SPECTRAL ASYMPTOTICS OF THE DIRICHLET-TO-NEUMANN MAP ON MULTIPLY CONNECTED DOMAINS IN $\mathbb{R}^d$

P. D. Hislop [1]

Department of Mathematics
University of Kentucky
Lexington, KY 40506–0027 USA

C. V. Lutzer

Department of Mathematics and Statistics
Rochester Institute of Technology
Rochester, NY 14623-5603 USA

Abstract

We study the spectral asymptotics of the Dirichlet-to-Neumann operator Λ_γ on a multiply-connected, bounded, domain in $\mathbb{R}^d$, $d \geq 3$, associated with the uniformly elliptic operator $L_\gamma = -\sum_{i,j=1}^d \partial_i \gamma_{ij} \partial_j$, where γ is a smooth, positive-definite, symmetric matrix-valued function on Ω. We prove that the operator is approximately diagonal in the sense that $\Lambda_\gamma = D_\gamma + R_\gamma$, where D_γ is a direct sum of operators, each of which acts on one boundary component only, and R_γ is a smoothing operator. This representation follows from the fact that the γ-harmonic extensions of eigenfunctions of Λ_γ vanish rapidly away from the boundary. Using this representation, we study the inverse problem of determining the number of holes in the body, that is, the number of the connected components of the boundary, by using the high-energy spectral asymptotics of Λ_γ.

September 12, 2001

[1] Supported in part by NSF grant DMS-9707049.

1 Introduction and Main Results

We study the spectral asymptotics of the Dirichlet-to-Neumann (DN) operator Λ_γ associated with a uniformly elliptic, second-order differential operator L_γ, on a bounded, multiply-connected region $\Omega \subset \mathbb{R}^d$, $d \geq 3$, with smooth boundary. The elliptic operator L_γ has the form

$$L_\gamma \equiv - \sum_{j,k=1}^{d} \partial_j \; \gamma_{jk} \; \partial_k, \tag{1.1}$$

where we assume that the $d \times d$-matrix-valued function $\gamma(x) = [\gamma_{jk}(x)]$ satisfies the following hypotheses:

H1. The real coefficients satisfy $\gamma_{jk}(x) = \gamma_{kj}(x) \in C^\infty(\overline{\Omega})$.
H2. There exist constants $0 < \lambda_0 \leq \lambda_1 < \infty$ such that for all $\xi \in \mathbb{R}^d$, we have

$$\lambda_0 \|\xi\|^2 \leq \sum_{j,k=1}^{d} \xi_j \xi_k \gamma_{jk}(x) \leq \lambda_1 \|\xi\|^2.$$

The DN map Λ_γ is defined as follows. Let $f \in C^\infty(\partial\Omega)$, and denote by $u_f(x)$ the unique solution of the Dirichlet problem

$$\begin{aligned} L_\gamma u(x) = 0, &\qquad x \in \Omega \\ u \mid \partial\Omega = f &\qquad x \in \partial\Omega. \end{aligned} \tag{1.2}$$

We then define

$$\Lambda_\gamma f \equiv \sum_{l,m=1}^{d} \left\{ \nu_l \gamma_{lm} \frac{\partial u_f}{\partial x_m} \right\} \Big| \; \partial\Omega, \tag{1.3}$$

where ν denotes the outward normal vector on $\partial\Omega$. The DN operator Λ_γ extends to a bounded map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. Furthermore, the DN operator Λ_γ is an unbounded, self-adjoint operator on $L^2(\partial\Omega)$ with a compact resolvent (cf. [21, 23]). Consequently, the L^2-spectrum of Λ_γ is discrete with no finite accumulation point. Let $\{\lambda_j \mid \lambda_1 = 0, \; \lambda_j \leq \lambda_{j+1}, j = 1, 2, \ldots\}$ denote the eigenvalues of the DN operator Λ_γ listed in nondecreasing order, including multiplicity. Let $\Lambda_{\gamma,\Omega}(x,\xi)$, for $(x,\xi) \in T^*\partial\Omega$, be the symbol of the first-order, elliptic pseudodifferential operator Λ_γ. The eigenvalues λ_j of Λ_γ satisfy classical Weyl asymptotics (cf. [8], chapter XXIX, or [20], chapter XII):

$$\lambda_j \sim (j / C(\partial\Omega, \Lambda_{\gamma,\partial\Omega}))^{1/(d-1)},$$

where

$$\begin{aligned} C(\partial\Omega, \Lambda_{\gamma,\partial\Omega}) &= (2\pi)^{-(d-1)} \text{ Vol } \{(x,\xi) \in T^*\partial\Omega\backslash\{0\} \mid \Lambda_{\gamma,\partial\Omega}(x,\xi) \leq 1\} \\ &= (2\pi)^{-(d-1)} \int_{\Lambda_{\gamma,\partial\Omega}(x,\xi)\leq 1} dx\ d\xi. \end{aligned}$$

The unique solution u_f of the Dirichlet problem (1.2) with boundary datum f is called the γ*-harmonic extension of* f. Our first result concerns the localization of the γ-harmonic extension of an eigenfunction of Λ_γ near the boundary. We say that a function g *decays rapidly*, written $g(m) = O(m^{-\infty})$, if $\lim_{m\to\infty} m^k g(m) = 0$, for every $k \in I\!N$.

Theorem 1.1. *Let* ϕ_m *be an eigenfunction of* Λ_γ *satisfying* $\Lambda_\gamma \phi_m = \lambda_m \phi_m$, *with* $\|\phi_m\|_{L^2(\partial\Omega)} = 1$, *and let* u_m *be the* γ*-harmonic extension of* ϕ_m *to* Ω. *For any compact* $K \subset \Omega$, $||u_m||_{H^1(K)} = O(m^{-\infty})$.

This result reflects the fact that the eigenfunctions of the DN operator Λ_γ become highly oscillatory as the eigenvalue increases, and hence the γ-harmonic extensions decay rapidly away from the boundary. We believe that the decay is actually of order $e^{-\text{dist}(K,\partial\Omega)|m|}$ in the case of an analytic boundary and analytic coefficients, but we have not been able to prove this. The localization result in Theorem 1.1 is the basis of our other results.

We are interested in the situation when $\partial\Omega$ consists of k mutually disconnected components, $\partial\Omega_j$, with each boundary component $\partial\Omega_j$ a smooth, connected, compact surface: $\partial\Omega = \cup_{j=1}^k \partial\Omega_j$. We label the boundary components so that $\partial\Omega_1$ is the boundary of the unbounded component of $I\!R^d\backslash\Omega$. We note that the regions $\text{Int}(\partial\Omega_j)$, bounded by the other boundary components $\partial\Omega_j$, are disjoint. They are contained in the bounded region interior to $\partial\Omega_1$. It follows that $L^2(\partial\Omega) = \oplus_{j=1}^k L^2(\partial\Omega_j)$. We write $\phi \in L^2(\partial\Omega)$ as the k-tuple $\phi = (\phi_1, \phi_2, \ldots, \phi_k)$, where $\phi_j = \phi \mid \partial\Omega_j$. The DN operator Λ_γ is a map between k-tuples of functions defined on the boundary. For each boundary component, we define the restriction operator $R_j : L^2(\partial\Omega) \to L^2(\partial\Omega_j)$ by $R_j\phi = \phi_j$, and the extension operator $E_j : L^2(\partial\Omega_j) \to L^2(\partial\Omega)$ by $E_j\phi = \psi$, where $\psi_i = 0$, when $i \neq j$, and $\psi_j = \phi$. It is easy to check that $C_j \equiv E_j R_j$ is an orthogonal projection on $L^2(\partial\Omega)$. The family of operators $\{C_j \mid j = 1, \ldots, k\}$ satisfies $C_j C_l = \delta_{jl} C_j$, and $Id = \sum_{j=1}^k C_j$, on $L^2(\partial\Omega)$. A pseudodifferential operator P on a smooth manifold X is said to be *smoothing* if $P : H^{-t}(X) \to H^s(X)$, for all $s, t \in I\!R^+$.

Theorem 1.2. *The DN operator* Λ_γ *admits a decomposition into two self-adjoint operators* $\Lambda_\gamma = D_\gamma + R_\gamma$, *where* $D_\gamma = \sum_{j=1}^k C_j \Lambda_\gamma C_j$, *and* R_γ *is a smoothing operator.*

Viewed as a map between k-tuples of functions on $\partial\Omega$, the DN operator Λ_γ can be represented as a $k \times k$-matrix. Theorem 1.2 indicates that, in this representation, the DN map Λ_γ is diagonal up to a smoothing error. The fact that Λ_γ and D_γ differ by a smoothing operator indicates that their spectra have the same asymptotics.

Theorem 1.3. *Let $\{\mu_l \mid l \in I\!N\}$ be the eigenvalues of D_γ, written in nondecreasing order, including multiplicity, and let $\{\lambda_l \mid l \in I\!N\}$ be the eigenvalues of Λ_γ written similarly. Then, we have $\lambda_l = \mu_l + O(l^{-\infty})$.*

We can make precise the nature of the operators $C_j \Lambda_\gamma C_j$ composing the diagonal operator D_γ. We define operators Λ_j on the boundary component Ω_j by $\Lambda_j \equiv R_j \Lambda_\gamma E_j : L^2(\partial\Omega_j) \to L^2(\partial\Omega_j)$. Since $R_j E_j = 1$ on $L^2(\partial\Omega_j)$, it is clear that the eigenvalues of $C_j \Lambda_\gamma C_j$ coincide with those of Λ_j. Although the operator Λ_j involves the other boundaries through Λ_γ, we will show that the effect of the other boundary components is small in the high-energy limit. One result in this direction is the following (see section 5 for more details). We extend $\gamma_{ij} \in C^\infty(\overline{\Omega})$ to be a smooth function on all of $I\!R^d$. We assume that the extended matrix of functions $\gamma = [\gamma_{ij}]$ remains symmetric and uniformly elliptic so that H2 remains valid for some constants $0 < \lambda_0 \leq \lambda_1 < \infty$. Furthermore, we assume

H3. There exists $0 < R < \infty$ so that $\Omega \subset B_R(0)$, and $\gamma_{ij}(x) = \delta_{ij}$, for $\|x\| > R$.

It follows from section 5 that the difference of two DN operators associated with various connected components of $\partial\Omega$, and constructed with two different extensions of γ, is a smoothing operator.

We introduce operators $\Lambda_j^\#$ that involve only the j^{th}-boundary component as follows. For $\partial\Omega_1$, we denote by $\Omega_1^\#$ the bounded component of $I\!R^d \backslash \partial\Omega_1$. For the other boundary components with $1 < j \leq k$, let $\Omega_j^\#$ be the unbounded component of $I\!R^d \backslash \partial\Omega_j$. We define operators $\Lambda_j^\# : L^2(\partial\Omega_j) \to L^2(\partial\Omega_j)$, acting on a single boundary component, by $\Lambda_j^\# f = \nu \cdot \gamma \nabla u_f^\# | \partial\Omega_j$, where $u_f^\#$ is the unique γ-harmonic extension of f to $\Omega_j^\#$, that decays at infinity for $1 < j \leq k$ (see the appendix, section 8), and ν is the outward normal to that region.

Theorem 1.4. *The difference $\Lambda_j - \Lambda_j^\#$, on $L^2(\partial\Omega_j)$, is a smoothing operator. Consequently, the difference of the m^{th} eigenvalues of these two operators vanishes like $O(m^{-\infty})$.*

We now discuss the application of these results to the inverse problem of determining the connectivity of a body Ω from the high-energy asymptotics of the DN operator Λ_γ. By the *weighted measure* of the boundary component $\partial\Omega_j$, we mean the constant, $C(\partial\Omega_j, \Lambda_{\gamma,\partial\Omega})$, similar to the one appearing in the Weyl eigenvalue asymptotics, given by $C(\partial\Omega_j, \Lambda_{\gamma,\partial\Omega}) = (2\pi)^{-(d-1)}$ Vol $\{(x,\xi) \in T^*\partial\Omega_j \backslash \{0\} \mid \Lambda_{\gamma,\partial\Omega}(x,\xi) \leq 1\}$, where $\Lambda_{\gamma,\partial\Omega}(x,\xi)$ is the symbol of Λ_γ. In the case that $\gamma_{ij} = \delta_{ij}$, we have $C(\partial\Omega_j, \Lambda_{1,\partial\Omega}) = |\partial\Omega_j| \, (\Gamma((d-1)/2+1)(4\pi)^{(d-1)/2})^{-1}$.

Theorem 1.5. *The high-energy spectrum of Λ_γ determines a lower bound on the number of connected components of $R^d \backslash \Omega$, and also determines the weighted measure of each boundary component (not counting multiplicities).*

The phrase "not counting multiplicities" means that the asymptotics of $\sigma(\Lambda_\gamma)$ may not indicate the existence of more than one boundary component with the same weighted measure.

This result is outside the domain of nondestructive evaluation since in order to determine the asymptotics of the eigenvalues of Λ_γ, boundary data on the interior surfaces $\partial\Omega_j, j \neq 1$, must be specified. In order to model a more realistic situation for which the methods of nondestructive evaluation can be applied, we can modify the problem as follows. We assume that the voids $\text{Int}(\partial\Omega_j)$, for $j \neq 1$, are filled with perfectly insulating material. This implies that the normal component of the current across each interior boundary component $\partial\Omega_j$, for $j \neq 1$, satisfies $\nu \cdot \gamma\nabla u \mid \partial\Omega_j = 0$. We further suppose that boundary data on the outer surface $\partial\Omega_1$ is specified. We can then define a corresponding DN operator for this mixed-problem on $L^2(\partial\Omega_1)$, and ask if the high-energy asymptotics of the spectrum of this operator allow us to determine the number of interior components. However, we prove that a result similar to Theorem 1.1 holds in this case also. That is, we prove that the γ-harmonic extensions of the eigenfunctions of the DN operator for the mixed-problem localize near $\partial\Omega_1$. Consequently, the number of connected components of $\partial\Omega$ cannot be determined from the high-energy asymptotics of the spectrum of this DN operator.

The results of this initial investigation for nondestructive evaluation are negative in that the high-energy asymptotics of the spectrum, obtained by measurements external to the body, are not sufficient to determine the connectivity of a body. It might be possible, however, to use all the eigenvalues in order to determine the connectivity. One result in this direction is due to J. Edward [1]. Using the zeta function associated with the eigenvalues of the DN operator, Edward proved that the disk in $\mathbb{R}^2$ of radius 1 is determined by the spectrum of the DN operator in the sense that any other simply-connected, bounded region in the plane with boundary measure 2π is isomorphic to the disk under Euclidean motions. (In fact, a stronger result is known in this two-dimensional case. The first eigenvalue of the DN operator determines the disk [24] among all simply connected regions with the same boundary measure). Quite recently, another approach has been taken by Lassas and Uhlmann [11] who proved that the DN map, restricted to a nonempty, open, real-analytic subset of the boundary, determines a compact, connected, real-analytic Riemannian manifold in dimensions $d \geq 3$, and the conformal class of a smooth, connected, compact Riemannian surface. This result was refined by Lassas, Taylor, and Uhlmann [12] who proved that for $d \geq 3$, the DN operator, restricted to a nonempty, open, subset of the boundary determines the complete, connected, real-analytic Riemannian manifold (not necessarily compact, but with compact, nonempty boundary). Consequently, the DN map restricted to Dirichlet data supported on a piece of the boundary determines the connectivity in the real analytic case. The methods of these papers, however, do not indicate how to determine the connectivity from the DN map acting on functions supported on a piece of the boundary. In this paper, we show that the spectral asymptotics of the DN map (employing Dirichlet data on the entire boundary) do determine the connectivity, and that a lower bound on the connectivity can be calculated from these asymptotics.

We mention some related works concerning the location of discontinuities within a body by measurements on the exterior surface. Isakov [9] proved that one can locate a discontinuity (supported on an open set) in the scalar conductivity within a body using the DN map associated with the exterior surface. There is some similarity between the contents of section 7 and the work of Friedman and Vogelius [4]. These authors consider, for the case of scalar conductivity, the question of locating small inhomogeneities of extreme conductivity in a conducting body using the DN operator.

The classic inverse problem of determining the scalar conductivity from the DN operator has been studied extensively, see, for example [10, 17, 19]. For a general discussion of inverse problems for isotropic and anisotropic materials, we refer the reader to the lecture notes of Uhlmann [22, 23].

Acknowledgments. We thank R. M. Brown, P. A. Perry, Z. Shen, and G. Uhlmann for many valuable remarks.

2 Preliminaries

We review some basic material needed in the proofs of the main theorems. The results on solutions to the Dirichlet problem can be found in many texts, cf. [2]. A nice account of the Dirichlet-to-Neumann operator can be found in [23]. We always assume that the boundary components are smooth and that the matrix γ satisfies hypotheses H1 and H2. We are concerned with the Dirichlet problem (1.2) for L_γ and Ω. The trace onto the boundary of Ω plays a key role.

Lemma 2.1. *Suppose $s > 1/2$. The restriction map $Tu = u|_{\partial\Omega}$, for $u \in H^s(\Omega) \cap C(\overline{\Omega})$, extends to a bounded linear map $T : H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$. The kernel of T is exactly $H_0^s(\Omega)$.*

We next need an estimate on the H^1-bound of a function on Ω in terms of its boundary value and the L^2-norm of its derivatives.

Lemma 2.2. *Suppose $\Gamma \subset \partial\Omega$ is a set of positive (d-1)-dimensional measure. Then there exists a constant C, dependent upon Γ, so that for any $u \in H^1(\Omega)$,*

$$||u||^2_{H^1(\Omega)} \leq C \left\{ ||Tu||^2_{L^2(\Gamma)} + \sum_{j=1}^{d} \left|\left| \frac{\partial u}{\partial x_j} \right|\right|^2_{L^2(\Omega)} \right\}. \tag{2.1}$$

We recall the main theorem on the solvability of the Dirichlet problem, which we write in its nonhomogeneous version. Let $f \in H^{1/2}(\partial\Omega)$, and $F \in H^{-1}(\Omega)$. We say that u_f is a solution to the Dirichlet problem in the weak sense if for any $\phi \in C_0^\infty(\Omega)$, we have,

$$\sum_{j,k=1}^{d} \int_\Omega \partial_j \phi \gamma_{jk} \partial_k u_f = \int_\Omega F\phi, \tag{2.2}$$

$$Tu_f = f. \tag{2.3}$$

Theorem 2.1. *The mapping $\mathcal{F}_\gamma : H^1(\Omega) \to H^{-1}(\Omega) \times H^{1/2}(\partial\Omega)$ defined by $\mathcal{F}_\gamma u = (L_\gamma u, Tu)$ is an isomorphism. Further, if $\mathcal{F}_\gamma u = (F, f)$, the function u satisfies the estimate*

$$||u||_{H^1(\Omega)} \leq C \left\{ ||F||_{H^{-1}(\Omega)} + ||f||_{H^{1/2}(\partial\Omega)} \right\}. \tag{2.4}$$

The unique function u_f that solves (2.2) with $F = 0$ will be called the γ-*harmonic extension* of f into Ω. Under the smoothness assumptions, the γ-harmonic extension of $f \in H^{1/2}(\partial\Omega)$ is actually in $C^\infty(\overline{\Omega})$.

Given Theorem 2.1, the DN operator can be defined as follows. Initially, we define Λ_γ on $H^{3/2}(\partial\Omega)$, to insure that the trace of the outward normal derivative of the γ-harmonic extension of f exists. We have

$$\Lambda_\gamma f = \sum_{l,m=1}^{d} \nu_l \gamma_{lm} \partial_m u_f \mid \partial\Omega, \tag{2.5}$$

where u_f is the γ-harmonic extension of f. To simplify the notation, we will write

$$\nu \cdot \gamma \nabla \equiv \sum_{l,m=1}^{d} \nu_l \gamma_{lm} \partial_m. \tag{2.6}$$

The domain of Λ_γ can be extend through a duality argument.

Theorem 2.2. *The linear operator Λ_γ defined by (2.5) extends to a bounded map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.*

As an operator on the Hilbert space $L^2(\partial\Omega)$, the DN operator is a non-negative, self-ajoint operator with compact resolvent. Consequently, the spectrum of Λ_γ, as an operator on $L^2(\partial\Omega)$, is discrete, and consists of eigenvalues λ_j, with $\lambda_j \to \infty$. We mention that in the simple case $\gamma_{ij} = \delta_{ij}$, the DN operator is, roughly, the operator $\sqrt{-\Delta_{\partial\Omega}}$, where $-\Delta_{\partial\Omega}$ is the Laplace-Beltrami operator on $\partial\Omega$, with the induced metric. We refer the reader to [21] and [13] for additional information about this representation.

Extensions of boundary data into unbounded sets containing Ω will be of interest. We extend $\gamma_{ij} \in C^\infty(\overline{\Omega})$ to be a smooth function on all of $\mathbb{R}^d$. We assume that the extended matrix of functions $\gamma = [\gamma_{ij}]$ remains symmetric and uniformly elliptic so that H2 remains valid for some constants $0 < \lambda_0 \leq \lambda_1 < \infty$. Furthermore, we assume

H3. There exists $0 < R < \infty$ so that $\Omega \subset B_R(0)$, and $\gamma_{ij}(x) = \delta_{ij}$, for $\|x\| > R$.

We will always assume this extension has been taken in later sections when we consider γ-harmonic functions outside of, and into the interior of, the region Ω. It follows from the analysis in section 5 that the main results of this paper are independent of the choice of this extension in the following sense. Suppose that γ_1 and γ_2 are two smooth extensions of γ satisfying H1, H2, and H3. Then, the differences of the DN operators, $\Lambda_j^{\#}(\gamma_1) - \Lambda_j^{\#}(\gamma_2)$, and $\Lambda_{\partial\Omega_j}(\gamma_1) - \Lambda_{\partial\Omega_j}(\gamma_2)$, associated with $\partial\Omega_j$, and defined with γ-harmonic extensions to the exterior, respectively, interior, regions (see section 5), are smoothing operators. Hence, each pair of DN operators has the same high energy spectral asymptotics.

Representation formulae for solutions of Dirichlet problems will play a central role in this analysis. For this, we need information on the Green's function G_γ, corresponding to the extended, elliptic operator L_γ on $I\!R^d$, and on the Dirichlet Green's function $G_{\gamma,\Omega}$, corresponding to the (extended) elliptic operator L_γ on an open domain (bounded or unbounded) $\Omega \subset I\!R^d, d \geq 3$, with $\partial\Omega \neq \emptyset$. Consequently, we state the following theorem for a general, open region $\Omega \subset I\!R^d$, for $d \geq 3$. The proof of parts of this theorem for real, symmetric γ_{ij} is contained in the paper of Littman, Stampacchia, and Weinberger [14]. Their results hold for $\gamma_{ij} \in L^\infty(I\!R^d)$, although we will state them here only under conditions H1–H3. The results of [14] were generalized to not necessarily symmetric γ_{ij} by Grüter and Widman [5]. In their paper, Grüter and Widman construct the Dirichlet Green's function for L^∞-coefficients on bounded, open domains. In a separate note [6], we show how this proof can be extended to general open regions (including $\Omega = I\!R^d, d \geq 3$) with, or without, smooth boundary. In the following theorem, the local Sobolev space $H^{1,s}_{loc}(I\!R^d)$ consists of those measurable functions f so that for any bounded, open region $\Omega \subset I\!R^d$, we have $f \in H^{1,s}(\Omega)$. The local Sobolev space $H^{1,s,0}_{loc}(\Omega)$, for an unbounded region $\Omega \subset I\!R^d$, with $\partial\Omega \in C^1$, consists of those functions $g \in H^{1,s}_{loc}(\Omega)$ for which $Tg = 0$, where $T : H^{1,s}_{loc}(\Omega) \to L^s(\partial\Omega)$ is the trace map.

Theorem 2.3. *Let $\Omega \subset I\!R^d$ be an open subset of $I\!R^d, d \geq 3$, with smooth boundary. We assume that the coefficients of the uniformly elliptic operator L_γ satisfy hypotheses H1, H2, and H3. There is a nonnegative function $G_{\gamma,\Omega} : \Omega \times \Omega \to I\!R^+ \cup \{\infty\}$, such that, for any $s \in [1, d/(d-1))$,*

1. *If $\Omega = I\!R^d$, we have $G_\gamma(x,y) \in H^{1,s}_{loc}(I\!R^d)$, for each fixed $y \in I\!R^d$;*

2. *If $\partial\Omega \neq \emptyset$, with Dirichlet boundary conditions, we have $G_{\gamma,\Omega}(x,y) \in H^{1,s,0}_{loc}(\Omega)$, for each fixed $y \in \Omega$.*

For all $\phi \in C^1_0(\Omega)$, this function satisfies,

$$\int_\Omega \nabla\phi \cdot \gamma\nabla G_{\gamma,\Omega}(x,\cdot)\ dy = \phi(x). \tag{2.7}$$

Furthermore, $G_{\gamma,\Omega}$ has the following properties:

(a.) *When $x \neq y$, $G_{\gamma,\Omega}(x,y) = G_{\gamma,\Omega}(y,x)$.*

(b.) *The function $G_{\gamma,\Omega}(x,y)$ is smooth for $x \neq y$.*

(c.) *There is a finite constant $K > 0$ such that*

$$K^{-1}\|x-y\|^{2-d} \leq G_{\gamma,\Omega}(x,y) \leq K\|x-y\|^{2-d}. \tag{2.8}$$

(d.) *There is a finite constant $K_1 > 0$ such that*

$$|\nabla G_{\gamma,\Omega}(x,y)| \leq K_1\|x-y\|^{1-d}. \tag{2.9}$$

(e.) *For the case when $\partial\Omega \neq \emptyset$, we have the Dirichlet boundary conditions: $G_{\gamma,\Omega}(\omega,y) = 0$, for $\omega \in \partial\Omega$ and $y \in \Omega$.*

In the following, we will write dy for the volume measure, $d\sigma$ for the induced surface measure, and $d\omega$ for the surface measure on the unit sphere S^{d-1}.

3 Rapid Decay of the γ-Harmonic Extensions

The Green's function G_γ for $I\!R^d, d \geq 3$, described in Theorem 2.3 allows us to write a representation formula for the solution to the Dirichlet problem (1.2) with boundary data $f \in H^{1/2}(\partial\Omega)$. This representation formula will be the basis of the proof of Theorem 1.1.

Proposition 3.1. *Suppose w_f is the γ-harmonic extension of $f \in H^{1/2}(\partial\Omega)$ into Ω, and G_γ is the function from Theorem 2.3 for $I\!R^d, d \geq 3$. Then*

$$w_f(x) = \int_{\partial\Omega} \{G_\gamma(x,\cdot)\Lambda_\gamma f - f\nu\cdot\gamma\nabla G_\gamma(x,\cdot)\}\ d\sigma. \tag{3.1}$$

PROOF: Fix $x \in \Omega$ and choose a function $\eta \in C_0^\infty(I\!R^d)$ that is identically one in a neighborhood of x and vanishes near $\partial\Omega$. Then $w_f = \eta w_f + (1-\eta)w_f$. Because w_f lies in the kernel of an elliptic operator with smooth coefficients, it is smooth away from $\partial\Omega$, so $\eta w_f \in C_0^\infty$. We use Theorem 2.3 to write

$$\begin{aligned} w_f(x) = (\eta w_f)(x) &= \int_\Omega \nabla G_\gamma(x,\cdot)\cdot\gamma\nabla(\eta w_f)\ dy \\ &= \int_\Omega \{\mathrm{div}\{G_\gamma(x,\cdot)\gamma\nabla(\eta w_f)\} - G_\gamma(x,\cdot)L_\gamma(\eta w_f)\}\ dy \\ &= -\int_\Omega G_\gamma(x,\cdot)L_\gamma(\eta w_f)\ dy \end{aligned} \tag{3.2}$$

according to the Divergence Theorem, since $\nabla(\eta w_f)$ is compactly supported away from $\partial\Omega$. We also know, because of (2.7) and part (b.) of Theorem 2.3 that

$$\begin{aligned}
&-\int_\Omega G_\gamma(x,\cdot)L_\gamma(1-\eta)w_f \, dy \\
&\quad = \int_\Omega \{(1-\eta)w_f L_\gamma G_\gamma(x,\cdot) - G_\gamma(x,\cdot)L_\gamma(1-\eta)w_f\} \, dy \\
&\quad = \int_{\partial\Omega} \{(1-\eta)w_f \nu\cdot\gamma\nabla G_\gamma(x,\cdot) - G_\gamma(x,\cdot)\nu\cdot\gamma\nabla(1-\eta)w_f\} \, d\sigma \\
&\quad = \int_{\partial\Omega} \{f\nu\cdot\gamma\nabla G_\gamma(x,\cdot) - G_\gamma(x,\cdot)\Lambda_\gamma f\} \, d\sigma, \qquad (3.3)
\end{aligned}$$

since η vanishes near $\partial\Omega$. Adding equations (3.2) and (3.3), we obtain

$$\begin{aligned}
w_f(x) &= -\int_\Omega G_\gamma(x,\cdot)L_\gamma(\eta w_f) \, dy - \int_\Omega G_\gamma(x,\cdot)L_\gamma(1-\eta)w_f \, dy \\
&\quad + \int_{\partial\Omega} \{G_\gamma(x,\cdot)\Lambda_\gamma f - f\nu\cdot\gamma\nabla G_\gamma(x,\cdot)\} \, d\sigma \\
&= \int_{\partial\Omega} \{G_\gamma(x,\cdot)\Lambda_\gamma f - f\nu\cdot\gamma\nabla G_\gamma(x,\cdot)\} \, d\sigma,
\end{aligned}$$

since w_f is γ-harmonic in Ω. ■

We can now prove Theorem 1.1. Let us recall that the eigenvalues $\{\lambda_j \mid \lambda_1 = 0, \lambda_j \le \lambda_{j+1}, j = 1,2,\ldots\}$ of Λ_γ form a nondecreasing sequence of nonnegative real numbers, and that $\lambda_j \sim j^{1/(d-1)}$.

Theorem 1.1. *Let ϕ_m be an eigenfunction of Λ_γ satisfying $\Lambda_\gamma\phi_m = \lambda_m\phi_m$, with $||\phi_m||_{L^2(\partial\Omega)} = 1$, and let u_m be the γ-harmonic extension of ϕ_m to Ω. For any compact $K \subset \Omega$, $\|u_m\|_{H^1(K)} = O(m^{-\infty})$.*

PROOF: Because $\partial\Omega \in C^\infty$, its outward normal vector is smooth. This fact, in conjunction with the regularity of $G_\gamma(x,\cdot)|_{\partial\Omega}$ for $x \in K$, and the self-adjointness of Λ_γ, allows us to use Proposition 3.1 to write

$$\begin{aligned}
u_n(x) &= \int_{\partial\Omega} \{G_\gamma(x,\cdot)\Lambda_\gamma\phi_m - \phi_m\nu\cdot\gamma\nabla G_\gamma(x,\cdot)\} \, d\sigma \\
&= \frac{1}{\lambda_m^p}\int_{\partial\Omega} \{G_\gamma(x,\cdot)\Lambda_\gamma^{p+1}\phi_m - \Lambda_\gamma^p\phi_m\nu\cdot\gamma\nabla G_\gamma(x,\cdot)\} \, d\sigma \\
&= \frac{1}{\lambda_m^p}\int_{\partial\Omega} \{\phi_m\Lambda_\gamma^{p+1}G_\gamma(x,\cdot) - \phi_m\Lambda_\gamma^p\nu\cdot\gamma\nabla G_\gamma(x,\cdot)\} \, d\sigma. \qquad (3.4)
\end{aligned}$$

We now use Hölder's inequality to write (3.4) as

$$
\begin{aligned}
|u_m(x)| &\le \frac{1}{\lambda_m^p}\left\{\|\Lambda_\gamma^{p+1} G_\gamma(x,\cdot)\|_{L^2(\partial\Omega)} + \left\|\Lambda_\gamma^p \nu\cdot\gamma\nabla G_\gamma(x,\cdot)\right\|_{L^2(\partial\Omega)}\right\} \\
&= \frac{C(x;p)}{\lambda_m^p}.
\end{aligned} \tag{3.5}
$$

Because the singularity of $G_\gamma(x,\cdot)$ depends only upon the distance from x to $\partial\Omega$, there is a finite constant $C_p > 0$ so that $|u_m(x)| \le C_p \lambda_m^{-p}$, for every $x \in K$. A similar inequality for $\|Du_n\|_{L^2(K)}$ is obtained by passing differentiation through the integral in Proposition 3.1. The classical spectral asymptotics, $\lambda_m \sim m^{p/(d-1)}$, and the fact that p was chosen arbitrarily, complete the proof. ∎

4 Approximate Diagonalization of Λ_γ

The rapid decay asserted by Theorem 1.1, in conjunction with the representation formula developed in Proposition 3.1, allows us to prove the decomposition theorem, Theorem 1.2. We recall that R_j is the restriction operator from $L^2(\partial\Omega)$ onto $L^2(\partial\Omega_j)$, and that E_j is the extension operator from $L^2(\partial\Omega_j)$ into $L^2(\partial\Omega)$. The operator $C_j \equiv E_j R_j$ is an orthogonal projector on $L^2(\partial\Omega)$.

Theorem 1.2. *The DN map Λ_γ admits a decomposition into two self-adjoint operators, $\Lambda_\gamma = D_\gamma + R_\gamma$, where $D_\gamma = \sum_{j=1}^k C_j \Lambda_\gamma C_j$, and R_γ is a smoothing operator.*

Proof: The decomposition follows from the orthogonality and completeness properties of the operators C_j. For any function $f \in D(\Lambda_\gamma)$,

$$
\Lambda_\gamma f = \sum_{j=1}^k \Lambda_\gamma C_j f = \sum_{j,l=1}^k C_j \Lambda_\gamma C_l f = D_\gamma f + R_\gamma f,
$$

where

$$
D_\gamma f = \sum_{j=1}^k C_j \Lambda_\gamma C_j f,
$$

and

$$
R_\gamma f = \sum_{j,l=1:l\neq j}^k C_j \Lambda_\gamma C_l f.
$$

Since the projections C_j preserve the domain of Λ_γ, it is clear that both operators D_γ and R_γ are self-adjoint. In order to prove that R_γ is smoothing, we prove the following result in Lemma 4.1. Let ϕ_n be an eigenfunction of Λ_γ satisfying $\Lambda_\gamma \phi_n = \lambda_n \phi_n$. The rapid decay of the γ-harmonic extension of ϕ_n away from the boundary implies that for all $j, l = 1, \ldots, k$,

$$R_l \Lambda_\gamma E_j R_j \phi_n = \lambda_n \delta_{lj} R_j \phi_n + O(n^{-\infty}). \tag{4.1}$$

We prove in Lemma 4.2 ahead that this result implies that both $\Lambda_\gamma^p R_\gamma$ and $R_\gamma \Lambda_\gamma^p$ extend to bounded operators on $L^2(\partial\Omega)$, for any $p \in \mathbb{Z}$. This immediately proves that R_γ is a bounded operator. To show that R_γ is smoothing, we note that

$$R_\gamma = \frac{1}{(1+\Lambda_\gamma)^s} \left((1+\Lambda_\gamma)^s R_\gamma\right) : L^2(\partial\Omega) \to H^s(\partial\Omega), \tag{4.2}$$

is bounded for any $s > 0$, and that

$$R_\gamma = (R_\gamma (1+\Lambda_\gamma)^t) \frac{1}{(1+\Lambda_\gamma)^t} : H^{-t}(\partial\Omega) \to L^2(\partial\Omega), \tag{4.3}$$

is bounded for any $t > 0$, so R_γ is smoothing. ■

Lemma 4.1. *Let ϕ_n be an eigenfunction of Λ_γ satisfying $\Lambda_\gamma \phi_n = \lambda_n \phi_n$ and $\|\phi_n\|_{L^2(\partial\Omega)} = 1$.. For all $j, l = 1, \ldots, k$, we have*

$$R_l \Lambda_\gamma E_j R_j \phi_n = \lambda_n \delta_{lj} R_j \phi_n + O(n^{-\infty}). \tag{4.4}$$

Proof: We need to compute the image of $E_j R_j \phi_n$ under the DN map Λ_γ. Let u_n be the γ-harmonic extension of ϕ_n to Ω. We denote by w_n the γ-harmonic extension of $E_j R_j \phi_n$ to Ω. This is computed from u_n as follows. Let $\xi_j \in C^\infty(\overline{\Omega})$, with $\xi_j \equiv 1$ in a neighborhood of $\partial\Omega_j$, and $\xi_j \equiv 0$ in a neighborhood of $\partial\Omega \backslash \partial\Omega_j$. The γ-harmonic extension w_n of $E_j R_j \phi_n$ to Ω can be written as

$$w_n = \xi_j u_n - H_\Omega (L_\gamma \xi_j u_n), \tag{4.5}$$

where H_Ω denotes the solution operator of the inhomogeneous Dirichlet problem with Dirichlet boundary data. That is, $H_\Omega(F)$ is the function which solves

$$\begin{aligned} L_\gamma u &= F \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.6}$$

One can easily check that w_n is γ-harmonic and satisfies the proper boundary conditions. This representation of w_n greatly simplifies our calculation of the outward normal derivative at the boundary. Because $\xi_j \equiv 1$ near $\partial\Omega_j$,

$$\begin{aligned}\nu \cdot \gamma\nabla w_n \mid \partial\Omega_j &= [\nu \cdot \gamma\nabla u_n - \nu \cdot \gamma\nabla H_\Omega(L_\gamma \xi_j u_n)] \mid \partial\Omega_j \\ &= \lambda_n R_j \phi_n - [\nu \cdot \gamma\nabla H_\Omega(L_\gamma \xi_j u_n)] \mid \partial\Omega_j. \end{aligned} \tag{4.7}$$

Using the Green's function $G_{\gamma,\Omega}$ of Theorem 2.3 to solve (4.6), we may write

$$\begin{aligned} H_\Omega(L_\gamma \xi_j u_n)(x) &= \int_\Omega L_\gamma(\xi_j u_n) G_{\gamma,\Omega}(x,\cdot)\ dy \\ &= \int_{\mathrm{supp}(\nabla\xi_j)} u_n(L_\gamma \xi_j) G_{\gamma,\Omega}(x,\cdot)\ dy \\ &\quad -2 \sum_{l,m=1}^{d} \int_{\mathrm{supp}(\nabla\xi_j)} (\partial_l \xi_j)\gamma_{lm}(\partial_m u_n) G_{\gamma,\Omega}(x,\cdot)\ dy. \end{aligned} \tag{4.8}$$

Thus, we rewrite (4.7) as

$$\nu \cdot \gamma\nabla w_n \mid \partial\Omega_j \tag{4.9}$$

$$\begin{aligned} &= \lambda_n R_j \phi_n - \left[\int_{\mathrm{supp}(\nabla\xi_j)} u_n(L_\gamma \xi_j)\ \nu \cdot \gamma\nabla_x G_{\gamma,\Omega}(x,\cdot)\ dy\right] \mid \partial\Omega_j \\ &\quad +2 \left[\sum_{l,m=1}^{d} \int_{\mathrm{supp}(\nabla\xi_j)} (\partial_l \xi_j)\gamma_{ml}(\partial_m u_n)\ \nu \cdot \gamma\nabla_x G_{\gamma,\Omega}(x,\cdot)\ dy\right] \Bigg| \ \partial\Omega_j. \end{aligned} \tag{4.10}$$

Recall that ξ_j is constant in a neighborhood of $\partial\Omega_j$ so, for x near $\partial\Omega_j$, the local Green's function $G_{\gamma,\Omega}(x,y)$ is smooth as y ranges over $\mathrm{supp}(\nabla\xi_j)$. Since both ξ_j and $\nabla_x G_{\gamma,\Omega}(x,\cdot)$ are smooth functions on $\mathrm{supp}(\nabla\xi_j)$, and $\mathrm{supp}(\nabla\xi_j)$ is a compact subset of Ω, there is a constant $C(\xi_j;\gamma;\Omega)$ such that

$$\begin{aligned} &\left|\int_{\mathrm{supp}(\nabla\xi_j)} [u_n(L_\gamma \xi_j) - 2 \sum_{l,m=1}^{d} (\partial_l \xi_j)\gamma_{lm}(\partial_m u_n)]\ \nu \cdot \gamma\nabla_x G_{\gamma,\Omega}(x,\cdot)\ dy\right| \\ &\qquad \leq C(\xi_j;\gamma;\Omega) \int_{\mathrm{supp}(\nabla\xi_j)} (|u_n| + \|\nabla u_n\|)\ dy. \end{aligned} \tag{4.11}$$

Since $\mathrm{supp}(\nabla\xi_j) \subset \Omega$, we can apply Theorem 1.1, to conclude that the right-hand side of (4.11) is $O(n^{-\infty})$, that is, $\Lambda_j R_j \phi_n = \lambda_n R_j \phi_n + O(n^{-\infty})$. On the other hand, when we restrict to $\partial\Omega_i$, $i \neq j$, equations (4.5), (4.7) and (4.10) imply

$$\begin{aligned}
&\nu(x) \cdot \nabla w_n(x) | \partial\Omega_i \\
&\quad = \; - \int_{\mathrm{supp}(\nabla\xi_j)} \left[u_n (L_\gamma \xi_j) - 2 \sum_{l,m=1}^{d} (\partial_l \xi_j) \gamma_{lm} (\partial_m u_n) \right] \nu \cdot \gamma \nabla_x G_{\gamma,\Omega}(x,\cdot) \; dy \; |\partial\Omega_i.
\end{aligned} \tag{4.12}$$

Since x near $\partial\Omega_i$ is disjoint from $\mathrm{supp}(\nabla\xi_j)$, the second summand of (4.12) is $O(n^{-\infty})$. ∎

Lemma 4.2. *For any $p \in I\!N$, the operators $R_\gamma \Lambda_\gamma^p$ and $\Lambda_\gamma^p R_\gamma$ can be extended to bounded operators on $L^2(\partial\Omega)$.*

PROOF: Suppose that ψ is in the domain of Λ_γ^p, with $\|\psi\|_{L^2(\partial\Omega)} = 1$. Because $\{\phi_n\}$ is an orthonormal basis of eigenfunctions of Λ_γ for $L^2(\partial\Omega)$, we can write $\psi = \sum_n \beta_n \phi_n$, where $\beta_n = \langle \psi, \phi_n \rangle_{L^2(\partial\Omega)}$. Using the Cauchy-Schwarz inequality for sequences in $\ell^2(I\!N)$ and the linearity of Λ_γ^p and R_γ, we see that

$$\begin{aligned}
\|R_\gamma \Lambda_\gamma^p \psi\|_{L^2(\partial\Omega)} &\leq \sum_n |\beta_n| \lambda_n^p \|R_\gamma \phi_n\|_{L^2(\partial\Omega)} \\
&\leq \left(\sum_n \lambda_n^{2p} \|R_\gamma \phi_n\|_{L^2(\partial\Omega)}^2 \right)^{1/2} \equiv C(p).
\end{aligned} \tag{4.13}$$

The constant $C(p)$ is finite since, by Lemma 4.1, $\|R_\gamma \phi_n\|_{L^2(\partial\Omega)} = O(\lambda_n^{-k})$, for every $k \in I\!N$. This bound, and the fact that the domain of Λ_γ^p is dense in $L^2(\partial\Omega)$, allow us to extend $R_\gamma \Lambda_\gamma^p$ to a unique bounded operator on all of $L^2(\partial\Omega)$. Turning to the second operator, $\Lambda_\gamma^p R_\gamma$, for any $\psi \in D(\Lambda_\gamma^p)$, and for any $\xi \in L^2(\partial\Omega)$, we have

$$|\langle \Lambda_\gamma^p \psi, R_\gamma \xi \rangle| = |\langle R_\gamma \Lambda_\gamma^p \psi, \xi \rangle| \leq C(p) \, \|\psi\|_{L^2(\partial\Omega)} \, \|\xi\|_{L^2(\partial\Omega)}. \tag{4.14}$$

We conclude that $R_\gamma \xi$ is in the domain of Λ_γ^p. Since the domain of Λ_γ^p is a dense set in $L^2(\partial\Omega)$, it follows that $\Lambda_\gamma^p R_\gamma : L^2(\partial\Omega) \to L^2(\partial\Omega)$ can be extended to a bounded operator (see [7]). ∎

5 Spectral Asymptotics

In this section, we show that the approximate diagonalization formula for Λ_γ of Theorem 1.2 allows us to approximate the spectrum of Λ_γ by that of the diagonal operator $D_\gamma = \sum_{j=1}^{k} C_j \Lambda_\gamma C_j$. We then study the operator $C_j \Lambda_\gamma C_j$ in detail and prove that its spectrum is asymptotically close to the spectrum of a DN operator Λ_j associated only with the j^{th}-boundary component.

Theorem 1.3. *Let $\{\mu_l \mid l \in I\!N\}$ be the eigenvalues of D_γ listed in nondecreasing order, including multiplicity, and let $\{\lambda_l \mid l \in I\!N\}$ be the eigenvalues of Λ_γ, listed similarly. Then, we have $\lambda_l = \mu_l + O(l^{-\infty})$.*

PROOF: Let $\{\theta_j, j = 1, \ldots\}$ be an orthonormal basis of eigenvectors of D_γ. We denote by M_j the eigenspace spanned by the first j eigenvectors of D_γ. We use a variational formula (cf. [18], section XIII.1) to calculate λ_{j+1}:

$$
\begin{aligned}
\lambda_{j+1} &= \max_{\dim(V^\perp)=j} \left(\min_{f \in V \cap D(\Lambda_\gamma)} \{ \langle \Lambda_\gamma f, f \rangle_{L^2(\partial\Omega)} \mid \|f\|_{L^2(\partial\Omega)} = 1 \} \right) \\
&= \max_{\dim(V^\perp)=j} \left(\min_{f \in V \cap D(\Lambda_\gamma)} \{ \langle D_\gamma f, f \rangle_{L^2(\partial\Omega)} + \langle R_\gamma f, f \rangle_{L^2(\partial\Omega)} \mid \|f\|_{L^2(\partial\Omega)} = 1 \} \right) \\
&\geq \max_{\dim(V^\perp)=j} \left(\min_{\substack{f \in V \cap D(\Lambda_\gamma) \\ \|f\|=1}} \{ \langle D_\gamma f, f \rangle_{L^2(\partial\Omega)} \} - \max_{\substack{f \in V \cap D(\Lambda_\gamma) \\ \|f\|=1}} \{ |\langle R_\gamma f, f \rangle_{L^2(\partial\Omega)}| \} \right) \\
&\geq \min_{\substack{f \in M_j^\perp \cap D(\Lambda_\gamma) \\ \|f\|=1}} \{ \langle D_\gamma f, f \rangle_{L^2(\partial\Omega)} \} - \max_{\substack{f \in M_j^\perp \cap D(\Lambda_\gamma) \\ \|f\|=1}} \{ |\langle R_\gamma f, f \rangle_{L^2(\partial\Omega)}| \} \\
&= \mu_{j+1} - \max_{f \in M_j^\perp \cap D(\Lambda_\gamma)} \{ |\langle R_\gamma f, f \rangle_{L^2(\partial\Omega)}| \mid \|f\|_{L^2(\partial\Omega)} = 1 \}. \qquad (5.1)
\end{aligned}
$$

Note that, according to Lemma 4.2, $R_\gamma D_\gamma^q = R_\gamma(\Lambda_\gamma - R_\gamma)^q$ is a bounded operator for any $q \in I\!N$. It follows that, for any eigenfunction θ_j of D_γ, there is a constant $C(q) \equiv \|R_\gamma D_\gamma^q\|$, independent of the eigenvalue index j, such that $\|R_\gamma \theta_j\|_{L^2(\partial\Omega)} \leq C(q) \mu_j^{-q}$. For any $p \in I\!N$, we choose $q = q(p) \in I\!N$ sufficiently large so that, recalling the classical eigenvalue asymptotics, we have $\{\mu_j^{p-q}\} \in \ell^2(I\!N)$. It follows from this and Lemma 4.2 that

$$
\|R_\gamma D_\gamma^p \theta_j\|_{L^2(\partial\Omega)} = \mu_j^p \|R_\gamma \theta_j\|_{L^2(\partial\Omega)} \leq C(q) \mu_j^{p-q}, \qquad (5.2)
$$

where the constant $C(q)$, because of the way in which we choose q, depends on p, but is independent of j. Thus, the sequence, indexed by j,

$\{||R_\gamma D_\gamma^p \theta_j||_{L^2(\partial\Omega)}\} \in \ell^2(I\!N)$, for any natural number p.

Now we can estimate the second summand on the right of (5.1). We may write any vector $f \in M_j^\perp$, $\|f\| = 1$, in terms of its Fourier coefficients, $f = \sum_{k>j} f_k \theta_k$ so that,

$$\begin{aligned}
\|R_\gamma f\|_{L^2(\partial\Omega)} &\leq \sum_{k>j} \left| \frac{f_k}{\mu_k^p} \right| \|R_\gamma D_\gamma^p \theta_k\|_{L^2(\partial\Omega)} \\
&\leq \frac{1}{\mu_{j+1}^p} \sum_{k>j} \left| \left(\frac{\mu_{j+1}}{\mu_k} \right)^p f_k \right| ||R_\gamma D_\gamma^p \theta_k||_{L^2(\partial\Omega)} \\
&\leq \frac{1}{\mu_{j+1}^p} \left\{ \sum_{k>j} ||R_\gamma D_\gamma^p \theta_k||^2_{L^2(\partial\Omega)} \right\}^{1/2} \\
&\leq \frac{1}{\mu_{j+1}^p} \left\{ \sum_{k=1}^{\infty} ||R_\gamma D_\gamma^p \theta_k||^2_{L^2(\partial\Omega)} \right\}^{1/2} \\
&= \frac{c(p)}{\mu_{j+1}^p},
\end{aligned} \tag{5.3}$$

where we used the Cauchy-Schwarz inequality for sequences in $\ell^2(I\!N)$, the fact that $\|f\| = 1$, and the nondecreasing nature of the sequence $\{\mu_j\}$. Inequality (5.3) allows us to rewrite (5.1) as

$$\lambda_{j+1} \geq \mu_{j+1} - O(j^{-\infty}).$$

Similarly, by reversing the role of Λ_γ and D_γ, we find that $\mu_{j+1} \geq \lambda_{j+1} - O(j^{-\infty})$ and the result follows. ■

Having shown that the asymptotic behavior of the spectrum of Λ_γ is determined by D_γ, we next describe the nature of the spectrum of the operator $D_\gamma = \sum_{j=1}^k \; C_j \Lambda_\gamma C_j$. The operator $C_j \Lambda_\gamma C_j$ on $L^2(\partial\Omega)$ depends on data on the other boundary components through Λ_γ. Due to the localization of the γ-harmonic extensions of the eigenfunctions, though, this dependence is very weak at high-energy. Let us note an obvious fact. We define $\Lambda_j \equiv R_j \Lambda_\gamma E_j : L^2(\partial\Omega_j) \to L^2(\partial\Omega_j)$. Then, it is clear that the spectrum of $C_j \Lambda_\gamma C_j$ and Λ_j coincide. Given a boundary component $\partial\Omega_j$, we want to define a DN map associated solely with this surface. There are two natural ways to do this since the surface $\partial\Omega_j$ partitions $I\!R^d$ into two regions. We can consider either the γ-harmonic extension to the bounded region *interior* to the surface $\partial\Omega_j$, or the extension to the *exterior*, unbounded region.

In order to maintain the sense of the normal derivative used in the definition of Λ_γ, we distinguish the boundary $\partial\Omega_1$ from the other boundary components. For $\partial\Omega_1$, we denote by $\Omega_1^\#$ the bounded component of $I\!R^d \backslash \partial\Omega_1$. For $1 < j \leq k$, let $\Omega_j^\#$ be the unbounded component of $I\!R^d \backslash \partial\Omega_j$. We define $\Lambda_j^\# : L^2(\partial\Omega_j) \to L^2(\partial\Omega_j)$ by $\Lambda_j^\# f = \nu \cdot \gamma \nabla u_f^\# |_{\partial\Omega_j}$, where $u_f^\#$ is the γ-harmonic extension of f to $\Omega_j^\#$ decaying at infinity. We assume hypothesis H3 so that we have chosen a smooth extension of γ to $I\!R^d$ that coincides with the identity matrix outside a sufficiently large ball. The existence of such a γ-harmonic extension for the unbounded regions $\partial\Omega_j^\#$, $1 < j \leq k$ is verified in the appendix, section 8. The vector ν is the outward normal to the region. Consequently, the outward normal is the same as was used in the definition of Λ_γ on each surface $\partial\Omega_j$, $1 \leq j \leq k$. This allows us to understand one γ-harmonic extension as an approximation of the other.

Lemma 5.1. *Suppose $\phi_n^{(j)}$ is a normalized eigenfunction of $\Lambda_j^\#$ satisfying $\Lambda_j^\# \phi_n^{(j)} = \lambda_n \phi_n^{(j)}$. We then have,*

$$\langle \phi_m^{(j)}, \Lambda_j \phi_n^{(j)} \rangle_{L^2(\partial\Omega_j)} = \lambda_n \delta_{nm} + O(\lambda_n^{-\infty}) \cdot O(\lambda_m^{-\infty}),$$

where the symbol on the right means rapid decay in each eigenvalue separately.

Proof: Consider the inner product,

$$\langle \phi_m^{(j)}, \Lambda_j \phi_n^{(j)} \rangle_{L^2(\partial\Omega_j)} = \lambda_n \delta_{nm} + \langle \phi_m^{(j)}, (\Lambda_j - \Lambda_j^\#) \phi_n^{(j)} \rangle_{L^2(\partial\Omega_j)}.$$

Using the Divergence Theorem, with u_n as the γ-harmonic extension of $E_j \phi_n^{(j)}$ to Ω, we have

$$\begin{aligned}
\langle \phi_m^{(j)}, \Lambda_j \phi_n^{(j)} \rangle_{L^2(\partial\Omega_j)} &= \int_{\partial\Omega_j} \phi_m^{(j)} \nu \cdot \gamma \nabla u_n \, d\sigma = \int_{\partial\Omega} (E_j \phi_m^{(j)}) \nu \cdot \gamma \nabla u_n \, d\sigma \\
&= \int_{\partial\Omega} u_m \nu \cdot \gamma \nabla u_n \, d\sigma = \int_\Omega \nabla u_m \cdot \gamma \nabla u_n \, dx.
\end{aligned}$$

Let $u_n^\#$ denote the γ-harmonic extension of the eigenfunction $\phi_n^{(j)}$ to $\Omega_j^\#$. Because of the decay estimates on the γ-harmonic solutions to the exterior region $\Omega_j^\#$, given in Theorem 8.1, we may use the Divergence Theorem with the functions $u_n^\#$ on the unbounded region $\Omega_j^\#$ to write

$$\langle \phi_m^{(j)}, \Lambda_j^\# \phi_n^{(j)} \rangle = \int_{\partial\Omega_j} u_m^\# \nu \cdot \gamma \nabla u_n^\# \, d\sigma = \int_{\Omega_j^\#} \nabla u_m^\# \cdot \gamma \nabla u_n^\# \, dx.$$

Therefore, combining these two equations, we obtain,

$$\begin{aligned}\langle \phi_m^{(j)}, (\Lambda_j - \Lambda_j^{\#})\phi_n^{(j)} \rangle &= \int_\Omega \left(\nabla u_m \cdot \gamma \nabla u_n - \nabla u_m^{\#} \cdot \gamma \nabla u_n^{\#} \right) \, dx \\ &\quad - \int_{\Omega_j^{\#} \setminus \Omega} \nabla u_m^{\#} \cdot \gamma \nabla u_n^{\#} \, dx. \end{aligned} \tag{5.4}$$

Using the Divergence Theorem again, we see that the second integral of (5.4) can be rewritten as

$$\int_{\partial(\Omega_j^{\#} \setminus \Omega)} u_m^{\#} \nu \cdot \gamma \nabla u_n^{\#} \, d\sigma. \tag{5.5}$$

In Corollary 8.1, we derive the following representation formula for $u_n^{\#}$,

$$u_n^{\#}(x) = \int_{\partial \Omega_j} \phi_n^{(j)}(y) \mathcal{G}_{\Omega_j^{\#}}(x, y) \, d\sigma(y)$$

where $\mathcal{G}_{\Omega_j^{\#}}(\cdot, \cdot)$ is a smooth function away from the diagonal. Consequently, we can apply the techniques of Lemma 1.1 to conclude that $\|u_n^{\#}\|_{H^1(K)} = O(\lambda_n^{-\infty})$, for any compact set $K \subset \Omega_j^{\#}$. In particular, since $\partial(\Omega_j^{\#} \setminus \Omega)$ is removed from $\partial \Omega_j$, we can conclude that the integral in (5.5) is of order $O(\lambda_m^{-\infty}) \cdot O(\lambda_n^{-\infty})$.

To address the first integral of (5.4), we add zero:

$$\begin{aligned} \int_\Omega & \left(\nabla u_m \cdot \gamma \nabla u_n - \nabla u_m^{\#} \cdot \gamma \nabla u_n^{\#} \right) \, dx \\ &= \int_\Omega \left(\nabla u_m^{\#} \cdot \gamma (\nabla u_n - \nabla u_n^{\#}) + (\nabla u_m - \nabla u_m^{\#}) \cdot \gamma \nabla u_n \right) \, dx \\ &= \int_\Omega \left(\operatorname{div}((u_m - u_m^{\#}) \gamma \nabla u_n) + \operatorname{div}((u_n - u_n^{\#}) \gamma \nabla u_m^{\#}) \right) \, dx \\ &= - \int_{\partial \Omega \setminus \partial \Omega_j} \left(u_m^{\#} \nu \cdot \gamma \nabla u_n + u_n^{\#} \nu \cdot \gamma \nabla u_m^{\#} \right) \, d\sigma, \end{aligned} \tag{5.6}$$

since $(u_n - u_n^{\#})$ vanishes on $\partial \Omega_j$. Because $\partial \Omega \setminus \partial \Omega_j$ is a compact subset of $\Omega_j^{\#}$ that is removed from $\partial \Omega_j^{\#}$, as in the analysis of (5.5), we have that $\|u_m^{\#}\|_{H^1(K)} = O(m^{-\infty})$. To estimate $\Lambda_\gamma u_n$ restricted to $\partial \Omega \setminus \partial \Omega_j$, we use the method of proof of Lemma 4.1. Recall that u_n is the γ-harmonic extension of $E_j \phi_n^{(j)}$ to Ω. As in the (4.5), we have the representation

$$u_n = \xi_j u_n^{\#} - H_\Omega (L_\gamma \xi_j u_n^{\#}), \tag{5.7}$$

where ξ_j is 1 in a neighborhood of $\partial \Omega_j$, and H_Ω is the solution operator for (4.6). For $l \neq j$, we compute

$$\Lambda_\gamma u_n \mid \partial \Omega_l = -\nu \cdot \gamma \nabla H_\Omega (L_\gamma \xi_j u_n^{\#}) \mid \partial \Omega_l. \tag{5.8}$$

We express the solution operator H_Ω in terms of the Green's function as in (4.8) to arrive at the analog of (4.12) with w_n there replaced by u_n, and u_n there replaced by $u_n^\#$. Since $\mathrm{supp}(\nabla\xi_j)$ is disjoint from $\partial\Omega_l$, for $l \neq j$, the decay for the γ-harmonic function $u_n^\#$, away from $\partial\Omega_j$, establishes that the right side of (5.8) is $O(\lambda_n^{-\infty})$. As a consequence, the last term in (5.6) is $O(\lambda_n^{-\infty}) \cdot O(\lambda_m^{-\infty})$. This completes the proof. ■

Lemma 5.1 tells us the action of Λ_γ on functions $f \in D(\Lambda_\gamma)$, with $\mathrm{supp}\ f \subset \partial\Omega_j$, is very similar to the action on $R_j f$ of the Dirichlet-to-Neumann operator defined by way of a γ-harmonic extension to the *exterior* region $\Omega_j^\#$, $1 < j \leq k$, or to the *interior* region $\Omega_1^\#$. We make that statement somewhat more precise with the following lemma.

Lemma 5.2. *The operator difference $A_j \equiv \Lambda_j - \Lambda_j^\#$, is a smoothing operator.*

Proof: Let $\{\phi_n^{(j)}\}$ be the eigenfunctions of $\Lambda_j^\#$ with associated eigenvalues $\{\lambda_n\}$. Let $\psi \in L^2(\partial\Omega_j)$ be in the domain of $(\Lambda_j^\#)^p$, for some arbitrary but fixed p, and $||\psi||_{L^2(\partial\Omega_j)} = 1$. We expand ψ as

$$\psi = \sum_{n\geq 0} \beta_n \phi_n^{(j)}, \text{ where } \beta_n = \langle \psi, \phi_n^{(n)} \rangle_{L^2(\partial\Omega_j)}.$$

Using the linearity of A_j and $(\Lambda_j^\#)^p$, we see that

$$||A_j (\Lambda_j^\#)^p \psi||_{L^2(\partial\Omega_j)} \leq \sum_{n\geq 0} \beta_n \lambda_n^p ||A_j \phi_n^{(j)}||_{L^2(\partial\Omega_j)} \leq \left(\sum_{n\geq 0} \lambda_n^{2p} \|A_j \phi_n^{(j)}\|^2_{L^2(\partial\Omega_j)} \right)^{1/2}, \tag{5.9}$$

where the final inequality of (5.9) arises from the Cauchy-Schwarz inequality for sequences in $\ell^2(I\!N)$, and the fact that $||\psi|| = 1$. Let us attend to $||A_j \phi_n^{(j)}||^2_{L^2(\partial\Omega_j)}$:

$$||A_j \phi_n^{(j)}||^2_{L^2(\partial\Omega_j)} = \sum_{m\geq 0} |\langle A_j \phi_m^{(j)}, \phi_n^{(j)} \rangle|^2$$

and, according to Lemma 5.1, $|\langle A_j \phi_m^{(j)}, \phi_n^{(j)} \rangle|^2 = O(\lambda_n^{-\infty}) \cdot O(\lambda_m^{-\infty})$. Thus,

$$||A_j \phi_n^{(j)}||^2_{L^2(\partial\Omega_j)} = \sum_{m\geq 0} O(\lambda_n^{-\infty}) \cdot O(\lambda_m^{-\infty}) = O(\lambda_n^{-\infty}),$$

whence the right-hand side of (5.9) converges, independently of ψ. That is, $A_j(\Lambda_j^\#)^p$ is a bounded operator on the domain of $(\Lambda_j^\#)^p$, which is dense in $L^2(\partial\Omega_j)$. We extend the operator to all of $L^2(\partial\Omega_j)$ and denote this extension, also, by $A_j(\Lambda_j^{\#})^p$. As a bounded operator, we know $A_j(\Lambda_j^{\#})^p$ has a bounded adjoint, $(A_j(\Lambda_j^{\#})^p)^* = (\Lambda_j^{\#})^p A_j$. It follows that $((\Lambda_j^{\#})^p + 1)A_j : L^2(\partial\Omega_j) \to L^2(\partial\Omega_j)$ is a bounded operator and, thus,

$$A_j = \frac{1}{(\Lambda_j^{\#})^p + 1}((\Lambda_j^{\#})^p + 1)A_j : L^2(\partial\Omega_j) \to H^p(\partial\Omega_j).$$

The fact that p is arbitrary concludes the proof. ∎

One might naturally ask if the spectral asymptotics depends on the choice of the extension to the exterior of the regions bounded by $\partial\Omega_j$, for $1 < j \leq k$, and the interior of $\partial\Omega_1$. To answer this, we now consider extensions to the interior of the region bounded by $\partial\Omega_j$, for $1 < j \leq k$, and, similarly, the extension to the exterior region bounded by $\partial\Omega_1$. We denote by $\Lambda_{\partial\Omega_j}$ the Dirichlet-to-Neumann operator on $L^2(\partial\Omega_j)$ that is defined using a γ-harmonic extension to the bounded region enclosed by $\partial\Omega_j$, $1 < j \leq k$, and to the unbounded region exterior to $\partial\Omega_1$. We will show that, up to smoothing, it does not matter which operators, $\Lambda_j^{\#}$ or $\Lambda_{\partial\Omega_j}$, we choose to model the boundary. The next result constitutes the final step of this section. It will be used in the next section in order to extract geometric information concerning $\partial\Omega$ from the DN operator.

Lemma 5.3. *For any $1 \leq j \leq k$, the operator $\Lambda_j^{\#} - \Lambda_{\partial\Omega_j}$, on $L^2(\partial\Omega_j)$, is a smoothing operator.*

PROOF: In [13], Lee and Uhlmann express the ambient Riemannian metric of $\mathbb{R}^d$ in boundary local coordinates, and proceed to calculate the full symbol of the DN operator in terms of the induced metric tensor on $\partial\Omega$. Let us suppose that the outward normal vector to $\mathbb{R}^d \backslash \Omega_j^{\#}$ (the bounded region enclosed by $\partial\Omega_j$), written in boundary normal coordinates, is $\nu = -\frac{\partial}{\partial x_d}$. To choose $\nu = \frac{\partial}{\partial x_d}$, instead, is to define the DN operator by way of a γ-harmonic extension to $\Omega_j^{\#}$ and, as one might expect, a factor of negative one is introduced into the symbol. However, we cannot use the same alternating (d-1)-form in our representation of the DN operator as defined with an extension into $\Omega_j^{\#}$ as was used when our extension was into $\mathbb{R}^d \backslash \Omega_j^{\#}$ because this form induces an orientation on the manifold $\partial\Omega_j^{\#}$ under which it is *not* the boundary of $\Omega_j^{\#}$. To properly orient $\partial\Omega_j^{\#}$, we commute the differentials of the alternating form and, thus, the negative sign that arose from our choice of outward normal is absorbed. The conclusion follows from fact that the symbols of $\Lambda_j^{\#}$ and $\Lambda_{\partial\Omega_j}$ agree to all orders. ■

Corollary 5.1. *Suppose the spectrum of $\Lambda_{\partial\Omega_j}$ is denoted by $\{\lambda_n\}$, and the spectrum of Λ_j is written as $\{\mu_n\}$. Then, $\lambda_n = \mu_n + O(n^{-\infty})$.*

PROOF: We can write

$$\begin{aligned}\Lambda_{\partial\Omega_j} &= \{(\Lambda_{\partial\Omega_j} - \Lambda_j^{\#}) + (\Lambda_j^{\#} - \Lambda_j)\} + \Lambda_j \\ &= \Lambda_j + R_j, \end{aligned} \tag{5.10}$$

where R_j is a sum of smoothing operators, and so is a smoothing operator. Consequently, if we write eigenfunctions of $\Lambda_{\partial\Omega_j}$ as $\phi_n^{(j)}$, we have

$$\|R_j\phi_n^{(j)}\| = \|R_j\Lambda_{\partial\Omega_j}^{p}\Lambda_{\partial\Omega_j}^{-p}\phi_n^{(j)}\| \le C(p)\lambda_n^{-p}. \tag{5.11}$$

That is, we have $||(\Lambda_{\partial\Omega_j} - \Lambda_j)\phi_n^{(j)}||_{L^2(\partial\Omega)} = O(n^{-\infty})$. We have a similar estimate if $\phi_n^{(j)}$ is replaced by an eigenfunction of Λ_j. The proof of Theorem 1.3 can now be applied, and the result follows. ■

6 Splitting of $\sigma(\Lambda_\gamma)$ and Geometric Interpretation

Let us summarize of our analysis up to this point. The Dirichlet-to-Neumann operator Λ_γ was first approximated by a diagonal operator, $D_\gamma = \sum_{j=1}^k C_j\Lambda_\gamma C_j$. While the operators $C_j\Lambda_\gamma C_j$ depend only on the boundary data on $\partial\Omega_j$, they still depend on the other boundary components $\partial\Omega_k, k \neq j$, through Λ_γ. We first replaced $C_j\Lambda_\gamma C_j$, acting on $L^2(\partial\Omega)$, by Λ_j, acting on $L^2(\partial\Omega_j)$. The operator Λ_j has the same spectrum as $C_j\Lambda_\gamma C_j$, but still depends on the other boundary components. However, each Λ_j is closely approximated by a similar operator $\Lambda_j^{\#}$ that is defined by way of a γ-harmonic extension to a region with only one boundary component. These operators $\Lambda_j^{\#}$, or, equivalently, the operators $\Lambda_{\partial\Omega_j}$, have the same spectral asymptotics as the $C_j\Lambda_\gamma C_j$, and are independent of $\partial\Omega_i$, for $i \neq j$. In each step, we have moved toward a simpler approximation of Λ_γ while maintaining the asymptotics of its spectrum. It is this sequence of approximations that lead us to the following assertion.

Theorem 6.1. *The spectrum of Λ_γ determines a lower bound on the number of components of $R^d\backslash\Omega$ and, further, also determines the weighted measure of each boundary component (not counting multiplicities).*

In the statement of this theorem, the phrase "not counting multiplicities" is intended to say that the asymptotics of $\sigma(\Lambda_\gamma)$ may not indicate the existence of more than one boundary component with the same weighted measure.

PROOF: Theorem 1.3 tells us the asymptotics of $\sigma(\Lambda_\gamma)$ are determined by $\sigma(D_\gamma)$, and we know $\sigma(D_\gamma) = \bigcup_{j=1}^k \sigma(C_j \Lambda_\gamma C_j) = \bigcup_{j=1}^k \sigma(\Lambda_j)$. Because $\Lambda_j^\# = \Lambda_j - A_j$, where $|\langle \phi_n^{(j)}, A_j \phi_n^{(j)} \rangle_{L^2(\partial\Omega_j)}| = (\lambda_n^{-\infty})$, for eigenfunctions $\phi_n^{(j)}$ of $\Lambda_j^\#$, the proof of Theorem 1.3 gives us that $\sigma(\Lambda_j^\#) \sim \sigma(\Lambda_j)$. It follows from Corollary 5.1 that $\sigma(\Lambda_j^\#) \sim \sigma(\Lambda_{\partial\Omega_j})$. Let $\sigma(\Lambda_{\partial\Omega_j}) = \{\lambda_m^j\}$, listed in nondecreasing order. The classical eigenvalue asymptotics give us

$$\lambda_m^j \sim \left(\frac{m}{C(\partial\Omega_j, \Lambda_{\gamma,\partial\Omega})} \right)^{1/(d-1)},$$

where,

$$C(\partial\Omega_j, \Lambda_{\gamma,\partial\Omega}) = (2\pi)^{-(d-1)} \operatorname{Vol} \{(x,\xi) \in T^*\partial\Omega_j \backslash \{0\} \mid \Lambda_{\gamma,\partial\Omega}(x,\xi) \leq 1\}.$$

Thus, we have the following equivalence:

$$\sigma(\Lambda_\gamma) \sim \sigma(D_\gamma) = \cup_{j=1}^k \sigma(\Lambda_j) \quad \text{and} \quad \sigma(\Lambda_j) \sim \sigma(\Lambda_j^\#) \sim \sigma(\Lambda_{\partial\Omega_j}).$$

That is to say, asymptotically, $\sigma(\Lambda_\gamma)$ splits into at most k disjoint sets, each of which is asymptotic to $(m/C(\partial\Omega_j, \Lambda_{\gamma,\partial\Omega}))^{1/(d-1)}$, for some j, $1 \leq j \leq k$. From this behavior, we can determine whether there *are* voids in Ω and, if so, bound the volume of each from above. However, if $i \neq j$, and $\partial\Omega_i$ and $\partial\Omega_j$ have the same $(d-1)$-dimensional weighted measure, the asymptotics of $\sigma(\Lambda_i)$ and $\sigma(\Lambda_j)$ are the same. Because of this, $\sigma(\Lambda_\gamma)$ may not indicate the existence of *both* holes, but only that there is *some* part of $\partial\Omega$ with the appropriate weighted measure. ■

Let us remark that this theorem implies that the high energy asymptotics of the spectrum of the DN map Λ_γ clearly indicate if the region is simply- or multiply-connected.

7 Perfect Insulators

The problem of the previous chapter was geometric in nature, because the DN map Λ_γ is defined by the voltage to current mapping on the *entire* boundary, including the interior boundaries. Clearly, if our goal is to detect imperfections in the interior of Ω, that is to say, if we do not already know such defects exist, then we surely cannot take measurements to help us calculate the spectrum based on fixing potentials on the inaccessible, interior boundaries. The following section reformulates our question so that only measurements on the outer boundary, $\partial\Omega_1$, are required for calculation. This is more in keeping with the methods of nondestructive evaluation.

Suppose the body Ω is interrupted not by voids but by a finite collection of smoothly bounded perfect insulators. As in the previous section, $\partial\Omega_1$ is understood to be the boundary of the reference region $\Omega_1^{\#}$, the bounded component of $\mathbb{R}^d \backslash \partial\Omega_1$, and $\partial\Omega_j$, $1 < j \leq k$, will denote the boundary of an insulator. In this setting, no current will pass through $\partial\Omega_j$, $1 < j \leq k$. We maintain hypotheses H1 and H2 in this section and, as before, write L_γ for the divergence form operator $L_\gamma = -\nabla \cdot \gamma \nabla$. We model the potential that is induced throughout Ω by a voltage f, imposed on $\partial\Omega_1$ only, as the solution to the mixed boundary-value problem

$$\begin{aligned} L_\gamma u &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega_1 \\ \nu \cdot \gamma \nabla u &= 0 \text{ on } \partial\Omega_j, \text{ for } 1 < j \leq k. \end{aligned} \tag{7.1}$$

We now prove the existence of weak solutions to the mixed problem (7.1). We write $L_\gamma[g]$ for the quadratic form associated with L_γ, that is,

$$L_\gamma[g] \equiv \int_\Omega \nabla g \cdot \gamma \nabla g \; dx. \tag{7.2}$$

Suppose that a solution, $v_f \in C^2(\overline{\Omega})$ exists. Let w be any other extension of f in $H^1(\Omega)$. Writing $w = v_f + (w - v_f)$, we see that

$$L_\gamma[w] = \int_\Omega \nabla w \cdot \gamma \nabla w \; dx = L_\gamma[w - v_f] + 2 \int_\Omega \nabla v_f \cdot \gamma \nabla (w - v_f) + L_\gamma[v_f].$$

But, because v_f solves (7.1) and $(w - v_f)$ vanishes on $\partial\Omega_1$, we know that

$$\int_\Omega \nabla v_f \cdot \gamma \nabla (w - v_f) = \int_\Omega \operatorname{div}((w - v_f)\gamma \cdot \nabla v_f) \; dx = \int_{\partial\Omega} (w - v_f) \nu \cdot \gamma \nabla v_f \; d\sigma = 0,$$

from which is follows that $L_\gamma[w] \geq L_\gamma[v_f]$. Thus, we search for a solution to (7.1) by minimizing the energy functional $L_\gamma[w]$, defined in (7.2), over a set $K_f^1(\Omega)$, defined as follows.

Definition 7.1. *For any $s > 1/2$, we define $K_f^s(\Omega) = \{w \in H^s(\Omega) \mid w = f$ on $\partial\Omega_1$ in the trace sense$\}$.*

Theorem 7.1. *The mixed problem (7.1) has a unique solution in the weak sense for any $f \in H^{1/2}(\partial\Omega_1)$.*

PROOF: The hypothesis that $f \in H^{1/2}(\partial\Omega)$ allows us to assert that $K^1_f(\Omega)$ is not empty, since the Dirichlet data $E_1 f$ on $\partial\Omega$ has a unique γ-harmonic extension. It is not immediately obvious that the energy functional L_γ achieves a minimum on $K^1_f(\Omega)$. However, it is bounded from below by H2 so an infimum exists. Let λ be this infimum. Then there is a sequence of functions $\{v_j\} \subset K^1_f(\Omega)$ such that $L_\gamma[v_j] \geq L_\gamma[v_{j+1}]$ and $\lim_{j\to\infty} L_\gamma[v_j] = \lambda$. It is easy to check that

$$L_\gamma\left[\frac{u+w}{2}\right] + L_\gamma\left[\frac{u-w}{2}\right] = \frac{1}{2}\{L_\gamma[u] + L_\gamma[w]\}, \tag{7.3}$$

whence

$$L_\gamma\left[\frac{v_j - v_k}{2}\right] = \frac{1}{2}\{L_\gamma[v_j] + L_\gamma[v_k]\} - L_\gamma\left[\frac{v_j + v_k}{2}\right]. \tag{7.4}$$

Because $(v_j + v_k)/2 \in K^1_f(\Omega)$, we know that $L_\gamma\left[\frac{v_j+v_k}{2}\right] \geq \lambda$. Consequently, (7.4) implies that

$$L_\gamma\left[\frac{v_j - v_k}{2}\right] \leq \frac{1}{2}\{L_\gamma[v_j] + L_\gamma[v_k]\} - \lambda. \tag{7.5}$$

Clearly, the right-hand side of (7.5) tends to zero as j and k tend to infinity. It follows from Lemma 2.2 that $\{v_j\}$ is a Cauchy sequence in $H^1(\Omega)$ so there is a function $v \in H^1(\Omega)$ to which it converges. The continuity of the trace operator ensures that $Tv = f$, as desired and, for any $\phi \in K^1_0(\Omega)$, the functional $L_\gamma[v + t\phi]$ takes its minimum at $t = 0$. Therefore,

$$0 = \left.\frac{d}{dt}\right|_{t=0} \int_\Omega \nabla(v + t\phi) \cdot \gamma\nabla(v + t\phi)\ dx = 2\int_\Omega \nabla v \cdot \gamma\nabla\phi\ dx, \tag{7.6}$$

so v is γ-harmonic in the weak (or distributional) sense. According to elliptic regularity (cf. [2]), the weak solution v is smooth in Ω, and $L_\gamma v = 0$ in the classical sense. Therefore, for any $\phi \in K^1_0(\Omega)$,

$$\int_\Omega \operatorname{div}(\phi\gamma \cdot \nabla v)\ dx = \int_\Omega (\nabla\phi \cdot \gamma\nabla v + \phi L_\gamma v)\ \ dx. \tag{7.7}$$

The first summand of the right-hand side of (7.7) is zero according to (7.6) and the second vanishes because $L_\gamma v \equiv 0$ in Ω. So we write, formally,

$$\int_{\partial\Omega\setminus\partial\Omega_1} (T\phi)\nu \cdot \gamma\nabla v\ d\sigma = \int_{\partial\Omega} (T\phi)\nu \cdot \gamma\nabla v\ d\sigma = \int_\Omega \operatorname{div}(\phi \cdot \gamma\nabla v)\ dx = 0.$$

Because the range of the trace map, T, is dense in $L^2(\partial\Omega)$, we conclude that the outward normal derivative of v on $\partial\Omega_j$, $1 < j \leq k$, is zero as defined in this weak sense.

Finally, we prove the solution to (7.1) is unique in $H^1(\Omega)$. Suppose, to the contrary, that two solutions exist in $H^1(\Omega)$, v and w. Then, the difference $(v - w) \in K_0^1(\Omega)$, so from the Divergence Theorem,

$$L_\gamma[v - w] = \int_\Omega \nabla(v - w) \cdot \gamma \nabla v \; dx - \int_\Omega \nabla(v - w) \cdot \gamma \nabla w \; dx = 0.$$

Therefore, $(v - w)$ is constant as a function in $L^2(\Omega)$. Because $v = w$ on $\partial\Omega_1$, it follows that $v = w$ over all of Ω. ∎

We will denote the solution to (7.1) by v_f. Though we know already that $v_f \in H^1(\Omega)$, because we are interested in the Dirichlet-to-Neumann operator, we must determine minimal conditions under which the existence of an outward normal derivative of v_f is guaranteed to exist at $\partial\Omega_1$, if not classically, at least as a function in $L^2(\partial\Omega_1)$.

Theorem 7.2. *If $f \in H^{3/2}(\partial\Omega_1)$, then the function $\frac{\partial}{\partial\nu} v_f$ exists in $L^2(\partial\Omega_1)$.*

PROOF: Let $\xi \in C^\infty(I\!R^d)$ such that ξ vanishes in a neighborhood of $\partial\Omega_j$, $1 < j \leq k$, and is identically one in a neighborhood of $\partial\Omega_1$. Then

$$v_f = \xi u_f + (1 - \xi) v_f + \xi(v_f - u_f),$$

where u_f is the γ-harmonic extension of $E_1 f$ to Ω. The outward normal derivative of u_f has been established under the given hypotheses, and the outward normal derivative of $(1 - \xi)v_f$ is trivial since $1 - \xi$ is identically zero near $\partial\Omega_1$. It remains to establish the existence of the outward normal derivative of $w_f \equiv \xi(v_f - u_f)$. This function w_f solves

$$\begin{aligned} L_\gamma w &= L_\gamma \xi(v_f - \xi u_f) = L_\gamma w_f \text{ in } \Omega \\ w &= 0 \text{ on } \partial\Omega_j,\ 1 \leq j \leq k. \end{aligned} \tag{7.8}$$

Because ξ is identically one in a tubular neighborhood of $\partial\Omega_1$, the function $L_\gamma w_f$ is supported in a neighborhood of Ω disjoint from $\partial\Omega_1$. Using the Dirichlet Green's function from Theorem 2.3, we can represent the function w_f as

$$w_f(x) = \int_\Omega G_{\gamma,\Omega}(x, y)(L_\gamma \xi(v_f - u_f))(y) \; dy.$$

The existence of the outward normal derivative of w_f for each $x^* \in \partial\Omega_1$ now follows from the smoothness properties of the Green's function away from the diagonal. ■

Theorems 7.1 and 7.2 allow us to define the Dirichlet-to-Neumann operator $\Lambda_\gamma^\flat : H^{3/2}(\partial\Omega_1) \to H^{1/2}(\partial\Omega_1)$ by $\Lambda_\gamma^\flat f = \nu \cdot \gamma\nabla v_f|\partial\Omega_1$. In fact, we may extend its definition to $H^{1/2}(\partial\Omega)$ and assert its self-adjointness (on the appropriate domain) with only minor adjustments to the previous proofs. It seems only natural to ask whether the results of the previous section extend to this new setting.

The main result of this section is that the asymptotics of $\sigma(\Lambda_\gamma^\flat)$ give us no information regarding the existence of perfect insulators in the interior of Ω. This will follow from the previous arguments and from Lemma 7.2. This result asserts the rapid localization of the eigenfunctions of $\Lambda_\gamma^\flat$ near the boundary of Ω_1, and away from the other boundary components $\partial\Omega_j$, for $j = 1, \ldots, k$. This is interesting because the definition of $\Lambda_\gamma^\flat$, through the mixed problem (7.1), *does* involve the other boundary components.

7.1 The Green's Function for the Case of Perfect Insulators

In order to apply the techniques of the previous section, we must provide an appropriate Green's function $G_{\gamma,\Omega}(x, y)$ for (7.1). We begin with Green's identity which, for a γ-harmonic function u, can be written

$$u(x) = \int_{\partial\Omega} \left(G_\gamma(x, y)\nu \cdot \gamma\nabla u(y) - u(y)\nu \cdot \gamma\nabla G_\gamma(x, y)\right) \, d\sigma(y), \qquad (7.9)$$

where $G_\gamma(x, y)$ is the Green's function for $I\!R^d, d \geq 3$ of Theorem 2.3. As it stands, (7.9) is unhelpful because we have no a priori knowledge of $u|_{\partial\Omega_j}$, for $1 < j \leq k$, or of $\nu \cdot \gamma\nabla u$ on $\partial\Omega_1$. To resolve this problem, we appeal to the following technical Lemma (whose proof we will postpone):

Lemma 7.1. *Suppose that $f \in H^{1/2}(\partial\Omega_1)$ and $g_j \in H^{1/2}(\partial\Omega_j)$, for $1 < j \leq k$. Then there exists a solution in $H^1(\Omega)$ to the mixed problem*

$$\begin{array}{rcl} L_\gamma w &=& 0 \text{ in } \Omega \\ w &=& f \text{ on } \partial\Omega_1 \\ \nu \cdot \gamma\nabla u &=& g_j \text{ on } \partial\Omega_j, \; 1 < j \leq k. \end{array} \qquad (7.10)$$

We take h_x to be the solution of (7.10) with $f = G_\gamma(x, \cdot)$ and $g_j = \nu \cdot \gamma\nabla G_\gamma(x, \cdot)|_{\partial\Omega_j}$. It follows from the Divergence Theorem that, for a γ-

harmonic function u,

$$0=\int_{\partial\Omega}(u\nu\cdot\gamma\nabla h_x-h_x\nu\cdot\gamma\nabla u)\ d\sigma. \tag{7.11}$$

If we add (7.11) to (7.9) we get

$$u(x)=\int_{\partial\Omega_1}f\nu\cdot\gamma\nabla G_{\gamma,\Omega}(x,\cdot)\ d\sigma, \tag{7.12}$$

where $G_{\gamma,\Omega}(x,y)=h_x(y)-G_\gamma(x,y)$. Hence, we have proved the following result.

Proposition 7.3. *The Green's function for (7.1), $G_{\gamma,\Omega}(x,y)$, can be represented as*

$$G_{\gamma,\Omega}(x,y)=h_x(y)-G_\gamma(x,y), \tag{7.13}$$

where h_x solves (7.10) with $f=G_\gamma(x,\cdot)$ and $g_j=\nu\cdot\gamma\nabla G_\gamma(x,\cdot)|_{\partial\Omega_j}$.

This Green's function enables us to prove the existence theorem for solutions of (7.1).

Theorem 7.4. *The solution of the mixed problem (7.1) is given by (7.12), where $G_{\gamma,\Omega}$ is the Green's function for Ω as described in Proposition 7.3.*

It remains only to prove Lemma 7.1.

Proof (of Lemma 7.1): We solve (7.10) by splitting it into two problems whose solutions will sum to the desired function:

$$\left.\begin{array}{rcl} L_\gamma w &=& 0 \text{ in } \Omega \\ w &=& f \text{ on } \partial\Omega_1 \\ \nu\cdot\gamma\nabla w &=& 0 \text{ on } \partial\Omega_j,\ 1<j\leq k \end{array}\right\} \tag{7.14}$$

and

$$\left.\begin{array}{rcl} L_\gamma w &=& 0 \text{ in } \Omega \\ w &=& 0 \text{ on } \partial\Omega_1 \\ \nu\cdot\gamma\nabla w &=& g_j \text{ on } \partial\Omega_j,\ 1<j\leq k \end{array}\right\}. \tag{7.15}$$

Problem (7.14) has already been addressed in Theorem 7.1, so we focus our attention on (7.15). Restricting ourselves once again to $w,\phi\in K_0^1(\Omega)$, we rewrite (7.15) in its weak formulation:

$$L_\gamma[\phi,w]=f(\phi), \tag{7.16}$$

where $L_\gamma[\phi, w] = \int_\Omega \nabla w \cdot \gamma \nabla \phi \, dx$, is the bilinear form associated with L_γ, and $f(\phi) = \int_{\partial\Omega} gT(\phi) \, d\sigma$. Because of H2, and the fact that functions in $K_0^1(\Omega)$ vanish on $\partial\Omega_1$, Lemma 2.2 implies that there is a constant C for which

$$L_\gamma[w, w] \geq C^{-1}||w||^2_{H^1(\Omega)},$$

for any $w \in K_0^1(\Omega)$. Further, it follows from H2 and Hölder's inequality that $L_\gamma[w, v] \leq \|w\|_{H^1(\Omega)} \; \|v\|_{H^1(\Omega)}$. These two bounds imply that the bilinear form L_γ over $K_0^1(\Omega)$ satisfies the hypotheses of the Lax-Milgram Theorem. Consequently, there exists a function $w \in K_0^1(\Omega)$ such that (7.16) is true for all $\phi \in K_0^1(\Omega)$. Thus, (7.15) has a solution in the weak sense. ∎

Proposition 7.5. *For any $x \in \Omega$, the Green's function for (7.1), $G_{\gamma,\Omega}(x, y)$, is a smooth function of $y \in \partial\Omega_1$.*

Proof: Since $G_{\gamma,\Omega}(x, y) = G_\gamma(x, y) - h_x(y)$, and $G_\gamma(x, \cdot)$ is smooth on $\partial\Omega_1$, for any $x \in \Omega$, we need only consider the function h_x and its behavior near $\partial\Omega_1$. Suppose $\xi \in C_0^\infty(I\!R^d)$ is identically one near $\partial\Omega_1$, vanishes in a neighborhood of $\partial\Omega_j$, $1 < j \leq k$, and whose support contains the point x. Then

$$h_x = \xi G_\gamma(x, \cdot) - R_\Omega(L_\gamma \xi G_\gamma(x, \cdot)),$$

where $R_\Omega(F)$ solves

$$\begin{aligned} L_\gamma w &= F \text{ in } \Omega \\ w &= 0 \text{ on } \partial\Omega_1 \\ \nu \cdot \gamma \nabla w &= \nu \cdot \gamma \nabla G_\gamma(x, \cdot), \text{ on } \partial\Omega_j, \; 1 < j \leq k. \end{aligned} \tag{7.17}$$

The function $R_\Omega(F)$ can be constructed as the sum of the solution to (7.10), where we take $f = 0$ and $g_j = \nu \cdot \gamma \nabla G_\gamma(x, \cdot)|_{\partial\Omega_j}$, and the solution to

$$\begin{aligned} L_\gamma w &= F \text{ in } \Omega \\ w &= 0 \text{ on } \partial\Omega_1 \\ \nu \cdot \gamma \nabla w &= 0 \quad \text{on } \partial\Omega_j, 1 < j \leq k. \end{aligned} \tag{7.18}$$

Equation (7.18) is solved in $K_0^1(\Omega)$ by applying the Lax-Milgram Theorem to sesquilinear form $L_\gamma[u, v] = \int_\Omega \nabla u \cdot \gamma \nabla v \, dx$, with $f(v) = \int_\Omega Fv \, dx$, as above.

Because ξ is identically one near $\partial\Omega_1$, there is a tubular neighborhood of $\partial\Omega_1$ in which $L_\gamma \xi G_\gamma(x, \cdot) = 0$. It follows from elliptic regularity that $R_\Omega(\xi G_\gamma(x, \cdot))$ is smooth on $\partial\Omega_1$ and, consequently, that $h_x|_{\partial\Omega_1}$ is the sum of smooth functions. ∎

7.2 Rapid Localization Revisited

In the previous subsection, we realized the solution of (7.1) as

$$v_f(x) = \int_{\partial\Omega_1} f G_{\gamma,\Omega}(x,\cdot)\ d\sigma = \langle f, G_{\gamma,\Omega}(x,\cdot)\rangle_{L^2(\partial\Omega_1)}, \tag{7.19}$$

where $G_{\gamma,\Omega}$ is the smooth Green's function for (7.1). We are now prepared to pass behaviors addressed in previous sections to this new setting.

Lemma 7.2. *Suppose that the eigenvalues of $\Lambda_\gamma^\flat$ are $\{\lambda_n\}$, indexed so that $\lambda_n \leq \lambda_{n+1}$, with associated eigenfunctions ϕ_n satisfying $\|\phi_n\| = 1$. Let $K \subset \overline{\Omega}\backslash\partial\Omega_1$ be a compact set. If v_n solves (7.1) with boundary data ϕ_n on $\partial\Omega_1$, we have $\|v_n\|_{H^1(K)} \in O(n^{-\infty})$.*

Proof: We follow the proof of Proposition 3.1. Using the representation formula (7.19), we have

$$\begin{aligned} |v_n(x)| &= |\langle \phi_n, G_{\gamma,\Omega}(x,\cdot)\rangle| \\ &= \lambda_n^{-j}\ |\langle (\Lambda_\gamma^\flat)^j \phi_n, G_{\gamma,\Omega}(x,\cdot)\rangle| \\ &= \lambda_n^{-j}\ |\langle \phi_n, (\Lambda_\gamma^\flat)^j G_{\gamma,\Omega}(x,\cdot)\rangle| \\ &\leq \lambda_n^{-j}\ \|(\Lambda_\gamma^\flat)^j G_{\gamma,\Omega}(x,\cdot)\|_{L^2(\partial\Omega)}, \end{aligned} \tag{7.20}$$

by the Cauchy-Schwarz inequality. Notice that $\|(\Lambda_\gamma^\flat)^j G_{\gamma,\Omega}(x,\cdot)\|_{L^2(\partial\Omega)}$ depends only upon x (i.e., it is independent of n). In fact, the dependence upon x lies in the singularity of G_γ which, in turn, is dependent upon the distance from x to $\partial\Omega_1$. Because K is compact, all of its members are removed from $\partial\Omega_1$ by some minimal distance, which results in a uniform constant. Thus,

$$|v_n(x)| \leq C(K,j)\lambda_n^{-j}.$$

The standard Weyl eigenvalue asymptotics [8, 20] allow us to assert the polynomial growth of λ_n^j in n, for sufficiently large j, and this completes gives the estimate on v_n. As for ∇v_n, it follows from (7.19) that

$$\nabla v_n(x) = \langle \phi_n, \nabla_x G_{\gamma,\Omega}(x,\cdot)\rangle,$$

and the same argument yields the decay of $|\nabla v_n|$. This completes the proof.∎

Lemma 7.2 says that γ-harmonic extensions of eigenfunction of $\Lambda_\gamma^\flat$, as defined in this setting, localize rapidly near the boundary $\partial\Omega_1$, as the eigenvalue index increases. This is important for the following reason: suppose that $f \in H^{1/2}(\partial\Omega_1)$ and v_f solves (7.1). Using the eigenfunctions of $\Lambda^\flat_\gamma$ as a basis for $L^2(\partial\Omega_1)$, we can write

$$f = \sum_{n\geq 0} \hat{f}(n)\phi_n, \text{ where } \hat{f}(n) = \langle f, \phi_n\rangle_{L^2(\partial\Omega_1)}.$$

It follows from the definition of v_n that

$$v_f = \sum_{n\geq 0} \hat{f}(n) v_n. \tag{7.21}$$

At high energy, the γ-harmonic extensions v_n of the eigenfunctions of $\Lambda^\flat_\gamma$ are localized near the boundary $\partial\Omega_1$. Since $f \in L^2(\partial\Omega_1)$ implies that $\hat{f}(n) \in \ell^2(I\!N)$, we infer from equation (7.21) that information concerning the existence of perfect insulators inside Ω cannot be extracted from the high-energy asymptotics of the spectrum of $\Lambda^\flat_\gamma$. This is observation is restated mathematically by the following theorem. Recall that $\Omega_1^\#$ is the bounded component of $I\!R^d \backslash \partial\Omega_1$, and that the DN operator $\Lambda_1^\#$ is defined on $\partial\Omega_1$ using the γ-harmonic extension to $\Omega_1^\#$.

Theorem 7.6. *The operators $\Lambda^\flat_\gamma$ and $\Lambda_1^\#$ differ by a smoothing operator.*

PROOF: Suppose the eigenfunctions of $\Lambda_1^\#$ are $\{\phi_n^{(1)}\}$ with associated eigenvalues $\{\lambda_n\}$. We will prove the theorem by showing that

$$\langle(\Lambda^\flat_\gamma - \Lambda_1^\#)\phi_n^{(1)}, \phi_m^{(1)}\rangle_{L^2(\partial\Omega_1)} = O(n^{-\infty})\cdot O(m^{-\infty}) \tag{7.22}$$

and applying the proof of Lemma 5.2. Let $u_n^\#$ be the γ-harmonic extension of $\phi_n^{(1)}$ to $\Omega^\#$, and let v_n be the γ-harmonic extension of $\phi_n^{(1)}$ to Ω as in Lemma 7.2. We begin, as before, by writing

$$\langle \Lambda_1^\# \phi_n, \phi_m\rangle_{L^2(\partial\Omega_1)} = \int_{\Omega_1^\#} \nabla u_n^\# \cdot \gamma\nabla u_m^\# \, dx,$$

and

$$\langle \Lambda^\flat_\gamma \phi_n, \phi_m\rangle_{L^2(\partial\Omega_1)} = \int_\Omega \nabla v_n \cdot \gamma\nabla v_m \, dx.$$

The difference can be expressed as

$$\begin{aligned}\langle(\Lambda^\flat_\gamma - \Lambda_1^\#)\phi_n, \phi_m\rangle_{L^2(\partial\Omega_1)} &= \int_\Omega \left(\nabla v_n \cdot \gamma\nabla v_m - \nabla u_n^\# \cdot \gamma\nabla u_m^\#\right) dx \\ &\quad - \int_{\Omega_1^\#\backslash\Omega} \nabla u_n^\# \cdot \gamma\nabla u_m^\# \, dx. \end{aligned}\tag{7.23}$$

Because $\Omega_1^{\#}\backslash\Omega$ is a compact subset of $\Omega_1^{\#}$, the second integral of (7.23) has the desired rate of decay by Lemma 7.2. We address the first integral of (7.23) by adding zero:

$$\begin{aligned}
&\int_\Omega \left(\nabla v_n \cdot \gamma \nabla v_m - \nabla u_n^{\#} \cdot \gamma \nabla u_m^{\#}\right)\, dx \\
&\qquad = \int_\Omega \left\{\nabla v_n \cdot \gamma \nabla (v_m - u_m^{\#}) + \nabla (v_n - u_n^{\#}) \cdot \gamma \nabla u_m^{\#}\right\}\, dx \\
&\qquad = \int_\Omega \left\{\mathrm{div}((v_m - u_m^{\#})\gamma \nabla v_n) + \mathrm{div}((v_n - u_n^{\#})\gamma \nabla u_m^{\#})\right\}\, dx \\
&\qquad = \int_{\partial\Omega\backslash\partial\Omega_1} (v_n - u_n^{\#})\nu \cdot \gamma \nabla u_m^{\#}\, d\sigma.
\end{aligned}$$

Because $\partial\Omega\backslash\partial\Omega_1$ is (clearly) removed from $\partial\Omega_1$, we know that $|u_m^{\#}(x)|$ and $|\frac{\partial u_m^{\#}}{\partial x_j}(x)|$ are $O(\lambda_m^{-\infty})$ for each $x \in \partial\Omega\backslash\partial\Omega_1$. Further, beginning with the representation of v_n as described by (7.12), we apply the techniques of the proof of Lemma 7.2 to conclude that $\|v_n\|_{L^2(\partial\Omega\backslash\partial\Omega_1)} = O(n^{-\infty})$. This proves (7.22), and the result follows. ∎

8 Appendix: Existence of γ-harmonic Extensions to Unbounded Domains

Let $\Omega \subset I\!R^d$, for $d \geq 3$, be an *unbounded* region such that $\partial\Omega$ is smooth, and its complement Ω^c is a compact region. In this section, we give a proof of the existence of a unique solution to the boundary value problem

$$\begin{aligned} L_\gamma u &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega, \end{aligned} \tag{8.1}$$

and

$$|u(x)| = O(\|x\|^{2-d}), \text{ as } \|x\| \to \infty. \tag{8.2}$$

This result is used in section 5.

Let L_γ be the second-order, uniformly elliptic operator defined in (1.1) with coefficients γ_{ij} satisfying H1, H2, and H3 with Ω replaced by Ω^c. We use Theorem 2.3 on the existence of the Dirichlet Green's function $G_{\gamma,\Omega}$ for unbounded regions Ω, with $\partial\Omega \neq \emptyset$, and for bounded, open regions. By using this Green's function, we prove the following theorem on the existence of solutions to the boundary-value problem (8.1)–(8.2) (cf. [15]).

Theorem 8.1. *Suppose Ω is an open, unbounded region in $I\!R^d$, $d \geq 3$, with a smooth, nonempty boundary, whose complement Ω^c is a compact set. Assume that the coefficients satisfy hypotheses H1–H3 (with Ω replaced by Ω^c in H3). Then, for any $f \in C(\partial\Omega)$, there is a unique function $u_f \in H^1(\Omega)$ satisfying*

$$\begin{aligned} L_\gamma u &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \end{aligned} \tag{8.3}$$

and

$$\begin{aligned} |u(x)| &= O(\|x\|^{2-d}), \\ |\nabla u(x)| &= O(\|x\|^{1-d}). \end{aligned} \tag{8.4}$$

as $\|x\| \to \infty$.

PROOF: We construct the solution u by combining the solution to two boundary-value problems that we already know have solutions. First, we consider the boundary-value problem on $\Omega_{2R} \equiv B_{2R}(0)\backslash\Omega^c$, given by

$$\begin{aligned} L_\gamma w &= 0 \text{ in } \Omega_{2R} \\ w &= f \text{ on } \partial\Omega \\ w &= 0 \text{ on } \partial B_{2R}(0). \end{aligned} \tag{8.5}$$

The unique solution to this problem is constructed using the Dirichlet Green's function $G_{\gamma,\Omega_{2R}}$ of Theorem 2.3 for the bounded region Ω_{2R}. By the method of layer potentials, we have

$$w_f(x) = \int_{\partial\Omega} f(\omega)\nu \cdot \gamma(\omega)\nabla G_{\gamma,\Omega_{2R}}(x,\omega)\ d\sigma(\omega).$$

Next, we consider the boundary-value problem on $I\!R^d\backslash B_R(0)$,

$$\begin{aligned} \Delta v &= 0 \text{ in } I\!R^d\backslash B_R(0) \\ v &= w \text{ on } \partial B_R(0) \\ \frac{\partial v}{\partial \nu} &= \frac{\partial w_f}{\partial \nu} \text{ on } \partial B_R(0), \end{aligned} \tag{8.6}$$

with $v(x)$ satisfying

$$|v(x)| = O(\|x\|^{2-d}). \tag{8.7}$$

The unique solution to this problem can be constructed explicitly, cf. [3]. Let $G_R(x,y)$ be the Green's function for the Laplacian on $I\!R^d\backslash B_R(0)$, with Dirichlet boundary conditions on $B_R(0)$, and decaying at infinity like $\|x\|^{-(d-2)}$, for fixed $y \in I\!R^d\backslash B_R(0)$. This function can be constructed using the Kelvin transform and the Green's function for the Laplacian on $\mathbb{R}^d$. The solution to the second problem (8.6) can be represented by

$$v(x) = \int_{\partial B_R(0)} [G_R(x,\omega)\partial_\nu w(\omega) - w(\omega)\partial_\nu G_R(x,\omega)] \; d\sigma(\omega). \tag{8.8}$$

Note that the decay property (8.7) is explicit from this representation and the decay property of the Green's function.

Finally, we combine these two solutions. Let $\eta \geq 0$ be a smooth cut-off function having the properties that $\eta|B_R(0) = 1$, and supp $|\nabla\eta| \subset B_{2R}(0)\backslash B_R(0)$. Let us define a function $\psi \equiv \eta w + (1-\eta)v$. This function ψ satisfies

$$L_\gamma \psi = -[\Delta,\eta]w + [\Delta,\eta]v \equiv F, \tag{8.9}$$

since $\gamma_{ij} = \delta_{ij}$ for $\|x\| > R$. The function F has compact support in the interior of $B_{2R}(0)\backslash B_R(0)$. From Theorem 2.3, there exists a Green's function $G_{\gamma,\Omega}(x,y)$ for L_γ on the region Ω, with Dirichlet boundary conditions on the boundary $\partial\Omega$, and decaying at infinity. Using this Green's function, we construct a function

$$z(x) \equiv \int_{B_{2R}(0)\backslash B_R(0)} G_{\gamma,\Omega}(x,y)F(y) \; dy. \tag{8.10}$$

This function satisfies

$$L_\gamma z = F, \quad \text{in } \Omega, \quad \text{and } z \mid \partial\Omega = 0. \tag{8.11}$$

Consequently, the function $u \equiv \psi - z$ is a solution to the boundary-value problem (8.1). The manner in which u is constructed, and Theorem 2.3, allow us to compute the decay rate of the solution u. For $\|x\| > 2R$, the solution is

$$u(x) = v(x) - z(x). \tag{8.12}$$

It is clear from parts (c.) and (d.) of Theorem 2.3, and the properties of G_R, that (8.4) holds. It remains only to prove that this function is unique among all those that vanish at infinity. To the contrary, suppose there is another such function, v, solving (8.1) and vanishing at infinity. Then $\phi = (u - v)$ is γ-harmonic in Ω, satisfies $\phi \mid \partial\Omega = 0$, and vanishes at infinity. It follows from Moser's form of Liouville's Theorem [16] that $\phi = 0$. ■

Given the existence of a unique solution to (8.1) decaying at infinity as in (8.2), we can now prove a representation formula for the solution.

Corollary 8.1. *Suppose $\Omega \subset I\!R^d, d \geq 3$ is an open region in $I\!R^d$ with smooth, nonempty boundary whose complement is a compact set. Then, there is a function $\mathcal{G}_{\gamma,\Omega} : \Omega \times \partial\Omega \to I\!R$ such that*

$$u(x) = \int_{\partial\Omega} \mathcal{G}_{\gamma,\Omega}(x,\omega) f(\omega) \; d\sigma(\omega)$$

is the unique solution to

$$\begin{aligned} L_\gamma u &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega, \end{aligned} \tag{8.13}$$

for $f \in C(\partial\Omega)$, among all functions that tend to zero at infinity. Furthermore, the solution satisfies the decay estimate (8.2).

PROOF: Let u_f be the unique solution to (8.1) constructed in Theorem 8.1. Using the Dirichlet Green's function for Ω, as given in Theorem 2.3, we derive a representation formula for the solution to (8.1). Let $R >> 0$ be chosen. An application of the Divergence Theorem to the region $B_R(0) \backslash \Omega^c$ yields

$$\begin{aligned} u_f(x) &= \int_{\partial\Omega} f(\omega) \nu \cdot \gamma(\omega) \nabla G_{\gamma,\Omega}(\omega, x) \; d\sigma(\omega) \\ &\quad + \int_{\partial B_R(0)} [u(R\omega)\nu \cdot \nabla G_{\gamma,\Omega}(R\omega, x) - G_{\gamma,\Omega}(R\omega, x)\nu \cdot \nabla u(R\omega)] \; R^{d-1} \; d\omega, \end{aligned} \tag{8.14}$$

where $d\omega$ is the measure on the sphere S^{d-1}, and $\gamma = 1$ for R large enough. We now use the decay estimates on $G_{\gamma,\Omega}$ and $\nabla G_{\gamma,\Omega}$, given in Theorem 2.3, parts (c.) and (d.), and the decay estimate on u_f and ∇u_f in (8.2), to prove that the integral over $\partial B_R(0)$ vanishes as $R \to \infty$. As a result, we obtain

$$u_f(x) = \int_{\partial\Omega} f(\omega) \nu \cdot \gamma(\omega) \nabla G_{\gamma,\Omega}(\omega, x) \; d\sigma(\omega), \tag{8.15}$$

so that $\mathcal{G}_\Omega(x,\omega) = \nu \cdot \gamma(\omega) \nabla G_{\gamma,\Omega}(x,\omega)$. ∎

We can now define the Dirichlet-to-Neumann operator for $\partial\Omega$ by way of γ-harmonic extensions to the unbounded region Ω.

Proposition 8.2. *Assume Ω and L_γ satisfy the same properties as in Theorem 8.1, so that Ω^c is compact. If u_f and u_g are the γ-harmonic extensions* *of $f, g \in C(\partial\Omega)$ to Ω, as in Theorem 8.1, and ν is the outward normal to Ω, then*

$$\langle g, \Lambda_\gamma f\rangle_{L^2(\partial\Omega)} = \langle g, \nu\cdot\gamma\nabla u_f\rangle_{L^2(\partial\Omega)} = \int_\Omega \nabla u_g \cdot \gamma\nabla u_f \; dx.$$

PROOF: Choose $R >> 0$ so that $R^d \backslash \Omega^c \subset B_R(0)$, and write $\Omega_R = \Omega \cap B_R(0)$. We use the Divergence Theorem on this smoothly bounded region to write

$$\int_{\Omega_R} \nabla u_g \cdot \gamma\nabla u_f \; dx = \int_{\partial\Omega} g\nu\cdot\gamma\nabla u_f \; d\sigma + \int_{\partial B_R(0)} u_g \nu \cdot \gamma \nabla u_f \; d\sigma, \quad (8.16)$$

where ν is the outward normal to Ω_R. Properties (8.2) of the γ-harmonic extensions u_f and ∇u_g show that the second integral in (8.16) vanishes as $R \to \infty$. ■

Bibliography

[1] J. Edward: An Inverse Spectral Result for the Neumann Operator on Planar Domains, *J. Funct. Anal.* **111**, 312–322 (1993).

[2] L. C. Evans: Partial Differential Equations, Graduate Studies in Mathematics **19**, Providence RI: American Mathematical Society, 1998.

[3] G. Folland: Introduction to Partial Differential Equations, 2^{nd} edition, Princeton NJ: Princeton University Press, 1995.

[4] A. Friedman, M. Vogelius: Identification of Small Inhomogeneities of Extreme Conductivity by Boundary Measurements: a Theorem on Continuous Dependence, *Arch. Rat. Mech. Anal* **105**, 299–326 (1989).

[5] M. Grüter, K. Widman: The Green Function for Uniformly Elliptic Equations, *Manuscripta Mathematica* **37**, 303–342 (1982).

[6] P. D. Hislop, C. Lutzer: A Note on the Green's Function for Elliptic Operators on Unbounded Domains in $\mathbb{R}^d$, in preparation.

[7] P. D. Hislop, I. M. Sigal: Introduction to Spectral Theory, with applications to Schrödinger operators, New York: Springer-Verlag, 1996.

[8] L. Hörmander: The Analysis of Linear Partial Differential Operators IV, New York: Springer-Verlag, 1985.

[9] V. Isakov: On the Uniqueness of the Recovery of a Discontinuous Conductivity Coefficient, *Communications on Pure and Applied Mathematics* **41**, 865-977 (1988).

[10] R. Kohn, M. Vogelius: Determining Conductivity by Boundary Measurements, II, Interior Results, *Communications on Pure and Applied Mathematics* **38**, 643–667 (1985).

[11] M. Lassas, G. Uhlmann: On Determining a Riemannian Manifold from the Dirichlet-to-Neumann Map, to appear in *Ann. Scien. Ecole Norm. Sup.*

[12] M. Lassas, M. Taylor, G. Uhlmann: The Dirichlet-to-Neumann Map for Complete Riemannian Manifolds with Boundary, preprint 2001.

[13] J. M. Lee, G. Uhlmann: Determining the Anisotropic Real-Analytic Conductivities by Boundary Measurements, *Communications on Pure and Applied Mathematics* **XLII**, 1097–1112 (1989).

[14] W. Littman, G. Stampacchia, H. Weingerger: Regular Points for Elliptic Equations with Discontinuous Coefficients, *Ann. Scuola Norm. Pisa* **17**, 43–77 (1963).

[15] C. Miranda: Partial Differential Equations of Elliptic Type, Second Edition, New York: Springer-Verlag, 1970.

[16] J. Moser: On Harnack's Theorem for Elliptic Differential Equations, *Commun. Pure Appl. Math.* **14**, 577–591 (1961).

[17] A. Nachman: Reconstructions from Boundary Measurements, *Annals of Math.* **128**, 531–587 (1988).

[18] M. Reed, B. Simon: Methods of Modern Mathematical Physics, IV. Analysis of Operators, New York: Academic Press, 1978.

[19] J. Sylvester, G. Uhlmann: A Global Uniqueness Theorem for an Inverse Boundary Value Problem, *Annals of Math.* **125**. 153–169 (1987).

[20] M. E. Taylor: Pseudodifferential Operators, New Jersey: Princeton University Press, 1981.

[21] M. E. Taylor: Partial Differential Equations II: Qualitative Studies of Linear Equations, New York: Springer-Verlag, 1996.

[22] G. Uhlmann: Inverse Boundary Value Problems and Applications, *Astérisque* **207**, 153–211 (1992).

[23] G. Uhlmann: CBMS Conference Notes, University of Kentucky, 1995, to be published by SIAM.

[24] R. Weinstock: Inequalities for a Certain Eigenvalue Problem, *J. Rat. Mech. Anal.* **3**, 745–753 (1954).