Stability of the 2D Brown-Ravenhall operator [1]

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Abstract

We prove that the two-dimensional Brown-Ravenhall operator is bounded from below when the coupling constant is below a specified critical value — property also referred to as stability. As a consequence, the operator is then self-adjoint. The proof is based on the strategy followed in [9] and [5] with some relevant changes characteristic of the dimension. Our analysis also yields a sharp Kato inequality.

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1 Introduction and Results

We consider here the two dimensional Brown-Ravenhall operator [1] :

$$\mathbf{B} = \Lambda_+(D - \frac{\delta}{|x|})\Lambda_+ \quad (\delta > 0). \tag{1.1}$$

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In (1.1), the notation is as follows :

- D is the free Dirac operator :

$$D = c\boldsymbol{\sigma} \cdot \frac{\hbar}{i}\nabla + mc^2\sigma_3 = \frac{c\hbar}{i}\sigma_1\frac{\partial}{\partial x_1} + \frac{c\hbar}{i}\sigma_2\frac{\partial}{\partial x_2} + mc^2\sigma_3$$

  where $\hbar$ is the Planck constant, m the electron mass, c is the velocity of light and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ and σ_3 are the Pauli matrices :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- $\Lambda_+ := \chi_{(0,+\infty)}(D)$, where $\chi_{(0,+\infty)}$ is the characteristic function of the interval $(0, +\infty)$, is the projection of $L^2(\mathbb{R}^2, \mathbb{C}^2)$ onto the positive spectral subspace of D.

The underlying Hilbert space in which $\mathbf{B}$ acts is

$$\mathcal{H}_+ = \Lambda_+(L^2(\mathbb{R}^2, \mathbb{C}^2)).$$

Due to the algebraic properties of the Pauli matrices, the Dirac operator D in $\mathbb{R}^2$ is a symmetric operator and satisfies

$$D^2 = -\hbar^2 c^2 \Delta + m^2 c^4.$$

Moreover, it has similar properties to that in $\mathbb{R}^3$ (see [11] for example) : it is a self-adjoint operator in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ with domain $\mathcal{D}(D) = H^1(\mathbb{R}^2, \mathbb{C}^2)$ and its spectrum is $(-\infty, -m] \cup [m, +\infty)$. In particular, it is unbounded from below, so that it is difficult to define the ground state of atoms and molecules. In order to study the stability of relativistic systems from the mathematical point of view, various models have been introduced. In the simplest one, D is replaced by the positive definite Hamiltonian $\sqrt{-\hbar^2 c^2 \Delta + m^2 c^4}$. Another alternative is the Brown-Ravenhall operator $\mathbf{B}$ introduced in [1]. If (1.1) is to model the 1-electron atom, then we take $\delta = Ze^2$, where Z and $-e$ are the nuclear and the electron charge respectively.

In the three dimensional situation, several recent papers have been devoted to the study of the Brown-Ravenhall operator. If we consider the many particle case (N electrons in the field of K nuclei), it takes the following form :

$$\mathbf{B}_{N,K} = \Lambda_{+,N}(\sum_{n=1}^{N} D^{(n)} - V_c)\Lambda_{+,N} \tag{1.2}$$

and acts in the N-fold antisymmetric tensor product $\mathcal{H}_N$ of $\Lambda_+(L^2(\mathbb{R}^3, \mathbb{C}^4))$. In (1.2), $D^{(n)}$ denotes the free 3 dimension Dirac operator acting on the n-th particle, $\Lambda_{+,N}$ is the projection onto $\mathcal{H}_N$ and the potential V_c is the usual Coulomb interaction of the particles

$$V_c(x_1, \cdots, x_N) = -\sum_{j=1}^{N}\sum_{k=1}^{K} \frac{Z_k e^2}{|x_j - R_k|} + \sum_{1\leq j<l\leq N} \frac{e^2}{|x_j - x_l|} + \sum_{1\leq j<k\leq K} \frac{Z_j Z_k e^2}{|R_j - R_k|} \tag{1.3}$$

where x_j, R_k denote the coordinates of the jth electron and kth nucleus respectively, and Z_k the charge of the kth nucleus.

The case of K nuclei and 1 electron was treated by Evans and Balinsky in [3]. They show that the Brown-Ravenhall operator $\mathbf{B}_{1,K}$ satisties $\mathbf{B}_{1,K} \geq const.K$ if for all $1 \leq k \leq K$, $\alpha Z_k \leq \gamma_c = \frac{2}{(\frac{2}{\pi}+\frac{\pi}{2})}$ and $\alpha \leq \gamma_c/(4 + 2\sqrt{1+\frac{\pi}{2}})$. Here $\alpha = e^2/c\hbar$ denotes Sommerfeld's fine structure constant. On taking $\alpha^{-1} \approx 137$, the critical charge Z_c in [3] is 124 and hence all known elements are include in the range. The case of K nuclei and N electrons, but zero mass, was subsequently treated by Hoever and Siedentop in [8] where it is shown that there exists an α_0 such that $\mathbf{B}_{N,K} \geq 0$ if for all $1 \leq k \leq K$, $\alpha Z_k \leq \gamma_c$ and $\alpha \leq \alpha_0$. Therefore, they get stability up to $Z_k \leq 88$ ($k = 1, \cdots, K$) on taking $\alpha^{-1} \approx 137$. They do not include all known elements but this exceeds the maximal atomic number one would get by considering the zero mass operator $\mathbf{H}_{N,K} = \sum_{n=1}^{N} \sqrt{-\Delta_n} + V_c$ (see [9]).

In the case of one electron and one nucleus, the results are more complete. Indeed it is proved by Evans, Perry and Siedentop [5] that the operator $\mathbf{B}_{1,1}$ is bounded from below if and only if $\alpha Z \leq \gamma_c$. Burenkov and Evans [2] and Tix [12, 13] proved that $\mathbf{B}_{1,1}$ is in fact strictly positive in this range αZ.

In this paper, we are interested in the mathematical properties of the two-dimensional Brown-Ravenhall operator (1.1). We particularly focus on the problem of the boundedness from below which is related to the stability of relativistic systems in 2 dimension. More precisely, we are interested in finding the largest value of δ such that the sesquilinear form $(\psi, \mathbf{B}\psi)$, for $\psi \in \mathcal{H}_+$ a smooth rapidly decaying Dirac spinor, is bounded from below. The result is the following :

Theorem 1.1 *The operator* $\mathbf{B}$ *is bounded from below if and only if*

$$\delta \leq \delta_c := \frac{\hbar c}{2\left(\frac{\Gamma(1/4)^4}{8\pi^2} + \frac{8\pi^2}{\Gamma(1/4)^4}\right)}. \tag{1.4}$$

where $\Gamma(\cdot)$ *is the usual Gamma function.*

Remark 1.2 *An approximate value of* $\delta_c/\hbar c$ *is* 0.189. *If we describe the one-electron atom by writing* $\delta = Ze^2$, *the Brown-Ravenhall operator* $\mathbf{B}$ *is then bounded from below for a critical charge* $Z_c = 25$ *on taking* $\alpha = \frac{e^2}{\hbar c} \approx 1/137$.

To prove Theorem 1.1, we use a similar method to that in [5, 9]. As we will see the analysis is a bit different in 2 dimension. Essentially, the boundedness below of $\mathbf{B}$ will be deduced from an inequality of Hilbert [7] (see also Remark 2.3). Our analysis also yields a sharp Kato inequality.

Theorem 1.3 *The following sharp inequality holds :*

$$\sqrt{-\Delta} \geq \frac{4\pi^2}{\Gamma(1/4)^4} \frac{1}{|x|}. \tag{1.5}$$

Remark 1.4 *An approximate value of the constant involved in (1.5) is* 0.124*.*

Two proofs of Theorem 1.3 are presented in section 3. One is based on the techniques used to prove Theorem 1.1 and one (more direct) is inspired by the work of [9].

2 Proof of Theorem 1.1

We will follow the same strategy as Lieb and Yau [9] and Evans, Perry and Siedentop [5]. Let us recall the different steps. The idea is to work in the *momentum representation.* The Fourier transform $\mathcal{F}$ is defined as follows :

$$(\mathcal{F}f)(\mathbf{p}) := \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}^2} e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{z}} f(\mathbf{z}) d\mathbf{z}, \quad f \in \mathcal{S}(\mathbb{R}^2)$$

where $\mathbf{p} = (p_1, p_2)$. The projection Λ_+ takes, in the momentum representation, the following simple form

$$\Lambda_+(\mathbf{p}) := \mathcal{F}\Lambda_+\mathcal{F}^{-1} = \frac{1}{2} I_{\mathbb{R}^2} + \frac{1}{2E(\mathbf{p})} D(\mathbf{p})$$

where $I_{\mathbb{R}^2}$ is the unit 2×2 matrix and $E(\mathbf{p}) = (c^2|\mathbf{p}|^2 + m^2c^4)^{1/2}$ is the positive eigenvalue of the hermitian 2×2 matrix

$$D(\mathbf{p}) := \mathcal{F}D\mathcal{F}^{-1} = c\boldsymbol{\sigma} \cdot \mathbf{p} + m\sigma_3 = \begin{pmatrix} mc^2 & c(p_1 - ip_2) \\ c(p_1 + ip_2) & -mc^2 \end{pmatrix}.$$

The normalized eigenvector of $D(\mathbf{p})$ corresponding to $E(\mathbf{p})$ is

$$\xi(\mathbf{p}) = \frac{1}{N(\mathbf{p})} \begin{pmatrix} E(\mathbf{p}) + mc^2 \\ c(p_1 + ip_2) \end{pmatrix} \in \mathbb{C}^2,$$

where $N(\mathbf{p}) = (2E(\mathbf{p})(E(\mathbf{p}) + mc^2))^{1/2}$. The Fourier transform $\mathcal{F}\psi$ of any spinor ψ in the positive spectral subspace of D can then be written in the following form

$$(\mathcal{F}\psi)(\mathbf{p}) = u(\mathbf{p})\xi(\mathbf{p}), \tag{2.6}$$

where $u \in L^2(\mathbb{R}^2)$. Conversely any Dirac spinor of the form (2.6) is in the image of $\mathcal{H}_+$ under the Fourier transform. To complete the description, note that the Fourier transform of the Coulomb potential remains Coulombian :

$$\mathcal{F}(\frac{1}{|\cdot|})(\mathbf{p}) = \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{|\mathbf{p}|}.$$

The first step is a *reduction of spinors*. We will write $\langle \cdot, \cdot \rangle$ for the inner product in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ or in $L^2(\mathbb{R}^2, \mathbb{C})$ as appropriate. Using the above description, for any ψ in the positive spectral subspace, we then obtain that

$$\begin{aligned} \langle \psi, \mathbf{B}\psi \rangle &= \int E(\mathbf{p})|u(\mathbf{p})|^2 d\mathbf{p} - \frac{\delta}{2\pi\hbar} \int\int u(\mathbf{p})\bar{u}(\mathbf{p}')K(\mathbf{p}, \mathbf{p}')d\mathbf{p}d\mathbf{p}' \\ &:= \langle u, \mathbf{b}u \rangle \end{aligned} \tag{2.7}$$

where the kernel K is the 2×2 matrix-valued function given by

$$K(\mathbf{p}, \mathbf{p}') = \frac{(E(\mathbf{p}) + mc^2)(E(\mathbf{p}') + mc^2)) + c^2\mathbf{p}\bar{\mathbf{p}}'}{N(\mathbf{p})N(\mathbf{p}')|\mathbf{p} - \mathbf{p}'|}. \tag{2.8}$$

The upper bar appearing in (2.8) denotes the complex conjugate ($\bar{\mathbf{p}}' = p_1' - ip_2'$). The transition from $\psi \in L^2(\mathbb{R}^2, \mathbb{C}^2)$ to the reduced spinor $u \in L^2(\mathbb{R}^2, \mathbb{C})$ through the introduction of the operator $\mathbf{b}$ is possible because we are working in $\mathcal{H}_+$. The equality (2.7) defines a self-adjoint operator $\mathbf{b}$ if we can show that the form $\langle u, \mathbf{b}u \rangle$ is bounded from below. Of course $\langle \psi, \mathbf{B}\psi \rangle$ is bounded from below if and only if $\langle u, \mathbf{b}u \rangle$ is bounded from below. In that case, (2.7) will define a self-adjoint operator $\mathbf{b}$. We will refer to this operator as the Brown-Ravenhall operator. It is associated to the original operator $\mathbf{B}$ via (2.7).

The second step is to study the operator $\mathbf{b}$ by a *partial wave decomposition.* For any $u \in L^2(\mathbb{R}^2, \mathbb{C})$, we have that

$$u(\mathbf{p}) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{r}} a_k(r) e^{ik\theta} \tag{2.9}$$

where $r = \sqrt{p_1^2 + p_2^2}$, $\theta = Arg(p_1 + ip_2)$ $[2\pi]$ and where $a_k(\cdot)$ are complex functions defined on $\mathbb{R}^+$. So that

$$\sum_{k \in \mathbb{Z}} \int_0^{+\infty} |a_k(r)|^2 dr = \int_{\mathbb{R}^2} |u(\mathbf{p})|^2 d\mathbf{p} (:= \|u\|_2).$$

We will write $e(r) := (c^2 r^2 + m^2 c^4)^{1/2}$ and $n(r) := (2e(r)(e(r) + mc^2))^{1/2}$. Using the expansion (2.9) of u, the operator $\mathbf{b}$ is splitted as follows.

Lemma 2.1 *For any spinor u, we have that*

$$\langle u, \mathbf{b} u \rangle = \langle a_0, b_0 a_0 \rangle + 2 \sum_{k \geq 1} \langle a_k, b_k a_k \rangle, \tag{2.10}$$

such that for any $k \in \mathbb{N}$,

$$\langle a_k, b_k a_k \rangle = \int_0^{+\infty} e(r) |a_k(r)|^2 dr - \frac{\delta}{\pi \hbar} \int_0^{+\infty} \int_0^{+\infty} a_k(r) \bar{a_k}(r') K_k(r, r') dr dr'. \tag{2.11}$$

Moreover, the kernels are given by

$$K_k(r, r') = \beta_1(r, r') \sqrt{rr'} I_k(r, r') + \beta_2(r, r') \sqrt{rr'} I_{k+1}(r, r')$$

where

$$\beta_1(r, r') := \frac{(e(r) + mc^2)(e(r') + mc^2)}{n(r) n(r')}, \quad \beta_2(r, r') := \frac{c^2 r r'}{n(r) n(r')}$$

and

$$I_k(r, r') := \int_0^{2\pi} \frac{\cos(k\theta)}{\sqrt{r^2 + r'^2 - 2rr' \cos(\theta)}} d\theta. \tag{2.12}$$

Proof : Inserting the expansion (2.9) of u, in the expression (2.7), yields

$$\langle u, \mathbf{b}u\rangle = \sum_{k\in\mathbb{Z}} \int_0^{+\infty} e(r)|a_k(r)|^2 dr$$
$$-\frac{\delta}{4\pi^2\hbar} \sum_{k,k'\in\mathbb{Z}} \int_0^{+\infty}\int_0^{+\infty}\int_0^{2\pi}\int_0^{2\pi} a_k(r)\bar{a_k}(r')K_{k,k'}(r,r',\theta,\theta')dr dr' d\theta d\theta'$$

with

$$K_{k,k'}(r,r',\theta,\theta') = \frac{e^{ik\theta}e^{-ik'\theta'}}{\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}}\left[\frac{\sqrt{rr'}(e(r)+mc^2)(e(r')+mc^2)}{n(r)n(r')} + \frac{c^2(rr')^{3/2}e^{i\theta}e^{-i\theta'}}{n(r)n(r')}\right].$$

Using now the fact that for any $q\in]0,1[$ and any $l,l'\in\mathbb{Z}$,

$$\int_0^{2\pi}\int_0^{2\pi} \frac{e^{il\theta}e^{-il'\theta'}}{\sqrt{1-q\cos(\theta-\theta')}}d\theta d\theta' = 4\pi\delta_{l,l'}\int_0^{2\pi}\frac{\cos(l\theta)}{\sqrt{1-q\cos(\theta)}}d\theta \tag{2.13}$$

and the parity of the cosine appearing in the right hand side of (2.13), we get the lemma.

∎

Let us give now a result which permits to describe more precisely the properties of the kernels K_k involved in each channel k.

Lemma 2.2 *For any $k\in\mathbb{N}$, we have that*

$$I_k(r,r') = \frac{2}{\sqrt{rr'}}Q_{k-1/2}\left(\frac{1}{2}(\frac{r}{r'}+\frac{r'}{r})\right) \tag{2.14}$$

where Q_s is the Legendre function of the second kind. Consequently, $(I_k)_{k\in\mathbb{N}}$ is a positive decreasing sequence.

Remark 2.3 *As in the three dimensional case [5], the kernels involved in each channel k are described in terms of Legendre functions of the second kind. Nevertheless, in our situation, they are not indexed by integers (but by half integers). Then, since we do not* *have the Neumann decomposition for Legendre polynomials (see [14]), the calculation of the critical constant δ_c in Theorem 1.1 has to be done in a different way. We shall use the Mellin transform (see below).*

Proof : Let $q = \frac{2rr'}{r^2+r'^2}$ and write

$$\sqrt{r^2+r'^2} I_k(r,r') = \int_0^{2\pi} \frac{e^{ik\theta}}{(1-q\cos(\theta))^{1/2}} d\theta. \tag{2.15}$$

Introduce now the following analytic function on $U = \mathbb{C} \setminus \mathbb{R}^+$:

$$f(z) = \sqrt{\frac{2}{q}} \frac{z^{k-\frac{1}{2}}}{\sqrt{(z-z_-)(z-z_+)}}$$

where $z_\pm = \frac{1\pm\sqrt{1-q^2}}{q}$ are both positive numbers satisfying $0 < z_- < 1 < z_+$. On integrating f on the usual "lock" path in U, we get that

$$\int_0^{2\pi} \frac{e^{ik\theta}}{(1-q\cos(\theta))^{1/2}} d\theta = J_k^1(q) + J_k^2(q)$$

where

$$J_k^1(q) := \sqrt{\frac{2}{q}} \int_0^{z_-} \frac{z^{k-\frac{1}{2}}}{\sqrt{(z-z_-)(z-z_+)}} dz + \sqrt{\frac{2}{q}} \int_{z_-}^1 \frac{z^{k-\frac{1}{2}}}{i\sqrt{(z-z_-)(z_+-z)}} dz \tag{2.16}$$

and

$$J_k^2(q) := \sqrt{\frac{2}{q}} \int_0^{z_-} \frac{z^{k-\frac{1}{2}}}{\sqrt{(z-z_-)(z-z_+)}} dz + \sqrt{\frac{2}{q}} \int_{z_-}^1 \frac{z^{k-\frac{1}{2}}}{-i\sqrt{(z-z_-)(z_+-z)}} dz. \tag{2.17}$$

By addition of (2.16) and (2.17) and setting $t = 1/q$, we get

$$\frac{\sqrt{rr'}}{2} I_k(r,r') = \int_0^{t-\sqrt{t^2-1}} \frac{z^{k-\frac{1}{2}}}{\sqrt{z^2-2tz+1}} dz. \tag{2.18}$$

The lemma is therefore proved since the right hand side of (2.18) is the Legendre function of the second kind $Q_{k-1/2}$ (see [14] p. 334). Moreover, since $t-\sqrt{t^2-1}$ is strictly smaller than 1 for $t > 1$, it follows from (2.18) that the sequence $(I_k)_{k\in\mathbb{N}}$ is decreasing. ∎

The following result implies that the channel $k = 0$ in (see equations (2.10)-(2.11)) makes the biggest contribution to the energy :

Corollary 2.4

$$\begin{aligned} & \inf\{\langle u, \mathbf{b}u\rangle / u \in L^2(\mathbb{R}^2, (1+|\mathbf{p}|^2)^{1/2} d\mathbf{p}), \|u\|_2 = 1\} \\ = \ & \inf\{\langle f, b_0 f\rangle / f \in L^2(\mathbb{R}^+, (1+r^2)^{1/2} dr), \|f\|_2 = 1\}. \end{aligned}$$

Proof : The idea is the same as in [5]. Let us first note that using the positivity of the I_k's, for any $a \in L^2(\mathbb{R}^+, dr)$, $\langle a, b_k a\rangle \geq \langle |a|, b_k |a|\rangle$. Thus we can restrict ourselves to positive functions when minimizing $\langle a, b_k a\rangle$. Now, since $(I_k)_{k\in\mathbb{N}}$ is a decreasing sequence (Lemma 2.2), we may assume in (2.10)-(2.11) that the coefficients a_k are zero unless $k = 0$ when minimizing. The channel corresponding to $k = 0$ then yields the lowest energy. ∎

Before giving the proof of Theorem 1.1, let us recall the definition of the Mellin transform and its behavior with regard to the properties of Legendre functions of the second kind that we shall need. The Mellin transform $\mathcal{M}$ is defined on $L^2((0, +\infty); \frac{dr}{r})$ by

$$(\mathcal{M}\psi)(w) := \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{-1-iw} \psi(r) dr, \quad w \in \mathbb{R}. \tag{2.19}$$

It is a unitary map from $L^2((0, +\infty); \frac{dr}{r})$ onto $L^2(\mathbb{R})$. The Mellin transform of the Legendre functions of the second kind satisfies (see [10]) :

$$(\mathcal{M}Q_s\left(\frac{1}{2}(\cdot + \frac{1}{\cdot})\right))(w) := \frac{1}{2}\sqrt{\frac{\pi}{2}} \frac{\Gamma([s+1-iw]/2)\Gamma([s+1+iw]/2)}{\Gamma([s+2-iw]/2)\Gamma([s+2+iw]/2)} \tag{2.20}$$

where $\Gamma(\cdot)$ is the Gamma function.

Proof of Theorem 1.1 :

In this part, we underline the m-dependance and write $\mathbf{b}_m$ for $\mathbf{b}$. We will also denote by

$Q(\mathbf{b}_m)$ the form domain of $\mathbf{b}_m$. In view of Corollary 2.4, one can minimize $\langle u, \mathbf{b}_m u\rangle$ just by looking at the expression

$$\langle u, \mathbf{b}_m u\rangle = \int_0^{+\infty} e(r)a(r)^2 dr$$
$$-\frac{\delta}{\pi\hbar}\int_0^{+\infty}\int_0^{+\infty} a(r)a(r')\sqrt{rr'}\{\beta_1(r,r')I_0(r,r') + \beta_2(r,r')I_1(r,r')\}dr dr', \quad u \in Q(\mathbf{b}_m), \tag{2.21}$$

where $a(\cdot)$ is a non-negative function. We prove the theorem in two steps. We consider first the massless case which is homogeneous and then conclude with the case of non-zero mass by estimating the difference $\mathbf{b}_m - \mathbf{b}_0$ in a suitable way.

Massless case :

When $m = 0$, $\beta_1(r,r') = \beta_2(r,r') = 1/2$ so that, the expression (2.21) becomes :

$$\langle u, \mathbf{b}_0 u\rangle = \int_0^{+\infty} cra(r)^2 dr - \frac{\delta}{2\pi\hbar}\int_0^{+\infty}\int_0^{+\infty} a(r)a(r')\sqrt{rr'}\{I_0(r,r') + I_1(r,r')\}dr dr'. \tag{2.22}$$

The massless case being homogeneous, it allows us to use the Hilbert inequality (see [7]) to the second integral on the right hand side of (2.22). It follows that

$$\langle u, \mathbf{b}_0 u\rangle \geq (1 - \frac{\delta\Omega}{2\pi\hbar c})\int_0^{+\infty} cra(r)^2 dr \tag{2.23}$$

where

$$\Omega = \int_0^{\infty} \frac{1}{\sqrt{r}}\left(I_0(1,r) + I_1(1,r)\right) dr.$$

Using Lemma 2.2 and definition (2.19), Ω can be rewritten as follows :

$$\Omega = 2\sqrt{2\pi}\left\{(\mathcal{M}Q_{-1/2}\left(\frac{1}{2}(\cdot + \frac{1}{\cdot})\right))(0) + (\mathcal{M}Q_{1/2}\left(\frac{1}{2}(\cdot + \frac{1}{\cdot})\right))(0)\right\}.$$

Then using the property (2.20) combined with the facts that $\Gamma(5/4) = \dfrac{1}{4}\Gamma(1/4)$ and $\Gamma(3/4) = \dfrac{\pi\sqrt{2}}{\Gamma(1/4)}$, we get that

$$\Omega = \frac{\Gamma(1/4)^4}{2\pi} + \frac{32\pi^3}{\Gamma(1/4)^4}. \tag{2.24}$$

According to (2.23) and (2.24), we then have that $\langle u, \mathbf{b}_0 u\rangle \geq 0$ on $Q(\mathbf{b}_0) = L^2(\mathbb{R}^2; \sqrt{1+|\mathbf{p}|^2}d\mathbf{p})$ if and only if $\delta \leq \delta_c := \dfrac{\hbar c}{\frac{\Gamma(1/4)^4}{4\pi^2} + \frac{16\pi^2}{\Gamma(1/4)^4}}$.

<u>Mass case</u> :

The boundedness below of the operator $\mathbf{b}_m$ will follow directly from the masseless case by use of the following result (see [4, 13]) :

Lemma 2.5 *For any $m \in \mathbb{R}^+$, the operator $\mathbf{b}_m - \mathbf{b}_0$ admits a bounded extension in $L^2(\mathbb{R}^2)$. Moreover, we have that, for any $u \in L^2(\mathbb{R}^2)$,*

$$\|\mathbf{b}_m(u) - \mathbf{b}_0(u)\|_2 \leq mc^2(1 + \frac{3\delta\Omega}{\pi\hbar c})\|u\|_2.$$

Proof : Write first

$$K_k^m - K_k^0 = \sqrt{rr'}\left\{(\beta_1(r,r') - \frac{1}{2})I_k(r,r') + (\beta_2(r,r') - \frac{1}{2})I_{k+1}(r,r')\right\}.$$

By a straightforward calculation, using essentially the fact that

$$\sqrt{2}(e(r) + mc^2) \geq n(r) \geq \sqrt{2}e(r) \geq \sqrt{2}cr,$$

we have for any $r, r' \in \mathbb{R}^+$,

$$0 \leq \beta_1(r,r') - \frac{1}{2} \leq \frac{3}{4}mc\left(\frac{1}{r} + \frac{1}{r'}\right)$$

and

$$0 \leq \frac{1}{2} - \beta_2(r,r') \leq \frac{3}{2}mc\left(\frac{1}{r} + \frac{1}{r'}\right).$$

So that, using (2.10)-(2.11) and the fact that $0 \le e_m(r) - e_0(r) \le mc^2$, we have that for any rapidly decaying u,

$$|\langle u, \mathbf{b}_m u\rangle - \langle u, \mathbf{b}_0 u\rangle| \le mc^2\|u\|_2^2 + \frac{3mc\delta}{\pi\hbar}$$
$$\times \sum_k \int_0^{+\infty}\int_0^{+\infty} \left(\sqrt{\frac{r'}{r}} + \sqrt{\frac{r}{r'}}\right)\left(\frac{I_k(r,r')}{4} + \frac{I_{k+1}(r,r')}{2}\right)|a_k(r)||a_k(r')|dr dr'. \tag{2.25}$$

Splitting the integral (2.25) into two parts and applying the Cauchy-Schwarz inequality, we get for the first one

$$\int_0^{+\infty}\int_0^{+\infty} \sqrt{\frac{r'}{r}}\left(\frac{I_k(r,r')}{4} + \frac{I_{k+1}(r,r')}{2}\right)|a_k(r)||a_k(r')|dr dr'$$
$$\le \int_0^{+\infty}\int_0^{+\infty} |a_k(r)|^2 \sqrt{\frac{r'}{r}}\left(\frac{I_k(r,r')}{4} + \frac{I_{k+1}(r,r')}{2}\right)dr dr' \tag{2.26}$$

and for the second one

$$\int_0^{+\infty}\int_0^{+\infty} \sqrt{\frac{r}{r'}}\left(\frac{I_k(r,r')}{4} + \frac{I_{k+1}(r,r')}{2}\right)|a_k(r)||a_k(r')|dr dr'$$
$$\le \int_0^{+\infty}\int_0^{+\infty} |a_k(r')|^2 \sqrt{\frac{r}{r'}}\left(\frac{I_k(r,r')}{4} + \frac{I_{k+1}(r,r')}{2}\right)dr dr'. \tag{2.27}$$

Then, by a change of variables and Lemma 2.2, the right hand side in (2.26) becomes

$$\int_0^{+\infty}\int_0^{+\infty} |a_k(r)|^2 \sqrt{\frac{r'}{r}}\left(\frac{I_k(r,r')}{4} + \frac{I_{k+1}(r,r')}{2}\right)dr dr'$$
$$\le \frac{1}{2}\left(\int_0^{+\infty} \frac{1}{\sqrt{u}}(I_0(1,u) + I_1(1,u))du\right)\int_0^{+\infty} |a_k(r)|^2 dr. \tag{2.28}$$

Similarly for the right hand side in (2.27), and we have that

$$|\langle u, \mathbf{b}_m u\rangle - \langle u, \mathbf{b}_0 u\rangle| \le \left(mc^2 + \frac{3mc\delta\Omega}{\pi\hbar}\right)\|u\|_2^2.$$

And then for $\delta \leq \delta_c$, the above sesquilinear form (with form domain $Q(\mathbf{b}_m) = Q(\mathbf{b}_0)$) can be extended to a bounded sesquilinear form on $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$. It follows that $\mathbf{b}_m$ and $\mathbf{b}_0$ have the same domain and this in turn implies the result. ∎

3 Proof of Theorem 1.3

Proof 1 : We suppose here that $\hbar = 1$. Consider the operator $H = \sqrt{-\Delta} - \frac{1}{|x|}$. We use the same method as previously for the Brown-Ravenhall operator. For any $\psi \in L^2(\mathbb{R}^2, \mathbb{C})$, set

$$(\mathcal{F}\psi)(\mathbf{p}) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{r}} v_k(r) e^{ik\theta},$$

where the functions $v_k(\cdot)$ satisfy

$$\sum_{k \in \mathbb{Z}} \int_0^{+\infty} |v_k(r)|^2 dr = \int_{\mathbb{R}^2} |(\mathcal{F}\psi)(\mathbf{p})|^2 d\mathbf{p}.$$

Thus,

$$\langle \psi, H\psi \rangle = \sum_{k \in \mathbb{Z}} \langle v_k, d_k v_k \rangle \tag{3.29}$$

where, with I_k given by (2.12),

$$\langle v_k, d_k v_k \rangle = \int_0^{+\infty} r|v_k(r)|^2 dr - \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} \sqrt{rr'} v_k(r) \bar{v_k}(r') I_k(r, r') dr dr'. \tag{3.30}$$

As for the Brown-Ravenhall operator, we need only consider the channel corresponding to $k = 0$ and positive functions v_0. And then, we apply the Hilbert inequality to get

$$\langle v_0, d_0 v_0 \rangle \geq \left\{ 1 - \frac{1}{\pi} \int_0^{+\infty} \frac{1}{r} Q_{-1/2} \left(\frac{1}{2}(r + \frac{1}{r}) \right) dr \right\} \int_0^{+\infty} r|v_0(r)|^2 dr. \tag{3.31}$$

Since the kernel involved in (3.30) (particularly for $k = 0$) is moreover $a.e$ positive, the inequality (3.31) is sharp. The conclusion follows from (2.20). ∎

Proof 2 : We also suppose here that $\hbar = 1$. On working in the momentum representation, we have, for all $\psi \in L^2(\mathbb{R}^2)$,

$$\langle \psi, |x|^{-1}\psi\rangle = \frac{1}{2\pi}\int_{\mathbb{R}^4} \frac{\bar{u}(\mathbf{p})u(\mathbf{p}')}{|\mathbf{p}|^{1/2}|\mathbf{p}'|^{1/2}|\mathbf{p}-\mathbf{p}'|} d\mathbf{p}d\mathbf{p}'$$

where $u(\mathbf{p}) = |\mathbf{p}|^{1/2}\hat{\psi}(\mathbf{p})$. To prove the Theorem, we will show that the integral operator

$$Tu(\mathbf{p}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{u(\mathbf{p}')}{|\mathbf{p}|^{1/2}|\mathbf{p}'|^{1/2}|\mathbf{p}-\mathbf{p}'|} d\mathbf{p}'$$

is bounded in $L^2(\mathbb{R}^2)$. For any $\omega \in S^1$ and $r \in \mathbb{R}^+$, write $u_\omega(r) = ru(rw)$ so that

$$\|u\|_2^2 = \int_{S^1\times\mathbb{R}^+} |u_\omega(r)|^2 \frac{dr}{r} d\omega.$$

Write also T as follows

$$(Tu)_\omega(r) = \int_{S^1\times\mathbb{R}^+} A_{\omega,\omega'}(\frac{r'}{r})u_{\omega'}(r')\frac{dr'}{r'}d\omega' \tag{3.32}$$

with

$$A_{\omega,\omega'}(t) = \frac{1}{2\pi}\frac{t^{1/2}}{|\omega - t\omega'|}.$$

The kernel $A_{\omega,\cdot}(\cdot)$ is in $L^1(S^1\times\mathbb{R}^+, \frac{dt}{t}\otimes d\omega')$ and its L^1 norm doesn't depend on ω. Indeed, using Lemma 2.2 and (2.19)-(2.20), we have

$$\frac{1}{2\pi}\int_{S^1\times\mathbb{R}^+} A_{\omega,\omega'}(t)\frac{dt}{t}d\omega' = \frac{1}{\pi}\int_0^{+\infty}\frac{1}{t}Q_{-1/2}(\frac{1}{2}(t+\frac{1}{t}))dt = \frac{\Gamma(1/4)^4}{4\pi^2}. \tag{3.33}$$

Then, by simple use of Schur's Lemma, we get

$$\int_{S^1\times\mathbb{R}^+} |(Tu)_\omega(r)|^2\frac{dr}{r}d\omega \le \left(\frac{\Gamma(1/4)^4}{4\pi^2}\right)^2 \int_{S^1\times\mathbb{R}^+} |u_\omega(r)|^2\frac{dr}{r}d\omega = \left(\frac{\Gamma(1/4)^4}{4\pi^2}\right)^2 \|u\|_2^2.$$

Since the kernel in (3.32) is positive, this inequality is sharp. ∎

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