *Free boundary condition: Here we take*

$$W(x)\;=\;\frac{1}{2}\sum_{\substack{i\neq j\\ i,j\in\Lambda}}\omega_{i,j}(x_i,x_j).$$

- *Fixed boundary condition: Let $\xi\in\mathbb{R}^{\mathbb{Z}^d}$ with $\sup_{i\in\mathbb{Z}^d}|\xi_i|_2<\infty$. Then*

$$W(x)\;=\;\frac{1}{2}\sum_{\substack{i\neq j\\ \{i,j\}\cap\Lambda\neq\emptyset}}\omega_{i,j}(x_i,x_j),$$

  with the convention that $x_i=\xi_i$ if $i\notin\Lambda$. We also require that the following stronger decay estimate holds, for some $\epsilon>0$:

$$\sup_{x,y\in\mathbb{R}^n}|\partial_x^\alpha\partial_y^\beta\omega_{i,j}(x,y)|\;\le\;C\,\Pi_{k=1}^d(|i_k-j_k|+1)^{-1-\epsilon},\tag{1.6}$$

  for $|\alpha+\beta|_1=2$.

- *Edgewise boundary condition: We assume a nearest neighbour interaction, that is; $\omega_{i,j}=0$ if $|i-j|_1\neq 1$. For each $1\le k\le d$ we introduce two boundaries*

$$E_k^\pm\;=\;\{i\in\mathbb{Z}^d\,:\,i_k=\pm L_k\pm 1\quad\textit{and}\quad -L_{k'}\le i_{k'}\le L_{k'},\,k'\neq k\}.$$

Let $\xi_k^\pm \in \mathbb{R}^n$, $1 \le k \le d$ be given. The interaction is

$$W(x) = \frac{1}{2} \sum_{\substack{i \neq j \\ \{i,j\} \cap \Lambda \neq \emptyset}} \omega_{i,j}(x_i, x_j),$$

with the convention that $x_i = \xi_k^\pm$ if $x_i \in E_k^\pm$.

. *Periodic boundary condition: Here we view Λ as a torus. That is*

$$W(x) = \frac{1}{4} \sum_{\substack{i \neq j \\ i,j \in \Lambda}} \sum_{l_* \in \Lambda^*} \omega_{i+l_*,j}(x_i, x_j) + \omega_{i,j-l_*}(x_i, x_j),$$

where $\Lambda^* = \times_{k=1}^d (2L_k + 1)\mathbb{Z}$. Here one should in addition to Example 1.1 (5) assume the two bounds:

$$\begin{aligned} &\sup_{\Lambda \in \Gamma} \sup_{i \in \Lambda} \sum_{j \in \Lambda \setminus \{i\}} \sum_{l_* \in \Lambda^*} \frac{1}{2} \left(\sup_{x,y \in \mathbb{R}^n} |\nabla_x^k \nabla_y^l \omega_{i+l_*,j}(x,y)|_{2,2} + \sup_{x,y \in \mathbb{R}^n} |\nabla_x^k \nabla_y^l \omega_{j,i-l_*}(x,y)|_{2,2} \right) < a_1 \\ &\sup_{\Lambda \in \Gamma} \sup_{i \in \Lambda} \sum_{j \in \Lambda \setminus \{i\}} \sum_{l_* \in \Lambda^*} \sup_{x,y \in \mathbb{R}^n} |\nabla_x^\alpha \nabla_y^\beta \omega_{i+l_*,j}(x,y)| + \sup_{x,y \in \mathbb{R}^n} |\nabla_x^\alpha \nabla_y^\beta \omega_{j,i-l_*}(x,y)| < \infty, \end{aligned} \tag{1.7}$$

for $k + l = 2$ and $|\alpha + \beta|_1 = 3$.

Remarks.

1) The case of the fixed boundary condition can be interpreted, in terms of the corresponding Gibb's measure, as a conditional probability measure. In fact, a result of Dobrushin, see [D], identifies a class of conditional measures (associated to spins on a countable set, with values in a measure space), which comes from a unique probability measure. Such a system exhibits no phase transition. The extra assumption (1.6) on the interaction term will be used to get an explicit bound on the contribution to $V^{(\Lambda)}$ from the ξ_i's, uniformly in Λ.

2) The edgewise boundary condition has played a role in the construction of different Gibb's measures for the Ising model in the low-temperature regime.

3) If $\omega_{i,j} = 0$ when $|i - j|_1 \neq 1$, then the periodic boundary condition corresponds to nearest neighbour interaction on the torus. We will write $i \sim_\Lambda j$ for nearest neighbours on Λ, viewed as a torus. The condition (1.7) reduces to Example 1.1 (5) (and is thus automatically satisfied) in the case of translation invariant interactions, $\omega_{i,j} = \omega_{i-j}$. The condition Example 1.1 (5) can furthermore be viewed as a special case of (1.7), namely for $\Lambda = \mathbb{Z}^d$ and $\Lambda^* = \{0\}$.

In section 2 we prove the primary technical lemma and present our localization result for Laplace integrals with a unique global minimum. In section 3 we recall the connection between Laplace integrals and transfer operators, and we discuss dependence on boundary conditions. In section 4 we apply the ideas of section 2 to transfer operators. In section 5 we investigate the localization properties for a class of double-well potentials of the type given by Example 1.1. In section 6 we combine our results with those of Helffer and Sjöstrand to give complete low-temperature expansions of the free energy. In appendix A we verify that our example, in the single-well case, satisfies our general conditions, and in appendix B we argue that l^∞ neighbourhoods are the smallest neighbourhoods for which the Laplace integral localizes.

2. The Laplace integral: The case of a unique global minimum

Let Γ satisfy (Γ1-2). We will in this section work under the following conditions on the family of potentials $V^{(\Lambda)}$.

Condition 2.1. *Let $V^{(\Lambda)}$, $\Lambda \in \Gamma$, be a family of real functions. There exist $R_0 > 0$, $0 < \sigma_1 \leq \infty$, $0 < \sigma_2 < \infty$, $a_0 > 0$, $\omega_0 > 0$ and a family of non-negative constants $\{C(\Lambda)\}_{\Lambda\in\Gamma}$ such that*

(1) $\lim_{\Lambda\to\infty} C(\Lambda)/|\Lambda| = 0$.
(2) $V^{(\Lambda)} \in C^3(\mathbb{R}^\Lambda)$, $|V^{(\Lambda)}(0)| \leq C(\Lambda)$ *and* $|\nabla V^{(\Lambda)}(0)|_1 \leq C(\Lambda)$.
(3) $\nabla^2 V^{(\Lambda)}(0) \leq \omega_0 I$ *and* $V^{(\Lambda)}(x) \geq \sum_{\lambda\in\Lambda} \min\{\sigma_1|x_\lambda|_2, \sigma_2|x_\lambda|_2^2\} - C(\Lambda)$.
(4) *For any $x \in \mathbb{R}^\Lambda$ we have*

$$\nabla^2 V^{(\Lambda)}(x) \;=\; D^{(\Lambda)}(x) \;+\; A^{(\Lambda)}(x), \quad \textit{where} \quad D^{(\Lambda)}(x) \;=\; \times_{\lambda\in\Lambda} D_\lambda^{(\Lambda)}(x)$$

and the $D_\lambda^{(\Lambda)}(x)$'s are operators on $\mathbb{R}^{n_\Lambda(\lambda)}$. Furthermore $\|A^{(\Lambda)}(x)\|_{\infty,\infty} \leq a_0$ uniformly in $x \in \mathbb{R}^\Lambda$ and $\Lambda \in \Gamma$. We require $a_0 < \sigma_2$ (weak coupling).
(5) $|\langle \nabla^3 V^{(\Lambda)}(x), t_1 \otimes t_2 \otimes t_3\rangle| \leq C_3|t_1|_{p_1}|t_2|_{p_2}|t_3|_{p_3}$ *uniformly in x with $|x|_\infty \leq R_0$, $t_i \in \mathbb{R}^\Lambda$, $1 \leq p_i \leq \infty$, with $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$, and $\Lambda \in \Gamma$.*

Remark: The lower bound in (3), in the case $\sigma_1 = \infty$, reduces to $V^{(\Lambda)}(x) \geq \sigma_2|x|_2^2 - C(\Lambda)$. The constants $C(\Lambda)$ will, due to (1), not contribute to the thermodynamic limit. Their role will be to account for the fixed and edgewise boundary conditions. The condition (5) follows by symmetry and interpolation from the same estimate with $p_1 = p_2 = \infty$ and $p_3 = 1$.

We will in the following drop the superscript (Λ) from the notation. The boundary of a box $\Lambda = \Lambda(\underline{L})$, $\underline{L} \in \mathbb{N}^d$, will be denoted by:

$$\partial\Lambda \;=\; \{i \in \Lambda \;:\; \exists 1 \leq k \leq d \text{ s.t. } i_k = -L_k \text{ or } i_k = L_k\}$$

and the smallest side length

$$L_{\min} \;=\; \min\{L_k \;:\; 1 \leq k \leq d\}.$$

In terms of Example 1.1, we introduce the single-well problem

Example 2.2 (Single-well potentials). *Let $V = V^{(\Lambda)}$ be as in Example 1.1, with the following extra assumptions*

(1) $\mathcal{M}_i = \{0\}$ *for all* $i \in \mathbb{Z}^d$.
(2) *There exist $\tau_2 > 0$ and $\Sigma > 0$ such that $v_i(x) \geq \tau_2|x|_2^2$ and $\frac{1}{2}\nabla^2 v_i(0) \geq \Sigma I$, for all $i \in \mathbb{Z}^d$.*
(3) $|J| < J_0 = \min\{\frac{\tau_2}{a_1}, \frac{\Sigma}{(n+1)a_1}\}$.

This example fits with Condition 2.1, as stated in the following proposition. We defer its proof to appendix A.

Proposition 2.3. *Suppose V is as in Example 2.2. Then, V satisfies Condition 2.1 with $a_0 = n|J|a_1$, any $a_0 < \sigma_2 < \Sigma - |J|a_1$, some $0 < \sigma_1 \leq \infty$ and, depending on the boundary condition,*

. *Free or periodic boundary conditions:* $C(\Lambda) = 0$.
. *Fixed boundary condition:* $C(\Lambda) = C|J||\Lambda|L_{\min}^{-\epsilon}$.
. *Edgewise boundary condition:* $C(\Lambda) = C|J||\partial\Lambda|$.

We introduce certain l^∞-neighbourhoods with centers $\rho = \{\rho_\lambda\}_{\lambda\in\Lambda}$, $\rho_\lambda \in \mathbb{R}^{n_\Lambda(\lambda)}$, which were used (in the case $n_0 = 1$) in [Sj2], [He1], [HeSj1] and [HeSj2]. It is the central technical tool used in this paper.

Definition 2.4. *Let $0 < \epsilon < r$. We say ρ is a center, with respect to ϵ and r, if either $\rho_\lambda = 0$ or $|\rho_{\lambda,j}| \geq r$ for some $1 \leq j \leq n_\Lambda(\lambda)$. For each center ρ we associate a neighbourhood*

$$\Omega(\rho) \;:=\; \times_{\lambda\in\Lambda} \times_{j=1}^{n_\Lambda(\lambda)} I_{\lambda,j}(\rho),$$

where

$$I_{\lambda,j}(\rho) := \begin{cases} [-r,r] & ,\rho_\lambda = 0 \\ [\rho_{\lambda,j}-\epsilon, \rho_{\lambda,j}+\epsilon] & ,\exists 1 \le j \le n_\Lambda(\lambda) : \rho_{\lambda,j} \ne 0 \end{cases}$$

We will in applications use coverings of $\mathbb{R}^\Lambda$ by choosing ρ in a suitable subset of $\epsilon\mathbb{Z}^\Lambda$, where $\mathbb{Z}^\Lambda := \times_{\lambda\in\Lambda}\mathbb{Z}^{n_\Lambda(\lambda)}$. The following lemma is the central result used in our study of localization properties.

Lemma 2.5. *Suppose V satisfies Condition 2.1. Let $0 < r < \min\{R_0, \sigma_1/a_0\}$. There exist $0 < \epsilon < r$ and $\delta_0 > 0$, such that for any center ρ, of the type above, and $x \in \Omega(\rho)$, we have*

$$V(x) - V(x-\rho) \ge \delta_0 |\rho|_1 - C(\Lambda).$$

Proof. Let $0 < \epsilon < \min\{1, r\}$ and let ρ be a center as in Definition 2.4 and let $x \in \Omega(\rho)$. Write $x_0 \in \Omega(\rho)$ for the vector with

$$x_{0,\lambda} := \begin{cases} 0 & ,\rho_\lambda = 0 \\ x_\lambda & ,\exists 1 \le j \le n_\Lambda(\lambda) : \rho_{\lambda,j} \ne 0 \end{cases} \tag{2.1}$$

We write

$$V(x) - V(x-\rho) = V(x_0) - V(x_0-\rho) + R_\rho(x), \tag{2.2}$$

where $R_\rho(x) = V(x) - V(x_0) + V(x-\rho) - V(x_0-\rho)$, and compute using Taylor's formula

$$\begin{aligned} R_\rho(x) &= \int_0^1 \langle \nabla V(x_t) - \nabla V(x_t-\rho), x - x_0\rangle\, dt \\ &= \int_0^1\int_0^1 \langle \nabla^2 V(x_{t,s}(\rho))\rho, x - x_0\rangle\, ds\, dt. \end{aligned}$$

Here $x_t = tx + (1-t)x_0$ and $x_{t,s}(\rho) = sx_t + (1-s)\rho$. We now write $\nabla^2 V = D + A$ and notice that by the choice of x_0, see (2.1), and Condition 2.1 (4), we have

$$\langle D(x_{t,s}(\rho))\rho, x - x_0\rangle = \sum_{\lambda\in\Lambda}\langle D_\lambda(x_{t,s}(\rho))\rho_\lambda, x_\lambda - x_{0,\lambda}\rangle = 0.$$

In other words

$$|R_\rho(x)| \le a_0 |\rho|_1 |x-x_0|_\infty \le a_0\, r\, |\rho|_1. \tag{2.3}$$

The next step will be to replace $V(x_0-\rho)$ by $V_0(x_0-\rho)$, where $V_0(x) = \frac{1}{2}\langle\nabla^2 V(0)x, x\rangle$. For y, with $|y|_\infty < R_0$, we write

$$V(y) = V_0(y) + R(y), \quad \text{where} \quad R(y) = 3\int_0^1 (1-t)^2\langle \nabla^3 V(ty), y\otimes y\otimes y\rangle\, dt \tag{2.4}$$

and estimate

$$|R(y)| \le C_3 |y|_\infty |y|_2^2.$$

In particular

$$|R(x_0-\rho)| \le \epsilon\, C_3 |x_0-\rho|_2^2.$$

Combining this with (2.2), (2.3), (2.4) and Condition 2.1 (3), we now get

$$V(x) - V(x-\rho) \ge \min\{\sigma_1\,|x_0|_1\,,\,\sigma_2\,|x_0|_2^2\} - (\frac{1}{2}\,\omega_0 + \epsilon\,C_3)\,|x_0-\rho|_2^2 - a_0\,r\,|\rho|_1 - C(\Lambda). \tag{2.5}$$

By the choice of x_0, see (2.1), we have

$$\min\{\sigma_1\,|x_0|_1\,,\,\sigma_2\,|x_0|_2^2\} \ge \sigma_0\,|x_0|_1\,, \quad \text{where} \quad \sigma_0 = \min\{\sigma_1\,,\,r\,\sigma_2\}. \tag{2.6}$$

We abbreviate $\theta = \max\{1, \sigma_0^{-1}(\frac{1}{2}\omega_0 + C_3)\}$ and estimate for a λ with $\rho_{\lambda,j} \neq 0$, for some $1 \le j \le n_\Lambda(\lambda)$ (the other case being trivial by the choice of x_0):

$$\begin{aligned} |x_{0,\lambda}|_1 - \theta\,|x_{0,\lambda}-\rho_\lambda|_2^2 &\ge |\rho_\lambda|_1 - |x_{0,\lambda}-\rho_\lambda|_1 - \theta\,|x_{0,\lambda}-\rho_\lambda|_2^2 \\ &\ge |\rho_\lambda|_1 - n_0\,\epsilon - n_0\,\epsilon^2\,\theta \\ &\ge |\rho_\lambda|_1 - C\,\epsilon, \end{aligned} \tag{2.7}$$

for some $C > 0$. Since we consider only $r < \sigma_1/a_0$ and we have the weak coupling condition $a_0 < \sigma_2$, we find: $\sigma_0 - ra_0 > 0$ (see (2.6)). Choose $0 < \alpha < 1$ such that

$$\delta_0 := \alpha\,\sigma_0 - r\,a_0 > 0.$$

Secondly we choose $\epsilon > 0$ small enough such that

$$(1-\alpha)\,r > C\,\epsilon.$$

We now combine (2.5) and (2.7), and write $|\rho_\lambda|_1 = \alpha|\rho_\lambda|_1 + (1-\alpha)|\rho_\lambda|_1$, to obtain (since $|\rho_\lambda|_1 \ge r$)

$$V(x) - V(x-\rho) \ge \delta_0\,|\rho|_1 - C(\Lambda),$$

which concludes the proof. □

We will be interested in the partition function $Z_\Lambda(\beta)$, see (1.1), (for $\beta > 0$) and its localized version

$$Z_\Lambda^R(\beta) := \int_{B_R} e^{-\beta V^{(\Lambda)}(x)} d^\Lambda x, \tag{2.8}$$

where $B_R = \{x \in \mathbb{R}^\Lambda : |x|_\infty \le R\}$. We introduce a localized version of the free energy $F_\Lambda(\beta)$, see (1.3),

$$F_\Lambda^R(\beta) := \frac{\ln Z_\Lambda^R(\beta)}{|\Lambda|}.$$

As an application of Lemma 2.5 we get

Theorem 2.6. *Suppose $V^{(\Lambda)}$ satisfies Condition 2.1. Let $R > 0$. There exists $a > 0$ (independent of Λ) such that*

$$F_\Lambda^R(\beta) \le F_\Lambda(\beta) \le F_\Lambda^R(\beta) + n_0\,e^{-a\beta} + \beta\,\frac{C(\Lambda)}{|\Lambda|}.$$

In particular

$$\limsup_{\Lambda\to\infty} |\,F_\Lambda(\beta) - F_\Lambda^R(\beta)\,| \le n_0\,e^{-a\beta}.$$

Proof. Since

$$F_\Lambda^R(\beta) < F_\Lambda^{R'}(\beta) < F_\Lambda(\beta)$$

for $0 < R < R' < \infty$, it is sufficient to consider the second inequality. Fix $0 < r \le \min\{R, R_0, \sigma_1/a_0\}$ and let $\epsilon > 0$ and $\delta_0 > 0$ be given by Lemma 2.5. Let $\mathcal{S}$ be a set of centers, ρ, chosen such that $\mathcal{S} \subset \epsilon\mathbb{Z}^\Lambda$ and $\cup_{\rho\in\mathcal{S}}\Omega(\rho) = \mathbb{R}^\Lambda$.

We estimate

$$\begin{aligned} Z_\Lambda(\beta) &\le \sum_{\rho\in\mathcal{S}} \int_{\Omega(\rho)} e^{-\beta V(x)} d^\Lambda x \\ &\le e^{\beta C(\Lambda)} \sum_{\rho\in\mathcal{S}} e^{-\beta\delta_0|\rho|_1} \int_{\Omega(\rho)} e^{-\beta V(x-\rho)} d^\Lambda x \\ &\le e^{\beta C(\Lambda)} \sum_{\rho\in\mathbb{Z}^\Lambda} e^{-\beta\delta_0\epsilon|\rho|_1} Z_\Lambda^R(\beta). \end{aligned}$$

Let

$$N_\Lambda = \sum_{\lambda\in\Lambda} n_\Lambda(\lambda) \le n_0\,|\Lambda|, \tag{2.9}$$

see $(\Gamma 2)$. Using the identity

$$\sum_{\rho\in\mathbb{Z}^\Lambda} e^{-\beta\delta_0\epsilon|\rho|_1} = (1 - e^{-\beta\delta_0\epsilon})^{-N_\Lambda},$$

we conclude the result with $a = \epsilon\delta_0$. □

In terms of our example, Example 2.2, we note that the correction term (containing $C(\Lambda)$) only appear in the case of fixed or edgewise boundary conditions, and that it disappears in the thermodynamic limit if $\Lambda \to \infty$ in the sense of van Hove.

3. From Laplace integrals to transfer operators

In this section we investigate the connection between Laplace integrals and transfer operators. We work here with translation invariant spin-systems on $\mathbb{Z}^d$, with nearest neighbour interaction. In particular we will examine the influence of boundary conditions. Some of the results presented here are well known. We will restrict attention to the free energy and just mention that the spectral gap (between the two highest eigenvalues of $K^{(\Lambda)}$), in the thermodynamic limit, is related to the correlation length of the corresponding spin system. See [K], [He3] and [He4].

Example 3.1. *Let $V^{(\Lambda)}$ be as in Example 1.1. We make the extra assumptions*

(1) *Nearest neighbour interaction: $\omega_{i,j} = 0$, $|i-j|_1 \ne 1$.*
(2) *Translation invariance: $v_i = v$ and $\omega_{i,j} = \omega$ for all $i, j \in \mathbb{Z}^d$ with $|i-j|_1 = 1$.*

We note that (1) can easily be replaced by a finite range assumption, by raising the dimension n of the one-particle space (and in (2) we would require $\omega_{i,j} = \omega_{i-j}$). In this section we do not make any assumptions on the set of global minima $\mathcal{M}_i = \mathcal{M}$. It is bounded by the growth assumption in Example 1.1 (6). In particular we note that

$$v(x) \ge \tau'\,|x|_2^2 - v_0 \tag{3.1}$$

for any $\tau' < \tau$ and some $v_0 > 0$. We note that Example 1.1 (5), in the case of Example 3.1, implies

$$|\omega(x,y)| \le \frac{a_1}{2d}(|x|_2^2 + |y|_2^2). \tag{3.2}$$

In the following we discuss the expression of the Laplace integral (1.1) in terms of the associated transfer operator, see (1.2), for the example above. The case of the fixed boundary condition will be treated separately later because it breaks translation invariance at the boundary.

We write $\Lambda = \Lambda(\underline{L}) = \Lambda_\perp \times \Lambda_d$, where $\Lambda_\perp = \times_{k=1}^{d-1}[-L_k, L_k]$ and $\Lambda_d = [-L_d, L_d]$. We decompose $\Lambda \ni i = \{i_\perp, i_d\}$. For $x \in \mathbb{R}^\Lambda$ and $i_\perp \in \Lambda_\perp$ we write $x^{i_\perp} \in \mathbb{R}^{\Lambda_d}$ for the spin-distribution on Λ_d with $x^{i_\perp}_{i_d} = x_i$. Conversely for a family $\{x^{i_\perp}\}_{i_\perp \in \Lambda_\perp} \subset \mathbb{R}^{\Lambda_d}$, we write $x \in \mathbb{R}^\Lambda$ for the spin-distribution on Λ given by $x_i = x^{i_\perp}_{i_d}$.

We will now introduce, for each type of boundary condition, a self-energy $V_{\text{tr}}^{(\Lambda_\perp)}$ (for spin-distributions $x \in \mathbb{R}^{\Lambda_\perp}$) and two boundary terms $b_\pm^{(\Lambda_\perp)} : \mathbb{R}^{\Lambda_\perp} \to \mathbb{R}$.

The free boundary condition:

$$V_{\text{tr}}^{(\Lambda_\perp)}(x) \;=\; \frac{1}{2}\sum_{i\in\Lambda_\perp} v(x_i) \;+\; \frac{J}{4}\sum_{\substack{|i-j|_1=1\\ i,j\in\Lambda_\perp}} \omega(x_i,x_j) \quad \text{and} \quad b_\pm^{(\Lambda_\perp)}(x) \;=\; V_{\text{tr}}(x).$$

The edgewise boundary condition:

$$V_{\text{tr}}^{(\Lambda_\perp)}(x) \;=\; \frac{1}{2}\sum_{i\in\Lambda_\perp} v(x_i) \;+\; \frac{J}{4}\sum_{\substack{|i-j|_1=1\\ \{i,j\}\cap\Lambda_\perp\neq\emptyset}} \omega(x_i,x_j),$$

with the convention that $x_i = \xi_k^\pm$ if $i \in E_k^\pm = E_k^\pm(\Lambda_\perp)$, and

$$b_\pm^{(\Lambda_\perp)}(x) \;=\; V_{\text{tr}}(x) + \frac{J}{2}\sum_{i\in\Lambda_\perp} \omega(x_i,\xi_d^\pm) \;+\; \omega(\xi_d^\pm, x_i).$$

The periodic boundary condition:

$$V_{\text{tr}}^{(\Lambda_\perp)}(x) \;=\; \frac{1}{2}\sum_{i\in\Lambda_\perp} v(x_i) \;+\; \frac{J}{4}\sum_{i\sim_{\Lambda_\perp} j} \omega(x_i,x_j) \quad \text{and} \quad b_\pm \;=\; 0,$$

where $i \sim_{\Lambda_\perp} j$ means nearest neighbour with respect to the torus structure on $\Lambda_\perp$. (The boundary terms $b_\pm^{(\Lambda_\perp)}$ are not used in the periodic boundary condition case.)

In all cases we have the same interaction term

$$I^{(\Lambda_\perp)}(x,y) \;=\; \frac{1}{2}\sum_{i\in\Lambda_\perp} \omega(x_i,y_i) \;+\; \omega(y_i,x_i).$$

For simplicity we drop the superscripts (Λ) and $(\Lambda_\perp)$ from the notation in the following. We can now write, for free and, edgewise boundary conditions, using the form of the kernel (1.2) of the transfer operator,

$$\begin{aligned}
Z_\Lambda(\beta) \;&=\; \int_{\mathbb{R}^\Lambda} e^{-\beta V(x)} d^\Lambda x \\
&=\; \int_{\mathbb{R}^\Lambda} e^{-\beta b_-(x^{-L_d})} K(x^{-L_d}, x^{-L_d+1}) \cdots K(x^{L_d-1}, x^{L_d}) e^{-\beta b_+(x^{L_d})} d^{\Lambda_\perp} x^{-L_d} \cdots d^{\Lambda_\perp} x^{L_d} \\
&=\; \int_{\mathbb{R}^{\Lambda_\perp}} \int_{\mathbb{R}^{\Lambda_\perp}} e^{-\beta b_-(x)} K^{(2L_d)}(x,y) e^{-\beta b_+(y)} d^{\Lambda_\perp} x \, d^{\Lambda_\perp} y.
\end{aligned} \tag{3.3}$$

As for the periodic boundary condition we similarly have

$$\begin{aligned} Z_\Lambda(\beta) &= \int_{\mathbb{R}^{\Lambda_\perp}} K^{(2L_d+1)}(x,x)\, d^{\Lambda_\perp}x \\ &= \operatorname{tr} K^{(2L_d+1)}. \end{aligned} \tag{3.4}$$

In order to compare the two expressions we write, see (3.3),

$$\begin{aligned} &\int_{\mathbb{R}^{\Lambda_\perp}}\int_{\mathbb{R}^{\Lambda_\perp}} e^{-\beta b_-(x)} K^{(2L_d)}(x,y)\, e^{-\beta b_+(y)} d^{\Lambda_\perp}x\, d^{\Lambda_\perp}y \\ &= {\lambda_1}^{2L_d} \sum_{n=1}^{\infty} \left(\frac{\lambda_n}{\lambda_1}\right)^{2L_d} \langle e^{-\beta b_-}, \psi_n\rangle \langle \psi_n, e^{-\beta b_+}\rangle, \end{aligned} \tag{3.5}$$

where $\lambda_n = \lambda_n^{(\Lambda_\perp)}$ are the eigenvalues of K, labeled in decreasing order, and $\psi_n = \psi_n^{(\Lambda_\perp)}$ is a corresponding orthonormal basis of eigenfunctions. We estimate (using $\lambda_n/\lambda_1 \le 1$)

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n}{\lambda_1}\right)^{2L_d} \langle e^{-\beta b_-}, \psi_n\rangle \langle \psi_n, e^{-\beta b_+}\rangle \le \frac{\operatorname{tr} K}{\lambda_1} \left\|e^{-\beta b_-}\right\|_{L^2(\mathbb{R}^{\Lambda_\perp})} \left\|e^{-\beta b_+}\right\|_{L^2(\mathbb{R}^{\Lambda_\perp})}. \tag{3.6}$$

We need a priori estimates for $\operatorname{tr} K$ and λ_1. Using (3.1), with $\tau' < |J|a_1$ (see Example 1.1 (6)), and (3.2) we get

$$\begin{aligned} \operatorname{tr} K &= \int_{\mathbb{R}^{\Lambda_\perp}} e^{-2\beta V_{\mathrm{tr}}(x) - \beta J I(x,x)} d^{\Lambda_\perp}x \\ &\le \left(\int_{\mathbb{R}^n} e^{-\beta(v(x) - |J|a_1|x|_2^2)} d^n x\right)^{|\Lambda_\perp|} e^{\beta\gamma_{\mathrm{ex}}|\partial\Lambda_\perp|} \\ &\le \left(\int_{\mathbb{R}^n} e^{-\beta(\tau' - |J|a_1)|x|_2^2} d^n x\right)^{|\Lambda_\perp|} e^{\beta(v_0+\gamma_{\mathrm{ex}})|\Lambda_\perp|} \\ &=: C_1(\beta)^{|\Lambda_\perp|}. \end{aligned} \tag{3.7}$$

Here $\gamma_{\mathrm{ex}} = 0$ for free and periodic boundary conditions, while

$$\gamma_{\mathrm{ex}} = \frac{d-1}{d}|J|a_1 \max_{1\le k\le d-1}\{|\xi_k^-|_2^2, |\xi_k^+|_2^2\}$$

for the edgewise boundary condition.

One can bound $\|e^{-\beta b_\pm}\|_{L^2(\mathbb{R}^{\Lambda_\perp})}$ similarly. On the other hand we can compare with an uncoupled problem, using (3.2),

$$\begin{aligned} V_{\mathrm{tr}}(x) + V_{\mathrm{tr}}(y) + JI(x,y) &\le V_{\mathrm{tr}}(x) + V_{\mathrm{tr}}(y) + (2d)^{-1}|J|\,a_1\,\left(|x|_2^2 + |y|_2^2\right) \\ &\le \sum_{i\in\Lambda_\perp} \frac{1}{2}\left(v(x_i) + |J|\,a_1\,|x_i|_2^2 + v_i(y_i) + |J|\,a_1\,|y_i|_2^2\right) + \gamma_{\mathrm{ex}}\,|\Lambda_\perp| \\ &= \tilde V(x) + \tilde V(y), \end{aligned}$$

where $\tilde V(x) = \sum_{i\in\Lambda_\perp} \tilde v(x_i)$ and $\tilde v(x) = v(x) + |J|a_1|x|_2^2 + \gamma_{\mathrm{ex}}$. This implies the estimate

$$\begin{aligned} \lambda_1 &\ge \langle e^{-\beta\tilde V}, K\, e^{-\beta\tilde V}\rangle / \|e^{-\beta\tilde V}\|^2_{L^2(\mathbb{R}^{\Lambda_\perp})} \\ &\ge \|e^{-\beta\tilde V}\|^2_{L^2(\mathbb{R}^{\Lambda_\perp})} \\ &= \left\|e^{-\beta\tilde v}\right\|^{2|\Lambda_\perp|}_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{3.8}$$

Combining (3.3) with (3.5)–(3.8) we find that for any $\beta > 0$ (and for any boundary condition)

$$F_\Lambda(\beta) = \frac{\lambda_1}{|\Lambda_\perp|} + O(L_d^{-1}),$$

uniformly in $\Lambda_\perp$ and L_d, but pointwise in β. Recall $F_\Lambda(\beta) = \frac{\ln Z_\Lambda(\beta)}{|\Lambda|}$. Similarly (but simpler) one can derive the same relation for $F_\Lambda(\beta)$ in the case of the periodic boundary condition, see (3.4).

In particular this shows that, modulo an $O(L_d^{-1})$ error, we can replace the boundary condition in the d-direction by the periodic boundary condition. One can now repeat the argument (with mixed boundary conditions) going through all the directions 1 through d and conclude

$$F_\Lambda(\beta) = \frac{\lambda_1^{\mathrm{per}}}{|\Lambda_\perp|} + O(L_{\min}^{-1}),$$

uniformly in Λ. Here λ_1^{per} now refers to the first eigenvalue of the transfer operator with the periodic boundary condition (on $\Lambda_\perp$). We have

Proposition 3.2. *Let $V^{(\Lambda)}$ be as in Example 3.1. Then for any $\beta > 0$*

$$F_\Lambda(\beta) = \frac{\ln \lambda_1^{\mathrm{per}}}{|\Lambda_\perp|} + O(L_{\min}^{-1}),$$

uniformly in Λ.

Proof. We have already verified the first part of the result except for the case of the fixed boundary condition. Note that the case $\xi_i = \xi_0 \in \mathbb{R}^n$ falls under the edgewise boundary condition due to Example 3.1 (1). Let $F_\Lambda(\beta)$ be the free energy with fixed boundary condition given by $\xi \in \mathbb{R}^{\mathbb{Z}^d}$, $\sup_{i\in\mathbb{Z}^d} |\xi_i|_2 < \infty$. We use Taylor's formula and the Cauchy-Schwarz inequality to estimate, for any $\delta > 0$:

$$V_0(x) - \delta\,|J| \sum_{i\in\partial\Lambda} |x_i|^2 - C_\delta\,|\partial\Lambda| \le V(x) \le V_0(x) + \delta\,|J| \sum_{i\in\partial\Lambda} |x_i|^2 + C_\delta\,|\partial\Lambda|.$$

Here V_0 is the potential with the free boundary condition. Note that

$$|\partial\Lambda| \le 2 \sum_{1\le k\le d} \Pi_{l\ne k}(2L_l + 1).$$

The constant terms contribution to the free energy is thus of the order $O(L_{\min}^{-1})$. As for the $\delta|J|\sum_{i\in\partial\Lambda} |x_i|_2^2$ error, we introduce an explicit Λ-dependence in V by replacing ω at the boundary of Λ by the following interaction term: For $i \sim_\Lambda j$ with $i \in \partial\Lambda$ and $j \in E_k^\pm$ for some k (or vice versa) we take the new interaction term to be

$$\omega_{i,j}^\pm(x, y) = \pm\,\mathrm{sign}(J)\,\delta\,|x - y|^2.$$

We write $V^\pm$ for the corresponding potential with fixed (or edgewise) boundary condition $\xi_i = 0$, $F_\Lambda^\pm(\beta)$ for the corresponding free energies. These objects are well-defined if we choose $\delta > 0$ small enough. We estimate

$$F_\Lambda^+(\beta) \le F_\Lambda(\beta) \le F_\Lambda^-(\beta).$$

One can now construct transfer operators for $V^\pm$ (for $\delta > 0$ small enough) as above and show that

$$F_\Lambda^\pm(\beta) = \frac{\ln \lambda_1^{\mathrm{per}}}{|\Lambda_\perp|} + O(L_{\min}^{-1}),$$

which completes the proof. □

The question of existence of the thermodynamic limit has been treated in [He1], for Laplace integrals, and in [He3], for transfer operators. Both results are with the periodic boundary condition and $d = n = 1$. (Recall that the lattice dimension is reduced by one when one goes to the transfer operator picture). Both proofs, however, seem to generalize to any lattice dimension $d > 1$. The convergence rate is in both cases $O(L_{\mathrm{min}}^{-1})$. Our result above now combines with the results of Helffer to show that the thermodynamic limit exists for any boundary condition (considered here) and the convergence rate is $O(L_{\mathrm{min}}^{-1})$. We note that further assumptions has to be imposed in order to apply the results of Helffer. See [R] for a comprehensive treatment of thermodynamic limits.

4. The transfer operator: The case of a unique global minimum

We will in this section prove a localization result for the transfer operator (1.2). We apply it to a translation invariant version of the example considered in section 2. We do not in this paper see any simplification in the use of the transfer operator, but we hope that one can utilize the decrease in lattice dimension (from d to $d-1$) to study the correlation length (or inverse spectral gap $(\ln\lambda_1 - \ln\lambda_2)^{-1}$) for $d \geq 2$. This viewpoint was taken in [M], in the case of spin chains ($d = 1$), where $\Lambda_\perp = \{0\}$ and K is an operator on $L^2(\mathbb{R}^n)$, which is independent of Λ. This makes it possible to show localization results under much weaker conditions than the ones considered here.

Condition 4.1. *Let $J \in \mathbb{R}$, $V_{\mathrm{tr}} = V_{\mathrm{tr}}^{(\Lambda)}$ a family of potentials satisfying Condition 2.1 with $\sigma_1 = \infty$. Let $I = I^{(\Lambda)} \in C^3(\mathbb{R}^\Lambda_x \times \mathbb{R}^\Lambda_y)$ satisfy for some $b_0 > 0$ and $C_3 > 0$*

(1) *$I(0,0) = 0$ and $\nabla_x I(0,0) = \nabla_y I(0,0) = 0$.*
(2) *$\|\nabla^2_{(x,y)} I(x,y)\|_{\infty,\infty} \leq b_0$ uniformly in $x, y \in \mathbb{R}^\Lambda$.*
(3) *For $t_1, t_2, t_3 \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ we have $|\langle \nabla^3_{(x,y)} I(x,y), t_1 \otimes t_2 \otimes t_2\rangle| \leq C_3 |t_1|_{p_1} |t_2|_{p_2} |t_3|_{p_3}$, uniformly in $x, y \in \mathbb{R}^\Lambda$, $1 \leq p_i \leq \infty$ with $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$.*
(4) *$|J| < J_0 = \frac{2(\sigma_2(V_{\mathrm{tr}}) - a_0(V_{\mathrm{tr}}))}{3b_0}$.*

For V_{tr}, I and J satisfying Condition 4.1 we define the transfer operator K via its kernel (1.2). Note that in Proposition 2.3 we could have chosen $\sigma_2 = \tau_2 - |J|a_1$ and $\sigma_1 = \infty$ instead (if we take $J_0 = \frac{\tau_2}{(n+1)a_1}$ in Example 2.2 also).

The following lemma can now be proved in the same way as Proposition 2.3, using the estimate (3.2).

Lemma 4.2. *Suppose V is as in Example 2.2 with the extra conditions given in Example 3.1. Then for $|J| < \frac{d\tau_2}{(3/2+(d-1)(n+1))a_1}$ the corresponding V_{tr} and I given in section 3 satisfies Condition 4.1 with $\sigma_1 = \infty$ (and Λ replaced by $\Lambda_\perp$). Here $b_0 = \frac{1}{d} n a_1$, $\sigma_2 = \tau_2 - \frac{d-1}{d}|J|a_1$ and $a_0 = \frac{d-1}{d} n |J| a_1$.*

Remark: The restriction on $|J|$ comes about because σ_2 and a_0 (for V_{tr}) are $|J|$-dependent. The requirement Condition 4.1 (4) therefore implies an extra assumption on the coupling strength.

We introduce a potential $V' \in C^3(\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda)$ by

$$V'(x,y) \;=\; V_{\mathrm{tr}}(x) \;+\; V_{\mathrm{tr}}(y) \;+\; JI(x,y).$$

For $\Lambda \in \Gamma$ write $\Lambda' = \Lambda \times \{-1,+1\}$ and define $\Gamma' = \{\Lambda'\}_{\Lambda\in\Gamma}$ equipped with the partial ordering $\Lambda_1' \prec \Lambda_2' \Leftrightarrow \Lambda_1 \prec \Lambda_2$. We define $n_{\Lambda'}(\lambda,\sigma) := n_\Lambda(\lambda)$, $(\lambda,\sigma) \in \Lambda'$. In particular $\mathbb{R}^{\Lambda'} = \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ and $n_0(\Gamma') = n_0(\Gamma)$. Then

Lemma 4.3. *The potential V' satisfies Condition 2.1 with Γ replaced by Γ' and we have the following relation between constants: $\omega_0(V') = \omega_0(V_{\mathrm{tr}}) + |J|b_0$, $\sigma_2(V') = \sigma_2(V_{\mathrm{tr}}) - \frac{1}{2}|J|b_0$, $a_0(V') = a_0(V_{\mathrm{tr}}) + |J|b_0$, $R_0(V') = R_0(V_{\mathrm{tr}})$ and $C_3(V') = C_3(V_{\mathrm{tr}}) + |J|C_3(I)$.*

Proof. The lemma follows easily from the form of the Hessian

$$\nabla^2_{(x,y)} V'(x,y) \;=\; \begin{pmatrix} \nabla^2 V_{\mathrm{tr}}(x) \,+\, J\,\nabla_x^2 I(x,y) & J\,\nabla_x \nabla_y I(x,y) \\ & \\ J\,\nabla_x \nabla_y I(x,y) & \nabla^2 V_{\mathrm{tr}}(y) \,+\, J\,\nabla_y^2 I(x,y) \end{pmatrix}.$$

We verify the estimate on the off-diagonal part

$$A_{V'}(x,y) \;=\; \begin{pmatrix} A_{V_{\mathrm{tr}}}(x) & 0 \\ 0 & A_{V_{\mathrm{tr}}}(y) \end{pmatrix} \;+\; J\,\nabla^2_{(x,y)} I(x,y).$$

For $s = (s_1, s_2), t = (t_1, t_2) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ we estimate

$$\begin{aligned} |\langle s, A_{V'}(x,y)t\rangle| \;&\le\; |\langle s_1, A_{V_{\mathrm{tr}}}(x)t_1\rangle| \,+\, |\langle s_2, A_{V_{\mathrm{tr}}}(y)t_2\rangle| \,+\, |J|\,|\langle s, \nabla_{(x,y)} I(x,y)t\rangle| \\ &\le\; (a_0(V_{\mathrm{tr}}) \,+\, |J|\, b_0)\; |s|_1\, |t|_\infty \\ &=:\; a_0(V')\, |s|_1\, |t|_\infty. \end{aligned}$$

That the choice of $C_3(V')$ works is clear. By duality and interpolation, Condition 4.1 (2) implies that $|I(x,y)| \le \frac{1}{2} b_0(|x|_2^2 + |y|_2^2)$. We can thus take $\sigma_2(V') = \sigma_2(V_{\mathrm{tr}}) - \frac{1}{2} b_0 |J|$ and by Condition 4.1 (4) we find $a_0(V') < \sigma_2(V')$, which completes the proof. $\square$

We note that the Perron-Frobenius Theorem implies that the highest eigenvalue $\lambda(\beta) = \lambda^{(\Lambda)}(\beta)$ of K is non-degenerate and one can choose an eigenfunction $\psi_1 = \psi_1^{(\Lambda)}$ to be strictly positive.

We wish to compare the free energy $f_\Lambda(\beta)$, see (1.4), with a localized version. As for the Laplace integral in the previous section we consider for $R > 0$ the localized operator

$$K_R(x,y) \;=\; \chi_R(x)\, K(x,y)\, \chi_R(y), \tag{4.1}$$

where $\chi_R = 1$ on B_R and $\chi_R = 0$ otherwise. The localized operator again has a simple highest eigenvalue $\lambda_R(\beta) = \lambda_R^{(\Lambda)}(\beta)$ and corresponding eigenfunction $\varphi_1 \in L^2(B_R)$, which, when extended by 0, can be viewed as a function on the full space. The corresponding localized free energy is

$$f_\Lambda^R(\beta) \;=\; \frac{\ln \lambda_R(\beta)}{|\Lambda|}. \tag{4.2}$$

Note that the first eigenvalue is given by an expectation value of K in ψ_1, which has the form of a Laplace integral. Using this observation we get

Theorem 4.4. *Assume V_{tr}, I and J satisfies Condition 4.1. For any $R > 0$, there exists $a > 0$ (independent of Λ) such that*

$$f_\Lambda^R(\beta) \;\le\; f_\Lambda(\beta) \;\le\; f_\Lambda^R(\beta) \,+\, 2\, n_0\, e^{-a\beta}.$$

Proof. First we note that

$$\begin{aligned} \lambda_R(\beta) \;&=\; \langle \varphi_1 K_R \varphi_1 \rangle_{L^2(B_R)} \\ &=\; \langle \varphi_1, K \varphi_1 \rangle_{L^2(\mathbb{R}^\Lambda)} \\ &\le\; \|K\| \;=\; \lambda(\beta). \end{aligned}$$

As for the other inequality, fix r such that $0 < r < \min\{R, R_0(V'), \sigma_1(V')/a_0(V')\}$, see Lemma 4.3. We get an $\epsilon > 0$ from Lemma 2.5 applied with r and V'. Let $\mathcal{S}$ be a family of centers $\rho = (\rho_x, \rho_y)$ such that $\Omega(\rho)$ form a covering of $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ and $\mathcal{S} \subset \epsilon(\mathbb{Z}^\Lambda \times \mathbb{Z}^\Lambda)$. We now use Lemma 2.5 and the positivity of ψ_1 to estimate

$$\begin{aligned}\lambda(\beta) &= \int_{\mathbb{R}^\Lambda\times\mathbb{R}^\Lambda} e^{-\beta V'(x,y)} \psi_1(x)\, \psi_1(y)\, d^\Lambda x\, d^\Lambda y \\ &\le \sum_{\rho\in\mathcal{S}} e^{-\beta\delta_0|\rho|_1} \int_{\Omega(\rho)} e^{-\beta V'(x-\rho_x, y-\rho_y)} \psi_1(x)\, \psi_1(y)\, d^\Lambda x\, d^\Lambda y \\ &\le \sum_{\rho\in\mathbb{Z}^\Lambda\times\mathbb{Z}^\Lambda} e^{-a\beta|\rho|_1} \int_{B_R\times B_R} e^{-\beta V'(x,y)} (T_{\rho_x}\psi_1)(x)\, (T_{\rho_y}\psi_1)(y)\, d^\Lambda x\, d^\Lambda y.\end{aligned}$$

Here $a = \delta_0\epsilon$ and $(T_s\psi)(x) = \psi(x+s)$, $s \in \mathbb{R}^\Lambda$. We recognize the integral as an expectation value and estimate further using the Cauchy-Schwarz inequality

$$\begin{aligned}\lambda(\beta) &\le \sum_{\rho\in\mathbb{Z}^\Lambda\times\mathbb{Z}^\Lambda} e^{-a\beta|\rho|_1} \|K_R\|\, \|T_{\rho_x}\psi_1\|\, \|T_{\rho_y}\psi_1\| \\ &\le (1 - e^{-a\beta})^{-2N_\Lambda} \lambda_R(\beta),\end{aligned}$$

see (2.9). This completes the proof. □

5. A class of symmetric double well potentials

In this section we turn to a class of double well problems of the form given in Example 1.1. We write $\mathbb{R}^n \ni x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. The self-energies v_i will be assumed symmetric with respect to the x_1 coordinate.

Example 5.1 (Double-well potentials). *Let V have the form given in Example 1.1 with the periodic boundary condition and $\omega_{i,j} = 0$ if $|i-j|_1 \neq 1$ (nearest neighbour interaction). We assume $J > 0$ and there exist $\{\tilde\omega_{i,j}\}_{|i-j|_1=1} \subset C^3(\mathbb{R}_z \times \mathbb{R}^{n-1}_{x'} \times \mathbb{R}^{n-1}_{y'})$, $r_c > 0$, $\tau_2 > 0$ and $a_2 > 0$ such that*

1. *The v_i's are reflection symmetric: $v_i(x_1, x') = v_i(-x_1, x')$, for any $x = (x_1, x') \in \mathbb{R}^n$.*
2. *There exists $z_i^c \ge 2r_c$ such that $\mathcal{M}_i = \{-z_i^c, z_i^c\}$.*
3. *$v_i(x) \ge \tau_2(|x_1 - z_i^c|_2^2 + |x'|_2^2)$, $x_1 \ge 0$.*
4. *$\omega_{i,j}(x,y) = \tilde\omega_{i,j}(x_1 - y_1, x', y')$, $\tilde\omega_{i,j}(z, x', y') = \tilde\omega_{i,j}(-z, x', y')$ for $x', y' \in \mathbb{R}^{n-1}$.*
5. *$z\partial_z\tilde\omega(z, x', y') \ge 0$, for all $z \in \mathbb{R}$, $x', y' \in \mathbb{R}^{n-1}$ and $\inf_{z>r_c, x', y'\in\mathbb{R}^{n-1}} \partial_z\tilde\omega_{i,j}(z, x', y') \ge a_2$.*

Remark: In the light of section 3, we note that the restriction to the periodic boundary condition is (in the translation invariant case) not a significant one. The lower bound in (3) can, at infinity, be replaced by a linear lower bound in the case $\omega_{i,j} \ge 0$. The condition (5), which is a ferromagnetic assumption, assures that the wells decouple in a way similar to the Ising model. We note that the system is only required to behave as a ferromagnet in the first variable. An example of an interaction satisfying (4-5) is $\omega_{i,j}(x,y) = (x_1 - y_1)^2 + \hat\omega(x', y')$, where $\hat\omega$ is as in Example 1.1 (with d replaced by $d-1$).

From the inequality

$$\begin{aligned}V^{(\Lambda)}(x) &\ge \sum_{i\in\Lambda} v_i(x_i) + \frac{J}{4} \sum_{\substack{i\neq j \\ i,j\in\Lambda}} \sum_{\substack{l_*\in\Lambda^* \\ |i-j-l_*|_1=1}} \tilde\omega_{i+l_*,j}(0, x_i', x_j') + \tilde\omega_{i,j-l_*}(0, x_i', x_j') \\ &\ge \sum_{i\in\Lambda} (v_i(x_i) - a_1\, |J|\, |x_i'|_2^2),\end{aligned}$$

which follows from Example 1.1 (5), (1.7) and Example 5.1 (5), we conclude that for $J < \tau_2/a_1$, $V^{(\Lambda)}$ has precisely two global minima located at $z_c^{\pm} \in \mathbb{R}^{\Lambda}$, where $\{z_c^{\pm}\}_i = (\pm z_i^c, 0) \in \mathbb{R}^n$. We take the localized partition functions to be

$$Z_{\Lambda}^{\pm,R}(\beta) = \int_{x \in \mathbb{R}^{\Lambda}, |x - z_c^{\pm}|_{\infty} \leq R} e^{-\beta V^{(\Lambda)}(x)} d^{\Lambda}x,$$

Note that, due to reflection symmetry (invariance of V under the map: $x \to \tilde{x}$ where $\tilde{x}_{i,1} = -x_{i,1}$ and $\tilde{x}' = x'$), we have $Z_{\Lambda}^{+,R}(\beta) = Z_{\Lambda}^{-,R}(\beta)$.

A central point in our analysis is a connection with the Ising model for discrete spins on $\mathbb{Z}^d$. We label the spin sectors $S_{\sigma} = \{x \in \mathbb{R}^{\Lambda} : \sigma_i x_{i,1} \geq 0\}$ by discrete spins $\sigma \in \{-1,+1\}^{\Lambda}$ and write σ^+ and σ^- for the spins with $\sigma_i^{\pm} = \pm 1$, $i \in \Lambda$.

We note that the part of V coming from the self-energies v_i is invariant under the involution

$$P_{\sigma} : S_{\sigma^+} \to S_{\sigma}$$

given by

$$\{P_{\sigma}x\}_i = (\sigma_i x_{i,1}, x_i').$$

The interaction part of V however, will in general increase under the map P_{σ}^+. We will now investigate this increase.

Let $x \in S_{\sigma^+}$ and $\sigma \neq \sigma^+$. We wish to examine the sum

$$\sum_{\substack{i \neq j \\ i,j \in \Lambda}} \sum_{\substack{l_* \in \Lambda^* \\ |i-j-l_*|_1 = 1}} \omega_{i+l_*,j}\left((P_{\sigma}x)_i, (P_{\sigma}x)_j\right) + \omega_{i,j-l_*}\left((P_{\sigma}x)_i, (P_{\sigma}x)_j\right).$$

Take $i \neq j \in \Lambda$ and $l_* \in \Lambda^*$ such that $|i-j-l_*|_1 = 1$. We consider the case $\sigma_i = -1$ and $\sigma_j = +1$. By Taylor's formula we have

$$\omega_{i+l_*,j}\left((P_{\sigma}x)_i, x_j\right) = \omega_{i+l_*,j}(x_i, x_j) - 2\int_0^1 \partial_z \tilde{\omega}_{i+l_*,j}\left(x_t, x_i', x_j'\right) x_{i,1}\, dt$$

$x_t = (1-2t)x_{i,1} - x_{j,1}$. Since $\partial_z \tilde{\omega}_{i+l_*,j}$ is an odd function of z (the first coordinate) we find that

$$-\int_0^1 \partial_z \tilde{\omega}_{i+l_*,j}\left(x_t, x_i', x_j'\right) x_{i,1}\, dt = -\int_{t_0(x_{i,1}, x_{j,1})}^1 \partial_z \tilde{\omega}_{i+l_*,j}\left(x_t, x_i', x_j'\right) x_{i,1}\, dt,$$

where $t_0(z_1, z_2) = 0$ if $z_1 > z_2$ and $t_0(z_1, z_2) = 1 - \frac{z_2}{z_1}$ if $z_1 \leq z_2$ (recall that for all $i \in \Lambda$: $x_{i,1} \geq 0$). The next step will be to employ the lower bound in Example 5.1 (5). Let $a_{i,j}(x) = a_2$ if $x_{i,1}, x_{j,1} > r_c$ and $a_{i,j}(x) = 0$ otherwise. We can now estimate

$$\begin{aligned}\omega_{i+l_*,j}\left((P_{\sigma}x)_i, x_j\right) &\geq \omega_{i+l_*,j}(x_i, x_j) + x_{i,1}\left(1 - \max\left\{\frac{1}{2}, t_0(x_{i,1}, x_{j,1})\right\}\right) a_{ij}(x) \\ &\geq \omega_{i+l_*,j}(x_i, x_j) + \frac{1}{2} r_c\, a_{i,j}(x).\end{aligned}$$

The estimate above clearly holds with $\omega_{i+l_*.j}$ replaced by $\omega_{i,j-l_*}$ as well. For general $i \neq j \in \Lambda$ and $l_* \in \Lambda^*$ with $|i-j-l_*|_1 = 1$ we thus get

$$\omega_{i+l_*,j}\left((P_{\sigma}x)_i, (P_{\sigma}x)_j\right) \geq \omega_{i+l_*,j}(x_i, x_j) + \frac{1}{8}(\sigma_i - \sigma_j)^2 r_c\, a_{ij}(x)$$

as well as the same estimate with $\omega_{i+l_*,j}$ replaced by $\omega_{i,j-l_*}$. Now let $N(x) = \#\{i : 0 \le x_i \le r_c\}$. Then

$$\sum_{\substack{i\neq j\\ i,j\in\Lambda}} \sum_{\substack{l_*\in\Lambda^*\\ |i-j-l_*|_1=1}} \omega_{i+l_*,j}((P_\sigma x)_i,(P_\sigma x)_j) + \omega_{i,j-l_*}((P_\sigma x)_i,(P_\sigma x)_j)$$
$$\ge \sum_{\substack{i\neq j\\ i,j\in\Lambda}} \sum_{\substack{l_*\in\Lambda^*\\ |i-j-l_*|_1=1}} \Big[\omega_{i+l_*,j}(x_i,x_j) + \omega_{i,j-l_*}(x_i,x_j) + \frac{1}{4}(\sigma_i - \sigma_j)^2 r_c\, a_2\Big] - N(x)\, q_d\, r_c\, a_2, \tag{5.1}$$

where $q_d = 2d$ is the number of nearest neighbours on the lattice.

Let $\mathcal{S}^+$ be a set of centers $\rho \in S_{\sigma^+}$ constructed around z_c^+ (instead of 0) in such a way that the $\Omega(\rho)$'s cover S_{σ^+} and $\Omega(\rho) \subset S_{\sigma^+}$ (see Definition 2.4). We extend this to a covering of $\mathbb{R}^\Lambda$ by choosing

$$\mathcal{S} = \big\{P_\sigma\rho : \rho \in \mathcal{S}^+,\ \sigma \in \{-1,+1\}^\Lambda\big\} \quad \text{and} \quad \Omega(P_\sigma\rho) = P_\sigma\Omega(\rho).$$

We furthermore choose the boundaries of the $\Omega(\rho)$'s such that

$$N(\rho) = N(x) \quad \text{for all} \quad x \in \Omega(\rho).$$

This means that we require $2\epsilon < r_c$. We can now estimate

$$\begin{aligned}\int_{\mathbb{R}^\Lambda} e^{-\beta V(x)} d^\Lambda x &= \sum_{\rho\in\mathcal{S}} \int_{\Omega(\rho)} e^{-\beta V(x)} d^\Lambda x\\ &= \sum_{\rho\in\mathcal{S}^+} \sum_{\sigma\in\{-1,+1\}^\Lambda} \int_{\Omega(\rho)} e^{-\beta V(P_\sigma(x))} d^\Lambda x\\ &\le \sum_{\rho\in\mathcal{S}^+} e^{\frac{1}{4}\beta J N(\rho) q_d r_c a_2} \int_{\Omega(\rho)} e^{-\beta V(x)} d^\Lambda x \sum_{\sigma\in\{-1,+1\}^\Lambda} e^{-\gamma I_\Lambda(\sigma)},\end{aligned}$$

where $\gamma = \frac{1}{8}\beta r_c a_2$. Here $I_\Lambda(\sigma)$ is the Ising energy (with periodic boundary conditions) of the discrete spin distribution σ on the d-dimensional lattice Λ, without the overall constant coming from the self-energy,

$$I_\Lambda(\sigma) = \frac{J}{2} \sum_{i\sim_\Lambda j} (\sigma_i - \sigma_j)^2.$$

The sum appearing above is essentially the partition function for the d-dimensional Ising model at temperature γ.

$$Z_\Lambda^{\mathrm{Ising}}(\gamma) = \sum_{\sigma\in\{-1,+1\}^\Lambda} e^{-\gamma I_\Lambda(\sigma)}.$$

We write

$$\Lambda = \Lambda_\perp \times \Lambda_d \quad \text{where} \quad \Lambda_\perp = \times_{k=1}^{d-1}[-L_k, L_k] \quad \text{and} \quad \Lambda_d = [-L_d, L_d].$$

and decompose $i = \{i_\perp, i_d\}$. For $\sigma \in \{-1,+1\}^\Lambda$ and $i_\perp \in \Lambda_\perp$ we write $\sigma^{i_\perp} \in \{-1,+1\}^{\Lambda_d}$ for the spin-distribution with $\sigma^{i_\perp}_{i_d} = \sigma_i$. Conversely we write $\sigma \in \{-1,+1\}^\Lambda$ for the spin-distribution on Λ composed by $\Lambda_\perp$ spin-distributions $\sigma^{i_\perp} \in \{-1,+1\}^{\Lambda_d}$, i.e. $\sigma_i = \sigma^{i_\perp}_{i_d}$. We estimate by cutting the interaction between lines

$$I_\Lambda(\sigma) \ge \sum_{i_\perp\in\Lambda_\perp} I_{\Lambda_d}(\sigma^{i_\perp}).$$

This leads us to the estimate

$$\begin{aligned} Z_\Lambda^{\mathrm{Ising}}(\gamma) \;&\le \sum_{\{\sigma^{i_\perp}\}_{i_\perp\in\Lambda_\perp}\subset\{-1,+1\}^{\Lambda_d}} \Pi_{i_\perp\in\Lambda_\perp} e^{-\gamma I_{\Lambda_d}(\sigma^{i_\perp})} \\ &= Z_{\Lambda_d}^{\mathrm{Ising}}(\gamma)^{|\Lambda_\perp|}. \end{aligned} \tag{5.2}$$

It is well-known that

$$Z_{\Lambda_d}^{\mathrm{Ising}}(\gamma) \;=\; (1 + e^{-2\gamma J})^{|\Lambda_d|} \;+\; (1 - e^{-2\gamma J})^{|\Lambda_d|}. \tag{5.3}$$

This can be computed either using the transfer matrix method or a simple recursive argument, see [T].

As in the proof of Theorem 2.6 we estimate, using Lemma 2.5,

$$Z_\Lambda(\beta) \;\le\; Z_{\Lambda_d}^{\mathrm{Ising}}(\gamma)^{|\Lambda_\perp|} \sum_{\rho\in\mathcal{S}^+} e^{-\beta\left(\delta_0|\rho - z_c^+|_1 - \frac{1}{4}JN(\rho)q_d r_c a_2\right)} \int_{\Omega(z_c^+)} e^{-\beta V(x)} d^\Lambda x.$$

We now notice that $N(\rho) \le r_c^{-1}|\rho - z_c^+|_1$, see Example 5.1 (2). This implies the following result (noticing that the choice of splitting $\Lambda = \Lambda_\perp \times \Lambda_d$ was arbitrary)

Theorem 5.2. *Let V be as in Example 5.1. For each $0 < R \le r_c$ there exist $a > 0$ and $J_0 > 0$ (independent of Λ) such that for $0 < J < J_0$*

$$|F_\Lambda(\beta) \;-\; F_\Lambda^{\pm,R}(\beta)| \;\le\; n\, e^{-a\beta} \;+\; O(L_{\min}^{-1}).$$

The decay in the last term is pointwise in β.

Another approach to this problem seems reasonable. Namely, to apply the method of section 2 to each of the spin-sectors $\mathcal{S}_\sigma$ separately and then use the Ising model to sum up the contributions. We were however not able to overcome the technical difficulties. This method, if made to work, could probably be applied to a non-symmetric double-well problem and perhaps one could then show that the free energy is determined by a localization to the shallowest of the two global minima. See [M] for such an argument in fixed dimension.

We end this section with the double-well problem, from the point of view of transfer operators. We consider Example 5.1, plus isotropy; $v_i = v$ and $\omega_{i,j} = \omega$. For the corresponding V_{tr} and I, see section 3.

We use spin-pairs $\underline{\sigma} = (\sigma_x, \sigma_y)$, $\sigma_x, \sigma_y \in \{-1,+1\}^{\Lambda_\perp}$, to label the spin-sectors $S_{\underline{\sigma}} = S_{\sigma_x} \times S_{\sigma_y}$, where S_{σ_x} and S_{σ_y} are given as above. We introduce an involution $P_{\underline{\sigma}} : S_{\underline{\sigma}^+} \to S_{\underline{\sigma}}$, where $\underline{\sigma}^+ = (\sigma^+, \sigma^+)$,

$$P_{\underline{\sigma}}(x,y) \;=\; (P_{\sigma_x}x, P_{\sigma_y}y).$$

We take centers $\mathcal{T}_+$ of the form $\rho = (\rho_x, \rho_y)$ where $\rho_x, \rho_y \in \mathcal{S}_+$. We use neighbourhoods $\Theta(\rho) = \Omega(\rho_x) \times \Omega(\rho_y)$, where the Ω's form a covering, as above, of S_{σ^+}. Then $\mathcal{T} = \bigcap_{\underline{\sigma}} P_{\underline{\sigma}}\mathcal{T}_+$ and $\Theta(P_{\underline{\sigma}}\rho) = P_{\underline{\sigma}}\Theta(\rho)$. We have

$$\begin{aligned} \lambda(\beta) \;&= \int_{\mathbb{R}^{\Lambda_\perp}} \int_{\mathbb{R}^{\Lambda_\perp}} K(x,y)\, \psi_1(x)\, \psi_1(y)\, d^{\Lambda_\perp}x\, d^{\Lambda_\perp}y \\ &\le \sum_{\rho\in\mathcal{T}} \int_{\Theta(\rho)} K(x,y)\, \psi_1(x)\, \psi_1(y)\, d^{\Lambda_\perp}x\, d^{\Lambda_\perp}y \\ &= \sum_{\underline{\sigma}} \sum_{\rho\in\mathcal{T}_+} \int_{\Theta(\rho)} K(P_{\underline{\sigma}}(x,y))\, \psi_1(P_{\sigma_x}x)\, \psi_1(P_{\sigma_y}y)\, d^{\Lambda_\perp}x\, d^{\Lambda_\perp}y. \end{aligned}$$

Let $V'(x,y) = V_{\rm tr}(x) + JI(x,y) + V_{\rm tr}(y)$. Then $K(x,y) = e^{-\beta V'(x,y)}$. We estimate using (5.1) for $(x,y) \in S_{\underline{\sigma}^+}$

$$\begin{aligned} V'(P_{\underline{\sigma}}(x,y)) &= V_{\rm tr}(P_{\sigma_x} x) + J\,I(P_{\sigma_x} x, P_{\sigma_y} y) + V_{\rm tr}(P_{\sigma_y} y) \\ &\geq V'(x,y) + \sum_{i \sim_{\Lambda_\perp} j} \frac{Jc}{2} \left[(\sigma_{x,i} - \sigma_{x,j})^2 + (\sigma_{y,i} - \sigma_{y,j})^2\right] - J\,c\,[N(x) + N(y)], \end{aligned}$$

for some $c > 0$. (We used the simple estimate $I(P_{\sigma_x} x, P_{\sigma_y} y) \geq I(x,y)$.) This implies that

$$\lambda(\beta) \leq \sum_{\underline{\sigma}} e^{-\gamma I_{\Lambda_\perp}(\sigma_x)} e^{-\gamma I_{\Lambda_\perp}(\sigma_y)} \sum_{\rho \in \mathcal{T}_+} e^{\beta J c (N(\rho_x) + N(\rho_y))} \int_{\Theta(\rho)} K(x,y)\,\psi_1(P_{\sigma_x} x)\,\psi_1(P_{\sigma_y} y)\, d^{\Lambda_\perp} x\, d^{\Lambda_\perp} y,$$

where $\gamma = c\beta$. From Lemmas 4.2, 4.3 and 2.5, we get a $\delta_0 > 0$, such that for $\rho \in \mathcal{T}_+$ and $(x,y) \in \Theta(\rho)$

$$V'(x,y) - V'(x - \rho_x, y - \rho_y) \geq \delta_0 \left(|\rho_x|_1 + |\rho_y|_1\right).$$

Let (see also (1.4), (4.1) and (4.2))

$$f_{\Lambda_\perp}(\beta) = \frac{\lambda(\beta)}{|\Lambda_\perp|} \quad \text{and} \quad f^R_{\Lambda_\perp}(\beta) = \frac{\lambda_R(\beta)}{|\Lambda_\perp|},$$

where, as in section 4, $\lambda_R(\beta)$ is the first eigenvalue of the transfer operator K localized to the ball $B_R(z_c^+) = B_R + z_c^+$.

We can now proceed as in the proof of Theorem 4.4, using (5.2) and (5.3), and obtain

Theorem 5.3. *Let V be as in Example 5.1 and assume the system is translation invariant ($v_i = v$ and $\omega_{i,j} = \omega$). Then for any $0 < R < r_c$ there exist $a > 0$ and $J_0 > 0$ such that for $0 < J < J_0$*

$$|f_{\Lambda_\perp}(\beta) - f^R_{\Lambda_\perp}(\beta)| \leq 2\,n\,e^{-a\beta} + O(L_{\rm min}^{-1}).$$

The decay in the last term is pointwise in β.

Here $L_{\rm min} = \min\{L_k : 1 \leq k \leq d-1\}$. Note that this result combines with Proposition 3.2 to give a complicated proof of Theorem 5.2.

6. Complete low-temperature expansions

We will not state the result of this section in its most general form. Instead we will focus on our examples. In this way we avoid introducing Sjöstrand's standard function calculus, see [Sj3].

We consider

Example 6.1. *Assume $J > 0$ and V is as in either Example 2.2 or Example 5.1, with $n = 1$. Suppose furthermore that*

(1) $v_i \in C^\infty(\mathcal{M}_i(R_0))$, $\omega_{i,j} \in C^\infty(\mathcal{M}_i(R_0) \times \mathcal{M}_j(R_0))$.
(2) $\sup_{i \in \mathbb{Z}^d} \sup_{x \in \mathcal{M}_i(R_0)} |\partial^k v_i(x)| < \infty$, *for $k \geq 4$.*
(3) *For $k + l \geq 4$:*

$$\sup_{i \in \mathbb{Z}^d} \sum_{j (\neq i)} \sup_{x \in \mathcal{M}_i(R_0), y \in \mathcal{M}_j(R_0)} |\partial_x^k \partial_y^l \omega_{i,j}(x,y)| + \sup_{x \in \mathcal{M}_j(R_0), y \in \mathcal{M}_i(R_0)} |\partial_x^k \partial_y^l \omega_{j,i}(x,y)| < \infty.$$

Under these assumptions one can verify that $V = V^{(\Lambda)}$ is a standard function family in the sense of [Sj3]. We now use the results of Helffer, [He1], and Sjöstrand, [Sj3], in conjunction with Theorem 2.6 and Theorem 5.2 to obtain

Theorem 6.2. *Suppose V is as in Example 6.1 and J is small enough. For each $\Lambda = \Lambda(\underline{L}) \subset \mathbb{Z}^d$ there exists $\{F_k^{(\Lambda)}\}_{k\geq 0} \subset \mathbb{R}$ with $\sup_{\Lambda\in\Gamma} |F_k^{(\Lambda)}| < \infty$ such that: For any $N > 0$ there exists $\beta_0 > 0$ and $C > 0$ independent of Λ such that for $\beta > \beta_0$*

$$|F_\Lambda(\beta) - \sum_{k=0}^{N} F_k^{(\Lambda)} \beta^{-k}| \leq C\,\beta^{-N-1} + O(L_{\min}^{-\epsilon}).$$

The decay in the last term is pointwise in β. In particular

$$\limsup_{\Lambda\to\infty} |F_\Lambda(\beta) - \sum_{k=0}^{N} F_k^{(\Lambda)} \beta^{-k}| \leq C\,\beta^{-N-1}.$$

Remarks:

1) The construction of [He1] and [Sj3] had previously appeared in another context in [Sj2].

2) We will not be more precise about the statement J small enough. We just note that $J > 0$ can be of the same order of magnitude as in Theorem 2.6 and Theorem 5.2.

3) The expansion coefficients F_k can be written down (more or less) explicitly in terms of $\nabla^k V(0)$ in the case of Example 2.2 and $\nabla^k V(z_c^+)$ in the case of Example 5.1, see [M].

4) One can formulate a similar theorem for a class of transfer operators with V_{tr} a standard function family and $I(x,y) = \frac{1}{2}|x-y|^2$. See [He2].

A standard function calculus with weights has been introduced by Helffer and Sjöstrand to study convergence problems in the thermodynamic limit. One can, in the case of translation invariant systems with periodic boundary conditions, use this method to show that $F_\Lambda(\beta)$ and the expansion coefficients $F_k^{(\Lambda)}$ has a thermodynamic limit: $F(\beta) = \lim_{\Lambda\to\infty} F_\Lambda(\beta)$ and $F_k = \lim_{\Lambda\to\infty} F_k^{(\Lambda)}$. See [Sj3] and references therein. (The convergence rate for the $F_k^{(\Lambda)}$'s is exponential in $L_{\min}$). This has been applied to Schrödinger operators, see [Sj1], [HeSj1] and [HeSj2], Laplace integrals, see [He1], and transfer operators in the strong coupling limit $J \to \infty$, see [HeRa]. (The three problems are technically very similar). The result above now implies statements of the following type. For each $N > 0$, there exist $\beta_0 > 0$ and $J_0 > 0$ such that for $|J| < J_0$ and $\beta > \beta_0$

$$F(\beta) = \sum_{k=0}^{N} F_k \beta^{-k} + O(\beta^{-N-1}).$$

We will not supply a proof of this statement here or give precise assumptions on v and ω_{i-j}. Again, the arguments given in the literature are for potentials with a one-dimensional lattice structure, but the results should generalize without problems.

Appendix A. Proof of Proposition 2.3

This appendix is devoted to the proof of Proposition 2.3.

Proof. The $C(\Lambda)$'s given in the statement of the proposition clearly satisfies Condition 2.1 (1).

As for (2), (3) and (4) we start by analyzing the diagonal and off-diagonal part of the Hessian of the interaction term $W(x) = \frac{1}{2}\sum_{i\neq j} \omega_{i,j}(x_i - x_j)$. We pick A to be the matrix with ($n \times n$-matrix valued) entries $a_{i,i} = 0$, $i \in \Lambda$, and for $i, j \in \Lambda$ with $i \neq j$ we take $a_{i,j} = J\{\nabla^2 W(x)\}_{i,j}$. In the case of free, fixed and edgewise boundary conditions we have

$$a_{i,j} \;=\; \frac{J}{2}\,(\nabla_x \nabla_y \omega_{i,j}(x_i, x_j) \;+\; \nabla_x \nabla_y \omega_{j,i}(x_j, x_i))\,,$$

while

$$a_{i,j} = \frac{J}{4} \sum_{l_* \in \Lambda^*} \nabla_x \nabla_y \omega_{i+l_*,j}(x_i, x_j) + \nabla_x \nabla_y \omega_{i,j-l_*}(x_i, x_j) + \nabla_x \nabla_y \omega_{j+l_*,i}(x_j, x_i) + \nabla_x \nabla_y \omega_{j,i-l_*}(x_j, x_i),$$

in the case of periodic boundary conditions. Let $s, t \in \mathbb{R}^\Lambda$. Then, by Example 1.1 (5) and (1.7) (for $k = l = 1$),

$$\begin{aligned} |\langle s, At\rangle| \;&\le\; \sum_{i\in\Lambda} |s_i|_2 \sum_{j(\neq i)} |a_{i,j}|_{2,2} |t_j|_2 \\ &\le\; n|J| \sum_{i\in\Lambda} |s_i|_1 a_1 \sup_{j\in\Lambda} |t_j|_\infty \\ &=\; n|J| a_1 |s|_1 |t|_\infty. \end{aligned}$$

This shows that $\|A\|_{\infty,\infty} \le a_0 = n|J|a_1$. Note that we also have the bound

$$\|A\|_{2,2} \;\le\; |J|\, a_1. \tag{A.1}$$

The contribution from the self-energy part, $\sum v_i$, of the potential is easily accounted for. We concentrate on the interaction part:

$$W(x) \;=\; W(0) \;+\; \nabla W(0) \cdot x \;+\; \int_0^1 (1-t)\langle \nabla^2 W(tx)x, x\rangle. \tag{A.2}$$

We have already computed the off-diagonal part of $J\nabla^2 W$ above. As for the block-diagonal part we have three cases. The free boundary condition:

$$\{\nabla^2 W\}_{i,i} \;=\; \frac{1}{2} \sum_{j \in \Lambda\setminus\{i\}} \nabla_x^2 \omega_{i,j}(x_i, x_j) \;+\; \nabla_y^2 \omega_{j,i}(x_j, x_i).$$

The fixed and edgewise boundary condition:

$$\{\nabla^2 W\}_{i,i} \;=\; \frac{1}{2} \sum_{j \in \mathbb{Z}^d\setminus\{i\}} \nabla_x^2 \omega_{i,j}(x_i, x_j) \;+\; \nabla_y^2 \omega_{j,i}(x_j, x_i),$$

with the convention that $x_j = \xi_j$ (fixed) or $x_j = \xi_k^\pm$, $j \in E_k^\pm$, (edgewise) if $j \notin \Lambda$. The periodic boundary condition:

$$\begin{aligned} \{\nabla^2 W\}_{i,i} \;=\; \frac{1}{4} \sum_{j \in \Lambda\setminus\{i\}} \sum_{l_* \in \Lambda^*} (\nabla_x^2 \omega_{i+l_*,j}(x_i, x_j) \;+\; \nabla_x^2 \omega_{i,j-l_*}(x_i, x_j) \\ +\; \nabla_y^2 \omega_{i+l_*,j}(x_j, x_i) \;+\; \nabla_y^2 \omega_{j,i-l_*}(x_j, x_i)). \end{aligned}$$

By Example 1.1 (5), (1.7) and (A.1), we get uniformly in $\mathbb{R}^\Lambda$

$$\|\nabla^2 W(x)\|_{2,2} \;\le\; \sup_{\Lambda\in\Gamma} \sup_{i\in\Lambda} \|\{\nabla^2 W\}_{i,i}\|_{2,2} \;+\; J^{-1}\,\|A\|_{2,2} \;\le\; 2\, a_1. \tag{A.3}$$

(In particular this shows that we can choose $\omega_0 = \sup_{i\in\mathbb{Z}^d}(|\nabla^2 v_i(0)|_{2,2}+2|J|a_1)$. Note that $W(0)=0$ and $\nabla W(0)=0$ in the case of free and periodic boundary conditions. In these two cases we proceed as follows (see (A.2)): Choose $C(\Lambda)=0$. Pick a σ_2 satisfying $a_0<\sigma_2<\Sigma-|J|a_1$, which is possible by Example 2.2 (3). There exists $R_1>0$, see Example 1.1 (3), such that $v_i(x_i)-|J|a_1|x_i|_2^2 \geq \sigma_2|x_i|_2^2$, for $|x_i|_\infty \leq R_1$. Pick $\sigma_1 = R_1(\tau_2-|J|a_1)$, which is positive by Example 2.2 (3). Then, by (A.3) and (A.2) , the lower bound in Condition 2.1 (3) is satisfied.

We are left with estimating $W(0)$ and $\nabla W(0)$ in the case of fixed and edgewise boundary conditions. In the case of the edgewise boundary condition we have nearest neighbour interaction on a box and it is easy to verify that

$$|W(0)| + |\nabla W(0)|_1 \leq C\,|\partial\Lambda| \quad \text{and} \quad |\nabla W(0)\cdot x| \leq C_\delta\,|\partial\Lambda| + \delta\sum_{i\in\partial\Lambda} x_i^2,$$

for any $\delta>0$. We take $a_0<\sigma_2<\Sigma-|J|a_1$ and $C(\Lambda)=J_0(C+C_\delta)|\partial\Lambda|$, for small enough $\delta>0$. We can now choose σ_1 as above.

As for the fixed boundary condition we will need to use the stronger decay assumption (1.6). We introduce a foliation of Λ

$$\Lambda = \bigcup_{k=1}^{\nu_\Lambda} \partial_k\Lambda,$$

where $\nu_\Lambda = [(L_{\min}+1)/2]$ and

$$\partial_k\Lambda = \{i\in\Lambda : \min\{|i-j|_\infty : j\notin\Lambda\}=k\}.$$

Note that $\partial_1\Lambda=\partial\Lambda$. We write

$$W(0) = \sum_{k=1}^{\nu_\Lambda} W_k, \quad \text{where} \quad W_k = \frac{1}{2}\sum_{i\in\partial_k\Lambda}\sum_{j\notin\Lambda}\omega_{i,j}(0,\xi_j)+\omega_{j,i}(\xi_j,0). \tag{A.4}$$

By Taylor's formula, Example 1.1 (4) and (1.6) we get the estimate

$$|W_k| \leq C\sum_{i\in\partial_k\Lambda}\sum_{j\notin\Lambda}\Pi_{l=1}^d(|i_l-j_l|+1)^{-1-\epsilon}.$$

We use that $\Lambda^c = \cup_{l=1}^d\{j\in\mathbb{Z}^d : |j_l|>L_l\}$ (not a disjoint union) to estimate further

$$\begin{aligned}|W_k| &\leq C\sum_{i\in\partial_k\Lambda}\sum_{l=1}^d\sum_{j_l\notin[-L_l,L_l]}(|i_l-j_l|+1)^{-1-\epsilon}\\ &\leq C\,|\partial_k\Lambda|\,k^{-\epsilon}\\ &\leq C\,|\Lambda|\,(L_{\min}-k)^{-1}\,k^{-\epsilon}.\end{aligned} \tag{A.5}$$

Here we used that

$$\begin{aligned}|\partial_k\Lambda| &\leq \sum_{l=1}^d 2\Pi_{l'\neq l}(2(L_{l'}-k+1)+1)\\ &\leq 2\,d\,[\Pi_{l=1}^d(2(L_l-k)+1)]\,(2(L_{\min}-k+1)+1)^{-1}\\ &\leq d\,|\Lambda|\,(L_{\min}-k)^{-1}.\end{aligned}$$

Note that for any $\epsilon < 1$ (which we can assume) the function

$$f_s(t) \,=\, (s-t)^{-1}\, t^{-\epsilon}, \qquad t \,\in\, [1, s-1]$$

takes its maximum, for large enough s, at $t = \frac{\epsilon}{1+\epsilon}s$. Hence, $f_s(t) \leq C_\epsilon s^{-1-\epsilon}$ for large s. It thus follows, by (A.4) and (A.5), that for $L_{\min}$ large enough:

$$|W(0)| \,\leq\, C\,|\Lambda| \sum_{k=1}^{\nu_\Lambda} (L_{\min} - k)^{-1}\, k^{-\epsilon} \,\leq\, C\,|\Lambda|\, L_{\min}^{-\epsilon}.$$

Similarly we can estimate

$$|\nabla W(0)|_1 \,\leq\, C\,|\Lambda|\, L_{\min}^{-\epsilon} \quad \text{and} \quad |\nabla W(0)\cdot x| \,\leq\, C_\delta\,|\Lambda|\, L_{\min}^{-\epsilon} \,+\, \delta\,|x|_2^2,$$

for any $\delta > 0$. We take again $a_0 < \sigma_2 < \Sigma - |J|a_1$, and $C(\Lambda) = J_0(C + C_\delta)|\Lambda| L_{\min}^{-\epsilon}$, for $\delta > 0$ small enough. We choose σ_1 as above.

As for the remaining bound (5), it follows from Example 1.1 (3) and (5). This concludes the proof. □

Appendix B. A counter example

In this appendix we show that the Laplace integral (1.1) does not localize in any l^p-neighbourhood, with $p < \infty$. This shows that l^∞ neighbourhoods is the natural choice. Take $\Gamma = \mathbb{N}$ with the natural ordering and $n_m(i) = 1$. We choose a harmonic potential, namely $v_i(x_i) = x_i^2$ and $J = 0$. Let $B_r^p = \{x \in \mathbb{R}^m : |x|_p \leq r\}$. For $\sigma \in \{-1,+1\}^m$ we take centers $\rho^\sigma \in \mathbb{R}^m$ defined by $\rho_i^\sigma = \frac{\sigma_i}{m^{1/p}}$. In the following we fix $\delta < \frac{1}{2}$. For each center we take an l^p neighbourhood $\Omega(\rho^\sigma) = \{x \in \mathbb{R}^m : |x - \rho^\sigma|_p \leq \delta\} = B_\delta^p + \rho$. Note that $B_\delta^p \cap \Omega(\rho^\delta) = \emptyset$. We have the lower bound

$$\int_{\mathbb{R}^m} e^{-\beta V(x)} d^m x \geq \int_{B_\delta^p} e^{-\beta V(x)} d^m x + \sum_{\sigma \in \{-1,+1\}^m} \int_{\Omega(\rho^\sigma)} e^{-\beta V(x)} d^m x$$

We now estimate for $\sigma \in \{-1,+1\}^m$ and $x \in \Omega(\rho^\sigma)$

$$\begin{aligned} V(x) \,-\, V(x-\rho^\sigma) \,&=\, |\rho^\sigma|_2^2 \,+\, 2\,\rho^\sigma\cdot(x-\rho^\sigma) \\ &\leq\, m^{1-2/p} \,+\, 2\,\delta\,|\rho^\sigma|_q \\ &=\, (1 \,+\, 2\,\delta)\, m^{1-2/p}. \end{aligned}$$

Here $1/p + 1/q = 1$. Combining these two estimates we get

$$\int_{\mathbb{R}^m} e^{-\beta V(x)} d^m x \,\geq\, \left(1 \,+\, 2^m\, e^{-2\beta m^{1-2/p}}\right) \int_{B_\delta^p} e^{-\beta V(x)} d^m x.$$

This implies that

$$\liminf_{m\to\infty} \frac{1}{m} \left\{ \ln\left(\int_{\mathbb{R}^m} e^{-\beta V(x)} d^m x \right) \,-\, \ln\left(\int_{B_\delta^p} e^{-\beta V(x)} d^m x \right) \right\} \,\geq\, \ln 2,$$

which shows that the low-temperature localization fails to hold in an l^p ball if $p < \infty$.

In [So] a semiclassical localization problem for the Schrödinger operator with uniform control in high dimension was considered. The study pertained to the first eigenvalue in the case of globally convex potentials. Instead of using an l^∞ neighbourhood the author used l^2 neighbourhoods which scaled with the dimension. That is, of the form $\{x \in \mathbb{R}^m \,:\, m^{-1/2}|x|_2 \,\leq\, R\}$. Notice that, for $1 \leq p < \infty$,

$$\{x \,\in\, \mathbb{R}^m \,:\, |x|_\infty \,\leq\, R\} \,\subset\, \{x \,\in\, \mathbb{R}^m \,:\, m^{-1/p}|x|_p \,\leq\, R\}.$$

This shows that Theorem 2.6 imply localization results with respect to the type of dimensionally scaled l^p neighbourhoods considered in [So].

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