OPERATOR-THEORETIC INFRARED RENORMALIZATION AND CONSTRUCTION OF DRESSED 1-PARTICLE STATES IN NON-RELATIVISTIC QED

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Abstract. We consider the infrared problem in a model of a freely propagating, nonrelativistic charged particle of mass 1 in interaction with the quantized electromagnetic field. The hamiltonian $H(\sigma_0) = H_0 + gI(\sigma_0)$ of the system is regularized by an infrared cutoff $\sigma_0 \ll 1$, and an ultraviolet cutoff $\Lambda \sim 1$ in the interaction term, in units of the mass of the charged particle. Due to translation invariance, it suffices to study the hamiltonian $H_p(\sigma_0) := H(\sigma_0)|_{\mathcal{H}_p}$, where $\mathcal{H}_p$ denotes the fibre space of the conserved momentum operator associated to total momentum $p \in \mathbb{R}^3$. Under the condition that the coupling constant g is sufficiently small, there exists a constant $p_c \in [\frac{1}{20}, 1)$, such that for all p with $|p| \leq p_c$, the following statements hold:

(1) For every $\sigma_0 > 0$, $E_0[|p|, \sigma_0] :=$ inf spec $H_p(\sigma_0)$ is an eigenvalue with corresponding eigenvector $\Omega[p, \sigma_0] \in \mathcal{H}_p$.

(2) For all $\sigma_0 \geq 0$, $\partial_{|p|}^{\beta} \left(E_0[|p|, \sigma_0] - \frac{|p|^2}{2} \right) \leq O(g^{\frac{1}{6}})$ for $\beta = 0, 1, 2$.

(3) $\Omega[p, \sigma_0]$ is not an element of the Fock space $\mathcal{H}_p$ in the limit $\sigma_0 \to 0$, if $|p| > 0$.

Our proofs are based on the operator-theoretic renormalization group of V. Bach, J. Fröhlich, and I.M. Sigal [**2, 3**]. The key difficulty in the analysis of this system is connected to the strictly marginal nature of the leading interaction term, and a main issue in our exposition is to develop analytic tools to control its renormalization flow.

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CHAPTER 1

INTRODUCTION

We consider a non-relativistic, charged particle, of mass 1 (where $\hbar = c = 1$), propagating in Euclidean $\mathbb{R}^3$, which interacts with the quantized electromagnetic field. Due to translation invariance of the system, the total momentum of the particle and the boson field is conserved. We thus consider the Hilbert space of states as a direct integral

$$\mathcal{H} \;=\; \int_{\mathbb{R}^3}^{\oplus} d^3p\, \mathcal{H}_p \;,$$

where $\mathcal{H}_p$ is the fibre space of the conserved momentum operator, corresponding to total momentum p.

Due to the translational invariance of the system, it suffices to study $H_p(\sigma_0)$, the restriction of the hamiltonian of the system to $\mathcal{H}_p$, for a range of p's of physical interest. Here, σ_0 denotes an artificial, arbitrarily small, but fixed infrared cutoff in the interaction. Every $\mathcal{H}_p$ is isomorphic to the Fock space of the quantized electromagnetic field, tensored with $\mathbb{C}^2$, accounting for electron spin.

1.1. Physical motivation and main results

The questions we propose to answer are:

(1) Does $H_p(\sigma_0)$, for sufficiently small values of $|p|$, have an eigenvalue at the bottom of its spectrum, which accounts for a dressed 1-particle state (or an infraparticle state) ? For values $|p| \geq 1$ (in units of the mass of the charged particle), this cannot be true, due to the phenomenon of Cherenkov radiation. In fact, if $|p|$ approaches 1 (corresponding to the rest energy mc^2 of the charged particle, where m is its mass, and c is the speed of light), the infraparticle increasingly tends to reduce its kinetic energy by the emission of electromagnetic (Cherenkov) radiation. Consequently, if $|p|$ becomes too large, infspec$H_p(\sigma_0)$ ceases to be an eigenstate. The eigenvalue $E_0[|p|, \sigma_0]$, considered as a function of $|p|$, conjecturally becomes a **resonance** if $|p|$ approaches 1. Clarification of the latter is beyond the scope of this text.

(2) What is the nature of the ground state vector of $H_p(\sigma_0)$?

(3) Is the ground state energy of $H_p(\sigma_0)$ sufficiently smooth in p, that is, of class C^2, for all $\sigma_0 \geq 0$? This degree of differentiability is, for instance, required for the construction of scattering theory according to [**9, 10, 21**], and for the analysis of the semiclassical motion of charged particles [**23**].

The main physical results obtained in this work are to answer questions (1) and (3) in the affirmative. As to question (2), the ground state of $H_p(\sigma_0)$ is a 'dressed one-particle state', which is, for $|p| > 0$, contained in the Fock space $\mathcal{H}_p$ for all $\sigma_0 > 0$. However, in the

limit $\sigma_0 \to 0$, it ceases to be an element of $\mathcal{H}_p$, unless $|p| = 0$. Instead, it is then contained in a Hilbert space $\mathcal{H}_{IR}$ that carries a representation of the canonical commutation relations unitarily inequivalent to the Fock representation, [**9, 10**].

The mathematically precise formulation of the main theorems, is, together with a discussion of their physical interpretation, given in Chapter 3.

1.2. Operator-theoretic renormalization group

The analytical method that we choose for the study of the above problems is the operator-theoretic renormalization group (RG) method developed by Bach, Fröhlich, and Sigal [**2, 3**].

The renormalization group iteration produces a sequence of effective hamiltonians that can be parametrized by points in a certain Banach space. This Banach space consists of infinite sequences of 'kernels'.

The renormalization group iteration is based on a decomposition of the Fock space $\mathcal{H}_p$ into 'shells' associated with a dyadic decomposition of the spectrum of the free photon hamiltonian.

In every recursion step, a shell is 'integrated out', or 'decimated', and a rescaling transformation zooms the reduced problem to a standard size. The decimation is achieved by an application of the Feshbach map.

In a manner made precise by a variant of Feshbach's theorem for smooth Feshbach maps developed by V. Bach, J. Fröhlich and I.M. Sigal, [**4**], all effective hamiltonians are isospectral to one another. In particular, if the ground state vector of any one of them is known, the ground state vectors of all of them can be reconstructed.

Thus, the strategy consists of constructing a long, but finite sequence of effective hamiltonians, for which the last element has a known solution of the eigenvalue problem. This is possible because of the infrared cutoff σ_0 which has been inserted into the interaction. Then, by recursively using the Feshbach theorem, the eigenvalues at the bottom of every effective hamiltonian, together with the corresponding eigenvectors, can be reconstructed. This yields the eigenvalue at the bottom of the spectrum of $H_p(\sigma_0)$, and the associated eigenvector $\Omega[p; \sigma_0]$.

1.2.1. Main problems in the operator-theoretic RG. The problem of renormalizing the physical hamiltonian is more difficult than in [**2, 3**], because the boson form factor is assumed to exhibit the physical $\frac{1}{\sqrt{|k|}}$-singularity, and the electron is not confined to a compact region of space. The main difficulties are:

(i) The interaction operator is purely marginal. Analytical control of its radiative corrections by resumming perturbation expansions is very subtle and requires considerable effort.

(ii) If the sequence of effective hamiltonians converges it is given by an effective hamiltonian with non-vanishing interaction part, corresponding to a fixed point of the renormalization map. Unfortunately, the Banach contraction principle is insufficient to prove its existence, and the Schauder or Schäfer fixed point theorems (which are infinite dimensional analogues, and, as a matter of fact, consequences of the Brouwer fixed point theorem), whose

application requires the verification of certain compactness properties, imply rather weak results. A solution of this problem can presumably be obtained by the development of center manifold theorems in operator-theoretic renormalization group theory, but this issue is beyond the scope of the present work.

1.2.2. New techniques. Further development of efficient methods in the operator-theoretic renormalization group is as important for us as the proof of the above physical results. Two new main insights are reached. They concern the following issues:

(I) Gauge invariance of the physical system can, if present, be exploited, in order to show that various marginal quantities are, in fact, renormalized identically. To this end, we develop an operator-theoretic formulation of the **Ward-Takahashi identities**, which express the $U(1)$-gauge invariance of the system. The Ward-Takahashi identities heavily reduce the complexity in the task of simultaneously renormalizing several marginal quantities. However, they do not, per se, provide convergent bounds on the latter.

(II) Control of the renormalization of purely marginal operators in the operator-theoretic renormalization group method. The main insight here is that the radiative corrections of strongly marginal quantities are subject to nearly complete mutual cancellations. Consequently, the cumulative radiative correction to a corresponding quantity remains small in the scaling limit.

The cancellations are consequences of important concatenation properties of the smooth Feshbach map, which are used to produce identities that interrelate certain quantities X_k, X_n of key interest at arbitrary scales k, n. To obtain analytical control of these quantities, a nested recursion argument is applied. Passing from a given scale n to the scale $n+1$, the identities between X_0 and X_{n+1} are determined. Analytical control of X_{n+1} is obtained from a renormalization group-type sub-iteration at fixed n that 'integrates' the radiative corrections up to the actual scale, and which only involves the quantities X_k for $0 \leq k \leq n$.

Among the various approaches to the above points that are conceivable, we have tried to select efficient and simple methods that allow for potential generalizations.

1.3. Related recent results

Aspects of infrared divergences in the massless Nelson model that are closely related to the results presented here have recently been studied by J. Lörinczi, R.A. Minlos and H. Spohn [**19, 20**], by use of probabilistic methods. Also see [**1**] and [**6**]. A discussion of the dynamics of infraparticles in the semiclassical classical limit of nonrelativistic QED is given by S. Teufel and H. Spohn in [**23**]; the smoothness properties of the ground state energy are required in their work. There has been progress in the scattering theory of massless bosons in the Nelson model in work of A. Pizzo [**21**], which is to a large extent based on the work of J. Fröhlich in [**9**]. His results require C^2-smoothness of the ground state energy as a function of the conserved momentum. Furthermore, there are new results on this problem owing to C. Gérard, [**15**].

Furthermore, recent progress in Rayleigh scattering theory has been obtained by M. Griesemer, B. Schlein and J. Fröhlich, [**11, 12**], in which one is concerned with charged

particles that are, in contrast to the present case, confined to atoms or molecules. For properties of the ground state energy in systems of non-relativistic QED, cf. the recent work of M. Griesemer, E. H. Lieb and M. Loss, [**16**].

CHAPTER 2

THE MODEL HAMILTONIAN

We introduce a regularized model of a non-relativistic electron propagating in $\mathbb{R}^3$, coupled to the quantized photon field at zero temperature.

2.1. The physical system

Writing

$$\mathcal{H}_{el} \;=\; L^2(\mathbb{R}^3, d^3x) \;\otimes\; \mathbb{C}^2$$

for the Hilbert space of a single non-relativistic electron, and

$$\mathcal{H}_f \;=\; \bigoplus_{n\geq 0} (L^2(\mathbb{R}^3, d^3k) \;\otimes\; \mathbb{C}^2)^{\otimes_s n}$$

for the photon Fock space, where "$\otimes_s$" denotes the fully symmetrized tensor product, the Hilbert space of the coupled system is the tensor product space

$$\mathcal{H} \;=\; \mathcal{H}_{el} \;\otimes\; \mathcal{H}_f \;.$$

The factors $\mathbb{C}^2$ account for electron spin and photon polarization.

Moreover, writing $k \in \mathbb{R}^3$ for photon momenta, and $\lambda \in \{+,-\}$ for photon polarizations, $a^*_\lambda(k)$ and $a_\lambda(k)$ denote bosonic creation- and annihilation operators, respectively. These operator-valued distributions are subject to the canonical commutation relations

$$[a_{\lambda'}(k'), a^*_\lambda(k)] \;=\; (2\pi)^3 \delta_{\lambda,\lambda'} \delta^{(3)}(k-k') \;,\;\; [a^\sharp_{\lambda'}(k'), a^\sharp_\lambda(k)] \;=\; 0 \;.$$

For $f \in L^2(\mathbb{R}^3, d^3k)$, we define the operators

$$\begin{aligned} a_\lambda(f) \;&\equiv\; \int_{\mathbb{R}^3} d^3k \, a_\lambda(k) f^*(k) \\ a^*_\lambda(f) \;&\equiv\; \int_{\mathbb{R}^3} d^3k \, a^*_\lambda(k) f(k) \end{aligned}$$

on $\mathcal{H}_f$. There is a unique unit ray, the vacuum vector $\Omega_f \in \mathcal{H}_f$, which obeys

$$a_\lambda(f)\, \Omega_f \;=\; 0$$

for all $f \in L^2(\mathbb{R}^3, d^3k)$. $\mathcal{H}_f$ is the closure of the span of all vectors of the form

$$\prod_i a^*_{\lambda_i}(f_i) \Omega_f \;,$$

with $f_i \in L^2(\mathbb{R}^3, d^3k)$.

2.1.1. Momentum operator and Hamiltonian. The total momentum operator of the system is given by

$$\mathbf{P}_{tot} = \mathbf{P}_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes \mathbf{P}_f, \tag{2.1.1}$$

where $\mathbf{P}_{el}$ and $\mathbf{P}_f = \sum_\lambda \int d^3k \; k \; a^*_\lambda(k) a_\lambda(k)$ denote the electron- and photon momentum operators, respectively.

Units. We set $c = \hbar = 1$. The energy will henceforth be measured in units of the bare electron rest mass, which, from now on, is set to $m = 1$.

The hamiltonian of the system consists of the following operators.

$$H_f \; = \; \sum_{\lambda=\pm} \int_{\mathbb{R}^3} \; d^3k \; |k| \; a^*_\lambda(k) a_\lambda(k) \tag{2.1.2}$$

denotes the usual hamiltonian of the noninteracting electromagnetic field, in the Coulomb gauge.

The electromagnetic field operator contains an ultraviolet cutoff Λ, and an infrared cutoff σ_0, which is inserted in order to render the theory mathematically well-defined. Λ is a real number in $(2, 3)$, which we keep fixed in the entire discussion. We let κ_Λ denote a smooth function with support in $[0, \Lambda]$, which equals 1 on $[0, 1]$, and

$$| \, \partial_x \kappa_\Lambda(x) \, | \; \leq \; 2 \, \chi_{[0,\Lambda]}(x) \; , \tag{2.1.3}$$

which establishes the ultraviolet cutoff (where χ_I is the characteristic function of the interval I).

Furthermore, let $0 < \sigma_0 \ll 1$ denote a positive, but arbitrarily small real number, which we let tend to zero at the end of our discussion; $\bar{\kappa}_{\sigma_0}$ is defined by

$$\bar{\kappa}_{\sigma_0}[x] \; = \; \begin{cases} 1 & x > \sigma_0 \\ \dfrac{x}{\sigma_0} & x \leq \sigma_0 \end{cases} \tag{2.1.4}$$

for $x \in \mathbb{R}_+$. As a matter of fact, we could use any function of order $O(x^\epsilon)$ for $x \leq \sigma_0$, and $\epsilon > 0$, as long as it is monotonic, and $\bar{\kappa}_{\sigma_0}[x] > 0$ for all $x > 0$. In fact, a zero of $\bar{\kappa}_{\sigma_0}$ at some $x > 0$ would produce problems in the operator-theoretic formulation of the Ward-Takahashi identities, and if $\bar{\kappa}_{\sigma_0}$ were not monotonic, there would be complications in the discussion of the renormalization group flow at extremely low scales.

Other possible implementations of the Ward-Takahashi identities that do not require this restriction, but lead to other complications, are addressed in remark 6.9.

The interaction between the charged particle, and the quantized electromagnetic field is implemented by the operator

$$\mathbf{A}_f(\sigma_0) = \sum_{\lambda=\pm} \int_{\mathbb{R}^3} \frac{d^3k \; \bar{\kappa}_{\sigma_0}(|k|) \kappa_\Lambda(|k|)}{\sqrt{|k|}} \left(\epsilon_{1,0}[n_k, \lambda] \mathfrak{S}_{-k} \otimes a^*_\lambda(k) + \epsilon_{0,1}[n_k, \lambda] \mathfrak{S}_k \otimes a_\lambda(k) \right) \; , \tag{2.1.5}$$

which is linear in the electromagnetic vector potential in the Coulomb gauge, where $\epsilon_{M,N}[n_k, \lambda]$, with $M + N = 1$, are polarization vectors, and $n_k = \frac{k}{|k|}$. We have only indicated the dependence of $\mathbf{A}_f(\sigma_0)$ on the infrared cutoff, since Λ is kept fixed. The factor $\frac{1}{\sqrt{2}(2\pi)^3}$, which

usually appears in the definition of $\mathbf{A}_f$, is absorbed into the coupling constant, that is, the electron charge, (2.1.9).

The choice of the Coulomb gauge implies that $\{\epsilon_{M,N}[n_k,+],\epsilon_{M,N}[n_k,-],n_k\}$ is an orthonormal basis in $\mathbb{R}^3$, for all k and $M+N=1$. It thus also implies that

$$\mathbf{P}_f\cdot\mathbf{A}_f(\sigma_0) \;=\; \mathbf{A}_f(\sigma_0)\cdot\mathbf{P}_f\;. \tag{2.1.6}$$

The operator $\mathfrak{S}_k=\mathfrak{S}^*_{-k}$, with $k\in\mathbb{R}^3$, shifts the electron momentum by k, that is, it acts on $\mathbf{P}_{el}$-eigenvectors in terms of

$$\mathfrak{S}_k\,|q\rangle_{el} \;=\; |q+k\rangle_{el}\;.$$

We also define the operator

$$\mathbf{B}_f(\sigma_0) \;=\; i\;\sum_{\lambda=\pm}\int_{\mathbb{R}^3}\frac{d^3k\;\bar\kappa_{\sigma_0}(|k|)\kappa_\Lambda(|k|)}{\sqrt{|k|}}\left(\;k\wedge\epsilon_{1,0}[n_k,\lambda]\mathfrak{S}_{-k}\otimes a^*_\lambda(k)\;+\;h.c.\right)\;. \tag{2.1.7}$$

The cutoffs are incorporated in a manner that preserves the $U(1)$-gauge invariance of the system. This is of great importance for our analysis, since we will make extensive use of the associated Ward-Takahashi identities.

The Hamiltonian that we will consider is given by

$$H^{(\sigma_0)} \;=\; \frac{1}{2}\left(\mathbf{P}_{el}\otimes\mathbf{1}_f\;-\;g\mathbf{A}_f(\sigma_0)\right)^2\;+\;\frac{g}{2}\,\mathbf{S}\cdot\mathbf{B}_f(\sigma_0)\;+\;\mathbf{1}_{el}\otimes H_f\;. \tag{2.1.8}$$

Due to the cutoffs in the interaction, it is mathematically well-defined. Here, $\mathbf{S}:=(\sigma_1,\sigma_2,\sigma_3)$ stands for the triple of Pauli matrices, and

$$g\;:=\;\frac{e}{\sqrt{2}\;(2\pi)^3}\;, \tag{2.1.9}$$

where e denotes the bare charge of the electron. Due to the absence of antiparticle production in the non-relativistic limit, there is no charge renormalization (no vacuum polarization diagrams).

The regularized system is translation invariant, $[\mathbf{P}_{tot},H^{(\sigma_0)}]=0$. The Hilbert space $\mathcal{H}$ can be written as a direct integral

$$\mathcal{H}\;=\;\int^{\oplus}_{\mathrm{spec}\mathbf{P}_{tot}}d^3p\;\mathcal{H}_p\;,$$

where $\mathcal{H}_p$ are fibre Hilbert spaces associated to fixed values p of the conserved momentum, which are invariant under space- and time-translations. That is, the actions of the time evolution map $e^{itH^{(\sigma_0)}}$, and of the space translation map $e^{ix\cdot\mathbf{P}_{tot}}$ on $\mathcal{H}$ are fibre-preserving. Every $\mathcal{H}_p$ is isomorphic to $\mathbb{C}^2\otimes\mathcal{H}_f$. For any fixed value p of the total momentum of the system, the restriction of $H^{(\sigma_0)}$ to the fibre $\mathcal{H}_p$ is given by the operator

$$H^{(\sigma_0)}_p\equiv H|_{\mathcal{H}_p} \;=\; H_f\;+\;\frac{1}{2}\Big(p-\mathbf{P}_f-g\mathbf{A}_f(\sigma_0)\Big)^2\;+\;\frac{g}{2}\,\mathbf{S}\cdot\mathbf{B}_f(\sigma_0) \tag{2.1.10}$$

on $\mathbb{C}^2\otimes\mathcal{H}_f$. We restrict the value of p to $|p|\leq p_c$, in all that follows.

2.2. The simplified model of a spinless charged particle

In the rest of this work, we will focus on the simplified model hamiltonian $H_p(\sigma_0)$ obtained by omitting the electron spin, that is, by dropping the term involving the magnetic field operator. This is justified by the fact that the magnetic field operator has, in the absence of cutoffs, a scaling dimension 2. Our later discussion will show that all operators with scaling dimensions > 1 are irrelevant from the point of view of the renormalization group. Thus, $H_p(\sigma_0)$ and $H_p^{(\sigma_0)}$ belong to the same universality class of microscopic theories, which justifies the simplification.

As a preparation for later considerations, we write the hamiltonian in the form

$$H_p(\sigma_0) \;=\; E_{bare}[|p|] \;+\; T[\mathcal{O}_0] \;+\; W[\mathcal{O}] \;, \tag{2.2.1}$$

where

$$E_{bare}[|p|] \;=\; \frac{|p|^2}{2} \;+\; \frac{g^2}{2}\,\left\langle \mathbf{A}_f^2(\sigma_0)\right\rangle_{\Omega_f} \tag{2.2.2}$$

corresponds to the classical expression for the ground state energy, except for the Wick ordering correction corresponding to $\mathbf{A}_f^2(\sigma_0)$.

The operator

$$T[\mathcal{O}_0] \;:=\; H_f \;-\; |p|\,\mathbf{P}_f^{\|} \;+\; \frac{1}{2}\,\left(|\mathbf{P}_f^{\|}|^2 \;+\; |\mathbf{P}_f^{\perp}|^2\right) \;, \tag{2.2.3}$$

where we define

$$\mathcal{O}_0 \;:=\; (H_f, \mathbf{P}_f^{\|}, |\mathbf{P}_f^{\perp}|) \;, \tag{2.2.4}$$

commutes with H_f and $\mathbf{P}_f$, and is referred to as the **noninteracting hamiltonian**.

In the nonrelativistic limit, the condition that only values of p with $|p| < 1$ are considered, is natural. For technical reasons, we will impose the more restrictive requirement that

$$|p| \;\leq\; p_c \;, \tag{2.2.5}$$

for some sufficiently small $p_c \in [\frac{1}{20}, 1)$. This bound can be improved, but there is a physical phenomenon which prevents the main results of this work, which are presented in Chapter 3, to be valid for $|p| \geq 1$. We have already mentioned in the introductory chapter that Cherenkov radiation occurs in this limit. This point will be discussed in more detail in Chapter 3.

The bounds

$$(1-|p|)H_f \;\leq\; T[\mathcal{O}_0] \;\leq\; \left(\frac{3}{2}+|p|\right) H_f \;+\; \frac{1}{2}\,|\mathbf{P}_f|^2 \;, \tag{2.2.6}$$

are an obvious consequence of $|\mathbf{P}_f^{\|}|, |\mathbf{P}_f^{\perp}| \leq H_f$.

Furthermore, defining

$$\mathcal{O} \;:=\; (H_f, \mathbf{P}_f^{\|}, \mathbf{P}_f^{\perp}) \;, \tag{2.2.7}$$

the operator

$$W[\mathcal{O}] \;=\; \sum_{1 \leq M+N \leq 2} W_{M,N}[\mathcal{O}] \;, \tag{2.2.8}$$

is referred to as the interaction hamiltonian. $W_{M,N}$ denote Wick ordered operators with M creation and N annihilation operators, given by

$$W_1[\mathcal{O}] \;:=\; \sum_{M+N=1} W_{M,N} \;=\; -\,g\,(p-\mathbf{P}_f)\cdot\mathbf{A}_f(\sigma_0)\;\;, \tag{2.2.9}$$

$$W_2[\mathcal{O}] \;:=\; \sum_{M+N=2} W_{M,N} \;=\; \frac{g^2}{2}\;:\;\mathbf{A}_f^2(\sigma_0)\;:\;, \tag{2.2.10}$$

where $\mathbf{P}_f = \mathbf{P}_f^{\|} + \mathbf{P}_f^{\perp}$, and $n_p := \frac{p}{|p|}$.

The operators $W_{M,N}$ are referred to as Wick monomials of degree $M+N$. Their general structure is given by

$$\begin{aligned} W_{M,N}[\mathcal{O}] \;=\; & \sum_{\lambda^{(M)},\tilde\lambda^{(N)}} \int \prod_{i=1}^{M}\prod_{j=1}^{N} \frac{d^3k_i\;\bar\kappa_{\sigma_0}(|k_i|)}{\sqrt{|k_i|}}\,\frac{d^3\tilde k_j\;\bar\kappa_{\sigma_0}(|\tilde k_j|)}{\sqrt{|\tilde k_j|}}\;\times \\ & \times\, a^*_{\lambda^{(M)}}(k^{(M)})\,w_{M,N}\left[\mathcal{O};K^{(M,N)}\right]\,a^\sharp_{\tilde\lambda^{(N)}}(\tilde k^{(N)})\;, \end{aligned} \tag{2.2.11}$$

where the integral kernels $w_{M,N}$ are operators that commute with H_f.

We have here introduced the following convenient multiindex notation:

$$\begin{aligned} \lambda^{(M)} &:= (\lambda_1,\ldots,\lambda_M) \\ k^{(M)} &:= (k_1,\ldots,k_M), \end{aligned}$$

as we recall, and likewise for $\tilde\lambda^{(N)}$ and $\tilde k^{(N)}$. Moreover,

$$K^{(M,N)} \;:=\; \left(k^{(M)},\lambda^{(M)};\tilde k^{(N)},\tilde\lambda^{(N)}\right),$$

and

$$a^\sharp_{\lambda^{(M)}}(k^{(M)}) := \prod_{i=1}^{M} a^\sharp_{\lambda_i}(k_i).$$

In cases where we consider the dependence of the interaction operators with respect to the interaction kernels, the notation

$$W_{M,N}\left[\left[\;w_{M,N}[z;\mathcal{O};\cdot]\;\right]\right] \tag{2.2.12}$$

will be used.

For $M+N=1$, we have

$$w_{1,0}[\mathcal{O};z;k,\lambda] \;=\; w^*_{0,1}[\mathcal{O};k,\lambda] \;=\; -\,g\,\kappa_\Lambda(|k|)\,(p-\mathbf{P}_f)\cdot\epsilon_\lambda(k)\;, \tag{2.2.13}$$

and for $M+N=2$, by

$$\begin{aligned} w_{2,0}[\mathcal{O};k,\lambda,\tilde k,\tilde\lambda] &= w^*_{0,2}[\mathcal{O};k,\lambda,\tilde k,\tilde\lambda] \\ &= g^2\,\kappa_\Lambda(|k|)\,\kappa_\Lambda(|\tilde k|)\,\epsilon_\lambda(k)\;\cdot\;\epsilon_{\tilde\lambda}(\tilde k) \end{aligned} \tag{2.2.14}$$

and

$$w_{1,1}[\mathcal{O};k,\lambda,\tilde k,\tilde\lambda] \;=\; g^2\;\kappa_\Lambda(|k|)\kappa_\Lambda(|\tilde k|)\;\left(\epsilon_\lambda(k)^*\;\cdot\;\epsilon_{\tilde\lambda}(\tilde k)\;+\;\epsilon_\lambda(k)\;\cdot\;\epsilon^*_{\tilde\lambda}(\tilde k)\right)\;. \tag{2.2.15}$$

The interaction kernels and their derivatives with respect to operators and momenta are controlled by the following bounds.

LEMMA 2.1. *Let X denote either $|p|$, $|k_i|$ or $|\tilde{k}_j|$, and let $\alpha = (\alpha_i)_{i=1}^3$ be a multiindex with $|\alpha| = \sum \alpha_i \leq 2$. Then, there is a constant C independent of $g, |p|$, such that if $|\alpha| = 1$,*

$$\begin{aligned} |\, w_{M,N}[\mathcal{O}; k, \lambda]\,| &\leq g\,(|p| + |\mathbf{P}_f|)\,\kappa_\Lambda(|k|) \\ \|\, \partial^\alpha_{\mathcal{O}}\, w_{M,N}[\mathcal{O}; k, \lambda]\,\| &\leq C\, g\, \kappa_\Lambda(|k|) \\ \|\, \partial_X w_{M,N}[\mathcal{O}; k, \lambda]\,\| &\leq C\, g\, \chi_{[0,\Lambda]}(|k|) \end{aligned} \tag{2.2.16}$$

for $M + N = 1$, and

$$\begin{aligned} \Big\|\, w_{M,N}[\mathcal{O}; k, \lambda, \tilde{k}, \tilde{\lambda}]\,\Big\| &\leq g^2\, \kappa_\Lambda(|k|)\, \kappa_\Lambda(|\tilde{k}|) \\ \Big\|\, \partial^\alpha_{\mathcal{O}} w_{M,N}[\mathcal{O}; k, \lambda, \tilde{k}, \tilde{\lambda}]\,\Big\| &\leq C\, g^2\, \kappa_\Lambda(|k|)\, \kappa_\Lambda(|\tilde{k}|) \\ \Big\|\, \partial_X w_{M,N}[\mathcal{O}; k, \lambda, \tilde{k}, \tilde{\lambda}]\,\Big\| &\leq C\, g^2\, \chi_{[0,\Lambda]}(|k|)\, \chi_{[0,\Lambda]}(|\tilde{k}|) \end{aligned} \tag{2.2.17}$$

for $M + N = 2$. Moreover, if $|\alpha| + \beta \geq 2$, where $|\alpha| \leq 2$ and $\beta \leq 1$,

$$\partial^\beta_{|p|}\, \partial^\alpha_{\mathcal{O}}\, w_{M,N} \;=\; 0\;, \tag{2.2.18}$$

for $1 \leq M + N \leq 2$.

Proof. These estimates are obvious consequences of the assumptions that were imposed on $w_{M,N}$ and κ_Λ. ■

2.3. Scaling and scaling dimensions

The issue of scaling has key importance in renormalization group analysis of Wilson type. The purpose of this subsection is to introduce the **scaling transformation**.

For $f \in L^2(\mathbb{R}^3, d^3k)$, let us consider the unitary transformation

$$(\Gamma_\rho\, f)(k) \;=\; \rho^{-\frac{3}{2}} f(\rho^{-1}\, k)\;. \tag{2.3.1}$$

Lifting Γ_ρ to the Fock space $\mathcal{H}_p$, we obtain a unitary operator U_ρ.

Let $\mathcal{O}$ denote any operator that is homogenous under rescaling of photon momenta. The scaling dimension of an operator A is defined as the number $d(\mathcal{O})$, such that

$$Ad_{U_\rho}[A] \;\equiv\; U_\rho\, A\, U_\rho^* \;=\; \rho^{d(A)}\; A\;, \tag{2.3.2}$$

if A scales homogenously under rescaling of the photon momenta.

The photon number operator

$$N_f \;=\; \sum_{\lambda=\pm} \int d^3k \;\; a^*_\lambda(k)\, a_\lambda(k) \tag{2.3.3}$$

is scale invariant, which implies that $d(a^\sharp_\lambda) = \frac{-3}{2}$. Moreover, $d[\mathcal{O}^r_{(0)}] = 1$ for all r, and in the limit $\sigma_0 \to 0$ and $\Lambda \to \infty$, $d[\mathbf{A}_f] = 1$ and $d[\mathbf{B}_f] = 2$. The photon vacuum is scale invariant,

$$U_\rho\, \Omega_f \;=\; \Omega_f \quad \forall\, \rho \;\geq\; 0\;. \tag{2.3.4}$$

This follows by observing that

$$\begin{aligned} 0 &= U_\rho\, a_\lambda(k)\, U_\rho^*\, U_\rho\, \Omega_f \\ &= \rho^{-\frac{3}{2}}\, a_\lambda(\rho^{-1}\, k)\, U_\rho\, \Omega_f \end{aligned} \tag{2.3.5}$$

holds for all λ and k. Thus, uniqueness of the photon vacuum implies (2.3.4).

2.3.1. Derivatives of scaled operators. The following brief section concerns the scaling of derivatives with respect to $\mathcal{O}$. Let us assume f to be at least C^2 in $\mathcal{O}$ on the Hilbert subspace $P_\rho \mathcal{H}_p$ of $\mathcal{H}_p$, where

$$P_\rho \;\equiv\; \chi[0 \leq H_f \leq \rho] \;,$$

for $0 < \rho < 1$, and that

$$\|\partial_{\mathcal{O}}^\alpha f[\mathcal{O}]\|_{P_\rho \mathcal{H}_p} \;<\; c_\alpha$$

holds for $0 \leq |\alpha|$. The differential operator $\partial_{\mathcal{O}}^\alpha$ scales like

$$Ad_{U_\rho}[\partial_{\mathcal{O}}^\alpha U_\rho^*] \;=\; \rho^{-|\alpha|} \partial_{\mathcal{O}}^\alpha \;,$$

in other words, its scaling dimension is given by $-|\alpha|$.

It is clear that

$$\begin{aligned} \hat{f}[\mathcal{O}] \;&\equiv\; Ad_{U_\rho}[f[\mathcal{O}]] \\ &=\; U_\rho \, f[\mathcal{O}] \, U_\rho^* \\ &=\; f[\rho \, \mathcal{O}] \end{aligned}$$

is an operator acting on the Hilbert space $P_1 \mathcal{H}_p$, and that

$$\|\hat{f}[\mathcal{O}]\|_{P_1 \mathcal{H}_p} \;=\; \|f[\mathcal{O}]\|_{P_\rho \mathcal{H}_p}$$

by unitarity of U_ρ. For the derivatives of $\hat{f}$, we have

$$\begin{aligned} \partial_{\mathcal{O}}^\alpha \hat{f}[\mathcal{O}] \;&=\; \partial_{\mathcal{O}}^\alpha \, U_\rho \, f[\mathcal{O}] \, U_\rho^* \\ &=\; U_\rho \, \left(U_\rho^* \, \partial_{\mathcal{O}}^\alpha U_\rho \right) \, f[\mathcal{O}] \, U_\rho^* \\ &=\; U_\rho \, Ad_{U_\rho}^{-1}[\partial_{\mathcal{O}}^\alpha] \, f[\mathcal{O}] \, U_\rho^* \\ &=\; \rho^{|\alpha|} \, U_\rho \, (\partial_{\mathcal{O}}^\alpha f[\mathcal{O}]) \, U_\rho^* \\ &=\; \rho^{|\alpha|} \, Ad_{U_\rho}[\partial_{\mathcal{O}}^\alpha f[\mathcal{O}]] \;, \end{aligned}$$

because of

$$U_\rho^* \, \partial_{\mathcal{O}}^\alpha \, U_\rho \;=\; Ad_{U_\rho}^{-1}[\partial_{\mathcal{O}}^\alpha] \;=\; \rho^{|\alpha|} \partial_{\mathcal{O}}^\alpha \;.$$

Notice that the inverse of Ad_{U_ρ} appears here. Thus,

$$\begin{aligned} \|\partial_{\mathcal{O}}^\alpha \hat{f}[\mathcal{O}]\|_{P_1 \mathcal{H}_p} \;&\leq\; \rho^{|\alpha|} \|f[\mathcal{O}]\|_{P_\rho \mathcal{H}_p} \\ &\leq\; \rho^{|\alpha|} \, c_\alpha \;, \end{aligned} \tag{2.3.6}$$

since

$$\|Ad_{U_\rho}[\partial_{\mathcal{O}}^\alpha f[\mathcal{O}]]\|_{P_1 \mathcal{H}_p} \;=\; \|\partial_{\mathcal{O}}^\alpha f[\mathcal{O}]\|_{P_\rho \mathcal{H}_p} \;,$$

again by unitarity of U_ρ. In particular, we emphasize that

$$\partial_{\mathcal{O}}^\alpha \hat{f}[\mathcal{O}] \;\neq\; Ad_{U_\rho}[\partial_{\mathcal{O}}^\alpha f[\mathcal{O}]] \;!$$

The norm of the right hand side on $P_1 \mathcal{H}_p$ is bounded by $\rho^{-|\alpha|} \, c_\alpha$, not by $\rho^{|\alpha|} \, c_\alpha$ as in (2.3.6), which is **larger** than c_α if $|\alpha| > 0$. Notably, a growth of the bounds on derivatives of this kind would have catastrophic consequences for our analysis.

2.4. The marginal part: An exactly solvable toy model

For illustrative purposes, mainly in order to give a motivation for the main results that are presented in the next chapter, we now briefly describe a toy model obtained by discarding all operators in $H_p(\sigma_0)$ that are nonlinear in $\mathcal{O}$ or $\mathbf{A}_f(\sigma_0)$.

The point is that H_{lin}, by which we denote the marginal part in $H_p(\sigma_0)$ (consisting of operators with scaling dimension 1 in the absence of cutoffs), can be easily explicitly diagonalized. It comprises only operators with a scaling dimension 1, and shares key properties with the full hamiltonian $H_p(\sigma_0)$. By definition,

$$H_{lin} \;:=\; H_f \;-\; |p|\,\mathbf{P}_f^{\|} \;+\; g\,|p|\,\mathbf{A}_f^{\|}(\sigma_0)\;,$$

where $\mathbf{A}_f^{\|}(\sigma_0)$ denotes the component of the electromagnetic vector potential in the direction of the conserved momentum p. As before, this operator acts on the Fock space $\mathcal{H}_p$, with the Fock vacuum given by Ω_f. The creation and annihilation operators are also as before.

Diagonalization of H_{lin} is achieved by the Bogoliubov transformation

$$H_{lin} \;\longmapsto\; e^{S_{\sigma_0}}\,H_{lin}\,e^{-S_{\sigma_0}}$$

generated by

$$S_{\sigma_0} \;=\; g\,|p|\,\sum_\lambda \int \frac{d^3k\,\bar\kappa_{\sigma_0}[|k|]}{\sqrt{|k|}\,(|k|-|p|k^{\|})}\Big(\,w_{1,0}[k,\lambda]a_\lambda^*(k)\;-\;w_{0,1}[k,\lambda]a_\lambda(k)\,\Big)\;, \tag{2.4.1}$$

where

$$w_{M,N}(k,\lambda) \;=\; g\,|p|\,\kappa_\Lambda(|k|)\,\epsilon_{M,N}^{\|}[n_k,\lambda]\;.$$

We recall that $\epsilon_{M,N}^{\|}[n_k,\lambda]$ denotes the projection of the photon polarization vector in the direction of the conserved momentum p. A trivial calculation shows that

$$e^{S_{\sigma_0}}\,H_{lin}\,e^{-S_{\sigma_0}} \;=\; E_{lin}\;+\;H_f\;-\;|p|\,\mathbf{P}_f^{\|}\;, \tag{2.4.2}$$

where

$$E_{lin} \;=\; -\,g^2\,|p|^2\,\sum_\lambda \int \frac{d^3k\,\bar\kappa_{\sigma_0}^2[|k|]\,\kappa_\Lambda^2(|k|)}{|k|\,(|k|-|p|k^{\|})}\,\epsilon_{0,1}^{\|}[n_k,\lambda]\,\epsilon_{1,0}^{\|}[n_k,\lambda] \tag{2.4.3}$$

denotes the ground state energy of H_{lin}. One can easily convince oneself of

$$\big|\,\partial_{|p|}^\alpha E_{lin}\,\big| \;\leq\; O(g^2)$$

as long as $|p|<1$, for $\alpha=0,1,2$, and all $\sigma_0\geq 0$.

The associated ground state eigenvector is given by

$$\Psi[\sigma_0,p] \;=\; e^{S_{\sigma_0}}\,\Omega_f\;,$$

which can clearly be written in the form

$$\begin{aligned} e^{S_{\sigma_0}}\,\Omega_f \;=\;& \exp\left(-\frac{g^2\,|p|^2}{2}\,\sum_\lambda\int\frac{d^3k\,\bar\kappa_{\sigma_0}^2[|k|]\,\kappa_\Lambda^2(|k|)}{|k|\,(|k|-|p|k^{\|})^2}\right)\;\times \\ &\times\;\exp\left(g\,|p|\,\int\frac{d^3k\;\bar\kappa_{\sigma_0}[|k|]\,\kappa_\Lambda(|k|)}{\sqrt{|k|}\,(|k|-|p|k^{\|})}\,w_{1,0}[k,\lambda]a_\lambda^*(k)\right)\,\Omega_f\;. \end{aligned}$$

Hence, defining

$$\Omega[\sigma_0, p] \;:=\; \exp\left(\frac{g^2\,|p|^2}{2} \int \frac{d^3k\;\bar{\kappa}^2_{\sigma_0}[|k|]\;w_{0,1}[k,\lambda]\;w_{1,0}[k,\lambda]}{\left(|k| - |p|k^{\|}\right)^2} \right)\; \Psi[\sigma_0, p]\;,$$

and fixing the projection

$$\langle\; \Omega_f\;,\; \Omega[\sigma_0, p]\;\rangle \;=\; 1$$

for all $\sigma_0 > 0$,

$$\begin{aligned} &\|\; \Omega[\sigma_0, p]\;\|^2 \\ &\quad= \exp\left(g^2\,|p|^2 \sum_{\lambda} \int \frac{d^3k\;\bar{\kappa}^2_{\sigma_0}[|k|]\;\kappa^2_{\Lambda}(|k|)}{|k|\,\left(|k| - |p|k^{\|}\right)^2}\;\epsilon^{\|}_{0,1}[n_k,\lambda]\;\epsilon^{\|}_{1,0}[n_k,\lambda] \right)\;. \end{aligned}$$

In the absence of the infrared regularization, the integrand in the exponent diverges like $\frac{1}{|k|^3}$ in the limit $|k| \to 0$; due to the infrared regularization, there are constants $0 < c, C < \infty$ such that

$$\|\; \Omega[\sigma_0, p]\;\| \;\geq\; C \exp\left(c\; g^2\,|p|^2\,|\log \sigma_0| \right)\;.$$

In the limit $\sigma_0 \to 0$, the vector $\Psi[\sigma_0, p]$ leaves the Fock space $\mathcal{H}_p$. For every fixed $|p| > 0$, the ground state belongs to a Hilbert space that carries a representation of the canonical commutation relations that is unitarily inequivalent to the Fock representation. For different values of p, these representations are mutually inequivalent.

For $|p| = 0$, however, it is trivially clear that the ground state is given by $\Omega_f \in \mathcal{H}_p$. A slightly more detailed exposition of matters concerning this toy model can be found in [**17**].

CHAPTER 3

STATEMENT AND DISCUSSION OF THE MAIN THEOREMS

The central results proved in the present work are the following two main theorems.

THEOREM 3.1. *For any arbitrarily small, but fixed value of the infrared cutoff $\sigma_0 > 0$, and g sufficiently small, there exists a constant $p_c \in [\frac{1}{20}, 1)$, such that for all p with $|p| \leq p_c$,*

$$E_0[|p|, \sigma_0] \;:=\; \inf \mathrm{spec}\{H_p(\sigma_0)\} \tag{3.0.4}$$

is a unique eigenvalue with an associated eigenvector $\Omega[p;\sigma_0]$ that lies in $\mathcal{H}_p$. $E_0[|p|,\sigma_0]$ only depends on the absolute value of p because of rotational symmetry. Under the normalization condition

$$\langle \Omega_f \,,\, \Omega[p;\sigma_0] \rangle \;=\; 1 \;,$$

there are constants, $c, C > 0$ and $c', C' > 0$, such that

$$c\, \exp\left[\, c'\, g^2\, |p|^2\, |\log \sigma_0\,|\,\right] \leq \; \|\,\Omega[p;\sigma_0]\,\|^2 \;\leq\; C\, \exp\left[\, C'\, g^2\, |p|^2\, |\log\, \sigma_0\,|\right] \quad .$$

In particular, this implies that $H_p(\sigma_0)$ possesses a ground state vector in $\mathcal{H}_p$, if and only if $|p| = 0$.

Under the condition that the coupling constant g, and the absolute value $|p|$ of the conserved momentum are sufficiently small, this gives a description of the bottom of the joint energy-momentum spectrum of the present system, that is, of the translation invariant model of a spinless, massive charge, which interacts with the quantized electromagnetic field. In particular, this theorem provides the infraparticle state $\Omega[p;\sigma_0]$, and demonstrates how the infrared catastrophe arises. The next theorem gives us analytical control of the first and second derivatives of the infraparticle energy with respect to the conserved momentum, again under the assumption that both g and $|p|$ are small enough.

THEOREM 3.2. *Assume that $|p| \leq p_c$, and that g is sufficiently small. The ground state eigenvalue $E_0[|p|,\sigma_0]$ is of class C^2 with respect to the conserved momentum p, and*

$$\left|\; \partial^{\alpha}_{|p|} \left(E_0[|p|,\sigma_0] - \frac{|p|^2}{2} \right) \;\right| \;\leq\; O(g^{1-3\nu}) \tag{3.0.5}$$

for all $\sigma_0 \geq 0$, with $\alpha = 0, 1, 2$. Furthermore,

$$\partial^2_{|p|} \left(E_0[|p|,\sigma_0] \;-\; \frac{|p|^2}{2} \right) \;\leq\; 0 \;,$$

that is, the renormalized mass of the infraparticle is bounded from below by the bare mass of the charged particle, for all $\sigma_0 \geq 0$.

Let us next comment on the physical interpretation of these results.

3.1. The joint energy-momentum spectrum

Theorem 3.1 gives the following description of the nature of the joint energy-momentum spectrum $\mathrm{spec}(H(\sigma_0), \mathbf{P}_{tot})$ of the translation invariant system of a massive, spinless charge that interacts with the quantized electromagnetic field, on the Hilbert space $\mathcal{H} = \int^{\oplus} d^3p \mathcal{H}_p$. The model hamiltonian $H(\sigma_0)$ is obtained by omitting $\mathbf{B}_f(\sigma_0)$ and the electron spin in $H^{(\sigma_0)}$ which is defined in (2.1.8), and $\mathbf{P}_{tot}$ is defined in (2.1.1).

In the limit $g \to 0$ of vanishing interaction, the spectrum consists of the parabola $E[|p|] = \frac{|p|^2}{2}$ in the $(E, |p|)$-plane (where E denotes the energy, and p the conserved momentum), which borders to absolutely continuous spectrum. Thus,

$$\mathrm{spec}\left[\left(\lim_{g\to 0} H(\sigma_0)\ ,\ \mathbf{P}_{tot}\right)\right] = \left(\frac{|p|^2}{2} + \mathbb{R}_+\ ,\ p\right)_{|p|\in\mathbb{R}_+}\ ,$$

independently of σ_0. For every fixed value of p, the infimum of the spectrum, $\frac{|p|^2}{2}$, is an eigenvalue with an associated eigenvector given by

$$|p\rangle_{el} \otimes \Omega_f\ ,$$

that is, a generalized electron eigenstate of momentum p, tensored with the photon vacuum Ω_f. Its restriction to the Hilbert fibrespace $\mathcal{H}_p$ is simply Ω_f.

Turning on the interaction (by choosing some $g > 0$), at an arbitrarily small, but fixed value of the infrared cutoff $\sigma_0 > 0$, Theorem 3.2 implies that at least for values $|p| \leq p_c$ of the conserved total momentum,

$$\mathrm{spec}\left[\left(H(\sigma_0)\ ,\ \mathbf{P}_{tot}\right)\ \middle|\ |\mathbf{P}_{tot}| \leq p_c\right] = \left(\frac{|p|^2}{2} + O(g^{\frac{1}{6}}) + \mathbb{R}_+\ ,\ p\right)_{|p|\leq p_c}\ ,$$

uniformly in σ_0. That is, the spectrum of the interacting system is, for sufficiently small g and $|p|$, a small perturbation of that of the non-interacting one. According to Theorem 3.1, the infimum of the spectrum is, under these restrictions, still an eigenvalue. The corresponding eigenvector, however, is **not** a small perturbation of the eigenvector at $g = 0$. Instead, at fixed p, with $|p| \leq p_c$ and g sufficiently small, its restriction to the Hilbert fibrespace $\mathcal{H}_p$, given by $\Omega[p; \sigma_0]$, Theorem 3.1 asserts that there exist constants $c_1, c_2, C_1, C_2 > 0$, such that

$$c_1\ \exp\left(c_2\, g^2\, |p|^2\ |\log\,\sigma_0\,|\right) \leq\ \|\,\Omega[p;\sigma_0]\,\|^2\ \leq\ C_1\ \exp\left(C_2\, g^2\, |p|^2\ |\log\,\sigma_0\,|\right)\ ,$$

if its projection in the direction of Ω_f, the eigenstate for $g = 0$, is normalized to the value 1. Most importantly, the lower bound **diverges** in the limit $\sigma_0 \to 0$. Consequently, either $\Omega[p; \sigma_0]$ ceases to have a projection in the direction of Ω_f, or it ceases to be an element of $\mathcal{H}_p$ if the infrared regularization is removed.

In fact, the latter is the case.

$$\Psi[p;\sigma_0]\ :=\ \frac{\Omega[p;\sigma_0]}{\|\Omega[p;\sigma_0]\|}$$

corresponds to a generalized coherent state, which is contained in a Hilbert space orthogonal to $\mathcal{H}_p$ that carries a representation of the canonical commutation relations that is unitarily inequivalent to the Fock representation. Both Hilbert spaces are at this point regarded as Hilbert subspaces embedded in some infinite tensor product Hilbert space whose construction dates back to J. v. Neumann [**24**]. For more material on this issue, cf. for instance [**18**].

3.2. Dressed 1-particle states

Physically, the state $\Omega[p;\sigma_0]$ describes a massive, charged particle that is inevitably surrounded by a cloud of infinitely many photons. It is usually referred to as an **infraparticle** or a **dressed 1-particle state**.

As has already been recognized by Bloch and Nordsieck long ago, [**5**], the fact that the number of soft bosons is not relatively bounded by their energy is responsible for the infrared catastrophe.

To arrive at a better insight into the structure of the problem, let us add a brief remark about operator algebraic aspects, and refer to the work of J. Fröhlich [**9, 10**]. Let $\mathfrak{A}$ denote the $*$-algebra generated by $\{\mathbf{1}, a^*_\lambda(f), a_\lambda(g)\}$ for $f, g \in L^2(\mathbb{R}^3, d^3k)$.

A state on $\mathfrak{A}$ is a linear functional

$$\omega \; : \; \mathfrak{A} \; \longrightarrow \; \mathbb{C} \tag{3.2.1}$$

that is positive and normalized, that is, $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$, and $\omega(\mathbf{1}) = 1$.

For fixed σ_0 and p, let ω_{p,σ_0} denote the vector state defined by

$$\begin{aligned} \omega_{p,\sigma_0} \; : \; \mathfrak{A} \; &\longrightarrow \; \mathbb{C} \\ A \; &\longrightarrow \; \omega_{p,\sigma_0}(A) \; = \; \langle \Psi[p;\sigma_0] \; , \; A\,\Psi[p;\sigma_0] \rangle \; . \end{aligned}$$

For the related case of the massless Nelson model, J. Fröhlich proved in [**9**] that, in a suitable sense, there exists a well-defined limiting state, $\omega_p(A) = \lim_{\sigma_0 \to 0} \omega_{p,\sigma_0}(A)$ for all $A \in \mathfrak{A}$, and every sufficiently small value of $|p|$. The Gelfand-Naimark-Segal (GNS-) construction associated to the state ω_p yields a Hilbert space $\mathcal{H}_p^{(IR)}$ that carries a non-Fock representation of the CCR algebra if $|p| > 0$ (that is, a representation of the CCR algebra unitarily inequivalent to the Fock representation), together with a vacuum that physically describes a dressed 1-particle state. The same fact is expected to hold for the present system.

3.3. Smoothness of the infraparticle energy

The fact that the first and second derivatives of the ground state energy of $H_p(\sigma_0)$ with respect to $|p|$ is close to that of the classical expression $E_{cl}[|p|] = \frac{|p|^2}{2}$ is physically important. For instance, this result is highly relevant for the construction of scattering theory according to [**9, 10, 21**], or for the semiclassical dynamics of charged particles, [**23**]. In particular, it provides a formula for the **mass** of the dressed particle.

3.4. Cherenkov radiation

The fact that Theorems 3.1 and 3.2 are only valid for sufficiently small values of $|p|$ is (although improvements on the bounds on p_c are certainly possible) not only due to a lack of mathematical technology, but also a consequence of the phenomenon of Cherenkov radiation. In fact, if $|p|$ approaches 1 (corresponding to the rest energy mc^2 of the charged particle, where m is its mass, and c is the speed of light), the infraparticle increasingly tends to reduce its kinetic energy by the emission of electromagnetic (Cherenkov) radiation. One expects that if $|p|$ approaches 1, the eigenvalue $E_0[|p|, \sigma_0]$ at the bottom of spec$H_p(\sigma_0)$ turns into a resonance, in the sense of [**2, 3**]. Clarification of this point is beyond the scope of this

text.

3.5. The proof idea

The proof of Theorems 3.1 and 3.2 uses the operator-theoretic renormalization group method developed by V. Bach, J. Fröhlich, and I.M. Sigal in [**2, 3**]. The latter is based on the recursive application of the **smooth Feshbach map** and the **Feshbach theorem**. The smooth Feshbach map associates to an operator H that acts on a given Hilbert space $\mathcal{H}$ an operator H' that acts on a Hilbert subspace $\mathcal{H}' \subset \mathcal{H}$, where, in a sense made precise in the Feshbach theorem, H and H' are mutually isospectral. The mapping $\mathcal{L}[\mathcal{H}] \to \mathcal{L}[\mathcal{H}']$, $H \mapsto H'$ is referred to as '(Feshbach) decimation', since in the language of Wilson's renormalization group, it 'decimates' the 'degrees of freedom' in $\mathcal{H} - \mathcal{H}'$.

The Bach-Fröhlich-Sigal operator-theoretic renormalization group is a tool aimed at the analysis of spectral problems in quantum field theories. In our case, the key issue is to study the bottom of the spectrum of the hamiltonian $H_p(\sigma_0)$, which acts on the Hilbert fibre space $\mathcal{H}_p$ associated to the fixed value p of the conserved momentum.

The general setup of the method is given as follows. We introduce a dyadic decomposition of $\mathcal{H}_p$,

$$\mathcal{H}_p \;=\; \mathcal{H}_{-1} \,\oplus\, \left(\bigoplus_{n \geq 0} \Delta_n \mathcal{H} \right) \;,$$

where

$$\Delta_n \mathcal{H} \;:=\; \chi_{[2^{-(n+1)}, 2^{-n}]}[H_f]\, \mathcal{H}_p \;,$$

and

$$\mathcal{H}_{-1} \;:=\; \chi_{[1,\infty)}[H_f]\, \mathcal{H}_p \;.$$

Then, the strategy is to employ the smooth Feshbach map to iteratively decimate the degrees of freedom of the theory in $\mathcal{H}_{-1} \oplus \mathcal{H}_0 \oplus \Delta_1 \mathcal{H} \oplus \cdots \oplus \Delta_n \mathcal{H}$, for increasing n in steps of 1. The purpose is to analytically control the decimation of the degrees of freedom associated to the Hilbert space $\chi_{[\mu,\infty)}[H_f]\mathcal{H}_p$, in the limit $\mu \to 0$.

We will in fact follow the setup of the Wilson renormalization group, hence a **rescaling transformation** is employed between each application of Feshbach decimation. This has the effect that all operators H_n obtained in this process act on the fixed Hilbert space $P_1 \mathcal{H}_p = \chi_{[0,1]}[H_f]\mathcal{H}_p$.

All of these operators are written in Wick ordered normal form

$$H_n \;=\; \sum_{j \in \mathbf{N}_0} H_n^{(j)} \;,$$

where $H_n^{(j)}$ is the sum of Wick monomials in H_n containing k creation- and $j-k$ annihilation operators, for $k = 0, \ldots, j$. In the absence of cutoffs, $H_n^{(j)}$ is homogenous under pure scaling, and has the integral scaling dimension j.

Along with the rapid reduction of the initial Hilbert space, we then observe that under the concatenation of 'Feshbach decimation' and 'rescaling', which is referred to as the operation of '**renormalization**', the norms of the operators $H_n^{(j)}$ are reduced by different numerical fractions under the rescaling map, the latter depending on their scaling dimensions in the absence of cutoffs. If this fraction is smaller or equal $\sim \frac{1}{2}$ (up to errors of order $O(g)$),

we say that $H_n^{(j)}$ is **irrelevant**. If it is ~ 1, $H_n^{(j)}$ is **marginal**, and if the fraction is larger or equal ~ 2, $H_n^{(j)}$ is called **relevant**.

To organize the problem, we introduce a Banach space $\mathfrak{H}$, whose points, denoted by $\mathfrak{h}$, are in one-to-one correspondence to the operators H_n, and a map H from $\mathfrak{H}$ to $\mathcal{B}(P_1\mathcal{H}_p)$, the bounded operators acting on $P_1\mathcal{H}_p$. That is, associated to each H_n, there is a point $\mathfrak{h}^{(n)} \in \mathfrak{H}$, such that $H[\mathfrak{h}^{(n)}] = H_n$. We refer to $H[\mathfrak{h}^{(n)}]$ as **effective hamiltonians**.

The **renormalization map** $\mathcal{R}$ is obtained as follows. Let $H[\mathfrak{h}]$ denote an effective hamiltonian at some given scale, acting on $P_1\mathcal{H}_p$. From an application of the **Feshbach map**, which is introduced in Chapter 4 below, concatenated with a rescaling transformation, we obtain an operator $H[\hat{\mathfrak{h}}]$. Pulling back the map $H[\mathfrak{h}] \mapsto H[\hat{\mathfrak{h}}]$ to $\mathfrak{H}$ by way of $H : \mathfrak{H} \to \mathcal{B}(P_1\mathcal{H}_p)$, we obtain the renormalization map $\mathcal{R} : \mathfrak{H} \to \mathfrak{H}$.

The pair $(\mathfrak{H}, \mathcal{R})$ is interpreted as a discrete dynamical system, whose orbits are sequences $\{\mathfrak{h}^{(n)}\}_{n\geq 0}$, generated by n-fold iterating $\mathcal{R}$, applied to some initial value $\mathfrak{h}^{(0)}$. The index n is referred to as the **scale**. Thus, the focus shifts away from the 'elimination' of the Hilbert spaces $\Delta_n\mathcal{H}$, and moves to the study of a discrete dynamical system on a Banach space.

All effective hamiltonians are, according to Feshbach's theorem in the next chapter, mutually isospectral at the bottom of their spectra. In addition, the Feshbach theorem provides a means to uniquely determine the ground state eigenvector of any effective hamiltonian provided that the ground state eigenvector is known for only one of them.

The very first step, in which the components of $H_p(\sigma_0)$ in $\mathcal{H}_{-1}$ are decimated, 'calibrates' the problem by producing an initial condition $\mathfrak{h}^{(0)}$. Furthermore, $H[\mathfrak{h}^{(0)}]$ is isospectral to $H_p(\sigma_0)$ in the sense of the Feshbach theorem. Therefore, if we succeed in finding the solution of the eigenvalue problem for any of the $H[\mathfrak{h}^{(n)}]$'s, we can immediately obtain the corresponding result for $H_p(\sigma_0)$.

The strategy of our analysis indeed consists of finding a finite scale $n \sim N_{\sigma_0}$, for which the eigenvalue problem reduces to a case that has a known solution, and from which the quantities of interest can be reconstructed for $H_p(\sigma_0)$. This is precisely the rôle of the infrared cutoff in the present model; it provides a finite scale $N_{\sigma_0} < \infty$, at which the effective hamiltonian $H[\mathfrak{h}^{(N_{\sigma_0})}]$ has a purely irrelevant interaction part. For the solution of the eigenvalue problem for $H[\mathfrak{h}^{(N_{\sigma_0})}]$, one can use the results of Bach, Fröhlich and Sigal [**2, 3**], to produce a final condition, from which the ground state data of the physical hamiltonian can be reconstructed, basically by backwards iterating the Feshbach theorem.

The key difficulty in the proof of Theorem 3.1 is the fact that there are purely marginal operators. For their analysis, we first of all invoke an operator-theoretic version of the Ward-Takahashi identities of non-relativistic QED to reduce the level of complexity of the problem. Then, a nested renormalization group construction is invoked to analytically master their renormalization flow.

CHAPTER 4

ALGEBRAIC AND ANALYTIC PROPERTIES OF FESHBACH TRIPLES, WICK ORDERING AND THE DECIMATION MAP

A mathematical tool of extraordinary importance in our analysis is the so-called **smooth Feshbach map**. Combined with the **Feshbach theorem**, it has, in its initial form, been very successfully used in the construction of an operator-theoretic renormalization group for the rigorous analysis of quantum field theories in [**2, 3**].

The smooth Feshbach map is a variant that uses smooth cutoff operators (in the sense of the spectral theorem) instead of projectors. It has been developed by V. Bach, J. Fröhlich and I. M. Sigal in the presently unpublished work [**4**]. I thank these authors for allowing me to use their work for the exposition at hand.

There are several properties of the smooth Feshbach map of great analytical power. The first and foremost one is that of isospectrality, which is the main reason for its use in the construction of an operator-theoretic renormalization group. Furthermore, derivations of the smooth Feshbach map have a particular algebraic structure that enables us to construct an operator-theoretic generalization of the Ward-Takahashi identities in $U(1)-$gauge theory.

In the Bach-Fröhlich-Sigal operator-theoretic renormalization group, one considers the flow on the Banach space $\mathfrak{H}$, which parametrizes the effective hamiltonians of the theory, that is generated by a renormalization map. To this end, we write all effective hamiltonians in Wick ordered normal form. This is discussed in detail in this chapter.

4.1. Feshbach triples and the Feshbach theorem

We assume a closed operator H with dense domain on a separable Hilbert space $\mathcal{H}$ to be given, together with a spectral parameter $z \in \mathbb{C}$. Furthermore, we associate an operator $\tau[H-z]$ to $H-z$, and define $\omega[H-z] \equiv H - z - \tau[H-z]$, in a manner that $\mathfrak{Dom}\{\omega[H-z]\} \supseteq \mathfrak{Dom}\{\tau[H-z]\}$ are dense in $\mathcal{H}$.

Let $\chi, \bar{\chi}$ denote a pair of bounded operators acting on $\mathcal{H}$, constituting a C^∞-partition of unity,

$$\chi^2 + \bar{\chi}^2 = \mathbf{1} \,, \tag{4.1.1}$$

and let

$$\begin{aligned} P &\Leftrightarrow \mathrm{Ran}\{\chi\} \,, \\ \bar{P} &\Leftrightarrow \mathrm{Ran}\{\bar{\chi}\} \end{aligned}$$

denote the orthogonal projections onto their respective ranges. Likewise, let

$$P^\perp \equiv \mathbf{1} - P \quad , \quad \bar{P}^\perp \equiv \mathbf{1} - \bar{P}$$

denote their respective complementary orthogonal projections. We emphasize that $PP^{\perp} = 0$, but $P\bar{P} \neq 0$ (and likewise for $\bar{P}$). Furthermore, it is, in particular, assumed that any commutator between any pair in $\tau[H], \chi, \bar{\chi}, P$ and $\bar{P}$ vanishes.

Let us write H for $H - z$, to simplify the notation. Assuming that the restrictions of $\tau[H]$ and $\tau[H] + \bar{\chi}\omega[H]\bar{\chi}$ to $\mathrm{Ran}\{\bar{P}\}$ are both bounded invertible, we may define the restricted resolvent

$$\bar{R} \;:=\; (\tau[H] + \bar{\chi}\omega[H]\bar{\chi})^{-1} \tag{4.1.2}$$

on $\bar{P}\mathcal{H}$. We assume that

$$\begin{aligned} &\bar{R} \;\;,\;\; \bar{R}\,\bar{\chi}\,\omega[H]\,\chi \;,\; \chi\,\omega[H]\,\bar{\chi}\,\bar{R} \;, \\ &\chi\omega[H]\,\bar{\chi}\,\bar{R}\,\bar{\chi}\,\omega[H]\chi \;\;,\;\; \chi\,\omega[H]\,\chi \end{aligned} \tag{4.1.3}$$

all extend to bounded operators on $\mathcal{H}$. Hence, the operators

$$\begin{aligned} Q \;&:=\; \chi \;-\; \bar{\chi}\,\bar{R}\,\bar{\chi}\,\omega[H]\,\chi \;\;, \\ Q^{\sharp} \;&:=\; \chi \;-\; \chi\,\omega[H]\,\bar{\chi}\,\bar{R}\,\bar{\chi} \end{aligned} \tag{4.1.4}$$

are well-defined. Furthermore,

$$\begin{aligned} F_{\chi,\tau}[H] \;&:=\; \tau[H] \;+\; \chi\,\omega[H]\,\chi \;-\; \chi\,\omega[H]\,\bar{\chi}\,\bar{R}\,\bar{\chi}\,\omega[H]\,\chi \\ &=\; \tau[H] \;+\; \chi\,\omega[H]\,Q \\ &=\; \tau[H] \;+\; Q^{\sharp}\,\omega[H]\,\chi \end{aligned} \tag{4.1.5}$$

is a well-defined operator that acts on $P\mathcal{H}$. $F_{\chi,\tau}$ is called the **smooth Feshbach map**.

Definition 4.1. *The data $(Q^{\sharp}, Q, F_{\chi,\tau})$ will be referred to as the* **Feshbach triple** *associated to $(\chi, \mathcal{H})$.*

Feshbach triples possess a number of miraculous properties. First and foremost, there is the property of isospectrality in the following sense.

Theorem 4.1. *The Feshbach triple $(Q^{\sharp}, Q, F_{\chi,\tau})$ associated to $(\chi, \mathcal{H})$ has the following properties.*

i) Isospectrality: *H is bounded invertible on $\mathcal{H}$ if and only if $F_{\chi,\tau}[H]$ is bounded invertible on $P\mathcal{H}$. Then, the formula*

$$F_{\chi,\tau}[H]^{-1} \;=\; \chi H^{-1}\chi \;+\; \bar{\chi}\tau[H]^{-1}\bar{\chi} \;, \tag{4.1.6}$$

relates the respective inverses to each other.

ii) Projection of eigenvectors: *$0 \neq \psi \in \mathcal{H}$ satisfies $H\psi = 0$ if and only if $F_{\chi,\tau}[H]\,\chi\,\psi = 0$.*

iii) Reconstruction of eigenvectors: *$0 \neq \zeta \in P\mathcal{H}$ satisfies $F_{\chi,\tau}[H]\zeta = 0$ if and only if $0 \neq Q\zeta \in \mathcal{H}$ satisfies $HQ\zeta = 0$.*

Proof. The proof is reproduced from the preprint [**4**] by V. Bach, I.M. Sigal, and J. Fröhlich.

Next, we observe that, writing $F \equiv F_{\chi,\tau}[H]$ in brief, the identities

$$\begin{aligned} H\,Q &= \chi\,F\,, \\ Q^\sharp\,H &= F\,\chi \end{aligned} \tag{4.1.7}$$

and

$$\begin{aligned} F &= \tau[H] + \chi\,\omega[H]\,Q \\ &= \tau[H] + Q^\sharp\,\omega[H]\,\chi\,. \end{aligned}$$

are satisfied, as can be verified by a straightforward computation using the fact that $\chi^2+\bar\chi^2 = \mathbf{1}$.

(I) Let us suppose that $\tau[H]+\omega[H]$ is invertible. Then, we claim that

$$F^{-1} \equiv \chi\,H^{-1}\,\chi + \bar\chi\,\tau[H]^{-1}\,\bar\chi\,.$$

To prove this assertion, we observe that

$$\begin{aligned} &\left[\tau[H]^{-1}\bar P + \left(\bar P^\perp - \tau[H]^{-1}\bar P\,\chi\,\omega[H]\right)H^{-1}\,\chi\right]\,F \\ &\quad= \bar\chi\tau[H]^{-1}\bar\chi(\tau[H]+\chi\,\omega[H]\,Q) \\ &\qquad\quad +\chi\,H^{-1}\,H\,Q \\ &\quad= \bar\chi^2 + \tau[H]^{-1}\left[\chi\,\bar\chi - \bar\chi^2\,\chi\,\omega[H]\,\bar R\right]\bar\chi\,\omega[H]\,\chi \\ &\qquad\quad +\chi^2 - \chi\,\bar\chi\,\bar R\,\bar\chi\,\omega[H]\,\chi \\ &\quad= \mathbf{1} + \tau[H]^{-1}\left[\chi\,\bar\chi - \chi\,\bar\chi\underbrace{(\tau[H]+\bar\chi\omega[H]\bar\chi)\bar R}_{=\bar P} + \tau[H]\chi\bar\chi\bar R\right]\bar\chi\,\omega[H]\,\chi \\ &\qquad\quad -\chi\,\bar\chi\,\bar R\,\bar\chi\,\omega[H]\,\chi \\ &\quad= \mathbf{1}\,, \end{aligned}$$

hence the expression in discussion is indeed a left inverse of F. Likewise, one proves that it is also a right inverse. Thus, F is globally and uniquely invertible on $\mathcal{H}$. In particular, since $PFP^\perp = P^\perp FP = 0$, this also implies the invertibility of PFP on $\mathrm{Ran}\{P\}$.

Conversely, let us assume that the restriction of F to $\mathrm{Ran}\{P\}$ is bounded invertible. Then, defining

$$R \equiv Q\,F^{-1}\,Q^\sharp + \bar\chi\,\bar R\,\bar\chi\,,$$

it follows that

$$\begin{aligned} H\,R &= \chi\,Q^\sharp + H\,\bar\chi\,\bar R\,\bar\chi \\ &= \chi^2 - \chi^2\,\omega[H]\,\bar\chi\,\bar R\,\bar\chi + \bar\chi\,H\,\bar R\,\bar\chi \\ &\quad + \chi^2\,\omega[H]\,\bar\chi\,\bar R\,\bar\chi \\ &= \chi^2 + \bar\chi^2 \\ &= \mathbf{1}\,. \end{aligned}$$

Likewise, one can prove that $R\,H = \mathbf{1}$, hence $R = H^{-1}$.

(II) Assume that $\psi \in \mathcal{H}$, $\psi \neq 0$, solves $H\psi = 0$. Then,

$$\begin{aligned} F\,\chi\,\psi &= Q^\sharp\,H\,\psi \\ &= 0 \end{aligned}$$

is satisfied.

(III) Suppose that $\phi \in \mathrm{Ran}\{P\}$, $\phi \neq 0$, solves $F\phi = 0$. Then,

$$\begin{aligned} H\,Q\,\phi &= \chi\,F\,\phi \\ &= 0\,. \end{aligned}$$

This proves the theorem. ■

In all cases of interest for us, the hamiltonians in discussion will be of the form $H = T + W$, where T commutes with χ, while W contains no part that commutes with χ, and has a small norm bound relative to T. For the construction of the smooth Feshbach map, the operator $\tau[H]$ will be defined in a manner that it commutes with χ, as it must be, but

$$\omega[H] \;=\; H \;-\; \tau[H]$$

will in general also contain parts of T that commute with χ. The following expression for the Feshbach map will later turn out to be highly useful, because it conveniently separates the terms in $F_{\chi,\tau}[H]$ that depend on T and W.

LEMMA 4.1. *Let* $T' \equiv T - \tau[H]$, *and*

$$\begin{aligned} \Pi &\equiv \mathbf{1} \,-\, \bar\chi T' \bar\chi \bar R_0 \\ &= \bar P^\perp \,+\, \bar P\,\tau[H]\,\bar R_0\;, \end{aligned} \tag{4.1.8}$$

where

$$\bar R_0 \;=\; (\tau[H] \,+\, \bar\chi T' \bar\chi)^{-1}$$

on $\bar P\mathcal{H}$. *Then,*

$$F_{\chi,\tau}[H] \;=\; \tau[H] \,+\, \chi\,T'\,\Pi\,\chi \,+\, \chi\,\Pi\,(W \,-\, W\,\bar R\,W)\,\Pi\,\chi\;, \tag{4.1.9}$$

where clearly, $[\Pi, \chi] = [\Pi, \bar\chi] = 0$.

Proof. Using the **second resolvent identity**

$$\begin{aligned} \bar R &= \bar R_0 \,-\, \bar R_0\,\bar\chi W \bar\chi \bar R \\ &= \bar R_0 \,-\, \bar R\,\bar\chi W \bar\chi \bar R_0\;, \end{aligned} \tag{4.1.10}$$

and

$$\omega[H] \;=\; T' \,+\, W\;,$$

straightforward calculation shows that

$$\begin{aligned} F_{\chi,\tau}[H] &= \tau[H] + \chi T'\chi + \chi W\chi \\ &\quad - \chi\omega[H]\bar\chi\bar R\bar\chi\omega[H]\chi \\ &= \tau[H] + \chi T'\chi - \chi T'\bar\chi\bar R_0\bar\chi T'\chi \\ &\quad + \chi W\chi - \chi W\bar\chi\bar R\bar\chi W\chi \\ &\quad - \chi W\bar\chi\bar R_0\bar\chi T'\chi - \chi T'\bar\chi\bar R_0\bar\chi W\chi \\ &\quad + \chi T'\bar\chi\bar R_0\bar\chi W\bar\chi\bar R_0\bar\chi T'\chi \\ &\quad + \chi T'\bar\chi\bar R_0\bar\chi W\bar\chi\bar R\bar\chi W\chi + \chi W\bar\chi\bar R\bar\chi W\bar\chi\bar R_0\bar\chi T'\chi \\ &\quad - \chi T'\bar\chi\bar R_0\bar\chi W\bar\chi\bar R\bar\chi W\bar\chi\bar R_0\bar\chi T'\chi \\ &= \tau[H] + \chi T'\chi - \chi\, T'\bar\chi\bar R_0\bar\chi T'\chi \\ &\quad + \chi(\mathbf{1} - \bar\chi T'\bar\chi\bar R_0)W(\mathbf{1} - \bar\chi T'\bar\chi\bar R_0)\chi \\ &\quad - \chi(\mathbf{1} - \bar\chi T'\bar\chi\bar R_0)W\bar R W(\mathbf{1} - \bar\chi T'\bar\chi\bar R_0)\chi\,. \end{aligned} \tag{4.1.11}$$

This proves the claim. ■

4.2. Derivations

Next, we consider derivatives of the smooth Feshbach map with respect to scalar and operator-valued variables, and of its commutators with respect to operators and operator valued distributions. These are cases of **derivations**, and are collectively governed by Lemma 4.2, and Theorem 4.2 below.

LEMMA 4.2. *Let $\mathcal{D}$ denote a derivation on the bounded linear operators on $\mathcal{H}$. That is,*

$$\begin{aligned} \mathcal{D}[AB] &= \mathcal{D}[A]\,B + A\,\mathcal{D}[B] \\ \mathcal{D}[\lambda A + \mu B] &= \lambda\,\mathcal{D}[A] + \mu\,\mathcal{D}[B] \\ \mathcal{D}[\mathbf{1}] &= 0\,, \end{aligned} \tag{4.2.1}$$

for all $A, B \in \mathcal{L}[\mathcal{H}]$, and all $\lambda, \mu \in \mathbb{C}$. Then,

$$\begin{aligned} \mathcal{D}[F_{\chi,\tau}[H-z]] &= \mathcal{D}[\tau[H]] + \chi\,\omega[H]\,\bar\chi\,\bar R\,\mathcal{D}[\tau[H]]\,\bar R\,\bar\chi\,\omega[H]\,\chi \\ &+ Q^\sharp\,\mathcal{D}[\omega[H]]\,Q \\ &+ \mathcal{D}[\chi]\,(H-z)\,Q + Q^\sharp\,(H-z)\,\mathcal{D}[\chi] \\ &- 2\,\chi\,\omega[H]\,\bar\chi\,\bar R\,\frac{\mathcal{D}[\bar\chi]}{\bar\chi}\,\tau[H]\,\bar R\,\bar\chi\,\omega[H]\,\chi \end{aligned} \tag{4.2.2}$$

if $\mathcal{D}[\bar\chi], \bar\chi$, and $\tau[H]$ are pairwise commutative.

Proof. Let us first of all note that since

$$\begin{aligned} \bar R &= \tau[H]^{-1} - \tau[H]^{-1}\,\bar\chi\,\omega[H]\,\bar\chi\,\bar R \\ &= \tau[H]^{-1} - \bar R\,\bar\chi\,\omega[H]\,\bar\chi\,\tau[H]^{-1} \end{aligned} \tag{4.2.3}$$

(the **second resolvent identity**), and

$$[\mathcal{D}[\bar\chi], \tau[H]]\,,\ [\mathcal{D}[\bar\chi], \bar\chi] = 0\,,$$

the operator formally written as

$$\bar\chi\,\bar R\,\frac{\mathcal{D}[\bar\chi]}{\bar\chi}$$

has a well-defined meaning.

Next, for any invertible operator A,

$$\mathcal{D}[A^{-1}] \;=\; -\,A^{-1}\,\mathcal{D}[A]\,A^{-1}\;,$$

from $0 \,=\, \mathcal{D}[\mathbf{1}] \,=\, \mathcal{D}[A\,A^{-1}] \,=\, A\,\mathcal{D}[A^{-1}] \,+\, \mathcal{D}[A]\,A^{-1}$ (and acting from the left with A^{-1}).

Thus,

$$\begin{aligned}\mathcal{D}[\bar R] \;&=\; -\bar R\,\left(\mathcal{D}[\tau[H]] \,+\, \bar\chi\,\mathcal{D}[\omega[H]]\,\bar\chi\right)\,\bar R\\ &\quad +\,\bar R\,\left(\mathcal{D}[\bar\chi]\,\omega[H]\,\bar\chi \,+\, \bar\chi\,\omega[H]\,\mathcal{D}[\bar\chi]\right)\,\bar R\;,\end{aligned}\tag{4.2.4}$$

and recalling that

$$F_{\chi,\tau}[H] \;=\; \tau[H] \,-\, z \,+\, \chi\,\omega[H]\,\chi \,-\, \chi\,\omega[H]\,\bar\chi\,\bar R\,\bar\chi\,\omega[H]\,\chi\;,$$

one finds

$$\begin{aligned}\mathcal{D}[F_{\chi,\tau}[H]] \;&=\; \mathcal{D}[\tau[H]]\\ &\quad +\chi\,\omega[H]\,\bar\chi\,\bar R\,\mathcal{D}[\tau[H]]\,\bar R\,\bar\chi\,\omega[H]\,\chi\\ &\quad +\chi\,\mathcal{D}[\omega[H]]\,\chi\\ &\quad -\,\chi\,\mathcal{D}[\omega[H]]\,\bar\chi\,\bar R\,\bar\chi\,\omega[H]\,\chi \,-\, \chi\,\omega[H]\,\bar\chi\,\bar R\,\bar\chi\,\mathcal{D}[\omega[H]]\\ &\quad +\,\chi\,\omega[H]\,\bar\chi\,\bar R\,\bar\chi\,\mathcal{D}[\omega[H]]\,\bar\chi\,\bar R\,\bar\chi\,\omega[H]\,\chi\\ &\quad +\,\mathcal{D}[\chi]\,H\,Q \,+\, Q^\sharp\,H\,\mathcal{D}[\chi]\\ &\quad -\,\chi\,\omega[H]\,\left(\mathcal{D}[\bar\chi]\,\bar R\,\bar\chi \,+\, \bar\chi\,\bar R\,\mathcal{D}[\bar\chi]\right)\,\omega[H]\,\chi\\ &\quad -\,\chi\,\omega[H]\,\bar\chi\,\bar R\,\left(\mathcal{D}[\bar\chi]\,\omega[H]\,\bar\chi \,+\, \bar\chi\,\omega[H]\,\mathcal{D}[\bar\chi]\right)\,\bar R\,\bar\chi\,\omega[H]\,\chi\;.\end{aligned}\tag{4.2.5}$$

The terms involving $\mathcal{D}[\tau[H]]$ and $\mathcal{D}[\chi]$ are already in the form asserted in the lemma. The four terms involving $\mathcal{D}[\omega[H]]$ are easily seen to combine into

$$Q^\sharp\,\mathcal{D}[\omega[H]]\,Q\;.$$

The terms involving $\mathcal{D}[\bar\chi]$ can be treated as follows. The last line of (4.2.5) is the sum of two terms, one of which is given by

$$\begin{aligned}\chi\,\omega[H]\,\bar\chi\,\bar R\,\frac{\mathcal{D}[\bar\chi]}{\bar\chi}\,\bar\chi\,\omega[H]\,\bar\chi\,\bar R\,\bar\chi\,\omega[H]\,\chi \;&=\; \chi\,\omega[H]\,\bar\chi\,\bar R\,\mathcal{D}[\bar\chi]\,\omega[H]\,\chi\\ &\quad -\;\chi\,\omega[H]\,\bar\chi\,\bar R\,\frac{\mathcal{D}[\bar\chi]}{\bar\chi}\,\tau[H]\,\bar R\,\bar\chi\,\omega[H]\,\chi\;.\end{aligned}$$

The first term on the right hand side of the equality sign cancels one of the two summands on the second last line of (4.2.5). The same argument applies to the terms on the two last lines of (4.2.5) that have not been considered yet. Therefore, the terms in (4.2.5) that involve $\mathcal{D}[\bar\chi]$ combine to

$$-\,2\,\chi\,\omega[H]\,\bar\chi\,\bar R\,\frac{\mathcal{D}[\bar\chi]}{\bar\chi}\,\tau[H]\,\bar R\,\bar\chi\,\omega[H]\,\chi\;.$$

Applying the second resolvent identity (4.2.3) to the operators $\bar R$, one readily arrives at the claim of the lemma. ■

THEOREM 4.2. *Let $\mathcal{D}$, $\mathcal{D}'$ denote derivations as in Lemma 4.2. Then, if*

$$[\mathcal{D}^{(\prime)}[\chi]\,,\,\bar{\chi}] \;=\; 0$$

and

$$\mathcal{D}^{(\prime)}[\tau[H]] \;=\; 0\;,$$

then

$$\mathcal{D}[F_{\chi,\tau}[H]] \;=\; Q^\sharp\,\mathcal{D}[\omega[H]]\,Q \tag{4.2.6}$$

on $\mathcal{H}$. Furthermore,

$$\begin{aligned}\mathcal{D}'[\,\mathcal{D}[F_{\chi,\tau}[H-z]]] \;&=\; Q^\sharp\;\mathcal{D}'[\mathcal{D}[H-z]]\,Q\\ &-\; Q^\sharp\,\mathcal{D}'[\omega[H]]\,\bar{\chi}\,\bar{R}\,\bar{\chi}\,\mathcal{D}[\omega[H]]\,Q\\ &-\; Q^\sharp\,\mathcal{D}[\omega[H]]\,\bar{\chi}\,\bar{R}\,\bar{\chi}\,\mathcal{D}'[\omega[H]]\,Q\end{aligned} \tag{4.2.7}$$

on $\mathcal{H}$. In particular, if $\mathcal{D} = \partial_X$ for X denoting some (possibly operator valued) variable, satisfying the above conditions,

$$\partial_X[F_{\chi,\tau}[H]] \;=\; Q^\sharp\,\partial_X[H]\,Q \tag{4.2.8}$$

on $\mathcal{H}$, and if H is analytic in X on $\mathcal{H}$, then $F_{\chi,\tau}[H]$ is also analytic in X on $P\mathcal{H}$.

Proof. Formula (4.2.6) follows directly from Lemma 4.2.
Next, by use of Lemma 4.2, and (4.2.6), we obtain

$$\begin{aligned}\mathcal{D}'[\,\mathcal{D}[F_{\chi,\tau}[H-z]]] \;&=\; \mathcal{D}'[\,Q^\sharp\,\mathcal{D}[H]\,Q\,]\\ &=\; Q^\sharp\;\mathcal{D}'[\mathcal{D}[H]]\,Q\\ &\quad+\mathcal{D}'[Q^\sharp]\,H\,Q\;+\;Q^\sharp\,H\,\mathcal{D}'[Q]\;.\end{aligned}$$

Inserting

$$\begin{aligned}\mathcal{D}[Q] \;&=\; \bar{\chi}\,\bar{R}\,\bar{\chi}\,\mathcal{D}[\omega[H]]\,\bar{R}\,\bar{\chi}\,\omega[H]\,\chi\;-\;\bar{\chi}\,\bar{R}\,\bar{\chi}\,\mathcal{D}[\omega[H]]\,\chi\\ &=\; -\,\bar{\chi}\,\bar{R}\,\bar{\chi}\,\mathcal{D}[H]\,Q\;,\end{aligned}$$

using that $\mathcal{D}[\omega[H]] = \mathcal{D}[H]$ and $\mathcal{D}[\bar{\chi}] = \mathcal{D}[\chi] = 0$ for the special case under investigation, and likewise for $\mathcal{D}'$ and $Q^\sharp$, we arrive at

$$\begin{aligned}\mathcal{D}'[\,\mathcal{D}[F_{\chi,\tau}[H]]] \;&=\; \mathcal{D}'[\,Q^\sharp\,\mathcal{D}[H]\,Q\,]\\ &=\; Q^\sharp\;\mathcal{D}'[\mathcal{D}[H]]\,Q\\ &\quad-Q^\sharp\,\mathcal{D}'[H]\,\bar{\chi}\,\bar{R}\,\bar{\chi}\,\mathcal{D}[H]\,Q\\ &\quad-Q^\sharp\,\mathcal{D}[H]\,\bar{\chi}\,\bar{R}\,\bar{\chi}\,\mathcal{D}'[H]\,Q\;.\end{aligned}$$

This implies (4.2.7).

The assertion about analyticity follows from the following argument. Let H be analytic in $X = X_R + iX_I$, where the subscripts 'R' and 'I' denote the real- and imaginary parts, respectively. That is, with $\bar{X} \equiv X_R - iX_I$, H is differentiable with respect to X, and

$$\partial_{\bar{X}} H \;=\; 0\;.$$

Thus, using (4.2.6), $F_{\chi,\tau}[H]$ is differentiable in X, since

$$\partial_X F_{\chi,\tau}[H] \;=\; Q^\sharp\,(\partial_X H)\,Q\;, \tag{4.2.9}$$

where by assumption, $\partial_X H$ exists and is well-defined, and

$$\begin{aligned} \partial_{\bar{X}} F_{\chi,\tau}[H] &= Q^\sharp \, (\partial_{\bar{X}} H) \, Q \\ &= 0 \end{aligned} \tag{4.2.10}$$

implies that $F_{\chi,\tau}[H]$ is analytic in X. ■

The following theorem is central for the operator-theoretic Ward-Takahashi identities considered later.

THEOREM 4.3. *Let $\mathcal{D}_i$, $i = 1, \ldots, m$, denote derivations obeying*

$$\mathcal{D}_i[\chi] \ , \ \mathcal{D}_i[\bar{\chi}] \ , \ \mathcal{D}_i[\tau[H]] \ = \ 0$$

on $\mathfrak{Dom}[\mathcal{D}_i[P]] \supseteq \mathcal{H}$, for all i. Furthermore, let h_j, $j = 0, \ldots, m$, denote linear mappings $\mathcal{L}[\mathcal{H}] \to \mathcal{L}[\mathcal{H}]$ which leave $\mathcal{L}[P\mathcal{H}]$ invariant, such that

$$h_j \, |_{P\mathcal{H}} \, [Q^\sharp \, A \, Q] \ = \ Q^\sharp \, h_j[A] \, Q$$

holds for all j and $A \in \mathcal{L}[\mathcal{H}]$. Then, defining $\mathcal{D}_0 \equiv \mathbf{1}$, the relationship

$$\sum_i h_i[\, \mathcal{D}_i[H] \,] \ = \ 0 \tag{4.2.11}$$

implies that likewise,

$$\sum_i h_i[\, \mathcal{D}_i[F_{\chi,\tau}[H]] \,] \ = \ 0 \tag{4.2.12}$$

is satisfied.

Proof. Defining $\mathcal{D}_0 := \mathbf{1}$, we have

$$\begin{aligned} \sum_{i \geq 0} h_i[\, \mathcal{D}_i[F_{\chi,\tau}[H]] \,] &= \sum_i h_i[\, \mathcal{D}_i[F_{\chi,\tau}[H]] \,] \\ &= \sum_{i \geq 0} h_i[\, Q^\sharp \, \mathcal{D}_i[\omega[H]] \, Q \,] \\ &= Q^\sharp \left(\sum_{i \geq 0} h_i[\, \mathcal{D}_i[\omega[H]] \,] \right) Q \\ &= 0 \ , \end{aligned} \tag{4.2.13}$$

as claimed, since by assumption, $\mathcal{D}_i[H] = \mathcal{D}_i[\omega[H]]$. ■

4.3. Concatenation rules

Let us in this section formulate the concatenation rules for the consecutive action of the elements of a Feshbach triple.

For two mutually commuting tuples of cutoff operators $(\chi_i, \bar{\chi}_i)$, $i = 1, 2$, with

$$\chi_1 \, \chi_2 \;=\; \chi_2 ,$$

the relationship

$$\begin{aligned} F_{\chi_2,\tau_2}[F_{\chi_1,\tau_1}[H]]^{-1} \;&=\; F_{\chi_2,\tau_2}[H]^{-1} \\ &\quad + \bar{\chi}_2 \, \big(\tau_2(F_{\chi_1,\tau_1}[H])^{-1} \;-\; \tau_2(H)^{-1}\big) \, \bar{\chi}_2 \end{aligned} \tag{4.3.1}$$

follows immediately from the inversion formula (4.1.6) for smooth Feshbach maps. Thus, the following theorem is clear.

THEOREM 4.4. *The concatenation rule*

$$F_{\chi_2,\tau_2}[F_{\chi_1,\tau_1}[H]] \;=\; F_{\chi_2,\tau_2}[H]$$

holds if

$$\tau_2[F_{\chi_1,\tau_1}[H]] \;=\; \tau_2[H]$$

is satisfied.

We will use this theorem in the following way. In the course of the operator-theoretic renormalization group, the situation will be that we successively determine $F_{\chi_1,\tau_1}[H]$ and $F_{\chi_2,\tau_2}[F_{\chi_1,\tau_1}[H]]$, where $\tau_1[H]$ and $\tau_2[F_{\chi_1,\tau_1}[H]]$ are defined in a prescribed manner. Thus, **defining** $\tau_2[H] \equiv \tau_2[F_{\chi_1,\tau_1}[H]]$, we obtain $F_{\chi_2,\tau_2}[H]$, which, according to the above theorem, corresponds to the concatenation of the two 'intermediate' smooth Feshbach maps. The intention will then be to prove certain algebraic identities satisfied by $F_{\chi_2,\tau_2}[H]$ that are important for the renormalization of the strongly marginal operators. The quintessence is that $F_{\chi_2,\tau_2}[H]$ is **independent** of the parameters in $F_{\chi_1,\tau_1}[H]$. Hence by exploiting the concatenation property, we control the cancellation of the latter.

For the operators $Q^\sharp, Q$, the following composition rules holds.

THEOREM 4.5. *The composition laws*

$$\begin{aligned} Q_{\chi_1,\tau_1}[H] \, Q_{\chi_2,\tau_2}[F_{\chi_1,\tau_1}[H]] \;&=\; Q_{\chi_2,\tau_2}[H] \\ Q^\sharp_{\chi_2,\tau_2}[F_{\chi_1,\tau_1}[H]] \, Q^\sharp_{\chi_1,\tau_1}[H] \;&=\; Q^\sharp_{\chi_2,\tau_2}[H] \end{aligned} \tag{4.3.2}$$

hold if $\tau_2[F_{\chi_1,\tau_1}[H]] = \tau_2[H]$. *In particular,*

$$\begin{aligned} A \, Q_{\chi_2,\tau_2}[H] \;&=\; A \, Q_{\chi_2,\tau_2}[F_{\chi_1,\tau_1}[H]] \\ Q^\sharp_{\chi_2,\tau_2}[H] \, A \;&=\; Q^\sharp_{\chi_2,\tau_2}[F_{\chi_1,\tau_1}[H]] \, A \end{aligned} \tag{4.3.3}$$

for any operator A that satisfies $A \, \bar{\chi}_1 = 0$.

Proof. Assume that H is invertible. From the general formula

$$\begin{aligned} H \, Q_{\chi,\tau}[H] \;&=\; \chi \, F_{\chi,\tau}[H] \\ Q^\sharp_{\chi,\tau}[H] \, H \;&=\; F_{\chi,\tau}[H] \, \chi \end{aligned} \tag{4.3.4}$$

follows that

$$\begin{aligned} H\,Q_{\chi_2,\tau_2}[H] &= \chi_2\,F_{\chi_2,\tau_2}[H] \\ &= \chi_1\,\chi_2\,F_{\chi_2,\tau_2}[F_{\chi_1,\tau_1}[H]] \\ &= \chi_1\,F_{\chi_1,\tau_1}[H]\,Q_{\chi_2,\tau_2}[F_{\chi_1,\tau_1}[H]] \\ &= H\,Q_{\chi_1,\tau_1}[H]\,Q_{\chi_2,\tau_2}[F_{\chi_1,\tau_1}[H]]\ , \end{aligned}$$

using $\chi_1\chi_2 = \chi_2$, and likewise for $Q^\sharp$. This implies the assertion. The fact that (4.3.3) holds if $A\,\bar{\chi}_1 = 0$ is immediate. ■

4.4. The Wick ordered normal form

In the given problem, we will exclusively study operators that can be represented as a series of Wick ordered monomials. In the case of the physical hamiltonian,

$$H_p(\sigma_0)\ -\ z\ =\ T[\mathcal{O}_0]\ -\ z\ +\ \sum_{1\leq M+N\leq 2} W_{M,N}[\mathcal{O}] \tag{4.4.1}$$

on the Hilbert space $\mathcal{H}_p$. The significance of the operators on the right hand side has already been explained in Chapter 2. An essential fact is that there is an estimate

$$(1-|p|)\,H_f\ +\ \frac{1}{2}|\mathbf{P}_f|^2\ \leq\ T[\mathcal{O}]\ \leq\ (1+|p|)H_f\ +\ \frac{1}{2}|\mathbf{P}_f|^2\ .$$

As will be shown later, the interaction is relatively bounded with respect to $T[\mathcal{O}]$.

The situation that we will typically encounter in the renormalization group analysis, is that we are given an operator of the form

$$H[z]\ =\ T[z;\mathcal{O}_0]\ -\ z\ +\ \sum_{M+N\geq 1} \chi_1\,W_{M,N}[z;\mathcal{O}]\,\chi_1 \tag{4.4.2}$$

on the Hilbert space $P_1\mathcal{H}_p = \chi[H_f < 1]\mathcal{H}_p$, which we refer to as an **effective hamiltonian**. (We recall the notations $\mathcal{O}_0 = (H_f, \mathbf{P}_f^{\|}, |\mathbf{P}_f^{\perp}|)$, and $\mathcal{O} = (H_f, \mathbf{P}_f^{\|}, \mathbf{P}_f^{\perp})$). χ_1 is, in the sense of the spectral theorem, given by a smooth operator-valued function $\chi_1[H_f]$, with $\mathrm{supp}\{\chi_1\} = [0,1]$,

$$\chi_1^2\ +\ \bar{\chi}_1^2\ =\ \mathbf{1}\ .$$

In addition, $\chi_1[x] = 1$ if $x \in [0, \frac{2}{3}]$, and $\chi_1[x]$ is strictly monotonic if $x \in (\frac{2}{3}, 1]$. The terms appearing in (4.4.2) are defined as follows.

(I) The parameter z is a complex number in the vicinity of the origin in $\mathbb{C}$ (assuming that $E_{bare}[|p|]$ is set to zero), and appropriately picked so that in all of our intended applications, the Feshbach map is well-defined.

(II) The operator $T[z;\mathcal{O}_0]$ denotes the noninteracting part of H, and commutes with H_f. It can be written as

$$T[z;\mathcal{O}_0]\ =\ H_f\ +\ a[z]\ \chi_1\,\mathbf{P}_f^{\|}\,\chi_1\ +\ \chi_1\,T_{n-l}[z;\mathcal{O}_0]\,\chi_1\ , \tag{4.4.3}$$

for $a[z] \in \mathbb{C}$. The operator $T_{n-l}[\mathcal{O}_0; z]$ is $O(H_f^2)$ in the limit $H_f \to 0$, of class C^2 in the variable H_f, and analytic in $\mathbf{P}_f$. In the cases to be considered, the values of the coefficient $a[z]$ are such that relative bounds

$$\gamma H_f \;\leq\; T[z; \mathcal{O}_0] \;\leq\; \Gamma H_f \tag{4.4.4}$$

hold on $P_1 \mathcal{H}_p$ for some $0 < \gamma < 1$, and $\Gamma > 1$, if z is appropriately picked in $\mathbb{R}$.

(III) The operator $W_{M,N}[z; \mathcal{O}]$ denotes the interaction term **of degree** $M + N \geq 1$ in $H[z]$, as we will say. It is a Wick monomial of the form

$$\begin{aligned} W_{M,N}[z; \mathcal{O}] \;\; = \;\; & \sum_{\lambda^{(M)}, \tilde{\lambda}^{(N)}} \int \prod_{i=1}^{M} \prod_{j=1}^{N} \frac{d^3 k_i \; \bar{\kappa}_\sigma(|k_i|)}{\sqrt{|k_i|}} \frac{d^3 \tilde{k}_j \; \bar{\kappa}_\sigma(|\tilde{k}_j|)}{\sqrt{|\tilde{k}_j|}} \;\times \\ & \times \, a^*_{\lambda^{(M)}}(k^{(M)}) \, w_{M,N} \left[z; \mathcal{O}; K^{(M,N)} \right] \, a^\sharp_{\tilde{\lambda}^{(N)}}(\tilde{k}^{(N)}) \;, \end{aligned} \tag{4.4.5}$$

containing M creation, and N annihilation operators, where

$$\sigma \;\geq\; \sigma_0$$

is referred to as the **running infrared cutoff**.

We recall the following multiindex notation:

$$\begin{aligned} \lambda^{(M)} \;\; &:= \;\; (\lambda_1, \ldots, \lambda_M) \\ k^{(M)} \;\; &:= \;\; (k_1, \ldots, k_M), \end{aligned}$$

as we recall, and likewise for $\tilde{\lambda}^{(N)}$ and $\tilde{k}^{(N)}$. Moreover,

$$K^{(M,N)} \;:=\; \left(k^{(M)}, \lambda^{(M)}; \tilde{k}^{(N)}, \tilde{\lambda}^{(N)} \right),$$

and

$$a^\sharp_{\lambda^{(M)}}(k^{(M)}) := \prod_{i=1}^{M} a^\sharp_{\lambda_i}(k_i).$$

In cases where we consider the dependence of the interaction operators with respect to the interaction kernels, the notation

$$W_{M,N} \left[\left[\; w_{M,N}[z; \mathcal{O}; \cdot] \; \right] \right] \tag{4.4.6}$$

will be used.

The **integral kernels** ('**interaction kernels**')

$$w_{M,N} \left[z; \mathcal{O}; K^{(M,N)} \right]$$

are operator valued C^2-functions of $\mathcal{O}$, and commute with H_f. They are fully symmetric functions with respect to the photon momenta $k_1, \ldots, k_M$ on the one hand, and $\tilde{k}_1, \ldots, \tilde{k}_N$, on the other hand. Furthermore, they are of class C^1 with respect to $|k_i|$, $|\tilde{k}_j|$, z, $|p|$, and **analytic** in $\mathbf{P}_f$.

4.4.1. Derivatives with respect to scalars and $\mathcal{O}$. Derivatives with respect to complex scalars of the Wick ordered operators $H = T + \chi_1 \sum W_{M,N} \chi_1$ are defined in the obvious manner.

Derivatives with respect to components $\mathcal{O}_0$ and $\mathcal{O}$ are given as follows. The appropriate definition of $\partial_{\mathcal{O}^r} W_{M,N}[z;\mathcal{O}]$, compatible with the operation of Wick ordering, is given by

$$\partial_{\mathcal{O}^r} W_{M,N}\left[[w_{M,N}[z;\mathcal{O};\cdot]\,]\right] \;=\; W_{M,N}\left[[\partial_{\mathcal{O}^r} w_{M,N}[z;\mathcal{O};\cdot]\,]\right],$$

that is, the derivative acts directly on the integral kernel of $W_{M,N}$. One then straightforwardly verifies that

$$\begin{aligned} &\partial_{\mathcal{O}^r} \,:\, (\; f_0[z;\mathcal{O}] W_{M_1,N_1}[z;\mathcal{O}] \cdots W_{M_L,N_L}[z;\mathcal{O}] f_L[z;\mathcal{O}]\;) \,: \\ &=: \partial_{\mathcal{O}^r} (\; f_0[z;\mathcal{O}] W_{M_1,N_1}[z;\mathcal{O}] \cdots W_{M_L,N_L}[z;\mathcal{O}] f_L[z;\mathcal{O}]\;) \,: \end{aligned} \tag{4.4.7}$$

holds for all C^1-functions $f_i[z;\mathcal{O}]$, which commute with H_f. The expression on the second line is evaluated by using the standard product rule, under preservation of the order of operators in the product.

4.5. (Smooth) Feshbach decimation

It is an important fact for the Bach-Fröhlich-Sigal operator-theoretic renormalization group that for a given effective hamiltonian $H[z]$ in the Wick ordered normal form (4.4.2), which acts on the Hilbert space $\mathcal{H}$, its smooth Feshbach image $F_{\chi,\tau}[H[z]]$, which acts on $P\mathcal{H} \subset \mathcal{H}$, can, for a unique choice of $\tau[H[z]]$, again be written in Wick ordered normal form, but under a modification of T and $w_{M,N}$. The process in which this Wick ordered expression for $F_{\chi,\tau}[H[z]]$ is generated is called **Feshbach decimation**.

4.5.1. Definition of the smooth Feshbach projection. The first step in the definition of the smooth Feshbach map consists of determining an appropriate operator $\tau[H]$. To this end, we notice that the operator $H_f - z$ differs from the other operators occurring in $H[z]$ by the fact that it is not cut off by operators χ_1 from the left and right. To preserve this property, we assume that

$$\tau[H[z]] \;=\; (1 + \Delta_{H_f}[z])\, H_f \;-\; z \;+\; \Delta E_0[z]$$

for unknown scalars $\Delta_{H_f}[z], \Delta E_0[z]$, which remain to be determined.

Then, with

$$\bar{\chi} \;\equiv\; \bar{\chi}_\rho\, \bar{\chi}_1 \;,$$

we have

$$\begin{aligned} F_{\chi_\rho,\tau}[H[z]] \;&=\; \tau[H[z]] \;+\; \chi_\rho\, \omega[H[z]]\, \chi_\rho \\ &\quad -\, \chi_\rho\, \omega[H[z]]\, \bar{\chi}_\rho\, \bar{R}\, \bar{\chi}_\rho\, \omega[H[z]]\, \chi_\rho \;, \end{aligned} \tag{4.5.1}$$

where

$$\begin{aligned} \omega[H[z]] \;&=\; -\,\Delta_{H_f}[z]\, H_f \;-\; \Delta E_0[z] \\ &\quad +\, \chi_1 \left[a[z] \mathbf{P}_f^{\|} \;+\; T_{n-l}[z;\mathcal{O}_0] \;+\; W[z;\mathcal{O}] \right] \chi_1 \;, \end{aligned}$$

and

$$\bar{R} \;=\; (\tau[H[z]] \;+\; \bar{\chi}_\rho\, \omega[H[z]]\, \bar{\chi}_\rho)^{-1}$$

on $\bar{P}\mathcal{H}$.

4.5.1.1. *The correction of z.* The number $\Delta E_0[z]$ is determined by the condition that

$$\left\langle F_{\chi_\rho,\tau}[H[z]]\right\rangle_{\Omega_f} \;=\; \Delta E_0[z] \,-\, z \;.$$

This implies that

$$\chi_\rho \,\left[-\Delta E_0[z] \,-\, \omega[H[z]]\,\bar{\chi}_\rho\,\bar{R}\,\bar{\chi}_\rho\,\omega[H[z]]\,\right]\,\chi_\rho$$

has a vanishing Ω_f-expectation value, that is,

$$\Delta E_0[z] \;=\; -\,\left\langle W\,\bar{\chi}_\rho\,\bar{R}\,\bar{\chi}_\rho\,W\right\rangle_{\Omega_f}\;, \tag{4.5.2}$$

where the right hand side depends on ΔE_0 via $\bar{R}$.

4.5.1.2. *The coefficient of H_f.* The quantity $\Delta_{H_f}[z]$ is obtained from

$$\left\langle \partial_{H_f} F_{\chi_\rho,\tau}[H[z]]\right\rangle_{\Omega_f} \;=\; 1+\Delta_{H_f}[z]\;.$$

This implies that the Wick ordered expression for

$$\chi_\rho \,\left[-\Delta_{H_f}[z]\,H_f \,-\, \omega[H[z]]\,\bar{\chi}_\rho\,\bar{R}\,\bar{\chi}_\rho\,\omega[H[z]]\,\right]\,\chi_\rho$$

contains no term proportional to $\chi_\rho H_f \chi_\rho$, so that the term linear in H_f is given by $(1+\Delta_{H_f}[z])H_f$. Let us next derive the implicit relation that determines $\Delta_{H_f}[z]$.

Lemma 4.3. *Let*

$$z' \;\equiv\; \langle \partial_{H_f} F_{\chi_\rho,\tau}[H[z]](z-\Delta E_0)\rangle_{\Omega_f}\;.$$

Then,

$$\begin{aligned}\left\langle \partial_{H_f} F_{\chi_\rho,\tau}[H[z]]\right\rangle_{\Omega_f} \;&=\; \left\langle Q^\sharp(1+\partial_{H_f}(T_{n-l}+W))\,Q\right\rangle_{\Omega_f}\;\times \\ \times\left(1-\left\langle W\bar{\chi}\bar{R}\chi^2\bar{R}\bar{\chi}W\right\rangle_{\Omega_f}+\left\langle W\bar{\chi}\bar{R}\frac{\partial_{H_f}\bar{\chi}}{\bar{\chi}}(H_f-z')\bar{R}\bar{\chi}W\right\rangle_{\Omega_f}\right)^{-1}&\;. \end{aligned}\tag{4.5.3}$$

Proof. Application of Lemma 4.2 yields

$$\begin{aligned} 1+\Delta_{H_f}[z] \;&=\; \left\langle \partial_{H_f} F_{\chi_\rho,\tau}[H[z]]\right\rangle_{\Omega_f} \\ &=\; (1+\Delta_{H_f}[z])\left(1+\left\langle W\bar{\chi}\bar{R}^2\bar{\chi}W\right\rangle_{\Omega_f}\right) \\ &\quad -\,\Delta_{H_f}[z]\,\left(1+\left\langle W\bar{\chi}\bar{R}\bar{\chi}^2\bar{R}\bar{\chi}W\right\rangle_{\Omega_f}\right) \\ &\quad +\,\left\langle Q^\sharp(\partial_{H_f}(T_{n-l}+W))Q\right\rangle_{\Omega_f} \\ &\quad -\,\left\langle W\bar{\chi}\bar{R}\frac{\partial_{H_f}\bar{\chi}}{\bar{\chi}}\tau[H[z]]\bar{R}\bar{\chi}W\right\rangle_{\Omega_f}\;, \end{aligned}\tag{4.5.4}$$

which can be straightforwardly converted into

$$\begin{aligned}\left\langle \partial_{H_f} F_{\chi_\rho,\tau}[H[z]]\right\rangle_{\Omega_f} &= \frac{\left\langle Q^\sharp(1+\partial_{H_f}(T_{n-l}+W))Q\right\rangle_{\Omega_f}}{1-\left\langle W\bar\chi\bar R\chi^2\bar R\bar\chi W\right\rangle_{\Omega_f}} \\ &\quad - \frac{\left\langle W\bar\chi\bar R\frac{\partial_{H_f}\bar\chi}{\bar\chi}\tau[H[z]]\bar R\bar\chi W\right\rangle_{\Omega_f}}{1-\left\langle W\bar\chi\bar R\chi^2\bar R\bar\chi W\right\rangle_{\Omega_f}} \,. \end{aligned} \tag{4.5.5}$$

With

$$z' \;=\; \langle \partial_{H_f} F_{\chi_\rho,\tau}[H[z]]\rangle_{\Omega_f}(z-\Delta E_0)\;,$$

we thus have

$$\begin{aligned}\left\langle \partial_{H_f} F_{\chi_\rho,\tau}[H[z]]\right\rangle_{\Omega_f} &= \left\langle Q^\sharp(1+\partial_{H_f}(T_{n-l}+W))\,Q\right\rangle_{\Omega_f} \;\times \\ &\times\left(1-\left\langle W\bar\chi\bar R\chi^2\bar R\bar\chi W\right\rangle_{\Omega_f}+\left\langle W\bar\chi\bar R\frac{\partial_{H_f}\bar\chi}{\bar\chi}(H_f-z')\bar R\bar\chi W\right\rangle_{\Omega_f}\right)^{-1}\;,\end{aligned}$$

which proves the claim. ■

4.5.2. Neumann series expansion. Feshbach decimation is achieved in two steps:
1) Expansion of $\bar R$ in $F_{\chi,\tau}[H[z]]$ into a Neumann series.
2) Application of Wick ordering to the resulting expression.

In Chapter 7, we will apply the smooth Feshbach map to the physical hamiltonian $H_p(\sigma_0)$. In that case, the Hilbert space is given by $\mathcal{H}=\mathcal{H}_p$, and $F_{\chi_1,\tau}$ maps it to an operator acting on the Hilbert subspace $P_1\mathcal{H}_p$. χ_1 is a smooth function of H_f, with spectral support $[0,1]$, and in possession all properties required for the construction of $F_{\chi_1,\tau}[\mathcal{H}_p-z]$. In all subsequent applications of the Feshbach map, discussed in Chapter 8, $\mathcal{H}=P_1\mathcal{H}_p$, and $P=P_\rho$, with $\rho=\frac{1}{2}$. We will here discuss the decimation map for the latter cases. For $H_p(\sigma_0)$, the arguments are fully analogous, and will be demonstrated in detail in Chapter 7.

Let us assume an effective hamiltonian $H[z]$ of the form (4.4.2) to be given, which shall act on $P_1\mathcal{H}_p$.

Step 1) of Feshbach decimation is obtained as follows. Let $T'\equiv T-\tau[H[z]]$, and

$$\bar R_0[z] \;:=\; (\tau[H[z]] \,+\, \bar\chi_\rho\, T'\,\bar\chi_\rho)^{-1}$$

on $\bar P\mathcal{H}_p$. Expansion of $\bar R[z]$ in $F_{\chi_\rho,\tau}$ into a Neumann series yields

$$\bar R[z] \;=\; \bar R_0^{\frac{1}{2}}[z]\sum_{L\geq 0}\left(\bar R_0^{\frac{1}{2}}[z]\,\bar\chi\,W[z;\mathcal{O}]\,\bar\chi\,\bar R_0^{\frac{1}{2}}[z]\right)^L\;\bar R_0^{\frac{1}{2}}[z]$$

(for an appropriate branch of the square root if z has a nonzero imaginary part), which is absolutely convergent if the estimates

$$\begin{aligned}&\left\|\,\bar R_0^{\frac{1}{2}}[z]\;\bar\chi\,W[z;\mathcal{O}]\;\bar\chi\,\bar R_0^{\frac{1}{2}}[z]\,\right\|\;,\;\left\|\,\bar R_0^{\frac{1}{2}}[z]\;\bar\chi\,\omega[H[z]]\;\chi\,\right\|\;,\\ &\left\|\,\chi\,\omega[H[z]]\;\bar\chi\,\bar R_0^{\frac{1}{2}}[z]\,\right\| \;\ll\; 1\end{aligned} \tag{4.5.6}$$

are satisfied (the norms are evaluated on $\mathcal{H}$, not on $P\mathcal{H}$). Using (4.4.4), these estimates are, for appropriate values of z, indeed true. This is demonstrated in detail in Chapter 7 for the first application of Feshbach decimation, and in Chapter 8 for all other applications of it.

It is thus clear that defining

$$\bar{\chi} \;:=\; \bar{\chi}_\rho\,\chi_1 \;,$$

and using $\chi_\rho\,\chi_1 = \chi_\rho$, one can write

$$\begin{aligned} F_{\chi_\rho,\tau}[H[z]] \;&=\; \tau[H[z]] \,+\, \chi_\rho\,T'\Pi\chi_\rho \,+\, \chi_\rho\,\omega[H[z]]\,\chi_\rho \\ &-\; \sum_{L=2}^{\infty}(-1)^L \sum_{\substack{1\le M_i+N_i \\ i=1,\dots,L}} \chi_\rho\,\Pi\,W_{M_1,N_1}\,\bar{\chi}\,\bar{R}_0\,\bar{\chi}\,\cdots \\ &\qquad \cdots\,\bar{\chi}\,W_{M_{L-1},N_{L-1}}\,\bar{\chi}\,\bar{R}_0\,\bar{\chi}W_{M_L,N_L}\,\Pi\,\chi_\rho \;, \end{aligned} \tag{4.5.7}$$

where the operator Π is defined in (4.1.8). The next issue is to Wick order this expression.

4.5.3. Wick ordering. Applying Wick ordering to each summand in the Neumann series (4.5.7) for every fixed value of the summation index L, and collecting all Wick monomials that are equal in the number of creation and annihilation operators, we find

$$\begin{aligned} \mathcal{D}_\rho[H[z]] \;&=\; (1+\Delta_{H_f}[z])\,H_f \,-\, z \,+\, \Delta E_0[z] \\ &+\,(a[z]+\Delta a[z])\,\chi_\rho\,\mathbf{P}_f^{\|}\,\chi_\rho \\ &+\,\chi_\rho\,(T_{n-l}\Pi+\Delta T_{n-l})\,\chi_\rho \\ &+\,\sum_{M+N\ge 1}\chi_\rho\left(\Pi\,W_{M,N}\,\Pi \,+\, \Delta W_{M,N}\right)\chi_\rho \;. \end{aligned} \tag{4.5.8}$$

The operator $\mathcal{D}_\rho$ denotes the concatenation of the Feshbach map with Neumann series expansion and Wick ordering, and is referred to as the **Feshbach decimation map**. Pulling χ_ρ from the left and right through the creation and annihilation operators in $W_{M,N}+\Delta W_{M,N}$, one observes that

$$\begin{aligned} H_f \,+\, \sum_{i=1}^{M}\,|k_i| \;&\le\; \rho \;, \\ H_f \,+\, \sum_{j=1}^{N}\,|\tilde{k}_j| \;&\le\; \rho \end{aligned}$$

is satisfied by the momentum space arguments. Therefore, their support is contained in the simplicial subset

$$\left\{(k_1,\dots,k_M)\in\mathbb{R}^{3M}\;,\;(\tilde{k}_1,\dots,\tilde{k}_N)\in\mathbb{R}^{3N}\;\middle|\;\sum_{i=1}^{M}\,|k_i|\;,\;\sum_{j=1}^{N}\,|\tilde{k}_j|\le 1\right\} \tag{4.5.9}$$

of $\mathbb{R}^{3(M+N)}$. In particular, this shows that every photon momentum appearing in an interaction kernel of any effective hamiltonian always lies in a unit ball around the origin.

For any partitioning of the set $\{1,\dots,m+q\}$ into two disjoint sets consisting of m respectively p elements, and of the set $\{1,\dots,n+q\}$ into two disjoint sets consisting of n

respectively q elements, we define the operator valued function

$$W^{m,n}_{p,q}[z;\mathcal{O};K^{(m,n)}] \;:=\; \sum_{\mu^{(p)},\tilde{\mu}^{(q)}} \int \left(\prod_{i=1}^{p} d^3x_i\; \bar{\kappa}_\sigma(|x_i|)\; |x_i|^{-\frac{1}{2}}\right) \;\times$$
$$\times \quad \left(\prod_{j=1}^{q} d^3\tilde{x}_j\; \bar{\kappa}_\sigma(|\tilde{x}_j|)\; |\tilde{x}_j|^{-\frac{1}{2}}\right) \;\times$$
$$\times\; a^*_{\lambda^{(p)}}(x^{(p)})\; w_{m+p,n+q}[z;\mathcal{O};X^{(p,q)},K^{(m,n)}]\; a_{\tilde{\lambda}^{(q)}}(\tilde{x}^{(q)})\;. \tag{4.5.10}$$

The integral kernels are totally symmetric functions of the photon momenta. Hence, each of the $\begin{pmatrix} m+p \\ p \end{pmatrix}\begin{pmatrix} n+q \\ q \end{pmatrix}$ possible partitionings produce the same operator $W^{m,n}_{p,q}$.

The integral kernels $\Delta w_{M,N}$ of $\Delta W_{M,N}$ are given by

$$\Delta w_{M,N}[\cdots] \;=\; -\sum_{L=2}^{\infty}(-1)^L \sum_{\substack{2\geq M_i+N_i\geq 1\\ i=1,\ldots,L}} \delta_{M,\sum_{i=1}^L m_i}\delta_{N,\sum_{i=1}^L n_i} \;\times$$
$$\times \prod_{i=1}^{L}\begin{pmatrix} m_i+p_i \\ p_i \end{pmatrix}\begin{pmatrix} n_i+q_i \\ q_i \end{pmatrix}\; \mathcal{E}_L\left[\{\underline{m},\underline{n},\underline{p},\underline{q}\};z;\mathcal{O};K^{(M,N)}\right] \tag{4.5.11}$$

with $M_i := m_i + p_i$, and $N_i := n_i + q_i$, $\underline{m} := (m_1,\ldots,m_L)$, etc., and $M := |\underline{m}| := \sum m_i$, $N = |\underline{n}| = \sum n_i$. Furthermore,

$$\mathcal{E}_L\left[\{\underline{m},\underline{n},\underline{p},\underline{q}\};z;\mathcal{O};K^{(M,N)}\right] \;\equiv\; \left\langle\, \Pi W^{m_1,n_1}_{p_1,q_1}\bar{\chi}\bar{R}_0\bar{\chi}W^{m_1,n_1}_{p_1,q_1}\cdots\bar{\chi}W^{m_l,n_l}_{p_L,q_L}\Pi\,\right\rangle_{\mathcal{O},sym}\;, \tag{4.5.12}$$

is defined as the expression

$$\mathrm{Sym}_{\underline{m},\underline{n}}\left\langle\, \Pi\left[S_X^{(m_1,\ldots,m_L)}\mathcal{O};z\right] W^{m_1,n_1}_{p_1,q_1}\left[S_X^{(m_2,\ldots,m_L)}\mathcal{O};z;K^{(m_1,n_1)}\right]\times\right.$$
$$\times\,\bar{\chi}\,\bar{R}_0\,\bar{\chi}\,\left[S_X^{(m_2,\ldots,m_L;n_1)}\mathcal{O}_0;z\right]\;\cdots$$
$$\left.\cdots\,\bar{\chi}\,W^{m_L,n_L}_{p_L,q_L}\left[S_X^{(n_1,\ldots,n_{L-1})}\mathcal{O};z;K^{(m_L,n_L)}\right]\Pi\left[S_X^{(n_1,\ldots,n_L)}\mathcal{O};z\right]\,\right\rangle_{\Omega_f}\Bigg|_{X\to\hat{\mathcal{O}}}\;,$$

where $\mathrm{Sym}_{\underline{m},\underline{n}}$ fully symmetrizes its argument in all $M = |\underline{m}|$ momenta associated to creation, and all $N = \underline{n}$ momenta labeling annihilation operators, but not among one another.

The shift operator S_X is defined as follows. Let

$$X \;=\; (X^1,X^2,\mathbf{X}^3) \;\in\; \mathrm{Spec}\{(H_f,\mathbf{P}_f^{\|},\mathbf{P}_f^{\perp})\}$$
$$=\; [0,1]\times[-1,1]\times D_1(0)$$

denote the spectral variable associated to $(H_f, \mathbf{P}_f^{\|}, \mathbf{P}_f)$, where $D_1(0) \ni \mathbf{X}^3$ is the unit disc in $\mathbb{R}^2$. The shift operator $S_X^{(m_i,\ldots,m_L;n_1,\ldots,n_{i(-1)})}$ is defined by

$$\begin{aligned} S_X^{(m_i,\ldots,m_L;n_1,\ldots,n_{i(-1)})} \,:\, H_f \;&\longmapsto\; H_f \;+\; \sum_{j=i}^{L} |k_j| \;+\; \sum_{l=1}^{i(-1)} |\tilde{k}_l| \;+\; X^1 \,, \\ \mathbf{P}_f^{\|} \;&\longmapsto\; \mathbf{P}_f^{\|} \;+\; \sum_{j=i}^{L} k_j^{\|} \;+\; \sum_{l=1}^{i(-1)} \tilde{k}_l^{\|} \;+\; X^2 \,, \\ \mathbf{P}_f^{\perp} \;&\longmapsto\; \mathbf{P}_f^{\perp} \;+\; \sum_{j=i}^{L} k_j \;+\; \sum_{l=1}^{i(-1)} \tilde{k}_l \;+\; \mathbf{X}^3 \,. \end{aligned} \tag{4.5.13}$$

The operators ΔT and $\Delta w_{M,N}$ correspond to the respective radiative corrections of T and $w_{M,N}$ under Feshbach decimation. For a more detailed exposition of issues concerning Wick ordering, we refer the reader to the appendix of [**3**].

Definition 4.2. *If the expectation value $\mathcal{E}_L$ in (4.5.12) exhibits an index $i \in \{1,\ldots,L\}$, so that $p_i \neq 0$ or $q_i \neq 0$, it is called a* **loop contribution**. *Otherwise, if $p_i = q_i = 0$ for all i, it is called a* **tree level contribution**. *A momentum which labels a creation or annihilation operator within the expectation value is called an* **inner momentum** *(indices p_i, q_i). Otherwise it is called an* **external momentum** *(indices m_i, n_i).*

4.5.4. Estimates on $\| \mathcal{E}_L \|$. We remark that for any Borel function $f \in L^\infty(\mathrm{spec}\mathcal{O})$,

$$\| S_X^{(m_i,\ldots,m_L;n_1,\ldots,n_{i(-1)})} f[\mathcal{O}] \| \;\leq\; \|f\|_\infty \;=\; \|f[\mathcal{O}]\| \,, \tag{4.5.14}$$

since shifting $\mathcal{O}$ corresponds to a translation of the support of f in spec$\mathcal{O}$, which, of course, does not increase the supremum of $|f|$ in spec$\mathcal{O}$.

Therefore, $\| \mathcal{E}_L \|$ can be estimated by

$$\| \mathcal{E}_L \| \;\leq\; \|\Pi\|^2 \, \|\bar{R}_0\|^{L-1} \, \prod_{i=1}^{L} \| W_{p_i,q_i}^{m_i,n_i} \| \,, \tag{4.5.15}$$

whenever the norms on the right hand side are finite.

CHAPTER 5

THE OPERATOR-THEORETIC RENORMALIZATION GROUP

In this chapter, we introduce the detailed framework of the operator-theoretic renormalization group developed in [**2, 3**].

The setting of any renormalization group construction generally comprises a space $\mathfrak{H}$, whose points, denoted by $\mathfrak{h}$, parametrize the physical system at hand (coupling constants, for instance, or operators in the present case). There is always a natural scale that characterizes the given problem (an energy or momentum cutoff, for instance). Considering the dependence of $\mathfrak{h} \in \mathfrak{H}$ with respect to this scale, and varying it in discrete steps, a dynamical system is obtained on $\mathfrak{H}$, whose flow map is referred to as the renormalization map $\mathcal{R}$. The fixed points of this flow correspond to macroscopic universality classes of microscopic theories.

In the present case, $\mathfrak{H}$ is a Banach space of function sequences, whose points define certain bounded operators on $P_1\mathcal{H}_p$. The renormalization map $\mathcal{R}$ is obtained from concatenating three distinct operations, the decimation map $\mathcal{D}_\rho$ introduced in the previous chapter, a rescaling transformation $\mathcal{S}_\rho$, and a redefinition Z of the spectral parameter z. $\mathcal{D}_\rho$ removes all photon states with energies in the interval $[\rho, 1]$ by use of the Feshbach map, for some fixed real number $0 < \rho < 1$.

5.1. The Banach sequence space of Wick monomials

Let us now give a detailed definition of the parameter space $\mathfrak{H}$ of the present system.

All effective hamiltonians that we consider can be written as a series of Wick monomials

$$H[z] \;=\; T[z;\mathcal{O}_0] \,-\, z \,+\, \sum_{M+N\geq 1} \chi_1\, W_{M,N}[z;\mathcal{O}]\,\chi_1 \;,$$

where

$$T[z;\mathcal{O}_0] \;=\; H_f \,+\, \chi_1 \left(\underbrace{a[z]\,\mathbf{P}_f^{\|} \,+\, T_{n-l}[z;\mathcal{O}_0]}_{\equiv T'[z;\mathcal{O}_0]} \right) \chi_1 \;,$$

which is precisely of the form introduced in and after (4.4.2), and acts on the Hilbert space $P_1\mathcal{H}_p$.

z is the spectral parameter that emerges in the definition of the Feshbach map, and is picked in a $O(g)$-vicinity of $\{0\} \in \mathbb{C}$.

As before, $T[z;\mathcal{O}_0]$ denotes the noninteracting part of $H[z]$, and commutes with H_f, and $W_{M,N}[z;\mathcal{O}]$ accounts for the interaction operator of degree $M+N \geq 1$, which contains M creation, and N annihilation operators. We refer to Section 4.4 for their precise definition.

All effective hamiltonians $H[z]$ that emerge in the renormalization group construction are elements of $\mathcal{B}(P_1\mathcal{H}_p)$, the bounded operators on $P_1\mathcal{H}_p$, and exhibit this structure.

Therefore, we introduce a Banach sequence space

$$\mathfrak{H} \;=\; \mathbb{C} \,\oplus\, \mathfrak{T} \,\oplus\, \mathfrak{W}$$

with

$$\mathfrak{W} \;=\; \bigoplus_{M+N\geq 1} \mathfrak{W}_{M,N} \;,$$

whose points shall parametrize operators of the form $H[z]$.

The building blocks of the space $\mathfrak{H}$ are thus given as follows.

(i) $\mathbb{C}$ is the space of spectral parameters z. z is picked in a $O(g)$-vicinity of the origin in $\mathbb{C}$.

(ii) By definition of $\mathcal{O}_0 = (H_f, \mathbf{P}_f^{\|}, |\mathbf{P}_f^{\perp}|)$,

$$\mathrm{Spec}\{\mathcal{O}_0\} \;=\; [0,1] \times [-1,1] \times [0,1] \;.$$

The space

$$\mathfrak{T} \;=\; C^2(\mathrm{Spec}\{\mathcal{O}_0\}) \times C^1(\mathbb{C})$$

parametrizes the noninteracting hamiltonians.

Let $\alpha = (\alpha_i)_{i=1}^3$ denote a 3-component multiindex, with $\alpha_i \in \{0,1,2\}$, and

$$|\alpha| \;:=\; \sum \alpha_i \;.$$

For $Y = (Y^i)_{i=1}^3$, where Y^i can be either a scalar or a vector, let

$$\partial_Y^\alpha \;:=\; \prod_i \partial_{Y^i}^{\alpha_i} \;.$$

We introduce the norm

$$\| \; T \; \| \;:=\; \sup_{|\alpha|=0,1,2} \; \sup_{Y\in \mathrm{Spec}\{\mathcal{O}_0\}} \; | \; \partial_Y^\alpha T[z;Y] \; |$$

on $\mathfrak{T}$, where $T \in \mathfrak{T}$. Evidently, $(\mathfrak{T}, \| \cdot \| \,)$ is a Banach space.

(iii) Clearly, for $\mathcal{O} = (H_f, \mathbf{P}_f^{\|}, \mathbf{P}_f^{\perp})$,

$$\mathrm{Spec}\{\mathcal{O}\} \;=\; [0,1] \times [-1,1] \times D_1(0)$$

on $P_1 \mathcal{H}_p$, where $D_1(0) \ni Y^3$ is the unit disc in $\mathbb{R}^2$.

The space

$$\mathfrak{W}_{M,N} \;:=\; C^2(\mathrm{Spec}\mathcal{O}) \times C^1(\mathbb{C}) \times C^1_{sym}(\mathbb{R}^{3M}) \times C^1_{sym}(\mathbb{R}^{3N}) \tag{5.1.1}$$

parametrizes integral kernels of degree $M+N \geq 1$. $C^1_{sym}(\mathbb{R}^{3M})$ is the space of continuously differentiable functions $f : \underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{M} \to \mathbb{C}$ which are invariant under permutations of the M factors.

Let α again denote a 3-component multiindex. We introduce the norms

$$\| \, w_{M,N} \| \;:=\; \sup_{Y\in \mathrm{Spec}\{\mathcal{O}_0\}} \; \sup_{K^{(M,N)}} \; \Big| \; w_{M,N}[z;Y;K^{(M,N)}] \; \Big|$$

and

$$\| w_{M,N} \|^{(\zeta)} \;:=\; \sum_{|\alpha| \le \zeta} \| \; \partial_Y^\alpha w_{M,N}[z; Y; K^{(M,N)}] \; \|$$

for $w_{M,N} \in \mathfrak{W}_{M,N}$. For every fixed value of $M+N$ and ζ, the pair $(\mathfrak{W}_{M,N}, \; \| \cdot \|^{(\zeta)})$ is a Banach space. In particular, $\| \cdot \| = \| \cdot \|^{(0)}$.

Thus, a point in $\mathfrak{H}$ is a sequence

$$\mathfrak{h} \;=\; (z, T, \{w_{M,N}\}) \;\in\; \mathfrak{H} \; .$$

Next, we introduce a map $H : \mathfrak{H} \longrightarrow \mathcal{L}(P_1 \mathcal{H}_p)$, the linear operators on $P_1 \mathcal{H}_p$. Its action on $\mathfrak{h} \in \mathfrak{H}$ is given by

$$H \;:\; \mathfrak{h} \;\longmapsto\; T[z; \mathcal{O}_0] \;-\; z \;+\; \sum_{M+N \ge 1} \chi_1 \, W_{M,N} \left[[w_{M,N}[z; \mathcal{O}; \cdot \,]] \right] \, \chi_1 \; ,$$

where

$$\begin{aligned} W_{M,N} \left[[w_{M,N}[z; \mathcal{O}; \cdot \,]] \right] \;&:=\; W_{M,N}[z; \mathcal{O}] \\ &=\; \sum_{\lambda^{(M)}, \tilde{\lambda}^{(N)}} \int \prod_{i=1}^{M} \frac{d^3 k_i \; \bar{\kappa}_\sigma(|k_i|)}{\sqrt{|k_i|}} \prod_{j=1}^{N} \frac{d^3 \tilde{k}_j \; \bar{\kappa}_\sigma(|\tilde{k}_j|)}{\sqrt{|\tilde{k}_j|}} \;\times \\ &\qquad \times \, a^*_{\lambda^{(M)}}(k^{(M)}) \, w_{M,N} \left[z; \mathcal{O}; K^{(M,N)} \right] \, a^\sharp_{\tilde{\lambda}^{(N)}}(\tilde{k}^{(N)}) \end{aligned} \tag{5.1.2}$$

is the same as the expression (4.4.5), if $\sigma \;\le\; 1$ holds for the running infrared cutoff. Otherwise, we let

$$\begin{aligned} W_{M,N} \left[[w_{M,N}[z; \mathcal{O}; \cdot \,]] \right] \;&:=\; W_{M,N}[z; \mathcal{O}] \\ &=\; \sum_{\lambda^{(M)}, \tilde{\lambda}^{(N)}} \int \prod_{i=1}^{M} d^3 k_i \, \sqrt{|k_i|} \prod_{j=1}^{N} d^3 \tilde{k}_j \, \sqrt{|\tilde{k}_j|} \;\times \\ &\qquad \times \, a^*_{\lambda^{(M)}}(k^{(M)}) \, w_{M,N} \left[z; \mathcal{O}; K^{(M,N)} \right] \, a^\sharp_{\tilde{\lambda}^{(N)}}(\tilde{k}^{(N)}) \end{aligned} \tag{5.1.3}$$

if $\sigma \;>\; 1$.

Conversely, every interaction operator $W_{M,N}$ of the form (4.4.5) uniquely defines an associated interaction kernel $w_{M,N}$ in $\mathfrak{W}_{M,N}$. This correspondence is obtained by simply reading off the respective components of the Wick expansion of a given effective hamiltonian in $H(\mathfrak{H})$.

Thus, the inverse of H is given by

$$H^{-1} \left[T[z; \mathcal{O}_0] \;-\; z \;+\; \sum_{M+N \ge 1} \chi_1 W_{M,N}[z; \mathcal{O}] \, \chi_1 \right] \;=\; (z, T, \{w_{M,N}\}) \; . \tag{5.1.4}$$

5.2. A polydisc of effective hamiltonians

For the spectral analysis of $H_p(\sigma_0)$, we consider the flow generated by $\mathcal{R}$ on a certain Banach subspace of $\mathfrak{H}$, defined as follows.

Let ϵ, ξ be small, positive numbers. The **polydisc** $\mathfrak{V}_{\epsilon,\xi} \subset \mathfrak{H}$ consists of all $\mathfrak{h} = (z, T, \{w_{M,N}\}) \in \mathfrak{H}$ possessing the following properties.

(i) The spectral parameter z is contained in $[-\epsilon, 0] \subset \mathbb{R}$ (it is picked **real** !).

(ii) The function

$$T[z;\cdot] \ : \ \mathrm{Spec}\{\mathcal{O}_0\} \ \longrightarrow \ \mathbb{R}$$

obeys the bounds

$$\| T \| \ \leq \ 2 \ ,$$

and can be decomposed into

$$T[z;\mathcal{O}_0] \ = \ H_f \ + \ a[z]\,\chi_1 \mathbf{P}_f^{\parallel} \chi_1 \ + \ \chi_1 T_{n-l}[z;\mathcal{O}_0]\chi_1 \ ,$$

where the operator T_{n-l} contains all terms in T that are $O(H_f^2)$ in the limit $H_f \to 0$. In particular, $T[z;\mathcal{O}_0]$ is analytic in $\mathbf{P}_f$.

(iii) For each value of $M + N \geq 1$,

$$\| \ w_{M,N} \ \|^{(2)} \ \leq \ \epsilon\, \xi^{M+N} \ .$$

We recall that the definition of $\| \cdot \|^{(2)}$ involves derivatives up to second order with respect to the spectral parameters associated to $\mathcal{O}$.

H maps $\mathfrak{V}_{\epsilon,\xi}$ to the bounded operators on $P_1\mathcal{H}_p$. This is a consequence of the following lemma.

LEMMA 5.1. *For fixed $M + N$,*

$$\| \ W_{M,N}[[\ w_{M,N}[z;\mathcal{O};\cdot] \ ]] \ \| \ \leq \ (2\sqrt{\pi})^{M+N} \ \| \, w_{M,N} \|$$

on $P_1\mathcal{H}_p$.

Proof. Cf. the appendix of [**3**]. ■

Hence, for $(z, T, \{w_{M,N}\}) \in \mathfrak{V}_{\epsilon,\xi}$, we have

$$\begin{aligned} \| \, H[\mathfrak{h}] \, \| \ &\leq \ |z| \ + \ \| \, T \, \| \ + \ \epsilon \sum_{M+N \geq 1} (2\sqrt{\pi}\xi)^{M+N} \\ &< \ 3 \ . \end{aligned}$$

Thus, $H[\mathfrak{h}]$ is a bounded operator on $P_1\mathcal{H}_p$ for all $\mathfrak{h} \in \mathfrak{V}_{\epsilon,\xi}$.

5.3. Definition of the renormalization map

The renormalization map

$$\begin{aligned} \mathcal{R} \;:\; \mathfrak{V}_{\epsilon,\xi} &\longrightarrow \mathfrak{V}_{\epsilon,\xi} \\ \mathfrak{h} &\longmapsto \hat{\mathfrak{h}} \;, \end{aligned}$$

is the pullback of $H[\mathfrak{h}] \mapsto H[\hat{\mathfrak{h}}]$ to $\mathfrak{H}$, that is,

$$\begin{array}{ccc} \mathfrak{V}_{\epsilon,\xi} & \xrightarrow{\mathcal{R}} & \mathfrak{V}_{\epsilon,\xi} \\ \downarrow H & & \downarrow H \\ \mathcal{B}(P_1\mathcal{H}_p) & \xrightarrow{Z\circ\mathcal{S}_\rho\circ\mathcal{D}_\rho} & \mathcal{B}(P_1\mathcal{H}_p) \end{array} \tag{5.3.1}$$

is a commutative diagram. The map $H[\mathfrak{h}] \mapsto H[\hat{\mathfrak{h}}]$ is obtained from concatenating the three distinct operations, which we discuss in detail below: Feshbach decimation $\mathcal{D}_\rho$, the rescaling transformation $\mathcal{S}_\rho$, and the shift of the spectral parameter Z.

Remark. The fact that $\mathcal{R}$ maps $\mathfrak{V}_{\epsilon,\xi}$ into itself is a central issue whose verification requires much effort in the subsequent analysis.

5.3.1. Feshbach decimation. Let $\rho \in (0,1)$ denote some fixed, real number. In the present work, our choice will always be $\rho = \frac{1}{2}$. $P_\rho = \chi[H_f < \rho]$ is a selfadjoint projector $P_1\mathcal{H}_p \to P_\rho\mathcal{H}_p$. For $\mathfrak{h} = (z, T, \{w_{M,N}\}) \in \mathfrak{V}_{\epsilon,\xi}$, the discussion in Section (4.5.3) has shown that the Feshbach decimation map $\mathcal{D}_\rho$ is well defined. We have seen that the image of $H[\mathfrak{h}]$ under

$$\mathcal{D}_\rho \;:\; \mathcal{B}(P_1\mathcal{H}_p) \;\longrightarrow\; \mathcal{B}(P_\rho\mathcal{H}_p)$$

can be written as

$$\begin{aligned} \mathcal{D}_\rho[H[\mathfrak{h}]] \;&=\; (1+\Delta_{H_f}[z])H_f \;+\; (\Delta E_0 - z) \;+\; \chi_\rho(T'\,\Pi + \Delta T')\chi_\rho \\ &\quad+ \sum_{M+N\geq 1} \chi_\rho(\Pi\, W_{M,N}\,\Pi + \Delta W_{M,N})\chi_\rho \;, \end{aligned}$$

where $T' = T - z - \tau[H[\mathfrak{h}]]$,

$$T'[z;\mathcal{O}_0] \;=\; a[z]\,\mathbf{P}_f^{\shortparallel} \;-\; \Delta_{H_f}[z]\,H_f \;-\; \Delta E_0[z] \;+\; T_{n-l}[z;\mathcal{O}_0]\;.$$

T_{n-l} comprises all operators in T that are nonlinear with respect to the components of $\mathcal{O}_0$.

5.3.2. The rescaling map. The rescaling map is obtained by concatenating the scaling transformation Ad_{U_ρ} of Section 2.3 with multiplication by a specific numerical factor.

It is a map

$$\mathcal{S}_\rho \;:\; \mathcal{B}(P_\rho\mathcal{H}_p) \;\longrightarrow\; \mathcal{B}(P_1\mathcal{H}_p)\;,$$

which acts on $A \in \mathcal{B}(P_\rho\mathcal{H}_p)$ by

$$\mathcal{S}_\rho[A] \;=\; (1+\Delta_{H_f}[z])^{-1}\,\rho^{-1}\;Ad_{U_\rho}[A]\;.$$

We recall that

$$(Ad_{U_\rho}a^\sharp)(k) \;=\; \rho^{-\frac{3}{2}}\;a^\sharp(\rho^{-1}\,k)\;, \tag{5.3.2}$$

and in particular, that $Ad_{U_\rho}[\mathcal{O}^r_{(0)}] = \rho\,\mathcal{O}^r_{(0)}$ for the components $\mathcal{O}^r_{(0)}$ of $\mathcal{O}_{(0)}$.

$\mathcal{S}_\rho$ is defined in this manner in order to keep the coefficient of H_f in T_{lin} fixed at the value 1 in each application of $\mathcal{R}$, thus the division by the factor $\rho(1+\Delta_{H_f}[z])$. As a consequence,

$$\hat{a}[Z[z]] \;:=\; \frac{a[z]+\Delta a[z]}{1+\Delta_{H_f}[z]}$$

corresponds to the renormalized coefficient of $\mathbf{P}^{\|}_f$.

5.3.3. Shift of the spectral parameter. Under $\mathcal{S}_\rho \circ \mathcal{D}_\rho$, the spectral parameter z is mapped to

$$z \;\longmapsto\; \hat{z} \;:=\; Z[z] \;:=\; (1+\Delta_{H_f}[z])^{-1}\;\rho^{-1}\,(z-\Delta E_0[z]) \;\;.$$

$\hat{z}$ is the renormalized spectral parameter.

5.3.4. Renormalization of the interaction kernels. ¿From Feshbach decimation, we have

$$w_{M,N} \;\longrightarrow\; \tilde{w}_{M,N} \;:=\; \Pi[z;S^{(M)}\mathcal{O}_0]w_{M,N}\Pi[z;S^{(N)}\mathcal{O}_0] \;+\; \Delta w_{M,N}\;,$$

where the kernel on the left hand side of the arrow acts on $P_1\mathcal{H}_p$, and the corrected kernel on the right hand side acts on $P_\rho\mathcal{H}_p$. The formula for $\Delta w_{M,N}$ is given in (4.5.11). We recall that $S^{(M)}$ shifts the components of $\mathcal{O}_0$ by the photon momenta $(k_1,\ldots,k_M)$.

Application of rescaling produces the bounded operators

$$\begin{aligned}\hat{w}_{M,N} \;&=\; \mathcal{S}_\rho[\tilde{w}_{M,N}] \\ &=\; \begin{cases} \rho^{M+N-1}\,(1+\Delta_{H_f}[z])^{-1}\,Ad_{U_\rho}[\tilde{w}_{M,N}] & \text{if } \sigma \le 1 \\ \rho^{2(M+N)-1}\,(1+\Delta_{H_f}[z])^{-1}\,Ad_{U_\rho}[\tilde{w}_{M,N}] & \text{if } \sigma > 1\;, \end{cases}\end{aligned}$$

on $P_1\mathcal{H}_p$. In the case $\sigma \le 1$, the factor ρ^{M+N-1} is obtained in the following manner: The integration measure in (5.1.2), given by

$$\prod_{i=1}^{N}\prod_{j=1}^{N} dk_i\,dk_j\,|k_i|^{\frac{1}{2}}\,|\tilde{k_j}|^{\frac{1}{2}}\;,$$

contributes a factor $\rho^{\frac{5}{2}(M+N)}$ under the action of Ad_{U_ρ}, since the photon momenta are modified by $k \mapsto \rho k$. Furthermore, since the scaling dimension of a creation-and an annihilation operator is $-\frac{3}{2}$, cf. (5.3.2), there is a factor $\rho^{-\frac{3}{2}(M+N)}$ from the action of Ad_{U_ρ} on $a^\sharp(k)$ in (5.1.2). Finally, an overall factor $\rho^{-1}(1+\Delta_{H_f}[z])^{-1}$ is contained in the definition of $\mathcal{S}_\rho$ (to fix the coefficient of H_f in T_{lin} at the constant value 1, as we recall).

The additional factor ρ^{M+N} in the case $\sigma > 1$ arises because the integration measure now includes a factor $\prod |k_i|^{\frac{1}{2}}|\tilde{k_j}|^{\frac{1}{2}}$, cf. (5.1.2), instead of its inverse $\prod |k_i|^{-\frac{1}{2}}|\tilde{k_j}|^{-\frac{1}{2}}$ in the case $\sigma \le 1$ considered above, cf. (5.1.3). This is due to the fact that for $\sigma > 1$, the infrared regularization $\bar{\kappa}_\sigma(x)$ is linear in x, and absorbed into the integration measure.

Substituting the spectral parameter by $z \to Z^{-1}[\hat{z}]$, one arrives at the renormalized interaction kernels, given by $\hat{w}_{M,N}[\hat{z};\mathcal{O};K^{(M,N)}]$.

Let $Y = (Y_1, \ldots, Y_r)$ denote a tuple of operators and parameters are homogenous of degree 1 under rescaling. Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a multiindex with $\alpha_i \in \mathbf{N}_0$ and $|\alpha| = \sum \alpha_i$. Then,

$$\partial_Y^\alpha \hat{w}_{M,N} \;=\; \begin{cases} \rho^{M+N+|\alpha|-1}\,(1+\Delta_{H_f}[z])^{-1}\,Ad_{U_\rho}[\partial_Y^\alpha\,\tilde{w}_{M,N}] & \text{if } \sigma \leq 1 \\ \rho^{2(M+N)+|\alpha|-1}\,(1+\Delta_{H_f}[z])^{-1}\,Ad_{U_\rho}[\partial_Y^\alpha\,\tilde{w}_{M,N}] & \text{if } \sigma > 1\ . \end{cases}$$

The origin of the additional factors $\rho^{|\alpha|}$ has been explained in the discussion given in Section 2.3.

Finally, the running infrared cutoff is renormalized by $\sigma \to \hat{\sigma} = \frac{\sigma}{\rho}$.

5.3.5. Construction of $\mathcal{R}$. Substituting z by

$$z[\hat{z}] \;=\; Z^{-1}[\hat{z}]$$

in $\mathcal{S}_\rho \circ \mathcal{D}_\rho[H[\mathfrak{h}] - z]$, we arrive at

$$H[\hat{\mathfrak{h}}] \;=\; H_f \;-\; \hat{z} \;+\; \chi_1\,\hat{T}'\,\chi_1 \;+\; \sum_{M+N\geq 1} \chi_1\,\hat{W}_{M,N}\,\chi_1\ .$$

Applying H^{-1}, we obtain the point $\hat{\mathfrak{h}} = (\hat{z}, \hat{T}, \{\hat{w}_{M,N}\})$ in $\mathfrak{H}$ (we recall once more that the issue is to prove that indeed, $\hat{\mathfrak{h}} \in \mathfrak{V}_{\epsilon,\xi} \subset \mathfrak{H}$). This completes the construction of the renormalization map $\mathcal{R} : \mathfrak{h} \mapsto \hat{\mathfrak{h}}$ in the manner exhibited by the diagram (5.3.1).

CHAPTER 6

RENORMALIZATION OF STRONGLY MARGINAL OPERATORS

The circumstance that the system considered in the work at hand comprises purely marginal operators distinguishes it from the majority of rigorous renormalization group studies that are present in the literature, where the marginal relevant and irrelevant cases are typically considered. The flow of operators in systems of the latter kind is dominated by the leading order radiative corrections in perturbation theory. The renormalization of strongly marginal quantities, however, is characterized by almost complete mutual cancellations of all radiative corrections. In the context of smooth Feshbach renormalization, these cancellations follow from the exact concatenation formulae for Feshbach triples that were proved in earlier sections. Our strategy will be to exploit these concatenation laws, together with other algebraic properties of the smooth Feshbach map, in order to derive identities that interrelate certain quantities of central interest between arbitrary scales. The analytic part, which produces bounds on the quantities of interest, will be carried out recursively, by use of a nested renormalization group scheme.

As it will turn out, the renormalization of strongly marginal operators in the present theory can be reduced to the renormalization of the coefficient $a[z]$ of $\mathbf{P}_f^{\parallel}$. This task is particularly subtle, and it is also especially important, for two main reasons:

(1) In the limit $n \to \infty$, $a_n[z_n]$ tends - as our analysis will show - to a finite, non-zero value. The physical hamiltonian $H_p(\sigma_0)$ exhibits an eigenvalue $E_0[|p|, \sigma_0]$ at the bottom of its spectrum. The first and second derivatives of the ground state energy $E_0[|p|, \sigma_0]$ of the physical hamiltonian $H_p(\sigma_0)$ with respect to $|p|$, for $|p| \leq p_c$, are, for $\alpha = 0, 1$, given by

$$\partial_{|p|} E_0[|p|, \sigma_0] \;=\; -\lim_{n\to\infty} \left(\eta_n[z_n]^{-1}\, a_n[z_n]\big|_{z_n = E_{0,n}} \right) \;,$$

where $E_{0,n}$ denotes the ground state eigenvalue of the effective hamiltonian $H[\mathfrak{h}^{(n)}]$, and the quantity $\eta_n[z_n]$ has the form $1 + O(g^2)$. This is explained in detail below.

Letting $\Omega[p; \sigma_0]$ denote the ground state vector corresponding to $E_0[|p|, \sigma_0]$, normalized by $\langle \Omega[p; \sigma_0], \Omega_f \rangle = 1$,

$$\begin{aligned} \partial_{|p|}^2 E_0 \;&=\; \frac{\Big\langle \Omega[p;\sigma_0]\,,\, (\partial_{|p|}^2 H_p(\sigma_0))\Omega[p;\sigma_0] \Big\rangle_{\Omega_f}}{\langle \Omega[p;\sigma_0]\,,\, \Omega[p;\sigma_0]\rangle_{\Omega_f}} \\ &-\; \frac{\big\langle \partial_{|p|}\Omega[p;\sigma_0]\,,\, (H_p(\sigma_0) - E_0)\, \partial_{|p|}\Omega[p;\sigma_0]\big\rangle_{\Omega_f}}{\langle \Omega[p;\sigma_0]\,,\, \Omega[p;\sigma_0]\rangle_{\Omega_f}} \;. \end{aligned} \tag{6.0.3}$$

Satisfactory bounds on these derivatives are of central importance for various purposes, including the construction of scattering theory [**9, 10, 21**], and the analysis of the semi-classical

dynamics of charged particles [**23**].

(2) Via the Ward-Takahashi identities that we will present and discuss in much detail below, $a[z]$ controls $W_{M,N}$, for $M+N=1$. Thus, $a[z]$ is responsible for all marginal operators in the theory other than H_f in $T_{lin}[\mathfrak{h}]$.

6.1. The ground state energy

The renormalization group produces a sequence of effective hamiltonians $\{H[\mathfrak{h}^{(k)}]\}$, for $k \in \mathbf{N}_0$, where we recall that

$$H[\mathfrak{h}^{(k+1)}] \;=\; (1+\Delta_{H_f}^{(k)}[z_k])\rho^{-1} Ad_{U_\rho}\left[F_{\chi_\rho,\tau^{(k)}}[H[\mathfrak{h}^{(k)}]]\right] \;,$$

with

$$\begin{aligned}\tau^{(k)}[H[\mathfrak{h}^{(k)}]] \;&=\; (1+\Delta_{H_f}^{(k)}[z_k])\left(H_f \;-\; z_k \;+\; \Delta E_0^{(k)}[z_k]\right)\\ &=\; (1+\Delta_{H_f}^{(k)}[z_k])\,(H_f \;-\; \rho\, z_{k+1}) \;.\end{aligned}$$

In particular, if we define

$$\tau_n[H_p(\sigma_0)] \;\equiv\; \left(\prod_{i=0}^{n}(1+\Delta_{H_f}^{(i)}[z_i])\right)(H_f \;+\; \rho^n\, z_n) \;,$$

and assume that all $H[\mathfrak{h}^{(k)}]$ are well-defined, the operator

$$F_n[z] \;\equiv\; F_{\chi_{\rho^n},\tau_n}[H_p(\sigma_0)-z]$$

is also well-defined, with

$$\chi_{\rho^n} \;=\; \chi_1[\rho^{-n}H_f] \;,$$

and

$$H[\mathfrak{h}^{(n)}] \;=\; \rho^{-n}\left(\prod_{i=0}^{n}(1+\Delta_{H_f}^{(i)}[z_i])^{-1}\right) Ad_{U_\rho^n}\left[F_n[z]\right] \;, \tag{6.1.1}$$

due to Theorem 4.4. Likewise, the operators

$$Q_n^{(\sharp)}[z] \;\equiv\; Q^{(\sharp)}_{\chi_{\rho^n},\tau_n}[H_p(\sigma_0)-z]$$

are well-defined.

Proposition 6.1. *Assume that the Feshbach triple $(F_n[z], Q_n^\sharp[z], Q_n[z])$ associated to $(\chi_{\rho^n}, \mathcal{H}_p)$, is well-defined for all $n \in \mathbf{N}_0$ and z sufficiently close to the origin. Furthermore, let $\rho = \frac{1}{2}$. For the sequence of ground state eigenvalues of $H[\mathfrak{h}^{(n)}]$, assume that*

$$\lim_{n\to\infty} \|W[h^{(n)}]\| \;=\; 0 \;.$$

Then, defining $\mathcal{Z}_n \equiv Z_n \circ \cdots \circ Z_0$,

$$E_0 \;=\; \lim_{n\to\infty} \mathcal{Z}_n^{-1}(z_n = 0) \;, \tag{6.1.2}$$

and

$$\Omega[p;\sigma_0] \;=\; s - \lim_{n\to\infty} Q_n[E_0]\,\Omega_f \;. \tag{6.1.3}$$

In particular,

$$\partial_{|p|} E_0[|p|,\sigma_0] \;=\; \lim_{n\to\infty} \frac{\Big\langle Q_n^\sharp[E_0](\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0))\, Q_n[E_0]\Big\rangle_{\Omega_f}}{\Big\langle\, Q_n^\sharp[E_0]\, Q_n[E_0]\Big\rangle_{\Omega_f}} \;. \tag{6.1.4}$$

Furthermore, defining

$$\eta_n[z_n] \;:=\; \frac{\big\langle\, \partial_{H_f} F_n[z]\big\rangle_{\Omega_f}}{\Big\langle\, Q_n^\sharp[z]\, Q_n[z]\Big\rangle_{\Omega_f}} \;, \tag{6.1.5}$$

with

$$z_n[z] \;=\; \rho^{-n}\, \frac{z \;-\; \langle F_n[z]\rangle_{\Omega_f}}{\big\langle\, \partial_{H_f} F_n[z]\big\rangle_{\Omega_f}} \;, \tag{6.1.6}$$

and assuming that $z \to z_n$ is invertible, the coefficient of $\mathbf{P}_f^{\|}$ in $T_{lin}[\mathfrak{h}^{(n)}]$ is given by

$$(a_n[z_n])|_{z_n=E_{0,n}} \;=\; \eta_n[E_{0,n}]^{-1}\, \frac{\Big\langle Q_n^\sharp[E_0](\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0))\, Q_n[E_0]\Big\rangle_{\Omega_f}}{\Big\langle\, Q_n^\sharp[E_0]\, Q_n[E_0]\Big\rangle_{\Omega_f}} \;, \tag{6.1.7}$$

such that

$$\lim_{n\to\infty} (a_n[z_n]\eta_n[z_n])\Big|_{z_n=E_{0,n}} \;=\; -\,\partial_{|p|}\, E_0[|p|,\sigma_0] \;. \tag{6.1.8}$$

Furthermore,

$$\partial_{|p|}^2 \left(E_0[|p|,\sigma_0] \;-\; \frac{|p|^2}{2} \right) \;\le\; 0 \;,$$

that is, the renormalized mass of the infraparticle is bounded from below by the bare mass of the charged particle, for all $\sigma_0 \ge 0$.

Proof. The condition that the Feshbach triple $(F_n, Q_n^\sharp, Q_n)$ is well-defined for all $n \in \mathbf{N}_0$ will be proved in the renormalization group analysis of Chapter 9. The fact that the limits in the asserted expressions exist is also proved there.

Let us thus assume that the limits exist. In order to fix the coefficient of H_f in the Wick ordered expression for $F_n[E_0]$ at the value 1, we must divide by $\langle \partial_{H_f} F_n[E_0]\rangle_{\Omega_f}$. We therefore have

$$H[\mathfrak{h}_0^{(n)}] \;=\; \frac{\rho^{-n}}{\big\langle \partial_{H_f} F_n[E_0]\big\rangle_{\Omega_f}}\, Ad_{U_{\rho^n}}\,[F_n[E_0]] \;,$$

again by Theorem 4.2.

Let us first prove that the expressions given in (6.1.2) and (6.1.3) satisfy the eigenvalue equation

$$(H_p(\sigma_0) - E_0)\Omega[p;\sigma_0] \;=\; 0$$

for $H_p(\sigma_0)$. To this end, the general formula (4.3.4) implies

$$\begin{aligned}
&(H_p(\sigma_0) - \mathcal{Z}_n^{-1}(0))\, Q_n[\mathcal{Z}_n^{-1}(0)] \\
&\quad = \chi_{\rho^n} F_n[\mathcal{Z}_n^{-1}(0)] \\
&\quad = \rho^n \chi_{\rho^n} \langle \partial_{H_f} F_n[\mathcal{Z}_n^{-1}(0)]\rangle_{\Omega_f} Ad_{U^*_{\rho^n}} \left[H[\mathfrak{h}^{(n)}]|_{z_n=0}\right] \\
&\quad = \rho^n \chi_{\rho^n} \langle \partial_{H_f} F_n[\mathcal{Z}_n^{-1}(0)]\rangle_{\Omega_f} Ad_{U^*_{\rho^n}} \left[H_f + \chi_1(T'[\mathfrak{h}^{(n)}] + W[\mathfrak{h}^{(n)}])|_{z_n=0}\chi_1\right] \ .
\end{aligned}$$

Thus, if $\lim_{n\to\infty} \|W[\mathfrak{h}^{(n)}]\| = 0$, it is evident that

$$\begin{aligned}
0 &= s - \lim_{n\to\infty} \left(H_p(\sigma_0) - \mathcal{Z}_n^{-1}(0)\, Q_n[\mathcal{Z}_n^{-1}(0)]\Omega_f\right) \\
&= \left(H_p(\sigma_0) - \lim_{n\to\infty} \mathcal{Z}_n^{-1}(0)\right) \left(s - \lim_{n\to\infty} Q_n[\mathcal{Z}_n^{-1}(0)]\Omega_f\right) \ .
\end{aligned}$$

This proves the claim.

The coefficient $a_n[z_n]|_{z_n=E_{0,n}}$ of $\mathbf{P}_f^{\|}$ in $T_{lin}[\mathfrak{h}^{(n)}]$ is obtained from multiplying

$$\left\langle \partial_{\mathbf{P}_f^{\|}} F_n[E_0]\right\rangle_{\Omega_f} \ = \ \left\langle Q_n^\sharp[E_0](\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0))Q_n[E_0]\right\rangle_{\Omega_f}$$

by $\left\langle \partial_{H_f} F_{\chi_{\rho^n},\tau_n}[H_p(\sigma_0)]\right\rangle_{\Omega_f}^{-1}$. Therefore,

$$\begin{aligned}
a_n[z_n]\big|_{z_n=E_{0,n}} &= \frac{\left\langle Q_n^\sharp[E_0](\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0))\, Q_n[E_0]\right\rangle_{\Omega_f}}{\left\langle\, \partial_{H_f} F_n[E_0]\right\rangle_{\Omega_f}} \\
&= \eta_n[E_{0,n}]^{-1} \frac{\left\langle Q_n^\sharp[E_0](\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0))\, Q_n[E_0]\right\rangle_{\Omega_f}}{\left\langle\, Q_n^\sharp[E_0]\, Q_n[E_0]\right\rangle_{\Omega_f}} \ ,
\end{aligned}$$

as claimed.

Next, we demonstrate that

$$\begin{aligned}
\partial_{|p|} E_0[|p|,\sigma_0] &= \lim_{n\to\infty} \frac{\left\langle Q_n^\sharp[E_0](\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0))\, Q_n[E_0]\right\rangle_{\Omega_f}}{\left\langle\, Q_n^\sharp[E_0]\, Q_n[E_0]\right\rangle_{\Omega_f}} \\
&= -\lim_{n\to\infty} \frac{\left\langle Q_n^\sharp[E_0](\partial_{|p|} H_p(\sigma_0))Q_n[E_0]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[E_0]\, Q_n[E_0]\right\rangle_{\Omega_f}} \ ,
\end{aligned}$$

where, passing to the second line, we have used $\partial_{|p|} H_p(\sigma_0) = -\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0)$.

To this end, let us consider the eigenvalue problem for the ground state energy of $H_p(\sigma_0)$. We recall that

$$s - \lim_{n\to\infty} Q_n[E_0]\, \Omega_f \ = \ \Omega[p;\sigma_0] \ .$$

Hence, defining

$$\Psi[p;\sigma_0] \ := \ \frac{\Omega[p;\sigma_0]}{\|\Omega[p;\sigma_0]\|}$$

on $\mathcal{H}_p$, we see that

$$\lim_{n\to\infty} \frac{\left\langle Q_n^\sharp(\partial_{|p|}H_p(\sigma_0))Q_n[E_0]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[E_0]\,Q_n[E_0]\right\rangle_{\Omega_f}} \;=\; -\left\langle \Psi[p;\sigma_0]\;,\;(\partial_{|p|}H_p(\sigma_0))\Psi[p;\sigma_0]\right\rangle\;.$$

On the other hand, differentiating the eigenvalue equation

$$(H_p(\sigma_0)\;-\;E_0[|p|,\sigma_0])\;\Psi[p;\sigma_0]\;=\;0 \tag{6.1.9}$$

with respect to $|p|$, we find

$$(H_p(\sigma_0)\;-\;E_0[|p|,\sigma_0])\;\partial_{|p|}\,\Psi[p;\sigma_0]\;=\;-\left(\partial_{|p|}\,(H_p(\sigma_0)\;-\;E_0[|p|,\sigma_0])\right)\;\Psi[p;\sigma_0]\;.$$

Taking the scalar product with respect to $\Psi[p;\sigma_0]$, we obtain

$$\left\langle \Psi[p;\sigma_0]\;,\;(\partial_{|p|}H_p(\sigma_0))\,\Psi[p;\sigma_0]\right\rangle\;=\;\partial_{|p|}E_0[|p|,\sigma_0]\;,$$

using that

$$\begin{aligned}\left\langle \partial_{|p|}\Psi[p;\sigma_0]\;,\;\Psi[p;\sigma_0]\right\rangle \;&=\; \frac{1}{2}\,\partial_{|p|}\,\underbrace{\left\langle \Psi[p;\sigma_0]\;,\;\Psi[p;\sigma_0]\right\rangle}_{=\,1}\\ &=\; 0\;.\end{aligned}$$

Finally, the formula

$$\begin{aligned}\partial_{|p|}^2E_0 \;&=\; \lim_{n\to\infty} \frac{\left\langle Q_n^\sharp[E_0](\partial_{|p|}^2H_p(\sigma_0))Q_n[E_0]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[E_0]Q_n[E_0]\right\rangle_{\Omega_f}}\\ &-\; \lim_{n\to\infty} \frac{\left\langle (\partial_{|p|}Q_n^\sharp[E_0])\,(H_p(\sigma_0)-E_0)\,\partial_{|p|}Q_n[E_0]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[E_0]Q_n[E_0]\right\rangle_{\Omega_f}}\;,\end{aligned}$$

together with $\partial_{|p|}^2H_p(\sigma_0)=1$, immediately implies that

$$\partial_{|p|}^2\left(E_0[|p|,\sigma_0]\;-\;\frac{|p|^2}{2}\right)\;\leq\;0\;.$$

This proves the claim. ■

To estimate the quotient

$$\eta_n[z]\;=\;\frac{\left\langle\,\partial_{H_f}F_n[z]\right\rangle_{\Omega_f}}{\left\langle\,Q_n^\sharp[z]\,Q_n[z]\right\rangle_{\Omega_f}}\;,$$

we employ formula (4.5.3). Using the fact that $\partial_{H_f}(T_{n-l}+W)=0$ for $H_p(\sigma_0)=T_{lin}+T_{n-l}+W$, one straightforwardly obtains

$$\begin{aligned}\left\langle \partial_{H_f} F_n[z]\right\rangle_{\Omega_f} &= \frac{\left\langle Q_n^\sharp[z]\, Q_n[z]\right\rangle_{\Omega_f}}{1-\left\langle W\bar\chi_{\rho^n}\bar R_n\chi_{\rho^n}^2\bar R_n\bar\chi_{\rho^n}W\right\rangle_{\Omega_f}}\ \times \\ &\times\left(1\ -\ \frac{\left\langle W\bar\chi_{\rho^n}\bar R_n\frac{\partial_{H_f}\bar\chi_{\rho^n}}{\bar\chi_{\rho^n}}\tau_n[H_p(\sigma_0)]\bar R_n\bar\chi_{\rho^n}W\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z]\, Q_n[z]\right\rangle_{\Omega_f}}\right)\ , \qquad (6.1.10)\end{aligned}$$

hence

$$\begin{aligned}\eta_n[z] &= \frac{1}{1-\left\langle W\bar\chi_{\rho^n}\bar R_n\chi_{\rho^n}^2\bar R_n\bar\chi_{\rho^n}W\right\rangle_{\Omega_f}}\ \times \\ &\times\left(1\ -\ \frac{\left\langle W\bar\chi_{\rho^n}\bar R_n\frac{\partial_{H_f}\bar\chi_{\rho^n}}{\bar\chi_{\rho^n}}\,\tau_n[H_p(\sigma_0)]\,\bar R_n\bar\chi_{\rho^n}W\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z]\, Q_n[z]\right\rangle_{\Omega_f}}\right)\ . \qquad (6.1.11)\end{aligned}$$

A very important element of the analysis in subsequent chapters will be to prove, by means of the renormalization group iteration, that

$$\left|\left\langle W\bar\chi_{\rho^n}\bar R_n\chi_{\rho^n}^2\bar R_n\bar\chi_{\rho^n}W\right\rangle_{\Omega_f}\right|\ \le\ O(\epsilon_0)\ ,$$

and that

$$\left|\frac{\left\langle W\bar\chi_{\rho^n}\bar R_n\frac{\partial_{H_f}\bar\chi_{\rho^n}}{\bar\chi_{\rho^n}}\tau_n[H_p(\sigma_0)]\bar R_n\bar\chi_{\rho^n}W\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z]\, Q_n[z]\right\rangle_{\Omega_f}}\right|\ \le\ O(\epsilon_0)\ ,$$

both uniformly in n. This implies that $\eta_n[z]=1+O(\epsilon_0)$, uniformly in n. For the proof of these estimates, the concatenation laws of section 4.3 have central importance.

6.2. Analytical control of $\eta_n[z]$ and cancellation phenomena

One of the key insights that makes the subsequent analysis possible is that the quantity

$$\eta_n[z]\ =\ \frac{\left\langle\,\partial_{H_f}F_n[z]\right\rangle_{\Omega_f}}{\left\langle\, Q_n^\sharp[z]\, Q_n[z]\right\rangle_{\Omega_f}}\ ,$$

does in fact solely depend on the effective hamiltonian of the scale $n-1$, as stated in the following proposition.

PROPOSITION 6.2. *Let $H[\mathfrak{h}]$ denote the effective hamiltonian at some arbitrary, but fixed scale, acting on the Hilbert space $P_1\mathcal{H}_p$, and define*

$$\begin{aligned}\eta[\mathfrak{h}] \;\equiv\; & \Big(1 + \big\langle W[\mathfrak{h}]\,\bar\chi\,\bar R[\mathfrak{h}]\,\chi_\rho^2\,\bar R[\mathfrak{h}]\,\bar\chi\,W[\mathfrak{h}]\big\rangle_{\Omega_f} \\ & + \left\langle W[\mathfrak{h}]\,\bar\chi\,\bar R[\mathfrak{h}]\,\frac{\partial_{H_f}\bar\chi_\rho}{\bar\chi_\rho}\,(H_f-\rho\hat z)\,\bar R[\mathfrak{h}]\,\bar\chi\,W[\mathfrak{h}]\right\rangle_{\Omega_f}\Big)^{-1}\,, \end{aligned} \tag{6.2.1}$$

where $\bar\chi := \bar\chi_\rho\chi_1$, and where $\hat z$ is the spectral parameter associated with $\hat{\mathfrak{h}} = \mathcal{R}[\mathfrak{h}]$. Then,

$$\begin{aligned}\eta[\mathfrak{h}^{(n-1)}] &= \eta_n[z] \\ &= \frac{\big\langle\,\partial_{H_f}F_n[z]\big\rangle_{\Omega_f}}{\Big\langle\,Q_n^\sharp[z]\,Q_n[z]\Big\rangle_{\Omega_f}}\,.\end{aligned}$$

where the relationship between the spectral parameters z_{n-1} (for $\mathfrak{h}^{(n-1)}$), z_n and z is determined by (6.1.6).

Proof. Let us separately discuss the overlap term (due to $\chi^2_{\rho^n}\bar P_n$)

$$\big\langle W\,\bar\chi_{\rho^n}\,\bar R_n\,\chi^2_{\rho^n}\,\bar R_n\,\bar\chi_{\rho^n}\,W\big\rangle_{\Omega_f}\;,$$

and the term

$$\frac{\Big\langle W\bar\chi_{\rho^n}\bar R_n\frac{\partial_{H_f}\bar\chi_{\rho^n}}{\bar\chi_{\rho^n}}\tau_n[H_p(\sigma_0)]\bar R_n\bar\chi_{\rho^n}W\Big\rangle_{\Omega_f}}{\Big\langle Q_n^\sharp[z]\,Q_n[z]\Big\rangle_{\Omega_f}}\,,$$

comprising the $\bar\chi_{\rho^n}$-derivative, which occur in $\eta_n[z_n]$.

LEMMA 6.1. *Let $H[\mathfrak{h}^{(k)}] = T[\mathfrak{h}^{(k)}] - z_k + \chi_1 W[\mathfrak{h}^{(k)}]\chi_1$ denote the effective hamiltonian at the scale k, acting on $P_1\mathcal{H}_p$, for $0 \le k < n$. Then,*

$$\begin{aligned}&\big\langle W\,\bar\chi_{\rho^n}\,\bar R_n\,\chi^2_{\rho^n}\,\bar R_n\,\bar\chi_{\rho^n}\,W\big\rangle_{\Omega_f} \\ &\qquad = \big\langle W[\mathfrak{h}^{(n-1)}]\,\bar\chi\,\bar R[\mathfrak{h}^{(n-1)}]\,\chi_\rho^2\,\bar R[\mathfrak{h}^{(n-1)}]\,\bar\chi\,W[\mathfrak{h}^{(n-1)}]\big\rangle_{\Omega_f}\;,\end{aligned} \tag{6.2.2}$$

where $\bar\chi := \bar\chi_\rho\chi_1$, and where the expectation value on the left hand side of the equality sign is determined by $H_p(\sigma_0)$ and $\bar\chi_{\rho^n}[H_f] = \bar\chi[\rho^{-n}H_f]$.

Proof. Clearly,

$$\begin{aligned}&\big\langle W\,\bar\chi_{\rho^n}\,\bar R_n\,\chi^2_{\rho^n}\,\bar R_n\,\bar\chi_{\rho^n}\,W\big\rangle_{\Omega_f} \\ &\qquad = \lim_{\delta\to 0}\left\langle Q_n^\sharp[z]\,P_n\bar P_n\,\frac{\chi^2_{\rho^n}}{\bar\chi^2_{\rho^n}+\delta}\,P_n\bar P_n\,Q_n[z]\right\rangle_{\Omega_f}\;,\end{aligned}$$

where $P_n, \bar P_n$, are the projectors onto the ranges of $\chi_{\rho^n}, \bar\chi_{\rho^n}$ respectively. Now, by the concatenation law for $Q^\sharp, Q$ of Theorem 4.5,

$$\begin{aligned}Q_n[z] \;=\; & Q_{\chi_1,\tau_0}[H_p(\sigma_0)-z] \\ & Q_{\chi_\rho,\tau_1}[[F_{\chi_0,\tau_0}[H_p(\sigma_0)-z]]\,\cdots \\ & \cdots\,Q_{\chi_{\rho^n},\tau_n}[F_{\chi_{\rho^{n-1}},\tau_{n-1}}[\,\cdots F_{\chi_1,\tau_0}[H_p(\sigma_0)-z]\,\cdots]]\,,\end{aligned}$$

thus by the second part of Theorem 4.5,

$$P_n\,\bar{P}_n\,Q_n[z] \;=\; P_n\,\bar{P}_n\,Q_{\chi_{\rho^n},\tau_n}[F_{\chi_{\rho^{n-1}},\tau_{n-1}}[\cdots F_{\chi_0,\tau_0}[H_p(\sigma_0)-z]\cdots]]\,, \tag{6.2.3}$$

since $P_n\,\bar{P}_n\bar{\chi}_k = 0$ for all $k < n$. In this notation, the operator $\tau_k[H[\mathfrak{h}^{(k-1)}]]$ determines the Feshbach decimation map passing from $H[\mathfrak{h}^{(k-1)}]$ to $H[\mathfrak{h}^{(k)}]$ in the renormalization step.

Now clearly,

$$H[\mathfrak{h}^{(n-1)}] \;=\; \frac{\rho^{-n}}{\big\langle \partial_{H_f} F_{n-1}[z]\big\rangle_{\Omega_f}}\, Ad_{U_{\rho^n}}\,\Big[F_{\chi_{\rho^{n-1}},\tau_{n-1}}[\cdots F_{\chi_1,\tau_0}[H_p(\sigma_0)-z]\cdots]\Big]\;,$$

thus

$$\begin{aligned}(\,6.2.3) \;&=\; P_n\,\bar{P}_n\,Q_{\chi_{\rho^n},\tau_n}\Big[\rho^n\,\big\langle \partial_{H_f} F_{n-1}[z]\big\rangle_{\Omega_f}\; Ad_{U^*_{\rho^{n-1}}}[H[\mathfrak{h}^{(n)}]]\Big]\\ &=\; Ad_{U^*_{\rho^{n-1}}}\,\big[P_\rho\,\bar{P}_\rho\,Q_{\chi_\rho,\tau}[H[\mathfrak{h}^{(n-1)}]]\big]\;,\end{aligned}$$

since the $Q^\sharp, Q$-operators are **homogenous of degree 0** with respect to the arguments in square brackets.

Due to

$$Ad_{U^*_{\rho^{n-1}}}\,\left[\frac{\chi^2_{\rho^n}}{\bar{\chi}^2_{\rho^n}+\delta}\right] \;=\; \frac{\chi^2_\rho}{\bar{\chi}^2_\rho+\delta}\;,$$

we have

$$\begin{aligned}&\lim_{\delta\to 0}\left\langle Q^\sharp_n[z]\,P_n\bar{P}_n\,\frac{\chi^2_{\rho^n}}{\bar{\chi}^2_{\rho^n}+\delta}\,P_n\bar{P}_n\,Q_n[z]\right\rangle_{\Omega_f}\\ =\;&\lim_{\delta\to 0}\left\langle Q^\sharp_{\chi_\rho,\tau}[H[\mathfrak{h}^{(n)}]]\,P_\rho\bar{P}_\rho\,\frac{\chi^2_\rho}{\bar{\chi}^2_\rho+\delta}\,P_\rho\bar{P}_\rho\,Q_{\chi_\rho,\tau}[H[\mathfrak{h}^{(n)}]]\right\rangle_{\Omega_f}\\ =\;&\big\langle W[\mathfrak{h}^{(n)}]\,\bar{\chi}\,\bar{R}[\mathfrak{h}^{(n)}]\,\chi^2_\rho\,\bar{R}[\mathfrak{h}^{(n)}]\,\bar{\chi}\,W[\mathfrak{h}^{(n)}]\big\rangle_{\Omega_f}\;.\end{aligned}$$

This proves the lemma. ■

The identity expressed by this lemma is based on algebraic properties of Feshbach triples. To obtain bounds on the right hand side of (6.2.2), we must invoke the renormalization group iteration.

Let us next consider the term

$$\frac{\left\langle W\bar{\chi}_{\rho^n}\bar{R}_n\frac{\partial_{H_f}\bar{\chi}_{\rho^n}}{\bar{\chi}_{\rho^n}}\tau_n[H_p(\sigma_0)]\bar{R}_n\bar{\chi}_{\rho^n}W\right\rangle_{\Omega_f}}{\left\langle Q^\sharp_n[z]\,Q_n[z]\right\rangle_{\Omega_f}}\;, \tag{6.2.4}$$

which occurs in $\eta_n[z]$.

LEMMA 6.2. *Using the notations of the previous lemma, the identity*

$$\begin{aligned}
&\left\langle W\bar\chi_{\rho^n}\bar R_n\frac{\partial_{H_f}\bar\chi_{\rho^n}}{\bar\chi_{\rho^n}}\tau_n[H_p(\sigma_0)]\bar R_n\bar\chi_{\rho^n}W\right\rangle_{\Omega_f}\\
&\quad=\ \left\langle\partial_{H_f}F_n[z]\right\rangle_{\Omega_f}\ \times\\
&\quad\quad\times\ \left\langle W[\mathfrak{h}^{(n-1)}]\,\bar\chi\,\bar R[\mathfrak{h}^{(n-1)}]\frac{\partial_{H_f}\bar\chi_\rho}{\bar\chi_\rho}\,(H_f-\rho z_n)\,\bar R[\mathfrak{h}^{(n-1)}]\,\bar\chi\,W[\mathfrak{h}^{(n-1)}]\right\rangle_{\Omega_f} \qquad (6.2.5)
\end{aligned}$$

is satisfied, where $\bar\chi=\chi_\rho\bar\chi_\rho$, *and where* $H[\mathfrak{h}^{(n)}]$ *denotes the effective hamiltonian at the scale* n, *acting on* $P_1\mathcal{H}_p$.

Proof. The left hand side of the equality sign in (6.2.5) can be written as

$$\lim_{\delta\to 0}\left\langle Q^\sharp_n[z]\,\frac{\partial_{H_f}\bar\chi_{\rho^n}}{\bar\chi^3_{\rho^n}+\delta}\,\tau_n[H_p(\sigma_0)]^{-1}\,P_n\,\bar P_n\,Q_n[z]\right\rangle_{\Omega_f}\ ,$$

where, as in the proof of Lemma 6.1, $P_n,\bar P_n$, are the projectors onto the ranges of $\chi_{\rho^n},\bar\chi_{\rho^n}$ respectively. Notice that all operators apart from $Q^\sharp,Q$ in the expectation value are mutually commutative. By the arguments demonstrated in the first part of the proof of Lemma 6.1, based on the concatenation rules for $Q^\sharp,Q$, and

$$Ad_{U^*_{\rho^{n-1}}}\left[\frac{\partial_{H_f}\bar\chi_{\rho^n}}{\bar\chi^3_{\rho^n}+\delta}\right]\ =\ \rho^{-(n-1)}\,\frac{\partial_{H_f}\bar\chi_\rho}{\bar\chi^3_\rho+\delta}\ ,$$

as well as

$$\begin{aligned}
Ad_{U_{\rho^{-(n-1)}}}\left[\tau_n[H_p(\sigma_0)]\right]\ &=\ \left\langle\partial_{H_f}F_n[z]\right\rangle_{\Omega_f}\ \times\\
&\quad\times\ \left(\rho^{n-1}\,H_f\ -\ z\ +\ \left\langle F_{\chi_{\rho^n},\tau_n}[H_p(\sigma_0)-z]\right\rangle_{\Omega_f}\right)\\
&=\ \rho^{n-1}\,\left\langle\partial_{H_f}F_n[z]\right\rangle_{\Omega_f}\,(H_f-\rho z_n)\ , \qquad (6.2.6)
\end{aligned}$$

cf. (6.1.6), the left hand side of (6.2.5) equals

$$\begin{aligned}
&\lim_{\delta\to 0}\left\langle Q^\sharp_{\chi_\rho,\tau}[H[\mathfrak{h}^{(n-1)}]]\,\frac{\partial_{H_f}\bar\chi}{\bar\chi^3+\delta}\,(H_f-\rho z_n)\,P_\rho\bar P_\rho\,Q_{\chi_\rho,\tau}[H[\mathfrak{h}^{(n-1)}]]\right\rangle_{\Omega_f}\\
&\quad=\ \left\langle\partial_{H_f}F_n[z]\right\rangle_{\Omega_f}\ \times\\
&\quad\quad\times\ \left\langle W[\mathfrak{h}^{(n-1)}]\,\bar\chi\,\bar R[\mathfrak{h}^{(n)}]\,\frac{\partial_{H_f}\bar\chi_\rho}{\bar\chi_\rho}\,(H_f-\rho z_n)\,\bar R[\mathfrak{h}^{(n)}]\,\bar\chi\,W[\mathfrak{h}^{(n-1)}]\right\rangle_{\Omega_f}\ ,
\end{aligned}$$

as claimed. ■

Thus,

$$\begin{aligned}
&\frac{\left\langle W\bar\chi_{\rho^n}\bar R_n\frac{\partial_{H_f}\bar\chi_{\rho^n}}{\bar\chi_{\rho^n}}\tau_n[H_p(\sigma_0)]\bar R_n\bar\chi_{\rho^n}W\right\rangle_{\Omega_f}}{\left\langle\,Q^\sharp_n[z]\,Q_n[z]\right\rangle_{\Omega_f}}\\
&\quad=\ \eta_n[z]\,\left\langle W[\mathfrak{h}^{(n)}]\,\bar\chi\,\bar R[\mathfrak{h}^{(n-1)}]\frac{\partial_{H_f}\bar\chi_\rho}{\bar\chi_\rho}\,(H_f-\rho z_n)\,\bar R[\mathfrak{h}^{(n-1)}]\,\bar\chi\,W[\mathfrak{h}^{(n-1)}]\right\rangle_{\Omega_f}\ . \qquad (6.2.7)
\end{aligned}$$

Inserting this expression into the formula (6.1.11), and solving it for $\eta_n[z]$, we arrive at the asserted formula. This proves the proposition. ■

6.2.1. Cancellation phenomena in the renormalization of purely marginal quantities. The quantity $\eta_n[z]$ does is, for $n > 0$, independent of $\mathfrak{h}^{(k)}$, for all $0 \leq k \leq n-2$. The task of controlling it analytically is therefore almost trivial. From the point of view of the 'global' argument in the proof of the above proposition, by which $H[\mathfrak{h}^{(n)}]$ is directly related to $H_p(\sigma_0)$ via the concatenation properties of the smooth Feshbach map, this is easy to understand.

However, from the 'local' renormalization group point of view, in which $H[\mathfrak{h}^{(k)}]$ is only compared to $H[\mathfrak{h}^{k+1}]$, for all $k = 0, \ldots, n-1$, this result looks far from obvious. Passing from the scale k to the scale $k+1$, the ratio

$$\frac{\eta_{k+1}[z]}{\eta_k[z]} \;=\; \frac{\langle \partial_{H_f} F_{\chi_\rho,\tau}[H[\mathfrak{h}^{(k)}]]\rangle_{\Omega_f}}{\langle Q^\sharp[\mathfrak{h}^{(k)}](\partial_{\mathbf{P}_f^\parallel} H[\mathfrak{h}^{(k)}])Q[\mathfrak{h}^{(k)}]\rangle_{\Omega_f}}$$

determines the renormalized value of $a[z]$. Looking at the right hand side of this expression, it seems far from evident that the product

$$\eta_n[z] \;=\; \prod_{k=0}^{n-1} \frac{\langle \partial_{H_f} F_{\chi_\rho,\tau}[H[\mathfrak{h}^{(k)}]]\rangle_{\Omega_f}}{\langle Q^\sharp[\mathfrak{h}^{(k)}](\partial_{\mathbf{P}_f^\parallel} H[\mathfrak{h}^{(k)}])Q[\mathfrak{h}^{(k)}]\rangle_{\Omega_f}}$$

should only depend on $\mathfrak{h}^{(n-1)}$. The fact that all contributions from intermediate scales mutually cancel each other in this product is absolutely crucial in the renormalization of the purely marginal quantity $a[z]$.

This insight in fact suggests that these cancellation phenomena are typical for the renormalization of purely marginal interactions in any quantum field theory. To explain this statement, let us for simplicity consider the renormalization of a marginal coupling $g_k \geq 0$ in some given quantum field theory, where $k \in \mathbf{N}$ labels the scale. The correction of the running coupling constant g_k under a renormalization step is at most of order $O(g_k^2)$. The familiar scenario where

$$g_{k+1} \;=\; g_k(1 \pm c_k \epsilon_k^2)$$

holds for some $c_k > 0$ (i.e. a definite sign), appears in the cases of marginal relevance and marginal irrelevance. One easily concludes there that either $g_k \to \infty$, or $g_k \to 0$ (UV/IR asymptotic freedom). In particular, it is not possible that $g_n = g_0(1 + O(g_0))$ for all $n \in \mathbf{N}$. However, the latter condition defines purely marginal interactions.

In a theory exhibiting purely marginal interactions, the mechanism at work must thus necessarily be different. The fact that, as in the above cases of marginal relevance and irrelevance,

$$g_{k+1} \;=\; g_k \,+\, \Delta g_k$$

is given, where $|\Delta g_k| = O(g_k^2)$, cannot be circumvented. But in contrast to the situation there, Δg_k does not have a definite sign ! This makes it possible that

$$\begin{aligned} g_n \;&=\; g_0 \,+\, \sum_{k=0}^{n-1} \Delta g_k \\ &=\; g_0(1 + O(g_0^2)) \end{aligned}$$

holds, due to lots of mutual cancellations among the Δg_k's. It is clear that perturbation theory to any arbitrary finite order is insufficient to analytically control the cancellations in these sums. If the error term were $O(g_k^\nu)$ for some $\nu \geq 2$, one would have an error of order $\geq c \sum_{k=0}^{n-1} g_k^\nu$ at the scale n, which diverges as $n \to \infty$.

Thus, one can putatively only rely on finding 'global identities' that govern the cancellations, as is the case in the present work.

6.3. Renormalization of $a[z]$

As has been remarked, it suffices, due to the Ward-Takahashi identities, to bound the parameter $a[z]$ and its derivative with respect to $|p|$ in order to analytically control all marginal operators of the theory apart from H_f in $T[z; \mathcal{O}_0]$ (which we choose as the 'reference operator'). Thus, proving sufficiently small upper bounds on $|\partial_{|p|}^\beta a[z]|$, for $\beta = 0, 1$, is the key point of the present renormalization group study.

Under the assumption that $\eta[\mathfrak{h}] = 1 + O(\epsilon_0^2)$ can be proved by the renormalization group analysis, bounds on $\partial_{|p|} E_0[|p|, \sigma_0]$ are implied by bounds on $a_n[z_n]$. We will now explain our strategy to determine

$$\begin{aligned} a_n[z_n] &= \frac{\Big\langle Q_n^\sharp[z[z_n]]\,(\partial_{\mathbf{P}_f^\parallel} H_p(\sigma_0))\,Q_n[z[z_n]]\Big\rangle_{\Omega_f}}{\big\langle\, \partial_{H_f} F_n[z[z_n]]\big\rangle_{\Omega_f}} \\ &= |p|\,\eta_n[z[z_n]]^{-1} \\ &\quad - \eta_n[z[z_n]]^{-1} \frac{\Big\langle Q_n^\sharp[z[z_n]](\mathbf{P}_f^\parallel + g\mathbf{A}_f^\parallel(\sigma_0))\,Q_n[z[z_n]]\Big\rangle_{\Omega_f}}{\Big\langle Q_n^\sharp[z[z_n]]\,Q_n[z[z_n]]\Big\rangle_{\Omega_f}}\,, \end{aligned} \tag{6.3.1}$$

where we recall the shorthand

$$\begin{aligned} Q_n^{(\sharp)}[z] &\equiv Q^{(\sharp)}_{\chi_{\rho^n},\tau_n}[H_p(\sigma_0) - z]\,, \\ F_n[z] &\equiv F_{\chi_{\rho^n},\tau_n}[H_p(\sigma_0) - z]\,, \end{aligned}$$

in order to analytically control the coefficient $a_n[z_n]$ of $\mathbf{P}_f^\parallel$ in $T_{lin}[\mathfrak{h}^{(n)}]$. We recall that $(a_n[z_n]\eta[\mathfrak{h}^{(n-1)}]^{-1})|_{z_n = E_{0,n}}$ tends to $\partial_{|p|} E_0[|p|, \sigma_0]$ in the limit $n \to \infty$.

Proposition 6.3. *Assume that there exist constants $C, C' < \infty$ that are independent of n, such that*

$$\big\langle Q_n^\sharp[z]\,H_f\,Q_n[z]\big\rangle_{\Omega_f} \;\leq\; C\,\epsilon_0^2\,\big\langle Q_n^\sharp[z]\,Q_n[z]\big\rangle_{\Omega_f}\,, \tag{6.3.2}$$

and

$$\big|\eta_n[z[z_n]]^{-1} - 1\big| \;\leq\; C'\,\epsilon_0^2 \tag{6.3.3}$$

hold. Then,

$$|a_n[z_n] + |p|| \;\leq\; \sqrt{\epsilon}_0$$

is satisfied for any $n \geq 0$.

Proof. Clearly,

$$\begin{aligned} \left| \left\langle Q_n^\sharp[z[z_n]]\, \mathbf{P}_f^{\shortparallel}\, Q_n[z[z_n]] \right\rangle_{\Omega_f} \right| &\leq \left\langle Q_n^\sharp[z[z_n]]\, \left| \mathbf{P}_f^{\shortparallel} \right|\, Q_n[z[z_n]] \right\rangle_{\Omega_f} \\ &\leq \left\langle Q_n^\sharp[z[z_n]]\, H_f\, Q_n[z[z_n]] \right\rangle_{\Omega_f} \ . \end{aligned} \tag{6.3.4}$$

Furthermore,

$$\begin{aligned} \left| \left\langle Q_n^\sharp[z[z_n]]\, \mathbf{A}_f^{\shortparallel}(\sigma_0)\, Q_n[z[z_n]] \right\rangle_{\Omega_f} \right| &\leq 2 \left\langle Q_n^\sharp[z[z_n]]\, \mathbf{A}_{f,-}^{\shortparallel}(\sigma_0)\, Q_n[z[z_n]] \right\rangle_{\Omega_f} \\ &\leq 2\, b\, \left\langle Q_n^\sharp[z[z_n]]\, Q_n[z[z_n]] \right\rangle_{\Omega_f}^{\frac{1}{2}} \times \\ &\qquad \times \left\langle Q_n^\sharp[z[z_n]]\, H_f\, Q_n[z[z_n]] \right\rangle_{\Omega_f}^{\frac{1}{2}} \ , \end{aligned} \tag{6.3.5}$$

where $\mathbf{A}_{f,-}^{\shortparallel}(\sigma_0)$ is the part in $\mathbf{A}_f^{\shortparallel}(\sigma_0)$ linear in annihilation operators, and the constant b is bounded by

$$b \ \leq \ \left(\int_{|k| \leq \Lambda} d^3k\, \frac{1}{|k|} \right)^{\frac{1}{2}} \ ,$$

where Λ denotes the ultraviolet cutoff introduced into the interaction in $H_p(\sigma_0)$, as we recall. This is due to the standard inequality

$$(\phi\,,\, a_\lambda(f)\,\phi) \ \leq \ \left(\int d^3k\, \frac{|f(k)|^2}{|k|} \right)^{\frac{1}{2}} \ \|\phi\|\ (\phi\,,\, H_f\,\phi)^{\frac{1}{2}}$$

for any $\phi \in \mathcal{H}_p$. Comparison with (6.3.1) shows that this immediately implies the claim. ■

The key point in the construction thus is to prove that (6.3.2) is indeed satisfied. To this end, we invoke a renormalization group iteration, which corresponds to an induction argument in n. Let

$$Q_0^{(\sharp)}[z] \ \equiv \ Q_{\chi_1,\tau_0}^{(\sharp)}[H_p(\sigma_0) - z] \ , \tag{6.3.6}$$

and let us introduce the notation

$$\langle Q^\sharp A Q \rangle_{(j;k)} \ := \ \left\langle Q^\sharp[\mathfrak{h}^{(k-1)}] Ad_{U_\rho} \left[\cdots Ad_{U_\rho} \left[Q^\sharp[\mathfrak{h}^{(j)}] A Q[\mathfrak{h}^{(j)}] \right] \cdots \right] Q[\mathfrak{h}^{(k-1)}] \right\rangle_{\Omega_f} \tag{6.3.7}$$

for any operator A acting on $\mathcal{H}_p$, and $0 \leq j < k$.

Lemma 6.3. *Assume that for any arbitrary, but fixed scale n,*

$$\epsilon_k \ \leq \ 2\,\epsilon_0 \ ,$$

and

$$\left| \frac{\left\langle Q^\sharp\, Q \right\rangle_{(k;n)}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right| \ \leq \ 2 \ , \tag{6.3.8}$$

for $0 \leq k < n$. In addition, assume that there exists a constant $B < \infty$ that is independent of n and ϵ_0, such that

$$\left\| Q^\sharp[\mathfrak{h}^{(0)}] \, \left(Q_0^\sharp[z]\, H_f\, Q_0[z] \ - \ H_f \right) \, Q[\mathfrak{h}^{(0)}] \right\| \ \leq \ B\, \epsilon_0^2 \tag{6.3.9}$$

and

$$\left\| Q^\sharp[\mathfrak{h}^{(k+1)}]\, U_\rho\, \left(Q^\sharp[\mathfrak{h}^{(k)}]\, H_f\, Q[\mathfrak{h}^{(k)}] \,-\, H_f \right)\, U_\rho^*\, Q[\mathfrak{h}^{(k+1)}] \right\| \;\leq\; B\,\epsilon_0^2 \tag{6.3.10}$$

for all $0 \leq k < n$. *Then,*

$$\left\langle Q_n^\sharp[z]\, H_f\, Q_n[z] \right\rangle_{\Omega_f} \;\leq\; 4\,B\,\epsilon_0^2\, \left\langle Q_n^\sharp[z]\, Q_n[z] \right\rangle_{\Omega_f} \;.$$

Proof. Clearly,

$$\left\langle Q_n^\sharp[z]\, H_f\, Q_n[z] \right\rangle_{\Omega_f} \;=\; \left\langle Q^\sharp\, \left(Q_0^\sharp[z]\, H_f\, Q_0[z] \right)\, Q \right\rangle_{(0;n)} \;. \tag{6.3.11}$$

The renormalization group recursion is defined as follows. Due to the condition

$$\left\| Q^\sharp[\mathfrak{h}^{(0)}]\, \left(Q_0^\sharp[z]\, H_f\, Q_0[z] \,-\, H_f P_1 \right)\, Q[\mathfrak{h}^{(0)}] \right\| \;\leq\; B\,\epsilon_0^2 \tag{6.3.12}$$

which is assumed to be satisified, one has

$$\left\langle Q_n^\sharp[z]\, H_f\, Q_n[z] \right\rangle_{\Omega_f} \;\leq\; \left\langle Q^\sharp\, H_f\, Q \right\rangle_{(0;n)} \,+\, B\,\epsilon_0^2\, \left\langle Q^\sharp\, Q \right\rangle_{(1;n)} \tag{6.3.13}$$

for a constant B that is independent of n. The assumption that

$$\epsilon_k \;\leq\; 2\,\epsilon_0 \;,$$

for general $0 \leq k \leq n$, is contained in a 'global' induction argument of which the statement of the lemma is only a part. Let us then consider the quantity

$$\left\langle Q^\sharp\, H_f\, Q \right\rangle_{(k;n)}$$

for $k \leq n-3$, which equals

$$\begin{aligned} &\left\langle Q^\sharp\, U_\rho\, H_f\, U_\rho^*\, Q \right\rangle_{(k+1;n)} \\ &+\, \left\langle Q^\sharp\, \left(Q^\sharp[\mathfrak{h}^{(k+1)}] U_\rho\, \left(Q^\sharp[\mathfrak{h}^{(k)}] H_f Q[\mathfrak{h}^{(k)}] - H_f P_\rho \right)\, U_\rho^* Q[\mathfrak{h}^{(k+1)}] \right)\, Q \right\rangle_{(k+2;n)} \;. \end{aligned}$$

By (6.3.10) and $U_\rho H_f U_\rho^* = \rho\, H_f$, this is bounded from above by

$$\rho\, \left\langle Q^\sharp\, H_f\, Q \right\rangle_{(k+1;n)} \,+\, B\,\epsilon_0^2\, \left\langle Q^\sharp\, Q \right\rangle_{(k+2;n)} \;. \tag{6.3.14}$$

Using the 'final condition'

$$\begin{aligned} \left\langle Q^\sharp\, H_f\, Q \right\rangle_{(n-1;n)} \;&\leq\; \rho\, \left\langle Q^\sharp[\mathfrak{h}^{(n-1)}]\, H_f\, Q[\mathfrak{h}^{(n-1)}] \right\rangle_{\Omega_f} \\ &\leq\; B\,\epsilon_0^2 \;, \end{aligned} \tag{6.3.15}$$

and backwards iterating the above estimates according to $n \to n-1 \to \cdots \to k \to \cdots \to 0$, one obtains

$$\begin{aligned} \left\langle Q_n^\sharp[z]\, H_f\, Q_n[z] \right\rangle_{\Omega_f} \;&\leq\; \sum_{k=0}^{n} \rho^{n-k} B\,\epsilon_0^2\, 2\, \left\langle Q_n^\sharp[z]\, Q_n[z] \right\rangle_{\Omega_f} \\ &=\; 4\,B\,\epsilon_0^2\, \left\langle Q_n^\sharp[z]\, Q_n[z] \right\rangle_{\Omega_f} \end{aligned} \tag{6.3.16}$$

for $\rho = \frac{1}{2}$, which is the asserted result. ■

6.4. Renormalization of $\partial_{|p|}a[z]$

In order to prove C^2-smoothness of the ground state energy $E_0[|p|,\sigma_0]$, it is necessary to analytically control first order derivatives of $H[\mathfrak{h}]$ with respect to $|p|$. Due to the pure marginality of $a[z]$, this task is complicated.

To begin with, let us recall that the smooth Feshbach map is defined by

$$\begin{aligned} F_n[z] &= \tau[H_p(\sigma_0)] + \chi_{\rho^n}\omega[H_p(\sigma_0)]\chi_{\rho^n} \\ &\quad - \chi_{\rho^n}\,\omega[H_p(\sigma_0)]\,\bar{\chi}_{\rho^n}\,\bar{R}\,\bar{\chi}_{\rho^n}\,\omega[H_p(\sigma_0)]\,\chi_{\rho^n} \end{aligned} \tag{6.4.1}$$

with

$$\tau[H_p(\sigma_0)] = \langle \partial_{H_f} F_n[z]\rangle_{\Omega_f}\, H_f - z' \tag{6.4.2}$$

and

$$\omega[H_p(\sigma_0)] = H_p(\sigma_0) - \big(\langle \partial_{H_f} F_n[z]\rangle_{\Omega_f} - 1\big)\, H_f - \Delta E_n[z]\,, \tag{6.4.3}$$

where

$$\Delta E_n[z] \equiv \langle F_n[z] + z\rangle_{\Omega_f} \tag{6.4.4}$$

and

$$z' = z - \Delta E_n[z]\,.$$

Furthermore, defining

$$z_n \equiv \langle \partial_{H_f} F_n[z]\rangle_{\Omega_f}^{-1}\,\rho^{-n}\,z'\,, \tag{6.4.5}$$

and assuming that $z \to z_n$ is invertible, we recall that

$$a_n[z_n] = \eta_n[z[z_n]]^{-1}\,\frac{\Big\langle Q_n^\sharp[z[z_n]](\partial_{\mathbf{P}_f^\parallel} H_p(\sigma_0))Q_n[z[z_n]]\Big\rangle_{\Omega_f}}{\Big\langle Q_n^\sharp[z[z_n]]Q_n[z[z_n]]\Big\rangle_{\Omega_f}}\,. \tag{6.4.6}$$

The explicit expression for $\partial_{|p|}a_n[z_n]$ is given in the following proposition.

Proposition 6.4. *Let n denote a fixed scale, and $z_n \in [-\epsilon,0] \subset \mathbb{R}_-$. Then, $\partial_{|p|}a_n[z_n]$ is explicitly given by*

$$\begin{aligned} \partial_{|p|}a_n[z_n] &= -\frac{\partial_{|p|}\eta_n[z_n]}{\eta_n[z_n]}\,a_n[z_n] + \eta_n[z[z_n]]^{-1} \\ &\quad - 2\,\eta_n[z[z_n]]^{-1}\,\frac{\Big\langle (\partial_{|p|}Q_n^\sharp[z[z_n]])\left(\partial_{\mathbf{P}_f^\parallel}H_p(\sigma_0)\right)\partial_{|p|}Q_n[z[z_n]]\Big\rangle_{\Omega_f}}{\Big\langle Q_n^\sharp[z[z_n]]Q_n[z[z_n]]\Big\rangle_{\Omega_f}} \\ &\quad + \rho^{n-1}\left(\Big\langle (\partial_{|p|}Q^\sharp[\mathfrak{h}^{(n-1)}])\,\bar{P}_\rho\frac{(H_f-\rho z_n)\chi_\rho^2}{\bar{\chi}_\rho^2}\bar{P}_\rho\,Q[\mathfrak{h}^{(n-1)}]\Big\rangle_{\Omega_f} + h.c.\right)\times \\ &\quad \times\eta_n[z[z_n]]^{-1}\,\frac{\langle\partial_{|p|}\partial_{H_f}F_n[z[z_n]]\rangle_{\Omega_f}}{\Big\langle Q_n^\sharp[z[z_n]]Q_n[z[z_n]]\Big\rangle_{\Omega_f}}\,\frac{\Big\langle(\partial_{|p|}Q_n^\sharp[z[z_n]])\,Q_n[z[z_n]]\Big\rangle_{\Omega_f}}{\Big\langle Q_n^\sharp[z[z_n]]Q_n[z[z_n]]\Big\rangle_{\Omega_f}} \\ &\quad + \rho^{n-1}\left(\big\langle(\partial_{|p|}Q^\sharp[\mathfrak{h}^{(n-1)}])\chi_\rho\partial_{|p|}H[\mathfrak{h}^{(n)}]\big\rangle_{\Omega_f} + h.c.\right)\,. \end{aligned} \tag{6.4.7}$$

In addition, for $|p| \le p_c$, let us assume that there exist constants $C, C', C'' < \infty$ independent of n and ϵ_0, such that

$$\begin{aligned}
\epsilon_k &\le 2\,\epsilon_0 \\
\left| \frac{\langle \partial_{|p|} \partial_{H_f} F_n[z[z_n]] \rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right| &\le C\,\epsilon_0^2\, n \\
\left| \frac{\left\langle (\partial_{|p|} Q_n^\sharp[z[z_n]])\, Q_n[z[z_n]] \right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right| &\le C\,\epsilon_0^2\, n \\
\left| \frac{\left\langle Q^\sharp\, Q \right\rangle_{(k;n)}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right| &\le 2 \\
\left\| (\partial_{|p|} Q[\mathfrak{h}_0^{(k)}]) U_\rho^* Q[\mathfrak{h}_0^{(k+1)}] \right\| &\le C'\,\epsilon_0\ , \\
\left| \partial_{|p|}^\beta \eta_n[z[z_n]] \right| &\le C''\,\epsilon_0^2 \\
Q^\sharp[\mathfrak{h}^{(k)}] H[\mathfrak{h}^{(k)}] Q[\mathfrak{h}^{(k)}] &\le \frac{7}{4}\, F_{\chi_\rho,\tau}[H[\mathfrak{h}^{(k)}]] \qquad (6.4.8)
\end{aligned}$$

for $\beta = 0, 1$, and all $0 \le k \le n$. Then, if $\left|\partial_{|p|} a_k[z_k] + 1\right| \le \sqrt{\epsilon}_0$ holds for all $0 \le k < n$, it follows that

$$\left|\partial_{|p|} a_n[z_n] + 1\right| \ \le\ \sqrt{\epsilon}_0 \qquad (6.4.9)$$

is satisfied for $k = n$.

Proof. Taking the derivative with respect to $|p|$ of (6.4.6),

$$\begin{aligned}
\partial_{|p|} a_n[z_n] \ =\ & -\frac{\partial_{|p|}\eta_n[z_n]}{\eta_n[z_n]}\, a_n[z_n] \\
& + \eta_n[z_n]^{-1}\, \frac{\left\langle Q_n^\sharp[z[z_n]] (\partial_{|p|} \partial_{\mathbf{P}_f^\parallel} H_p(\sigma_0)) Q_n[z[z_n]] \right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \\
& + \eta_n[z_n]^{-1}\, \frac{\left\langle (\partial_{|p|} Q_n^\sharp[z[z_n]]) \left(\partial_{\mathbf{P}_f^\parallel} H_p(\sigma_0) - a_n[z_n]\eta_n[z_n] \right) Q_n[z[z_n]] \right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \\
& + \eta_n[z_n]^{-1}\, \frac{\left\langle Q_n^\sharp[z[z_n]] \left(\partial_{\mathbf{P}_f^\parallel} H_p(\sigma_0) - a_n[z_n]\eta_n[z_n] \right) (\partial_{|p|} Q_n[z[z_n]]) \right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \qquad (6.4.10)
\end{aligned}$$

The first two terms on the right hand side are, due to

$$\partial_{|p|} \partial_{\mathbf{P}_f^\parallel} H_p(\sigma_0) \ =\ 1\ ,$$

determined by $\partial_{|p|}^{\beta}\eta_n[z[z_n]]$ alone, for $\beta = 0, 1$. The main problem consists of estimating the two last terms in (6.4.10).

Since both of them can be treated in the precise same manner, we will only focus on one of them, say the expression on the second last line. It will shown in the course of the renormalization group analysis in the subsequent chapters that for fixed n, all maps $z_k \mapsto z_{k+1}$, with $k = 0, \ldots, n-1$, are invertible (for all z_k in suitable domains). Hence, the map $z \to z_n$ is invertible, and

$$z[z_n] \;=\; \Delta E_n[z[z_n]] \,+\, \langle \partial_{H_f} F_n[z[z_n]] \rangle_{\Omega_f}\, \rho^n\, z_n$$

is well-defined. Taking the derivative with respect to $|p|$ on both sides of

$$(H_p(\sigma_0) \,-\, z[z_n])\, Q_n[z_n] \;=\; \chi_{\rho^n} F_n[z[z_n]] \;,$$

and recalling that

$$\partial_{|p|} H_p(\sigma_0) \;=\; -\,\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0) \;,$$

we find

$$\begin{aligned}
\Big(\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0)\Big)\, Q_n[z[z_n]] \;\;=\;\; & -\,(H_p(\sigma_0) - z[z_n])\,\partial_{|p|} Q_n[z[z_n]] \\
& + \langle \partial_{|p|} F_n[z[z_n]] \rangle_{\Omega_f} Q_n[z[z_n]] \\
& + \chi_{\rho^n} \partial_{|p|} F_n[z[z_n]] \;.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \eta_n[z_n]^{-1}\, \frac{\Big\langle (\partial_{|p|} Q_n^{\sharp}[z[z_n]])\, \Big(\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0) - a_n[z_n]\eta_n[z_n]\Big)\, Q_n[z[z_n]] \Big\rangle_{\Omega_f}}{\Big\langle Q_n^{\sharp}[z[z_n]] Q_n[z[z_n]] \Big\rangle_{\Omega_f}} \\
& = -\,\eta_n[z_n]^{-1}\, \frac{\big\langle (\partial_{|p|} Q_n^{\sharp}[z[z_n]])\, (H_p(\sigma_0) - z[z_n])\, \partial_{|p|} Q_n[z[z_n]] \big\rangle_{\Omega_f}}{\Big\langle Q_n^{\sharp}[z[z_n]] Q_n[z[z_n]] \Big\rangle_{\Omega_f}} \qquad (6.4.11) \\
& + \eta_n[z_n]^{-1}\, \frac{\big\langle (\partial_{|p|} Q_n^{\sharp}[z_n])\, \big(\partial_{|p|} \Delta E_n[z[z_n]] - a_n[z_n]\eta_n[z_n]\big)\, Q_n[z[z_n]] \big\rangle_{\Omega_f}}{\Big\langle Q_n^{\sharp}[z[z_n]] Q_n[z[z_n]] \Big\rangle_{\Omega_f}} \qquad (6.4.12) \\
& + \eta_n[z_n]^{-1}\, \frac{\big\langle (\partial_{|p|} Q_n^{\sharp}[z[z_n]]) \chi_{\rho^n} \partial_{|p|} F_n[z[z_n]] \big\rangle_{\Omega_f}}{\Big\langle Q_n^{\sharp}[z[z_n]] Q_n[z[z_n]] \Big\rangle_{\Omega_f}} \qquad (6.4.13)
\end{aligned}$$

Let us first focus on the term (6.4.12).

It is straightforward to deduce by means of Lemma 4.2, that

$$\begin{aligned}
\partial_{|p|} \Delta E_n[z[z_n]] \;\;=\;\; & \langle \partial_{|p|} \partial_{H_f} F_n[z[z_n]] \rangle_{\Omega_f} \Big\langle Q_n^{\sharp}[z[z_n]]\, \bar{P} \frac{(H_f - \rho^n\, z_n)}{\bar{\chi}_{\rho^n}^2} \bar{P}\, Q_n[z[z_n]] \Big\rangle_{\Omega_f} \\
& - \Big\langle Q_n^{\sharp}[z[z_n]]\, (\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0))\, Q_n[z[z_n]] \Big\rangle_{\Omega_f} \\
& - (\partial_{|p|} \Delta E_n[z[z_n]])\, \big\langle Q_n^{\sharp}[z[z_n]]\, \bar{P}\, Q_n[z[z_n]] \big\rangle_{\Omega_f} \;. \qquad (6.4.14)
\end{aligned}$$

Therefore, using

$$\left\langle Q_n^\sharp[z[z_n]]\, Q_n[z[z_n]]\right\rangle_{\Omega_f} \;=\; 1 \,+\, \left\langle Q_n^\sharp[z[z_n]]\, \bar{P}\, Q_n[z[z_n]]\right\rangle_{\Omega_f} \;,$$

we have

$$\partial_{|p|}\Delta E_n[z[z_n]] \;=\; -\,\frac{\left\langle Q_n^\sharp[z[z_n]]\,(\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0))\, Q_n[z[z_n]]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]]\, Q_n[z[z_n]]\right\rangle_{\Omega_f}}$$

$$+\,\langle \partial_{|p|}\partial_{H_f} F_n[z[z_n]]\rangle_{\Omega_f}\,\frac{\left\langle Q_n^\sharp[z[z_n]]\,\bar{P}\frac{(H_f-\rho^n\, z_n)\chi_{\rho^n}^2}{\bar{\chi}_{\rho^n}^2}\bar{P}\, Q_n[z[z_n]]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]]\, Q_n[z[z_n]]\right\rangle_{\Omega_f}} \;, \tag{6.4.15}$$

so that by (6.4.6),

$$\begin{aligned}
&= \frac{\left\langle (\partial_{|p|}Q_n^\sharp[z[z_n]])\,\bar{P}\frac{(H_f-\rho^n\, z_n)\chi_{\rho^n}^2}{\bar{\chi}_{\rho^n}^2}\bar{P}\, Q_n[z[z_n]]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]]\right\rangle_{\Omega_f}} \;\times \qquad (\,6.4.12) \\
&\quad \times \eta_n[z_n]^{-1}\,\langle \partial_{|p|}\partial_{H_f} F_n[z[z_n]]\rangle_{\Omega_f}\,\frac{\left\langle (\partial_{|p|}Q_n^\sharp[z[z_n]])\, Q_n[z[z_n]]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]]\right\rangle_{\Omega_f}} \\
&= \rho^{n-1}\left\langle (\partial_{|p|}Q^\sharp[\mathfrak{h}^{(n-1)}])\,\bar{P}_\rho\frac{(H_f-\,\rho z_n)\chi_\rho^2}{\bar{\chi}_\rho^2}\bar{P}_\rho\, Q[\mathfrak{h}^{(n-1)}]\right\rangle_{\Omega_f} \;\times \\
&\quad \times \eta_n[z[z_n]]^{-1}\,\frac{\langle \partial_{|p|}\partial_{H_f} F_n[z[z_n]]\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]]\right\rangle_{\Omega_f}}\,\frac{\left\langle (\partial_{|p|}Q_n^\sharp[z[z_n]])\, Q_n[z[z_n]]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]]\right\rangle_{\Omega_f}} \;.
\end{aligned}$$

Finally,

$$\begin{aligned}
&= \rho^{n-1}\,\eta_n[z[z_n]]^{-1}\,\langle \partial_{H_f} F_n[z[z_n]]\rangle_{\Omega_f} \;\times \qquad (\,6.4.13) \\
&\qquad\qquad \times \frac{\left\langle (\partial_{|p|}Q^\sharp[\mathfrak{h}^{(n-1)}])\chi_\rho\partial_{|p|}H[\mathfrak{h}^{(n)}]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]]\right\rangle_{\Omega_f}} \\
&= \rho^{n-1}\,\left\langle (\partial_{|p|}Q^\sharp[\mathfrak{h}^{(n-1)}])\chi_\rho\partial_{|p|}H[\mathfrak{h}^{(n)}]\right\rangle_{\Omega_f} \;. \qquad (6.4.16)
\end{aligned}$$

This proves the asserted expression for $\partial_{|p|}a_n[z_n]$.

The renormalization group analysis in the subsequent chapters will show that the bound

$$\left|\frac{\left\langle\langle \partial_{|p|}\partial_{H_f} F_n[z[z_n]]\rangle_{\Omega_f}\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]]\right\rangle_{\Omega_f}}\right| \;,\; \left|\frac{\left\langle (\partial_{|p|}Q_n^\sharp[z[z_n]])\, Q_n[z[z_n]]\right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]]\right\rangle_{\Omega_f}}\right| \le C\,\epsilon_0\, n \;,$$

indeed holds, where the constant is $O(1)$ with respect to n and g. Due to

$$\left\langle (\partial_{|p|}Q^\sharp[\mathfrak{h}^{(n-1)}])\,\bar{P}_\rho\frac{(H_f-\,\rho z_n)\chi_\rho^2}{\bar{\chi}_{\rho^n}^2}\bar{P}_\rho\, Q[\mathfrak{h}^{(n-1)}]\right\rangle_{\Omega_f} \;\le\; O(\epsilon_0) \;,$$

one has

$$|(\ 6.4.12)| \ \leq \ C\,\epsilon_0\,\rho^n\,n^2 \quad ,$$

which tends to 0 in the limit $n \to \infty$. Furthermore, one can straightforwardly show that

$$(\ 6.4.13) \ \leq \ \rho^n\,C\,\epsilon_0 \ .$$

In order to estimate (6.4.11), we invoke a renormalization group iteration. We define

$$A_0^{(n)} \ := \ \left(\frac{\left\langle (\partial_{|p|} Q_n^\sharp[z[z_n]])\,(H_p(\sigma_0) - z[z_n])\,\partial_{|p|} Q_n[z[z_n]] \right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right)^{\frac{1}{2}} \quad ,$$

and for general n,

$$A_k^{(n)} \ := \ \left(\frac{\left\langle (\partial_{|p|} Q^\sharp) H[\mathfrak{h}^{(k)}] \partial_{|p|} Q \right\rangle_{(k;n)}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right)^{\frac{1}{2}} \quad ,$$

using the notation introduced in (6.3.7), and

$$\mathfrak{h} = (z, T, \{w_{M,N}\}) \ .$$

To relate subsequent $A_k^{(n)}$'s for $k \to k+1$, for $0 \leq k < n$, to one another, we exploit the concatenation rules for $Q^\sharp, Q$,

$$\Omega[\mathfrak{h}_0] \ = \ Q[\mathfrak{h}_0]\,U_\rho^*\,\Omega[\hat{\mathfrak{h}}_0] \ ,$$

where U_ρ^* reverts the action of rescaling, and $\hat{\mathfrak{h}}_0 = \mathcal{R}[\mathfrak{h}_0]$. Next, we exploit that

$$\partial_{|p|}\Omega[\mathfrak{h}] \ = \ Q_n[E_0]\,U_\rho^*\,\partial_{|p|}\Omega[\hat{\mathfrak{h}}_0] \ + \ (\partial_{|p|} Q_n[E_0])\,U_\rho^*\,\Omega[\hat{\mathfrak{h}}_0],$$

and

$$\begin{aligned} Q^\sharp[\mathfrak{h}] H[\mathfrak{h}] Q[\mathfrak{h}] \ &\leq \ (1+\mu) F_{\chi,\tau}[H[\mathfrak{h}]] \\ &= \ (1+\mu)\langle \partial_{H_f} F_{\chi,\tau}[H[\mathfrak{h}]] \rangle_{\Omega_f} \rho\, Ad_{U_\rho^*}[H[\hat{\mathfrak{h}}_0]] \end{aligned}$$

for some $\mu < \frac{3}{4}$, and $|\langle \partial_{H_f} F_{\chi,\tau}[H[\mathfrak{h}]] \rangle_{\Omega_f}| = 1 + O(\epsilon^2)$. Thus, we find through the Cauchy-Schwarz inequality that

$$\begin{aligned} A_k^{(n)} \ &\leq \ \sqrt{\rho(1+\mu)\langle \partial_{H_f} F_{\chi,\tau}[H[\mathfrak{h}]] \rangle_{\Omega_f}}\, A_{k+1}^{(n)} \\ &\quad + \ \left\| \sqrt{H[\mathfrak{h}^{(k)}]} (\partial_{|p|} Q[\mathfrak{h}^{(k)}]) U_\rho^* Q[\mathfrak{h}^{(k+1)}] \right\| \ \sqrt{\frac{\langle Q^\sharp Q \rangle_{(k+2;n)}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}}} \\ &\leq \ \frac{9}{10}\, A_{k+1}^{(n)} \ + \ C\,\epsilon_0 \end{aligned}$$

for $\rho = \frac{1}{2}$. This follows from the assumption that

$$\left\| (\partial_{|p|} Q[\mathfrak{h}^{(k)}]) U_\rho^* Q[\mathfrak{h}^{(k+1)}] \right\| \ \leq \ O(\epsilon_0) \ ,$$

and

$$\frac{\left\langle Q^\sharp Q\right\rangle_{(k;n)}}{\left\langle Q_n^\sharp[z[z_n]]Q_n[z[z_n]]\right\rangle_{\Omega_f}} \;\leq\; 2$$

for all $0 \leq k < n$. Therefore,

$$A_0^{(n)} \;\leq\; C'\,\epsilon_0 \;,$$

for a constant C' that is independent of ϵ and n. By assumption, $\epsilon_k \leq 2\epsilon_0$ for all $k \geq 0$.

Collecting the above results, one arrives at an estimate of the form

$$|\partial_{|p|}a_n[z_n]| \;\leq\; (1 + C_1\epsilon_0)\,(1 \,+\, C_2\,\epsilon_0) \;.$$

Thus, we obtain

$$|\partial_{|p|}a_n[z_n]| \;\leq\; 1 \,+\, \sqrt{\epsilon_0} \;,$$

for $n \geq 0$.

This proves the proposition. ■

An immediate consequence is that due to

$$\partial_{|p|}^2 E_0[|p|,\sigma_0] \;=\; \lim_{n\to\infty} \partial_{|p|}\left(\eta_n[z[z_n]]\,a_n[z_n]\right) \;,$$

the bounds on $\partial_{|p|}a_n[z_n]$ obtained above immediately imply

$$\left|\partial_{|p|}^2\left(E_0[|p|,\sigma_0] - \frac{|p|^2}{2}\right)\right| \;\leq\; \sqrt{\epsilon_0}$$

for the ground state energy of $H_p(\sigma_0)$.

6.5. The Ward-Takahashi identities

Analytically controlling the renormalization flow of purely marginal interactions is usually a very difficult task. The key to the solution of this problem here is to exploit the $U(1)$-gauge invariance of the effective hamiltonians $H[\mathfrak{h}^{(n)}]$. This reduces the number of marginal operators to merely one in the present system, given by $a[z]$ (since the coefficient of the purely marginal operator H_f is kept fixed at the value 1). As has been demonstrated in the preceding sections, our approach to renormalizing $a[z]$ heavily uses the concatenation identities fulfilled by Feshbach triples.

On the quantum level, gauge invariance imposes certain constraints on the correlation functions of the theory. They manifest themselves in the form of relationships between the fully renormalized n-point functions of the theory, for varying n, which are in the context of QED referred to as Ward-Takahashi (W-T) identities. A standard textbook exposition can for instance be found in [**17**].

In the present operator-theoretic framework, our focus will not be on n-point functions; they do in fact not enter the discussion at all. Instead, we will prove a relationship satisfied by the effective hamiltonians at arbitrary scales, which represent (a generalization of) the W-T identities.

6.6. Ward-Takahashi identities in non-relativistic QED

The structure of the W-T identities in non-relativistic QED is the same as in the special relativistic theory. However, due to the lack of antiparticle production in the non-relativistic limit, the "spinor-algebraic" aspects can be simplified, and for the spinless model at hand, they are trivial. The derivation of the W-T identities of non-relativistic QED in the standard textbook way is accomplished in full analogy to the relativistic case, for which we refer for instance to [**17**]. They can, of course, also be obtained by properly taking the non-relativistic limit of the relativistic W-T identities in QED. We omit an exposition of these matters, and simply state the final result, since our discussion yields a different, Feshbach-theoretic proof.

To this end, let the scalar

$$\Sigma(p)$$

denote the **renormalized 2-point function** of the spinless fermion, and

$$\Gamma^i(p,k,p+k)$$

the **renormalized vertex function** in non-relativistic, zero-fermion-spin QED. The vertex function describes an incoming fermion of momentum p that absorbs a transverse photon of momentum k, after which it propagates on with a different momentum $p+k$. The index $'i'$ couples it to the components of the quantized electromagnetic vector potential in Coulomb gauge.

The Ward-Takahashi identities

$$k_i\Gamma^i(p,k,p+k) \;=\; \Sigma(p)-\Sigma(p+k)\;.$$

are fully analogous to the relativistic case. The renormalized vertex function is thus determined by the renormalized charge propagator. More relevant for us is its differential expression,

$$\Gamma^i(p,0,p) \;=\; -\,\partial_{p_i}\Sigma(p)\;, \tag{6.6.1}$$

obtained in the limit $|k|\to 0$, which will occur in this form in the operator-theoretic context.

In the context of non-relativistic QED, it is clear that

$$\Sigma(p) \;=\; E_0[|p|,\sigma_0]\;,$$

the **renormalized ground state energy** of $H_p(\sigma_0)$, and $g\Gamma^i(p,0,p)$ turns out to be the renormalized coefficient of the marginal interaction term. Thus,

$$\Gamma^i(p,0,p) \;=\; n^i_p\,\lim_{n\to\infty} a_n[E_{0,n}]\;,$$

and we observe that the Ward-Takahashi identities are equivalent to the statement of Proposition 6.1, given that its assumptions are satisfied.

Likewise, there are higher order Ward-Takahashi identities which relate n-point functions of the theory to n'-point functions, with $n'<n$, which we will not discuss.

In the present operator-theoretic formulation, we will verify that a particular relationship is fulfilled on the operator level. This is to say, in contrast to relations between particular kinds of expectation values, for instance. They are satisfied by all effective hamiltonians, and will be interpreted as a generalized form of Ward-Takahashi identities.

6.7. Ward-Takahashi identities in Feshbach renormalization

We will now return to the functional analytic setting introduced in the previous chapters, and formulate a variant of the W-T identities that is applicable in the operator theoretical renormalization scheme. The relations between n-point functions in the above discussion of the functional integral formulation of the problem are now translated into relations between partial derivatives of the effective hamiltonians with respect to operators, and commutators with creation and annihilation operators.

The strategy is based on the recursive application of Theorem 4.3. We first prove that the physical hamiltonian $H_p(\sigma_0)$ obeys a relationship of the form stated there. The consequence is that the effective hamiltonian $H[\mathfrak{h}^{(0)}]$ obeys the same relationship, thus also $H[\mathfrak{h}^{(1)}]$, and so on.

The starting point is the following fact.

LEMMA 6.4. *The operator*

$$\mathcal{D}^\sharp_{comm} \;:=\; \lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} \bar\kappa_\sigma^{-1}[|k|]\,\sqrt{|k|}\,[a^\sharp_\lambda(k)\,,\,\cdot]$$

satisfies $\mathcal{D}^\sharp_{comm}[f[H_f]] = 0$ *for all* f *of class* C^1.

Proof. The derivation rules are obviously satisfied by commutators, thus $\mathcal{D}^\sharp_{comm}$ is a derivation.

Next, for any C^1-function f,

$$\lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} \bar\kappa_\sigma^{-1}[|k|]\,|k|^{\frac12}\,[a_\lambda(k)\,,\,f[H_f]] \;\;=\;\; 0\;,$$

This is because the expression on the left hand side, of which the limit is taken, can be written as

$$\sigma\,|k|^{-\frac12}\;(f[H_f+|k|]\;-\;f[H_f])\;a_\lambda(k)\;,$$

since $\bar\kappa_\sigma[|k|] = \frac{1}{\sigma}|k|$ for small $|k|$. C^1-smoothness of f implies the claim. ■

Next, we define the two following special derivations, which we will in the sequel refer to as the '**Ward-Takahashi derivations**':

$$\mathcal{D}^*_{WT,n_k,\lambda,\sigma} \;:=\; g\,\epsilon_{1,0}[n_k;\lambda]\cdot\partial_{\mathbf{P}_f} \;+\; \lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} \bar\kappa_\sigma^{-1}[|k|]\sqrt{|k|}\;[\cdot, a^*_\lambda(k)]$$

$$\mathcal{D}_{WT,n_k,\lambda,\sigma} \;:=\; g\,\epsilon_{0,1}[n_k;\lambda]\cdot\partial_{\mathbf{P}_f} \;+\; \lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} \bar\kappa_\sigma^{-1}[|k|]\sqrt{|k|}\;[a_\lambda(k),\cdot]\;, \tag{6.7.1}$$

where $\bar\kappa_\sigma$ denotes the running infrared regulator. Then, the following proposition holds.

PROPOSITION 6.5. *The physical hamiltonian* $H_p(\sigma_0)$ *satisfies*

$$\mathcal{D}^\sharp_{WT,n_k,\lambda,\sigma_0}[H_p(\sigma_0)] \;=\; 0\;. \tag{6.7.2}$$

Proof. The cutoff regularized hamiltonian of the system in discussion is, as we recall, given by

$$H_p(\sigma_0) \;=\; E_{bare}[|p|] + \; H_f \;+\; \frac{1}{2}\,:\!\Big(\,p \;-\; \mathbf{P}_f + g\mathbf{A}_f(\sigma_0)\Big)^2\!:\;, \tag{6.7.3}$$

with

$$E_{bare}[|p|] \;=\; \frac{|p|^2}{2} \;+\; \frac{g^2}{2}\;\big\langle \mathbf{A}_f(\sigma_0)^2\big\rangle_{\Omega_f}\;,$$

and acts on the Hilbert space $\mathcal{H}_p$, where p denotes the conserved total momentum.

It can be easily checked that

$$\partial_{\mathbf{P}_f} H_p(\sigma_0) \;=\; -\,p \;+\; \mathbf{P}_f \;+\; g\;\mathbf{A}_f(\sigma_0)\;, \tag{6.7.4}$$

and from

$$\mathbf{A}_f(\sigma_0) \;=\; \sum_\lambda \int \frac{d^3k\;\bar\kappa_{\sigma_0}[|k|]\;\kappa_\Lambda[|k|]}{\sqrt{|k|}}\Big(\,\epsilon_{1,0}[n_k,\lambda]a^*_\lambda(k) + \epsilon_{0,1}[n_k,\lambda]a_\lambda(k)\Big)\;, \tag{6.7.5}$$

that

$$\begin{aligned}[H_p(\sigma_0), a^*_\lambda(k)] \;&=\; a^*_\lambda(k)\left(\frac{1}{2}\left((\mathbf{P}_f+k)^2 - \mathbf{P}_f^2\right) + |k| - |p|k^{\|}\right)\\ &\quad + g\;a^*_\lambda(k)\;k\cdot\mathbf{A}_f(\sigma_0) \;+\; \frac{1}{2}a^*_\lambda(k)\\ &\quad + g\;\frac{\bar\kappa_{\sigma_0}[|k|]}{\sqrt{|k|}}\;\epsilon_{1,0}[n_k;\lambda]\cdot(p \;-\; \mathbf{P}_f \;-\; g\;\mathbf{A}_f(\sigma_0))\;.\end{aligned}$$

Therefore,

$$\lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} \bar\kappa_{\sigma_0}^{-1}[|k|]\sqrt{|k|}\;\;[H_p(\sigma_0), a^*_\lambda(k)] \;=\; -\,g\;\epsilon_{1,0}[n_k;\lambda]\;\cdot\;\partial_{\mathbf{P}_f}H_p(\sigma_0)\;, \tag{6.7.6}$$

and likewise,

$$\lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} \bar\kappa_{\sigma_0}^{-1}[|k|]\sqrt{|k|}\;\;[a_\lambda(k), H_p(\sigma_0)] \;=\; -\,g\;\epsilon_{0,1}[n_k;\lambda]\;\cdot\;\partial_{\mathbf{P}_f}H_p(\sigma_0)\;, \tag{6.7.7}$$

for the commutator with $a_\lambda(k)$. ■

We notice that the limit $|k|\to 0$ is in full analogy to the corresponding limit of vanishing external photon momentum which was used to obtain (6.6.1).

According to the notation used in Theorem 4.3, let

$$\mathcal{D}_1 \;:=\; \partial_{\mathbf{P}_f}\;,$$

and

$$h_1 \;:=\; -\,g\,\epsilon_{0,1}[n_k,\lambda]\,\mathbf{1}\;.$$

Furthermore, let

$$\mathcal{D}_2 \;:=\; \lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} \bar\kappa_{\sigma_0}^{-1}[|k|]\sqrt{|k|}\;[a_\lambda(k)\,,\,\cdot\,]\;,$$

and

$$h_2 \;:=\; \mathbf{1}\;.$$

Theorem 4.3 straightforwardly implies that these relations also hold for

$$H[\mathfrak{h}^{(0)}] \;=\; \mathcal{D}_1[H_p(\sigma_0) - z]\;.$$

That is,

$$\mathcal{D}^\sharp_{WT,n_k,\lambda,\sigma_0}[H[\mathfrak{h}^{(0)}]] \;=\; 0\;. \tag{6.7.8}$$

6.8. The Ward-Takahashi identities in inductive form

Arguing by induction, this relationship can be extended to all effective hamiltonians produced in the renormalization group iteration. The following proposition establishes this.

PROPOSITION 6.6. *Let $H[\mathfrak{h}]$, with $\mathfrak{h} \in \mathfrak{V}_{\epsilon,\xi}$ for ϵ, ξ sufficiently small, be a given effective hamiltonian, which satisfies*

$$\mathcal{D}^\sharp_{WT,n_k,\lambda,\sigma}[H[\mathfrak{h}]] \;=\; 0\;,$$

σ denotes the running infrared cutoff. Then, its renormalized counterpart $H[\hat{\mathfrak{h}}]$, with $\hat{\mathfrak{h}} = \mathcal{R}[\mathfrak{h}]$, satisfies

$$\mathcal{D}^\sharp_{WT,n_k,\lambda,\hat{\sigma}}[H[\hat{h}]] \;=\; 0\;,$$

where $\hat{\sigma} = \frac{\sigma}{\rho}$.

Proof. This follows directly from Theorem 4.3, and the fact that T and $w_{M,N}$ are of class C^1, for all $M + N \geq 1$, with respect to the components of $\mathcal{O}$, since $\mathfrak{h} \in \mathfrak{V}_{\epsilon,\xi}$. ■

The Ward-Takahashi identities in the operator-theoretic renormalization group are used in Chapters 7, 8, for the purpose of proving that the marginal interaction kernels $w_{M,N}$, for $M + N = 1$, are fully determined by $a[z]$, the coefficient of the operator $\mathbf{P}^{\|}_f$ in $T_{lin}[\mathfrak{h}]$. This reduces the number of independent marginal operators, whose renormalization must be controlled, by one.

6.9. Remark on alternative formulations

Other operator-theoretic versions of the Ward-Takahashi identities certainly deserve consideration. For instance, one could consider coupling the charged particle to a classical external electromagnetic vector potential $\mathbf{A}^{(cl)}_f$, and study derivatives of the effective hamiltonians with respect to $\mathbf{A}^{(cl)}_f$. Likewise, one could have thought of comparing derivatives with respect to $|p|$, of effective hamiltonians at subsequent scales to each other.

The problem here is that in order to achieve maximal simplicity of the method, it is desirable only to involve operations into the formulation of the W-T identities which **commute with rescaling**. Neither of the two aforementioned tentative alternatives have this property, because the factor $(1 + \Delta_{H_f}[z])^{-1}$ is a function both of $|p|$ and $\mathbf{A}^{(cl)}_f$, in case the latter is not zero.

In our favored approach presented in Section 6.7, however, the operations $\partial_{\mathbf{P}^{\|}_f}$, and $[a^\sharp_\lambda(k)\,,\,\cdot]$ (commutation with creation- and annihilation operators) both commute with the

rescaling transformation $\mathcal{S}_\rho$. The price that we pay for this convenient property is that the infrared regularization $\bar{\kappa}_\sigma[|k|]$ cannot be picked entirely arbitrarily. However, since the latter is an artificial item of the given theory anyway, we shall accept this mild restriction.

CHAPTER 7

ANALYSIS OF THE FIRST DECIMATION STEP

7.1. The main Theorem

The purpose of this chapter is to determine the initial condition $\mathfrak{h}^{(0)} \in \mathfrak{V}_{\epsilon,\xi}$ (the polydisc defined in Section 5.2), for the renormalization group recursion. We recall the selfadjoint that the smooth cutoff operators are given by

$$\begin{aligned} \chi_1 &= \chi_1[H_f] \\ \chi_1^2 + \bar{\chi}_1^2 &= \mathbf{1} \ , \end{aligned}$$

and that the projectors onto the respective ranges are given by $P_1, \bar{P}_1$. Furthermore, $P_1^\perp = \mathbf{1} - P_1$, and $\bar{P}_1^\perp = \mathbf{1} - \bar{P}_1$.

For the spectral parameter, we choose

$$z \ \in \ [E_{bare}[|p|] - g \ , \ E_{bare}[|p|]] \ ,$$

where $E_{bare}[|p|]$ is the bare ground state energy defined in (2.2.2). We assume that the conserved total momentum satisfies

$$|p| \ \leq \ p_c \ ,$$

for some constant $p_c \in [\frac{1}{20}, 1)$, which is required to be sufficiently small for the results of this and the next chapter to hold. Cf. also the remark on Cherenkov radiation in the introductory chapter.

The Feshbach triple $(Q_1, Q_1^\sharp, F_{\chi_1,\tau})$ associated to $(\chi_1, \mathcal{H}_p)$ (cf. definition 4.1) is given by

$$\begin{aligned} Q_1[z] &:= \chi_1 - \bar{\chi}_1 \bar{R}[z; \mathcal{O}_0] \bar{\chi}_1 \omega[H_p(\sigma_0)] \chi_1 \\ Q_1^\sharp[z] &:= \chi_1 - \chi_1 \, \omega[H_p(\sigma_0)] \, \bar{\chi}_1 \bar{R}[z; \mathcal{O}_0] \bar{\chi}_1 \\ F_1[z] &\equiv F_{\chi_1,\tau}[H_p(\sigma_0)] \\ &= \tau[H_p(\sigma_0)] + \chi_1 \omega[H_p(\sigma_0)] \chi_1 \\ &\qquad - \chi_1 \omega[H_p(\sigma_0)] \bar{\chi} \bar{R}[z; \mathcal{O}] \bar{\chi}_1 \omega[H_p(\sigma_0)] \chi_1 \ , \end{aligned} \tag{7.1.1}$$

where

$$\bar{R}[z; \mathcal{O}] \ = \ (\tau[H_p(\sigma_0)] + \bar{\chi}_1 \omega[H_p(\sigma_0)] \bar{\chi}_1)^{-1}$$

on $\bar{P}_1 \mathcal{H}_p$, and

$$\begin{aligned} \tau[H_p(\sigma_0)] &= (1 + \Delta_{H_f}[z]) \, H_f - z + \Delta E_0[z] \\ \omega[H_p(\sigma_0)] &= H_p(\sigma_0) - z - \tau[H_p(\sigma_0)] \ , \end{aligned}$$

cf. the discussion in Section 4.5. The quantity $\Delta_{H_f}[z]$ is defined by the implicit relation

$$1 + \Delta_{H_f}[z] \ := \ \left\langle \partial_{H_f} F_{\chi_1,\tau}[H_p(\sigma_0) - z] \right\rangle_{\Omega_f} \ .$$

Furthermore,

$$\begin{aligned} T'[\mathfrak{h}] &\equiv T[\mathfrak{h}] - \tau[H[\mathfrak{h}]] \\ &= -\Delta_{H_f}[z]H_f + \Delta E_0[z] + \chi_1 \left(-|p|\mathbf{P}_f^{\|} + \lambda_{n-l}|\mathbf{P}_f|^2\right)\chi_1 , \end{aligned}$$

where

$$\lambda_{n-l} = \frac{1}{2}$$

(we introduce this parameter for later convenience). The element $\mathfrak{h}^{(0)} \in \mathfrak{V}_{\epsilon,\xi}$ is determined by

$$H[\mathfrak{h}^{(0)}] := (1 + \Delta_{H_f}[z])^{-1}\mathcal{D}_1[H_p(\sigma_0) - z] ,$$

where we recall that $\mathcal{D}_1$ is the decimation operator, given by the concatenation of the Feshbach map $F_1[z]$ with Neumann series expansion and Wick ordering, cf. Sections 4.5 and 4.5.3. $H[\mathfrak{h}^{(0)}]$ is referred to as the effective hamiltonian at the scale 0.

Neumann series expansion of $\bar{R}[z;\mathcal{O}_0]$ in $F_1[z]$ yields

$$\begin{aligned} F_1[z] &= \tau[H_p(\sigma_0)] + \chi_1 T' \Pi \chi_1 + \chi_1 \Pi W \Pi \chi_1 \\ &- \sum_{L=2}^{\infty} (-1)^L \sum_{\substack{1 \le M_i + N_i \le 2 \\ i=2,..,L-1}} \chi_1 \Pi \left(\sum_{1 \le M_1 + N_1 \le 2} W_{M_1,N_1} \right) \bar{\chi}_1 \bar{R}_0 \bar{\chi}_1 \cdots \end{aligned}$$

$$\cdots \bar{\chi} W_{M_{L-1},N_{L-1}} \bar{\chi} \bar{R}_0 \bar{\chi}_1 \left(\sum_{1 \le M_L + N_L \le 2} W_{M_L,N_L} \right) \Pi \chi_1 , \tag{7.1.2}$$

where

$$\begin{aligned} \Pi &= \mathbf{1} - \bar{\chi}_1 T' \bar{\chi}_1 \bar{R}_0 \\ &= \bar{P}_1^{\perp} + \bar{P}_1 \tau[H] \bar{R}_0 , \end{aligned} \tag{7.1.3}$$

cf. Lemma 4.1, and

$$T' = T - \tau[H_p(\sigma_0)] .$$

Wick ordering each of the summands in the Neumann series, and collecting all Wick monomials that are equal in the number of creation and annihilation operators, we find

$$\begin{aligned} \mathcal{D}_1[H_p(\sigma_0) - z] &= (1 + \Delta_{H_f}[z'])H_f + \Delta E_0[z'] - z' \\ &+ \chi_1 (T' \Pi + \Delta T')[z';\mathcal{O}_0] \chi_1 \\ &+ \sum_{M+N \ge 1} \chi_1 \left(\Pi W_{M,N} \Pi + \Delta W_{M,N} \right)[z';\mathcal{O}] \chi_1 , \end{aligned} \tag{7.1.4}$$

where again, $z' = z - E_{bare}[|p|]$.

The main result of this chapter is the following theorem.

THEOREM 7.1. *The point $\mathfrak{h}^{(0)} \in \mathfrak{H}$ lies in a polydisc $\mathfrak{V}_{\epsilon_0,\xi}$ (for definitions, cf. Section 5.2), where*

$$\begin{aligned} \epsilon_0 &= \xi^2 \\ \epsilon_0 \xi &= 2g , \end{aligned}$$

for g sufficiently small, and it is assumed that

$$|p| \;\leq\; p_c$$

holds for the conserved momentum.

The function $\eta[H_p(\sigma_0)]$ (cf. Proposition 6.2 for its definition) obeys the bound

$$\left|\partial_{|p|}^{\beta}\left(\eta[H_p(\sigma_0)]-1\right)\right| \;\leq\; O(g^2)\;,$$

with $\beta = 0, 1$, and the renormalized coefficient $\hat{a}[Z[z]]$ of $\mathbf{P}_f^{\|}$ in $T_{lin}[\hat{\mathfrak{h}}]$ satisfies

$$\begin{aligned}\lim_{|p|\to 0}\hat{a}[Z[z]] \;&=\; 0\;,\\ \left|\,\partial_{|p|}^{\beta}(\hat{a}[Z[z]]+|p|)\,\right| \;&\leq\; O(g^2)\;.\end{aligned}$$

For the operator $T_{n-l}[\mathfrak{h}^{(0)}]$, there exists a constant $\kappa_{n-l} < \infty$ independent of g, such that

$$\left|\partial_{|p|}^{\beta}\partial_{\mathcal{O}}^{\alpha}\left(T_{n-l}[\mathfrak{h}^{(0)}] \;-\; \hat{\lambda}_{n-l}\,|\mathbf{P}_f|^2\right)\right| \;\leq\; O(g^2) \;+\; \kappa_{n-l}\bar{P}_1\;,$$

for $0 \leq |\alpha| \leq 2$, $\beta = 0, 1$, where

$$\begin{aligned}\hat{\lambda}_{n-l} \;&=\; \frac{1}{2}\;,\\ |\partial_{|p|}\hat{\lambda}_{n-l}|\,,\,|\partial_z\hat{\lambda}_{n-l}| \;&=\; 0\;.\end{aligned}$$

In particular,

$$\left|\left(T_{n-l}[\mathfrak{h}^{(0)}] \;-\; \hat{\lambda}_{n-l}|\mathbf{P}_f|^2\right)\right| \;\leq\; O(g^2) + \left(\frac{|p|+\hat{\lambda}_{n-l}}{2} + O(\sqrt{\epsilon_0})\right)\bar{P}_1 \tag{7.1.5}$$

and

$$\begin{aligned}\left|\partial_{H_f}T_{n-l}[\mathfrak{h}^{(0)}]\right| \;&\leq\; O(\sqrt{\epsilon_0}) \;+\; (|p| \;+\; \hat{\lambda}_{n-l}) \;\times\\ &\times\left(2 \;+\; 3\,|\partial_{H_f}(\chi_1^2\bar{\chi}_1^2)| \;+\; 2(|p|+\hat{\lambda}_{n-l})\,|\partial_{H_f}\bar{\chi}_1^2|\right)\;.\end{aligned} \tag{7.1.6}$$

For $M + N = 1$,

$$\begin{aligned}u_{M,N}[\mathfrak{h}^{(0)}] \;&=\; \lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} w_{M,N}[\mathfrak{h}^{(0)}]|_{\mathcal{O}\to 0}\\ &=\; g\;\hat{a}[Z[z]]\;\epsilon_{M,N}^{\|}[n_k,\lambda]\;,\end{aligned}$$

and for Y denoting $|k|$, z, H_f or $\mathbf{P}_f^{\perp}$,

$$\begin{aligned}\|\;\bar{P}_1^{\perp}[H_f]\;\partial_X w_{M,N}[\mathfrak{h}^{(0)}]\;\bar{P}_1^{\perp}[H_f+|k|]\;\| \;&\leq\; O(g^2)\;,\\ \|\;\bar{P}_1^{\perp}[H_f]\;\partial_{\mathbf{P}_f^{\|}} w_{M,N}[\mathfrak{h}^{(0)}]\;\bar{P}_1^{\perp}[H_f+|k|]\;\| \;&\leq\; 2\,g\,\hat{\lambda}_{n-l} \;+\; O(g^2)\;.\end{aligned}$$

More generally, the following estimates hold for $0 \leq |\alpha| \leq 2$, $0 \leq \beta \leq 1$:

$$\|\;\partial_{|p|}^{\beta}\,\partial_X\,w_{M,N}[\mathfrak{h}^{(0)}]\;\|\;,\;\|\;\partial_{|p|}^{\beta}\partial_{\mathcal{O}}^{\alpha}w_{M,N}[\mathfrak{h}^{(0)}]\;\| \;\leq\; \epsilon_0\xi \quad \text{if } M+N=1\;,$$

while for $M + N \geq 2$,

$$\begin{aligned}\|\;\partial_{|p|}^{\beta}\,\partial_{\mathcal{O}}^{\alpha}\,w_{M,N}[\mathfrak{h}^{(0)}]\;\| \;&\leq\; \epsilon_0^{\frac{7}{4}}\,\xi^{M+N}\;,\\ \|\;\partial_{|p|}^{\beta}\,\partial_X\,w_{M,N}[\mathfrak{h}^{(0)}]\;\| \;&\leq\; \epsilon_0^{\frac{7}{4}}\,\xi^{M+N}\;.\end{aligned}$$

All bounds are uniform in $|p| \leq p_c$, and for sufficiently small $p_c \in [\frac{1}{20}, 1)$.

7.2. Well-definedness of the Feshbach triple

First of all, it must be verified that for the given choice of z, the Feshbach triple $(Q_1, Q_1^\sharp, F_{\chi_1,\tau})$ is well-defined.

For notational convenience, let us make a redefinition of the spectral parameter,

$$z \longrightarrow z' := z - E_{bare}[|p|] \in \mathbb{R}_- , \tag{7.2.1}$$

so that z' lies in the interval $[-g, 0]$. In all of the above expressions, z can be replaced by z' if $E_{bare}[|p|]$ is set to zero.

We will now prove that the quantities (4.1.3) needed for the definition of the Feshbach triple extend to bounded operators on $\mathcal{H}_p$, and start by showing that $\bar{R}[z;\mathcal{O}]$ is well-defined.

LEMMA 7.1. *For $z' = z - E_{bare}[|p|] \in [-g, 0]$,*

$$\left\| \bar{R}[z;\mathcal{O}_0] \right\| , \left\| \bar{R}_0[z;\mathcal{O}_0] \right\| \leq \frac{1}{1 - |p| + O(g^2)} \leq 2 \tag{7.2.2}$$

for all $|p| \leq p_c$, with $p_c \in [\frac{1}{20}, 1)$ sufficiently small.

Proof. Let, for $T' \equiv T - z - \tau[H_p(\sigma_0)] = \omega[H_p(\sigma_0)]|_{g\to 0}$,

$$\bar{R}_0[z';\mathcal{O}_0] = \bar{\chi}_1 \left(\tau[H_p(\sigma_0)] + \bar{\chi}_1 T' \bar{\chi}_1\right)^{-1} \bar{\chi}_1$$

denote the free resolvent, obtained from $\bar{R}[z;\mathcal{O}_0]$ by setting the coupling constant g equal to zero. Clearly,

$$\left\| \bar{R}_0[z';\mathcal{O}_0] \right\| \leq (1 - |p|)^{-1} \leq O(1) , \tag{7.2.3}$$

since

$$\tau[H_p(\sigma_0)] + \bar{\chi}_1 T' \bar{\chi}_1 \geq (1 - |p|) H_f .$$

¿From Lemma 7.2 below follows that

$$\left\| \bar{R}_0^{\frac{1}{2}} \chi_1 W \chi_1 \bar{R}_0^{\frac{1}{2}} \right\| \leq O(g) \ll 1$$

if $|p| \leq p_c$, hence $\bar{R}[z;\mathcal{O}]$ can be expanded into an absolutely convergent Neumann series, such that

$$\begin{aligned} \|\bar{R}\| &\leq \|\bar{R}_0^{\frac{1}{2}}\| \sum_{j\geq 0} \left(\|\bar{R}_0^{\frac{1}{2}} \chi_1 W \chi_1 \bar{R}_0^{\frac{1}{2}}\| \right)^j \|\bar{R}_0^{\frac{1}{2}}\| \\ &\leq \sum_{j\geq 0} O(g^j) \\ &= O(1) \end{aligned}$$

is satisfied for the given range of z, resp. z'. This proves the lemma. ■

LEMMA 7.2. *For the interaction operators in $H_p(\sigma_0)$, cf. (2.2.8),*

$$\begin{aligned} \left\| \bar{R}_0^{\frac{1}{2}}[z;\mathcal{O}_0]\bar{\chi}_1 \, W_1[\mathcal{O}] \, \bar{\chi}_1 \bar{R}_0^{\frac{1}{2}}[z;\mathcal{O}_0] \right\| &\leq O(g) \\ \left\| \bar{R}_0^{\frac{1}{2}}[z;\mathcal{O}_0] \, \bar{\chi}_1 W_2[\mathcal{O}] \, \bar{\chi}_1 \bar{R}_0^{\frac{1}{2}}[z;\mathcal{O}_0] \right\| &\leq O(g^2) \end{aligned} \tag{7.2.4}$$

holds on $\mathcal{H}_p$, for $|p| \leq p_c$.

Proof. The proof is given in the Appendix in Chapter 10. ■

LEMMA 7.3. *The interaction operators in $H_p(\sigma_0)$, cf. (2.2.8), obey*

$$\begin{aligned} \left\| \bar{R}_0^{\frac{1}{2}}[z;\mathcal{O}_0] \, \bar{\chi}_1 W_1[\mathcal{O}] \, \chi_1 \right\| &\leq O(g) \\ \left\| \bar{R}_0^{\frac{1}{2}}[z;\mathcal{O}_0] \, \bar{\chi}_1 W_2[\mathcal{O}] \, \chi_1 \right\| &\leq O(g^2) \end{aligned} \tag{7.2.5}$$

on $\mathcal{H}_p$, for $|p| \leq p_c$.

Proof. Cf. the Appendix in Chapter 10. ■

These bounds imply that the Feshbach triple $(Q_1, Q_1^\sharp, F_{\chi_1,\tau})$ is well-defined, cf. Section 4.1.

LEMMA 7.4. *There is a constant C that is independent of g, such that*

$$\|\partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha \Pi\| \leq C \tag{7.2.6}$$

holds on $P_1\mathcal{H}_p$, for $0 \leq |\alpha| \leq 2$ and $\beta = 0, 1$. In particular, with

$$\|\partial_{H_f}\bar{\chi}_1^2\| \, , \, \|\partial_{H_f}\bar{\chi}_1^2\| \leq 4 \, ,$$

the estimates

$$\begin{aligned} \|\Pi\| &\leq \frac{4}{3} \\ \|\chi_1 \, (\Pi - \mathbf{1}) \, \chi_1\| &\leq \frac{1}{3} \qquad (7.2.7) \\ \|\partial_{|p|}\Pi\| &\leq 2 \qquad (7.2.8) \\ |\partial_{H_f}(\chi_1 \Pi \chi_1)| &\leq 2 + 3\,|\partial_{H_f}(\chi_1^2\bar{\chi}_1^2)| \\ &\quad + 2(|p| + \lambda_{n-l})\,|\partial_{H_f}\bar{\chi}_1^2| \qquad (7.2.9) \end{aligned}$$

hold for $|p| \leq p_c$ with $p_c \in [\frac{1}{20}, 1)$ sufficiently small.

Proof. From

$$\left\| P_1(\partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha X) P_1 \right\| \leq c$$

for

$$X = \chi_1^2\bar{\chi}_1^2 \, , \, \bar{R}_0 \text{ and } T'[\mathfrak{h}]$$

(using $|\partial_{|p|}\Delta_{H_f}[z]|, |\partial_{|p|}\Delta E_0[z]| \leq O(g^2)$, as will be proved below, and Lemma 7.1) follows straightforwardly that there exists a constant $C < \infty$ such that (7.2.6) holds.

To prove (7.2.7), where (on $P_1\mathcal{H}_p$)

$$\Pi[\mathfrak{h}] - \mathbf{1} = -\bar{\chi}_1 T' \bar{\chi}_1 \bar{R}_0 \, ,$$

we use

$$\left\| P_1 \, \bar{R}_0[\mathfrak{h}] \, P_1 \right\| \;\leq\; \left(1 - |p| + O(g^2)\right)^{-1} \;,$$

from Lemma 7.1. Furthermore,

$$\|\chi_1^2 \bar{\chi}_1^2\| \;\leq\; \frac{1}{4} \,, \tag{7.2.10}$$

since $\max_{x\in[0,1]}\{x^2(1-x^2)\} = \frac{1}{4}$. Moreover, by definition of T' and $|\Delta_{H_f}[z]|, |\Delta E_0[z]| \leq O(g^2)$,

$$\|P_1 T' P_1\| \;\leq\; |p| \,+\, \lambda_{n-l} \,+\, O(g^2) \,.$$

Combining these estimates with $\lambda_{n-l} = \frac{1}{2}$, we find that

$$\begin{aligned} \|\chi_1 \, (\Pi \,-\, \mathbf{1}) \, \chi_1\| \;&\leq\; \frac{1}{4} \frac{|p| + \frac{1}{2} + O(g^2)}{1 - |p| + O(g^2)} \\ &\leq\; \frac{1}{3} \,, \end{aligned}$$

and from

$$\Pi \;=\; \bar{P}_1^{\perp} \,+\, \bar{P}_1 \, \tau[H_p(\sigma_0)] \, \bar{R}_0$$

we have

$$\begin{aligned} \|\Pi\| \;&\leq\; \max\left\{1 \,,\; \|\tau[H_p(\sigma_0)] \, \bar{P}_1 \bar{R}_0\|\right\} \\ &\leq\; (1 + O(g^2)) \max_{x\geq 1} \frac{x}{(1-|p|)x + O(g^2)} \\ &\leq\; \frac{4}{3} \,, \end{aligned}$$

for all $|p| \leq p_c$ if $p_c \in [\frac{1}{20}, 1)$ is picked sufficiently small, as claimed.

As to (7.2.8), we have

$$\begin{aligned} \|\partial_{|p|}\Pi\| \;&\leq\; \|(\partial_{|p|}\tau[H_p(\sigma_0)]) \, \bar{R}_0[\mathfrak{h}]\| \\ &\quad + \|\tau[H_p(\sigma_0)] \bar{R}_0^2 \partial_{|p|}(\tau[H_p(\sigma_0)] + \bar{\chi}_1 T' \bar{\chi}_1)\| \\ &\leq\; \max_{x\geq 1} \frac{(1 + O(g^2)) \, x^2}{((1-|p|)x + O(g^2))^2} \,+\, O(g^2) \\ &\leq\; 2 \end{aligned}$$

on $P_\rho \mathcal{H}_p$. This establishes the claim.

To prove (7.2.9), we use

$$\left\| P_1 (\partial_{H_f} T'[\mathfrak{h}]) P_1 \right\| \;\leq\; O(g^2) \,,$$

using the fact that $P_1\,(\partial_{H_f}\bar{\chi}_1)\,P_1 = 0$ since the spectral supports of $\partial_{H_f}\bar{\chi}_1$ and P_1 are mutually disjoint, as well as

$$\begin{aligned}
\left|P_1\left(\partial_{H_f}\bar{R}_0\right)P_1\right| &\leq \left\|P_1\bar{R}_0P_1\right\|^2\left|P_1\left(\partial_{H_f}(\tau[H_p(\sigma_0)]+\bar{\chi}_1T'\bar{\chi}_1)\right)P_1\right| \\
&\leq \frac{\left(1+O(g^2)+\|P_1T'P_1\|\,|\partial_{H_f}\bar{\chi}_\rho^2|\right)}{(1-|p|+O(g^2))^2} \\
&\leq \frac{\left(1+(|p|+\lambda_{n-l})|\partial_{H_f}\bar{\chi}_1^2|+O(g^2)\right)}{(1-|p|+O(g^2))^2}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\partial_{H_f}(\chi_1\,\Pi[\mathfrak{h}]\,\chi_1)\right| &\leq |\partial_{H_f}(\chi_1^2\bar{\chi}_1^2)|\,\|P_1T'P_1\|\,\|P_1\,\bar{R}_0\,P_1\| \\
&\quad + \|\chi_1^2\bar{\chi}_1^2\|\,\|P_1(\partial_{H_f}T')P_1\|\,\|P_1\,\bar{R}_0\,P_1\| \\
&\quad + \|\chi_1^2\bar{\chi}_1^2\|\,\|P_1T'P_1\|\,|P_1\,(\partial_{H_f}\bar{R}_0)\,P_1| \\
&\leq \frac{(1+|p|+\lambda_{n-l}+O(g^2))}{1-|p|+O(g^2)}|\partial_{H_f}(\chi_1^2\bar{\chi}_1^2)| \;+\; O(g^2) \\
&\quad + \frac{1}{4\,(1-|p|+O(g^2))^2}(1+O(g^2)) \\
&\quad + \frac{1}{4}\frac{|p|+\lambda_{n-l}}{(1-|p|+O(g^2))^2}\,|\partial_{H_f}\bar{\chi}_1^2| \\
&\leq 2 \;+\; 3\,|\partial_{H_f}(\chi_1^2\bar{\chi}_1^2)| \;+\; 2(|p|+\lambda_{n-l})\,|\partial_{H_f}\bar{\chi}_1^2|
\end{aligned}$$

for $|p| \leq p_c$ with $p_c \in [\frac{1}{20}, 1)$ sufficiently small. ■

7.3. Two technical Lemmata

The following two technical lemmata will be very helpful in the subsequent analysis given in this chapter.

LEMMA 7.5. *Let Y denote $|p|, z$ or any component of $\mathcal{O}$. Then,*

$$\begin{aligned}
\|\;\bar{\chi}_1\,\partial_Y H_p(\sigma_0)\,Q_1\;\| &\leq O(g)\,, \\
\|\;\bar{R}\,Q_1\;\| &\leq O(g)\,.
\end{aligned} \tag{7.3.1}$$

Proof. These estimates are straightforward consequences of Lemmata 7.2 and 7.3. ■

LEMMA 7.6. *Let $M+N \geq 0$, and let stands for z or $|k_r|, |\tilde{k}_s|$. Then, are constants $C_{\alpha,\beta}$ and $C_{X,\beta}$ independent of ϵ_0, ξ, such that*

$$\|\;\partial_{|p|}^{\beta}\partial_{\mathcal{O}}^{\alpha}\Delta w_{M,N}\;\| \;\leq\; C_{\alpha,\beta}\,\epsilon_0^2\;\xi^{M+N}\,,$$

for $0 \leq |\alpha| \leq 2$ and $\beta = 0, 1$, and

$$\|\;\partial_{|p|}^{\beta}\partial_{X}\Delta w_{M,N}\;\| \;\leq\; C_{X,\beta}\,\epsilon_0^2\;\xi^{M+N}\,,$$

for all $M+N \geq 0$. In particular, $\Delta w_{M,N}$ are **analytic** *in $\mathbf{P}_f$.*

Proof. Cf. the Appendix of Chapter 10. ■

7.4. Shift of the spectral parameter

By a redefinition of the spectral parameter, we eliminate the quantity

$$\begin{aligned}\Delta E_0[z] &= \langle F_{\chi_1,\tau}[H_p(\sigma_0)] + z\rangle_{\Omega_f} \\ &= -\left\langle W[\mathcal{O}]\,\bar\chi_1\,\bar R[z;\mathcal{O}]\,\bar\chi_1\,W[\mathcal{O}]\right\rangle_{\Omega_f} .\end{aligned}$$

Due to Lemma 7.3,

$$\begin{aligned}|\,\Delta E_0[z]\,| &\le \left\|\,\bar R^{\frac12}[z;\mathcal{O}]\,\bar\chi_1\,W[\mathcal{O}]\,P_1\,\right\|^2 \\ &\le O(g^2)\,.\end{aligned}$$

We define the map

$$Z\,:\,z\,\to\,\hat z\,:=\,Z[z]$$

by

$$\hat z\;=\;(1+\Delta_{H_f}[z])^{-1}\left(\,z\,-\,E_{bare}[|p|]\,-\,\Delta E_0[z]\right)\,,$$

which corresponds to a translation of the origin in $\mathbb{C}$ by $E_{bare}[|p|]+\Delta E_0[z]$. The quantity $\Delta_{H_f}[z]$ is defined in (7.1.2), and discussed below in Section 7.5.1.

Proposition 7.1. *The map $Z: z\to\hat z$ is of class C^1, and*

$$\partial_z Z[z]\;=\;\rho^{-1}(1+\Delta_{H_f}[z])^{-1}\;+\;O(\epsilon_0^2\,\xi^2)\,.$$

Thus, for any function h of class C^1 in z,

$$\left|\partial_{\hat z}h[Z^{-1}[\hat z]]\right|\;\le\;(1\;+\;O(\epsilon))\;|\partial_z h[z]|\;. \tag{7.4.1}$$

In addition,

$$|\,\partial_z^\alpha\Delta E_0[z]\,|\;\le\;O(\epsilon_0^2\,\xi^2)$$

for $\alpha=0,1$.

Proof. Cf. the proof of Proposition 8.1 below, but employing Lemmata 7.2 and 7.3 instead of the estimates used there. ■

7.5. Correction of the noninteracting hamiltonian

Next, let us focus on $(T'+\Delta T')[z;\mathcal{O}_0]$, which we separate into

$$-\Delta_{H_f}[z]H_f\;-\;\Delta E_0[z]\;+\;(-|p|+\Delta a[z])\mathbf{P}_f^{\|}\;+\;\left(\frac{|\mathbf{P}_f^2|}{2}\;+\;\Delta T_{n-l}[z;\mathcal{O}_0]\right)\,,$$

where $T_{n-l}+\Delta T_{n-l}$ comprises all operators in T' of order $O(H_f^2)$ in the limit $H_f\to 0$. We recall that $\mathcal{O}_0=(H_f,\mathbf{P}_f^{\|},|\mathbf{P}_f^{\perp}|)$, and that $T'+\Delta T'$ depends on $\mathbf{P}_f^{\perp}$ only via $|\mathbf{P}_f^{\perp}|$, due to the $O(2)$-rotation and reflection symmetry of the model, which is preserved under Feshbach decimation. There is no term linear in $|\mathbf{P}_f^{\perp}|$ because $\mathcal{D}_1[H_p(\sigma_0)-z]$ is analytic in $\mathcal{O}$ by Theorem 4.2. $|\mathbf{P}_f^{\perp}|$ is not analytic in $\mathbf{P}_f^{\perp}$, hence only even powers of $|\mathbf{P}_f^{\perp}|$ (which are, of course, analytic) appear in ΔT.

Let us next discuss $\Delta_{H_f}[z]$ and $\Delta a[z]$.

7.5.1. The coefficient of H_f. By Proposition 6.2 (i.e. setting $n = 0$), the coefficient 1 of H_f in $H_p(\sigma_0)$ is modified into

$$\begin{aligned} 1 + \Delta_{H_f}[z] &= \left\langle \partial_{H_f} F_{\chi_1,\tau}[H_p(\sigma_0)] \right\rangle_{\Omega_f} \\ &= \eta_0[z] \left\langle Q_1^\sharp Q_1 \right\rangle_{\Omega_f} , \end{aligned}$$

where

$$\begin{aligned} \eta_0[z] &= \Big(1 + \left\langle W \bar{\chi}_1 \bar{R} \chi_1^2 \bar{R} \bar{\chi}_1 W \right\rangle_{\Omega_f} \\ &\quad + \left\langle W \bar{\chi}_1 \bar{R} \frac{\partial_{H_f} \bar{\chi}_1}{\bar{\chi}_1} (H_f - \hat{z}) \bar{R} \bar{\chi}_1 W \right\rangle_{\Omega_f} \Big)^{-1} , \end{aligned}$$

as we recall.

LEMMA 7.7. *The quantity $\Delta_{H_f}[z]$ satisfies $|\partial_{|p|}^\beta \Delta_{H_f}[z]| \leq O(g^2)$, for $\beta = 0, 1$.*

Proof. This is easily obtained along the lines demonstrated above, using the fact that $\|\partial_{H_f} \bar{\chi}_1\| \leq 4$. ■

We require that the fully renormalized coefficient of H_f shall again be 1. Therefore, we divide $F_{\chi_1,\tau}[H_p(\sigma_0)]$ by $(1 + \Delta_{H_f}[z])$.

7.5.2. The coefficient of $\mathbf{P}_f^{\|}$. The renormalized expression for the coefficient of $\mathbf{P}_f^{\|}$ in $T_{lin}[\mathfrak{h}^{(0)}]$ is given by

$$\hat{a}[Z[z]] = \frac{a[z] + \Delta a[z]}{1 + \Delta_{H_f}[z]} . \tag{7.5.1}$$

This is obtained from Feshbach decimation,

$$\begin{aligned} a[z] + \Delta a[z] &= \left\langle \partial_{\mathbf{P}_f^{\|}} \Big(F_{\chi_1,\tau}[H_p(\sigma_0) - z] \Big) \right\rangle_{\Omega_f} \\ &= \left\langle Q_1^\sharp \left(\partial_{\mathbf{P}_f^{\|}} H_p(\sigma_0) \right) Q_1 \right\rangle_{\Omega_f} \\ &= -|p| \left\langle Q_1^\sharp Q_1 \right\rangle_{\Omega_f} + \left\langle Q_1^\sharp \left(\mathbf{P}_f^{\|} + g\mathbf{A}_f^{\|}(\sigma_0) \right) Q_1 \right\rangle_{\Omega_f} , \end{aligned} \tag{7.5.2}$$

and subsequent application of rescaling, yielding

$$\hat{a}[Z[z]] = -|p| \frac{\left\langle Q_1^\sharp Q_1 \right\rangle_{\Omega_f}}{1 + \Delta_{H_f}[z]} + \frac{\left\langle Q_1^\sharp \left(\mathbf{P}_f^{\|} + g\mathbf{A}_f^{\|}(\sigma_0) \right) Q_1 \right\rangle_{\Omega_f}}{1 + \Delta_{H_f}[z]} . \tag{7.5.3}$$

PROPOSITION 7.2. *The renormalized coefficient of $\mathbf{P}_f^{\|}$ in $H[\mathfrak{h}^{(0)}]$ is $O(|p|)$ in the limit $|p| \to 0$, satisfying*

$$|\partial_{|p|}(\hat{a}[Z[z]] + |p|)| \leq O(g^2) ,$$

uniformly in $|p|$ for $|p| \leq p_c$.

Proof. The projection of the vector

$$\left\langle Q_1^\sharp \left(\mathbf{P}_f + g\mathbf{A}_f(\sigma_0)\right) Q_1 \right\rangle_{\Omega_f}$$

in the direction of p defines the second term in (7.5.3).

In the limit $|p| \to 0$, $Q_1^{(\sharp)}$ is $O(3)$-symmetric with respect to spatial rotations. Hence

$$\lim_{|p|\to 0} \left\langle Q_1^\sharp \left(\mathbf{P}_f + g\mathbf{A}_f(\sigma_0)\right) Q_1 \right\rangle_{\Omega_f}$$

transforms like a vector, but is invariant under all linear $O(3)$-actions on $\mathbb{R}^3$. Therefore, it must be zero. This implies that $\hat{a}[Z[z]] \to 0$ in the limit $|p| \to 0$, as claimed.

Straightforward calculation shows that

$$\begin{aligned} \partial_{|p|}(\hat{a}[Z[z]] \,+\, |p|) \;&=\; -\,\frac{\partial_{|p|}\Delta_{H_f}[z]}{1+\Delta_{H_f}[z]}\,(\hat{a}[Z[z]] \,+\, |p|) \\ &\quad -\,\frac{\left\langle Q_1^\sharp \left(\partial_{|p|}H_p(\sigma_0)\right) \bar{R}\left(\partial_{\mathbf{P}_f^\parallel}H_p(\sigma_0)\right) Q_1 \right\rangle_{\Omega_f} \,+\, h.c.}{1+\Delta_{H_f}[z]}\,. \end{aligned} \tag{7.5.4}$$

Let us first control the first term on the right hand side of the equality sign, and bound $|(\hat{a}[Z[z]] + |p|)|$. Because of $\mathbf{P}_f^\parallel \Omega_f = 0$,

$$\begin{aligned} \left|\left\langle Q_1^\sharp \,\mathbf{P}_f^\parallel\, Q_1 \right\rangle\right| \;&=\; \left|\left\langle W[\mathcal{O}]\,\bar{\chi}_1\,\bar{R}[z;\mathcal{O}]\,\bar{\chi}_1\,\mathbf{P}_f^\parallel\,\bar{\chi}_1\,\bar{R}[z;\mathcal{O}]\,\bar{\chi}_1\,W[\mathcal{O}]\right\rangle\right| \\ &\leq\; O(g^2)\,, \end{aligned}$$

using

$$\left\|\mathbf{P}_f^\parallel\,\bar{\chi}_1\,\bar{R}[z;\mathcal{O}]\right\| \;\leq\; O(1)$$

for $|p| \leq p_c$.

Moreover,

$$\begin{aligned} g\left|\left\langle Q_1^\sharp\;\mathbf{A}_f^\parallel(\sigma_0)\;Q_1\right\rangle\right| \;&\leq\; g\left|\left\langle W[\mathcal{O}]\,\bar{\chi}_1\,\bar{R}[z;\mathcal{O}]\,\bar{\chi}_1\,\mathbf{A}_f^\parallel(\sigma_0)\right\rangle \,+\, h.c.\right| \\ &\quad + g\left|\left\langle W[\mathcal{O}]\,\bar{\chi}_1\,\bar{R}[z;\mathcal{O}]\,\bar{\chi}_1\,\mathbf{A}_f^\parallel(\sigma_0)\,\bar{\chi}_1\,\bar{R}[z;\mathcal{O}]\,\bar{\chi}_1\,W[\mathcal{O}]\right\rangle\right| \\ &\leq\; O(g^2)\,, \end{aligned}$$

using the estimates

$$\begin{aligned} \left\|\;\bar{R}^{\frac{1}{2}}[z;\mathcal{O}]\,\bar{\chi}_1\,\mathbf{A}_f^\parallel(\sigma_0)\,\bar{\chi}_1\,\bar{R}^{\frac{1}{2}}[z;\mathcal{O}]\;\right\| \;&\leq\; O(1)\,, \\ \left\|\;\bar{R}^{\frac{1}{2}}[z;\mathcal{O}]\,\bar{\chi}_1\,\mathbf{A}_f^\parallel(\sigma_0)\;\chi_1\right\| \;&\leq\; O(1)\,, \end{aligned}$$

which are proved in a manner fully analogous to Lemmata 7.2 and 7.3.

Hence, we arrive at

$$\left|\;\hat{a}[Z[z]] \,+\, |p|\;\right| \leq\; O(g^2) \quad,$$

uniformly in $|p| \leq p_c$, for $p_c \in [\frac{1}{20}, 1)$ sufficiently small.

To control the first term on the right hand side of the equality sign in (7.5.4), we use Lemma 7.7. Collecting all estimates,

$$\left|\partial_{|p|}(\hat{a}[Z[z]] \,+\, |p|)\right| \;\leq\; O(g^2)\,, \tag{7.5.5}$$

uniformly in $|p|$ for $|p| \leq p_c$. Taylor's theorem thus implies

$$|\hat{\delta}_a[Z[z]]| \;\leq\; O(g^2)\;.$$

This proves the proposition. ■

7.6. Correction of $T_{n-l}[\mathcal{O}_0]$

Let us next consider the renormalized expression for the operator T_{n-l}.

PROPOSITION 7.3. *The renormalized operator $T_{n-l}[\mathfrak{h}^{(0)}]$ is $O(H_f^2)$ in the limit $H_f \to 0$, and there exists a constant $\kappa_{n-l} < \infty$ independent of g, such that*

$$\left|\partial_{|p|}^{\beta}\partial_{\mathcal{O}}^{\alpha}\left(T_{n-l}[\mathfrak{h}^{(0)}] \;-\; \hat{\lambda}_{n-l}\,|\mathbf{P}_f|^2\right)\right| \;\leq\; O(g^2) \;+\; \kappa_{n-l}\bar{P}_1\;, \tag{7.6.1}$$

for $0 \leq |\alpha| \leq 2$, $\beta = 0, 1$, where

$$\begin{aligned} \hat{\lambda}_{n-l} \;&=\; \frac{1}{2}\;, \\ |\partial_{|p|}\hat{\lambda}_{n-l}|\,,\; |\partial_z\hat{\lambda}_{n-l}| \;&=\; 0\;. \end{aligned}$$

In particular,

$$\left|\left(T_{n-l}[\mathfrak{h}^{(0)}] \;-\; \hat{\lambda}_{n-l}|\mathbf{P}_f|^2\right)\right| \;\leq\; O(g^2) \;+\; \left(\frac{|p| + \hat{\lambda}_{n-l}}{2} \;+\; O(\sqrt{\epsilon}_0)\right)\bar{P}_1 \tag{7.6.2}$$

and

$$\begin{aligned} \left|\partial_{H_f}T_{n-l}[\mathfrak{h}^{(0)}]\right| \;&\leq\; O(\sqrt{\epsilon}_0) \;+\; (|p| \;+\; \hat{\lambda}_{n-l}) \;\times \\ &\quad\times\; \left(2 \;+\; 3\,|\partial_{H_f}(\chi_1^2\bar{\chi}_1^2)| \;+\; 2(|p| + \hat{\lambda}_{n-l})\,|\partial_{H_f}\bar{\chi}_1^2|\right)\;. \end{aligned} \tag{7.6.3}$$

are satisfied.

Proof.

¿From smooth Feshbach decimation, $\mathcal{D}_\rho : T_{n-l} \to T_{n-l} + \Delta T_{n-l}$, where ΔT_{n-l} is given as follows. By Lemma 4.1,

$$\begin{aligned} \mathcal{D}_1 \;:\; T_{n-l} \;&\longrightarrow\; P_1 T_{n-l} P_1 \;+\; T'\,(\Pi - \mathbf{1}) \\ &\quad + (\Delta w_{0,0} - \Delta w_{0,0}|_{\mathcal{O}\to 0} - \Delta_{H_f}[z]H_f - \Delta a[z]\mathbf{P}_f^{\|}) \\ &=\; P_1 T_{n-l} P_1 \;-\; \bar{\chi}_1^2 (T')^2 \bar{R}_0 \\ &\quad + (\Delta w_{0,0} - \Delta w_{0,0}|_{\mathcal{O}\to 0} - \Delta_{H_f}[z]H_f - \Delta a[z]\mathbf{P}_f^{\|}) \\ &\equiv\; P_1\,(T_{n-l} \;+\; \Delta T_{n-l})\,P_\rho \end{aligned}$$

on $P_1\mathcal{H}_p$, since

$$\Pi \;=\; \mathbf{1} \;-\; \bar{\chi}_1 T' \bar{\chi}_1 \bar{R}_0\;,$$

as we recall. The contribution to ΔT_{n-l} from $\Delta w_{0,0}$ can be estimated by

$$\frac{1}{2!}\;\sum_{|\alpha|=2}\|\,\partial_{\mathcal{O}_0^r}^{\alpha}\,\Delta w_{0,0}\|H_f^2 \;\leq\; O(\epsilon^2)\,H_f^2\;,$$

which is immediate from Lemma 7.6 and Taylor's theorem.

It thus remains to bound the operator

$$T'(\Pi - \mathbf{1}) \;=\; -\bar{\chi}_1^2 (T')^2 \bar{R}_0 \;, \tag{7.6.4}$$

together with its derivatives with respect to $\mathcal{O}$.

It is clear that

$$\begin{aligned} \big| P_1 \left(T_{n-l} \,-\, \lambda_{n-l} |\mathbf{P}_f|^2 \,+\, \Delta T_{n-l} \right) P_1 \big| \\ &\leq\; O(\epsilon) \,+\, \|P_1 T'[\mathfrak{h}] P_1\| \, |P_1 (\Pi[\mathfrak{h}] - \mathbf{1}) P_1| \\ &\leq\; O(\epsilon) \,+\, \frac{|p| + \hat{\lambda}_{n-l} + O(\sqrt{\epsilon_0})}{2} \bar{P}_1 \;. \end{aligned}$$

This is obtained by using Lemma 7.4, and the fact that

$$\begin{aligned} \|P_1 T' P_1\| \;&\leq\; O(\sqrt{\epsilon_0}) \,+\, |p| \|P_1 \mathbf{P}_f^{\|} P_1\| \,+\, \lambda_{n-l} \|P_\rho |\mathbf{P}_f|^2 P_\rho\| \\ &\leq\; O(\sqrt{\epsilon_0}) \,+\, |p| \,+\, \hat{\lambda}_{n-l} \;, \end{aligned}$$

where by definition, $\hat{\lambda}_{n-l} = \lambda_{n-l} = \frac{1}{2}$. Hence, rescaling by multiplication with $(1 + \Delta_{H_f}[z])^{-1}$ yields (7.6.2).

Let us next consider the derivative with respect to H_f. We have

$$\begin{aligned} \big| \partial_{H_f} \left(\chi_1 \left(T_{n-l} \,-\, \lambda_{n-l} |\mathbf{P}_f|^2 \,+\, \Delta T_{n-l} \right) \chi_1 \right) \big| \\ &\leq\; O(\sqrt{\epsilon_0}) \,+\, \|P_1 (\partial_{H_f} T') P_1\| \, \|\Pi\| \\ &\quad + \|P_1 T' P_1\| \, |\partial_{H_f} (\chi_1 \Pi \chi_1)| \\ &\leq\; O(\sqrt{\epsilon_0}) \,+\, (|p| + \hat{\lambda}_{n-l}) \, |\partial_{H_f} (\chi_1 \Pi \chi_1)| \;, \end{aligned}$$

since $\|P_1 (\partial_{H_f} T') P_1\| \leq O(\epsilon^2)$. Applying Lemma 7.4, and rescaling, one obtains the asserted bound.

For general $|\alpha| + \beta \geq 1$, the part in $\partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha \Delta T_{n-l}$ stemming $\Delta w_{0,0}$ has an operator norm bounded by $O(\epsilon^2)$, again by Lemma 7.6 and Taylor's theorem. Since

$$\begin{aligned} \|\bar{R}_0^{\frac{1}{2}} (\partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha T) \bar{R}_0^{\frac{1}{2}}\| \;&\leq\; O(1) \;, \\ \|\bar{R}_0^{\frac{1}{2}} (\partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha W) \bar{R}_0^{\frac{1}{2}}\| \;&\leq\; O(\epsilon_0^\beta \epsilon^{1-\beta} \xi) \;, \\ \|\partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha \chi_1^2\| \;&\leq\; O(1) \;, \end{aligned}$$

where the implicit constants only depend on $|p|$, it follows straightforwardly that there exists a bound of the asserted form (7.6.1). ■

7.7. The interaction kernels of degree 1

The modification of the interaction kernels $w_{0,1}$ and $w_{1,0}$ due to Feshbach decimation is in part controlled by the operator-theoretic Ward-Takahashi identities derived in Section 6.7. The result is that $\Delta w_{M,N}$, for $M + N = 1$, are corrected in the same manner as the coefficient of $\mathbf{P}_f^{\|}$.

7.7.1. The marginal interaction and the Ward-Takahashi identities. Let

$$\begin{aligned} u_{M,N}[z;k,\lambda] &:= \lim_{\substack{|k|\to 0 \\ n_k\ fixed}} w_{M,N}[\mathcal{O}\to 0;z;k,\lambda] \\ &= g\ |p|\ \epsilon^{\|}_{M,N}[n_k,\lambda] \end{aligned}$$

for $M+N=1$, where we recall that $\epsilon_{M,N}[n_k,\lambda]$ denote the photon polarization vectors. For its image under the renormalizion map, we have the following proposition.

PROPOSITION 7.4. *Let $M+N=1$. The renormalized expression for $u_{M,N}$ is given by*

$$\hat{u}_{M,N}[Z[z];n_k,\lambda] = -\,g\,\hat{a}[Z[z]]\,\epsilon^{\|}_{M,N}[n_k,\lambda]\ , \tag{7.7.1}$$

where $\hat{a}[Z[z]]$ is the coefficient of the operator $\chi_1\mathbf{P}^{\|}_f\chi_1$ in $T[\mathfrak{h}^{(0)}]$. Furthermore, for $0\le|\alpha|\le 2$ and $\beta=0,1$, the renormalized interaction kernels satisfy

$$\|\!|\ \partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}} w_{M,N}[\mathfrak{h}^{(0)}]\ |\!\| \ \le\ \epsilon_0\,\xi\ . \tag{7.7.2}$$

In particular,

$$\begin{aligned} \|\!|\ \bar{P}^\perp_1[H_f]\ \partial_X w_{M,N}[\mathfrak{h}^{(0)}]\ \bar{P}^\perp_1[H_f+|k|]\ |\!\| &\le O(g^2) \\ \|\!|\ \bar{P}^\perp_1[H_f]\ \partial_{\mathbf{P}^{\|}_f} w_{M,N}[\mathfrak{h}^{(0)}]\ \bar{P}^\perp_1[H_f+|k|]\ |\!\| &\le 2\,g\,\lambda_{n-l}+O(g^2)\ . \end{aligned}$$

for X denoting either the absolute value of the photon momentum $|k|$, z, H_f or $\mathbf{P}^\perp_f$.

Proof.

(I) Let us first prove (7.7.1). According to the discussion of the iterative Ward-Takahashi identities in Section 6.7, we focus on the relationship

$$\begin{aligned} \lim_{\substack{|k|\to 0 \\ n_k\ fixed}} \bar{\kappa}^{-1}_\sigma[|k|]\sqrt{|k|}\ \ [H_p(\sigma_0),a^*_\lambda(k)] &= -\,g\,\epsilon_{1,0}[n_k;z]\cdot\partial_{\mathbf{P}_f}H_p(\sigma_0)\ , \\ \lim_{\substack{|k|\to 0 \\ n_k\ fixed}} \bar{\kappa}^{-1}_\sigma[|k|]\sqrt{|k|}\ \ [a_\lambda(k),H_p(\sigma_0)] &= -\,g\,\epsilon_{0,1}[n_k;z]\cdot\partial_{\mathbf{P}_f}H_p(\sigma_0)\ , \end{aligned}$$

which can be straightforwardly checked by inserting the definition of $H_p(\sigma_0)$.

Next, we consider the correction of $u_{M,N}$ under Feshbach decimation $\mathcal{D}_1$, that is,

$$\Delta u_{M,N}[z;n_k,\lambda] = \lim_{\substack{|k|\to 0 \\ n_k\ fixed}} \bar{\kappa}^{-1}_\sigma[|k|]\sqrt{|k|}\ \Delta w_{M,N}[z;\mathcal{O}\to 0;k,\lambda]\ ,$$

where $\Delta w_{M,N}$ is obtained from the Wick monomial of degree $M+N=1$ in $\mathcal{D}_1[H_p(\sigma_0)-z]$ (its interaction kernel is given by $w_{M,N}+\Delta w_{M,N}$). In particular,

$$\Delta u_{1,0}[z;n_k,\lambda] = \lim_{\substack{|k|\to 0 \\ n_k\ fixed}} \bar{\kappa}^{-1}_\sigma[|k|]|k|^{\frac{1}{2}}\ \ [\Delta W_{1,0}[z;\mathcal{O}\to 0]\,,\,a_\lambda(k)]\ ,$$

and likewise for $\Delta u_{0,1}[z;n_k,\lambda]$.

In order to exploit the O(2)-symmetry of the system, we write this in the form

$$\begin{aligned} \lim_{\substack{|k|\to 0 \\ n_k\ fixed}} \bar{\kappa}^{-1}_\sigma[|k|]\sqrt{|k|}\ \ [a_\lambda(k),H_p(\sigma_0)] &= -\,g\,\epsilon^{\|}_{0,1}[n_k,\lambda]\partial_{\mathbf{P}^{\|}_f}H_p(\sigma_0) \\ &\quad -\,g\,\epsilon^{\perp}_{0,1}[n_k;\lambda]\cdot\partial_{\mathbf{P}^{\perp}_f}H_p(\sigma_0)\ , \end{aligned}$$

where the right hand side is split into contributions from directions parallel to p, and contributions perpendicular to it. Likewise, the same applies to the commutator with $a^*_\lambda(k)$.

The Ward-Takahashi identities are a consequence of Theorem 4.3, which implies that these relations also hold for $H[\mathfrak{h}^{(0)}] = \mathcal{D}_1[H_p(\sigma_0) - z]$. That is,

$$\begin{aligned}\lim_{\substack{|k|\to 0\\ n_k\ fixed}} & \bar\kappa_\sigma^{-1}[|k|]\sqrt{|k|}\ [a_\lambda(k), \mathcal{D}_1[H_p(\sigma_0) - z]] \\ &= -\, g\, \epsilon^{\|}_{0,1}[n_k,\lambda]\partial_{\mathbf{P}^{\|}_f}\mathcal{D}_1[H_p(\sigma_0) - z] \\ &\quad -\, g\, \epsilon^{\perp}_{0,1}[n_k,\lambda]\cdot\partial_{\mathbf{P}^{\perp}_f}\mathcal{D}_1[H_p(\sigma_0) - z]\ .\end{aligned}$$

Next, we take the vacuum expectation value with respect to the Fock vacuum,

$$\begin{aligned}\lim_{\substack{|k|\to 0\\ n_k\ fixed}} (w_{0,1} + \Delta w_{0,1})\,[\mathcal{O}\to 0; z; k, \lambda] \;&=\; -\, g\, \epsilon^{\|}_{0,1}[n_k,\lambda]\left\langle \partial_{\mathbf{P}^{\|}_f}\mathcal{D}_1[H_p(\sigma_0) - z]\right\rangle_{\Omega_f} \\ &\quad -\, g\, \epsilon^{\perp}_{0,1}[n_k,\lambda]\cdot\left\langle \partial_{\mathbf{P}^{\perp}_f}\mathcal{D}_1[H_p(\sigma_0) - z]\right\rangle_{\Omega_f}\ .\end{aligned}$$

The expectation value in the first term on the right hand side of the equality sign equals the corrected coefficient of $\mathbf{P}^{\|}_f$ in $T[z;\mathcal{O}]$ under Feshbach decimation,

$$\left\langle \partial_{\mathbf{P}^{\|}_f}\mathcal{D}_1[H_p(\sigma_0) - z]\right\rangle_{\Omega_f} \;=\; -\,|p| \;+\; \Delta a[z]\ ,$$

as one observes from comparison with (7.5.2). On the other hand, the expectation value in the second term,

$$\left\langle \partial_{\mathbf{P}^{\perp}_f}\mathcal{D}_1[H_p(\sigma_0) - z]\right\rangle_{\Omega_f}\ , \tag{7.7.3}$$

is a vector in the 1-plane perpendicular to p. Since both $\mathcal{D}_1[H_p(\sigma_0) - z]$ and Ω_f are invariant under O(2)-actions on $\mathbb{R}^3$ which keep p fixed, it is necessarily O(2)-invariant, and must thus be zero.

Finally, the rescaling transformation acts on $u_{M,N} + \Delta u_{M,N}$ by division by $(1 + \Delta_{H_f}[z])$, which gives

$$\hat u_{M,N}[Z[z]; n_k, \lambda] \;=\; g\,\hat a[Z[z]]\ \epsilon^{\|}_{M,N}[n_k,\lambda]\ .$$

This proves the assertion about $\hat u_{M,N}$.

(II) Next, let us verify the inequality

$$\|\!|\ \hat w_{M,N}\ |\!\| \;\le\; \epsilon_0\,\xi\ .$$

The left hand side is clearly bounded by

$$\begin{aligned}\|\!|\ \Pi\, w_{M,N}\,\Pi\ |\!\| \;+\; \|\!|\ \Delta w_{M,N}\ |\!\| \;&\le\; \|\Pi\|^2\, g\,(|p| + 1) \;+\; \|\!|\ \Delta w_{M,N}\ |\!\| \\ &\le\; \left(\frac{4}{3}\right)^2\ g\,(|p| + 1) \;+\; O(g^2) \\ &\le\; 2\,g \\ &=\; \epsilon_0\,\xi\ ,\end{aligned}$$

for $|p| \leq p_c$ for $p_c \in [\frac{1}{20}, 1)$ sufficient small, since Lemma 7.6 implies

$$\| \Delta w_{M,N} \| \ \leq \ O(g^2) \ ,$$

and $\|\Pi\| \leq \frac{4}{3}$ by Lemma 7.4.

(III) Next, considering that $\bar{P}_\rho^\perp \Pi = \bar{P}_\rho^\perp$, we have

$$\begin{aligned} \| \ \bar{P}_1^\perp[H_f] \, (\partial_Y \hat{w}_{M,N}) \, \bar{P}_1^\perp[H_f + |k|] \ \| \\ &\leq \ (1 + \Delta_{H_f}[z])^{-1} \| \ \bar{P}_1^\perp[H_f] \, (\partial_Y \, (\Pi[z; \mathcal{O}] w_{M,N} \Pi[z; S_k \mathcal{O}])) \, \bar{P}_1^\perp[H_f + |k|] \| \\ &\quad + (1 + \Delta_{H_f}[z])^{-1} \| \, \partial_Y \Delta w_{M,N} \| \\ &\leq \ (1 + \Delta_{H_f}[z])^{-1} \| \ \bar{P}_1^\perp[H_f] \, (\partial_Y w_{M,N}) \, \bar{P}_1^\perp[H_f + |k|] \| \\ &\quad + (1 + \Delta_{H_f}[z])^{-1} \| \, \partial_Y \Delta w_{M,N} \| \\ &\leq \ \| \ \bar{P}_1^\perp[H_f] \, (\partial_Y w_{M,N}) \, \bar{P}_1^\perp[H_f + |k|] \| \ + \ O(g^2) \ . \end{aligned}$$

The asserted bounds follow immediately, since $\| \, \partial_{\mathbf{P}_f^\parallel} w_{M,N} \| \ = g = 2 \, \lambda_{n-l} \, g$ for $Y = \mathbf{P}_f^\parallel$, and $\| \, \partial_Y w_{M,N} \| \ \leq O(g^2)$ for Y denoting any of the quantities accounted for by the variable X. ■

7.8. The interaction kernels of degree 2 and higher

PROPOSITION 7.5. *Let X stand for z or $|k_r|, |\tilde{k}_s|$. Then, for $0 \leq \beta \leq 1$ and $0 \leq |\alpha| \leq 2$, the renormalized integral kernels $\{\hat{w}_{M,N}\}$ in $\hat{\mathfrak{h}^{(0)}}$ satisfy*

$$\| \ \partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha \hat{w}_{M,N} \ \| \ \leq \ \epsilon_0^{\frac{7}{4}} \, \xi^{M+N} \ ,$$

and

$$\| \ \partial_{|p|}^\beta \partial_X \hat{w}_{M,N} \ \| \ \leq \ \epsilon_0^{\frac{7}{4}} \, \xi^{M+N} \ ,$$

uniformly in $|p|$, for $|p| \leq p_c$, and $M + N \geq 1$. In particular, $\hat{w}_{M,N}$ is **analytic** *in $\mathbf{P}_f$.*

Proof. This follows from

$$\begin{aligned} \| \ \partial_{|p|}^\beta \, \partial_{\mathcal{O}}^\alpha \, \hat{w}_{M,N} \ \| \ \leq \ \rho^{M+N-1+|\alpha|} \sum_{\alpha' + \alpha'' = \alpha} \left(\begin{array}{c} |\alpha| \\ |\alpha'|, |\alpha''| \end{array} \right) \times \\ \times \ \Big(2^\beta \| \partial_{|p|}^\beta \partial_{\mathcal{O}}^{\alpha'} \Pi \| \, \| \partial_{\mathcal{O}}^{\alpha''} \Pi \| \, \| \, \partial_{\mathcal{O}}^{\alpha - \alpha' - \alpha''} \, w_{M,N} \| \\ + \ \| \partial_{\mathcal{O}}^{\alpha'} \Pi \| \, \| \partial_{\mathcal{O}}^{\alpha''} \Pi \| \, \| \, \partial_{|p|}^\beta \partial_{\mathcal{O}}^{\alpha - \alpha' - \alpha''} \, w_{M,N} \| \ \Big) \\ + \ C_{\alpha,\beta} \, g^{M+N+2} \ , \end{aligned}$$

and

$$\begin{aligned} \| \ \partial_{|p|}^\beta \, \partial_X \, \hat{w}_{M,N} \ \| \ \leq \ \rho^{M+N} \, \| \, \partial_{|p|}^\beta \, \partial_X \, w_{M,N} \| \\ + \ C_{X,\beta} \, g^{M+N+2} \ , \end{aligned}$$

which is a straightforward consequence of Lemma 7.6. The constants $C_{\alpha,\beta}$, $C_{X,\beta}$ do not depend on ϵ, ξ. We remark that it is trivially, $\| \, w_{M,N} \| \ = 0$ if $M + N \geq 3$. ■

CHAPTER 8

ANALYSIS OF THE INDUCTION STEP

8.1. General framework

The purpose of this chapter is to study the renormalization map

$$\begin{aligned} \mathcal{R} \,:\, \mathfrak{V}_{2\epsilon_0,\xi} &\longrightarrow \mathfrak{V}_{2\epsilon_0,\xi} \\ \mathfrak{h} &\longmapsto \hat{\mathfrak{h}} \,, \end{aligned}$$

where we recall from the previous chapter that

$$\begin{aligned} \epsilon_0 \, \xi &= 2\, g \\ \epsilon_0 &= \xi^2 \,, \end{aligned}$$

such that $\xi \ll 1$ for $g \ll 1$, which we assume to be sufficiently small. The detailed constructions of $\mathcal{R}$ and of the polydisc $\mathfrak{V}_{\epsilon,\xi}$ are given in Chapter 5.

Let us pick a number

$$\epsilon \,\leq\, 2\epsilon_0 \,,$$

together with an arbitrary element

$$\mathfrak{h} \,\in\, \mathfrak{V}_{\epsilon,\xi} \,\subseteq\, \mathfrak{V}_{2\epsilon_0,\xi} \,. \tag{8.1.1}$$

The main purpose is to demonstrate that

$$\hat{\mathfrak{h}} \,\equiv\, \mathcal{R}[\mathfrak{h}] \,\in\, \mathfrak{V}_{\hat{\epsilon},\xi} \tag{8.1.2}$$

for a renormalized value $\hat{\epsilon} \leq \epsilon$.

The value of ξ is kept fixed ($\xi = (2g)^{\frac{1}{3}}$), while the number ϵ is a measure for the renormalization group flow of the electron charge g. Notably it does a priori **not depend on** $|p|$, a fact that plays a rôle in connection to the derivation of uniform bounds in $|p|$ at various places in the sequel.

8.2. General induction hypotheses for $H[\mathfrak{h}]$

Let us present the list of induction hypotheses imposed on $H[\mathfrak{h}]$. To begin with, the general structure of $H[\mathfrak{h}]$ is assumed to be of the form

$$H[\mathfrak{h}] \,=\, T[\mathfrak{h}] \,+\, \chi_1 \, W[\mathfrak{h}] \, \chi_1 \,,$$

with

$$T[\mathfrak{h}] \,=\, H_f \,-\, z \,+\, \chi_1 \, (a[z] \mathbf{P}_f^{\shortparallel} \,+\, T_{n-l}[\mathfrak{h}]) \, \chi_1 \,,$$

where $T_{n-l}[\mathfrak{h}]$ comprises all terms of order $O(H_f^2)$ in the limit $H_f \to 0$ that commute with H_f, and where $a[z]$ is a real scalar for $z \in \mathbb{R}$. Furthermore,

$$W[\mathfrak{h}] \,=\, \sum_{M+N \geq 1} W_{M,N}[\mathfrak{h}] \,,$$

where the Wick monomials $W_{M,N}[\mathfrak{h}]$ have the form given in (5.1.2) and (5.1.3).

Next, it is assumed that

$$H[\mathfrak{h}] \text{ is analytic in } \mathbf{P}_f^{\|} , \mathbf{P}_f^{\perp} .$$

This condition is immediately seen to be preserved by the induction step, due to consequence of Theorem 4.2. It is needed to determine the functional structure of $T_{lin}[\mathfrak{h}]$, the linear part of the noninteracting hamiltonian.

The absolute value of the conserved momentum p is assumed to be restricted to the values

$$|p| \leq p_c , \tag{8.2.1}$$

with some constant

$$p_c \in [\frac{1}{20}, 1) ,$$

which must be picked sufficiently small for the results in this chapter to hold. In the subsequent discussion, bounds that are **uniform in** $|p|$ will always be implicitly understood only for the range of values specified in (8.2.1).

The induction assumption on the symmetry properties of $H[\mathfrak{h}]$ are that for $|p| > 0$, $H[\mathfrak{h}]$ is $O(2)$-invariant with respect to rotations and reflections in $\mathbb{R}^3$ that leave p fixed. In the special case $|p| = 0$,

$$\lim_{|p|\to 0} H[\mathfrak{h}] \tag{8.2.2}$$

is fully $O(3)$-rotation and reflection invariant.

The smooth cutoff operators are assumed to obey the bounds

$$\begin{aligned} \|\partial_{H_f}(\chi_\rho^2 \bar{\chi}_\rho^2)\| , \ \|\partial_{H_f} \bar{\chi}_\rho^2\| &\leq 8 , \\ \|\partial_{H_f}(\chi_1^2 \bar{\chi}_1^2)\| , \ \|\partial_{H_f} \bar{\chi}_1^2\| &\leq 4 , \end{aligned} \tag{8.2.3}$$

where the spectral support of χ_1 is assumed to be given by $[\frac{2}{3}, 1]$.

8.3. Induction hypotheses for the spectral parameter

For the spectral parameter, we assume that

$$z \in [-\epsilon, 0] .$$

This ensures well-definedness of the Feshbach map. Furthermore, the interval is chosen appropriately such that its interior contains the infimum of the spectrum of $H[\mathfrak{h}]$. This is essential for the application of the reconstruction part of the Feshbach theorem, which will be invoked to determine the ground state eigenvalue of $H_p(\sigma_0)$, and the corresponding eigenvector.

8.4. Induction hypotheses for $T[\mathfrak{h}]$

For $|p| > 0$, the noninteracting hamiltonian $T[\mathfrak{h}]$ is invariant under linear $O(2)$-actions on $\mathbb{R}^3$ which leave p invariant. Therefore, it can only depend on $\mathbf{P}_f^\perp$ only via $|\mathbf{P}_f^\perp|$, it is thus a function of $\mathcal{O}_0 = (H_f, \mathbf{P}_f^\parallel, |\mathbf{P}_f^\perp|)$. In addition, it is assumed to be **analytic** in $\mathbf{P}_f$. Hence, it only depends on even powers of $|\mathbf{P}_f^\perp|$. We introduce the decomposition

$$T[\mathfrak{h}] \;=\; H_f \,-\, z \,+\, \chi_1 \left(a[z]\, \mathbf{P}_f^\parallel \,+\, T_{n-l}[\mathfrak{h}] \right) \chi_1 \;, \tag{8.4.1}$$

where we assume that

$$\left| \partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha \left(T_{n-l}[\mathfrak{h}] \,-\, \lambda_{n-l}\, |\mathbf{P}_f|^2 \right) \right| \;\leq\; \epsilon \,+\, \kappa_{n-l}\, \bar{P}_1 \;, \tag{8.4.2}$$

on $P_1 \mathcal{H}_p$, for $0 \leq |\alpha| \leq 2$, $\beta = 0, 1$, where

$$\begin{aligned} 0 \;&\leq\; \lambda_{n-l} \;\leq\; \frac{1}{2} \;, \\ |\partial_{|p|} \lambda_{n-l}| \;, \; |\partial_z \lambda_{n-l}| \;&=\; 0 \;, \end{aligned}$$

and where the constant κ_{n-l} is independent of ϵ. In particular,

$$\left| \left(T_{n-l}[\mathfrak{h}] \,-\, \lambda_{n-l} |\mathbf{P}_f|^2 \right) \right| \;\leq\; \epsilon \,+\, \left(\frac{|p| + \lambda_{n-l}}{2} \,+\, O(\sqrt{\epsilon_0}) \right) \bar{P}_1 \tag{8.4.3}$$

and

$$\begin{aligned} \left| \partial_{H_f} T_{n-l}[\mathfrak{h}] \right| \;&\leq\; O(\sqrt{\epsilon_0}) \,+\, (|p| \,+\, \lambda_{n-l}) \,\times \\ &\times \left(2 \,+\, 3\, |\partial_{H_f} (\chi_1^2 \bar{\chi}_1^2)| \,+\, 2(|p| + \lambda_{n-l})\, |\partial_{H_f} \bar{\chi}_1^2| \right) \;. \end{aligned} \tag{8.4.4}$$

A direct consequence is that

$$|(T_{n-l}[\mathfrak{h}] - \lambda_{n-l} |\mathbf{P}_f|^2)| \;\leq\; \frac{3}{2} \left(\frac{|p| + \lambda_{n-l}}{2} \,+\, O(\sqrt{\epsilon_0}) \right) H_f \;. \tag{8.4.5}$$

To show this, we observe that

$$P_1 \bar{P}_1 \;\leq\; \frac{3}{2} H_f \;, \tag{8.4.6}$$

since the spectral support of $P_1 \bar{P}_1$ (as a function of H_f) is the interval $[\frac{2}{3}, 1]$, so

$$\begin{aligned} |\bar{P}_1 (T_{n-l}[\mathfrak{h}] - \lambda_{n-l} |\mathbf{P}_f|^2) \bar{P}_1| \;&\leq\; \| \bar{P}_1 (T_{n-l}[\mathfrak{h}] - \lambda_{n-l} |\mathbf{P}_f|^2) \bar{P}_1 \|\, \bar{P}_1 \\ &\leq\; \frac{3}{2} \| \bar{P}_1 (T_{n-l} - \lambda_{n-l} |\mathbf{P}_f|^2) \bar{P}_1 \|\, H_f \;. \end{aligned}$$

Furthermore, since $|\mathbf{P}_f^\parallel|, |\mathbf{P}_f^\perp| \leq H_f$,

$$\begin{aligned} |\bar{P}_1^\perp (T_{n-l}[\mathfrak{h}] - \lambda_{n-l} |\mathbf{P}_f|^2) \bar{P}_1^\perp| \;&\leq\; \sum_{|\alpha|=1} \| \bar{P}_1^\perp (\partial_{\mathcal{O}}^\alpha (T_{n-l}[\mathfrak{h}] - \lambda_{n-l} |\mathbf{P}_f|^2)) \bar{P}_1^\perp \|\, H_f\, \bar{P}_1^\perp \\ &\leq\; O(\epsilon)\, H_f\, \bar{P}_1^\perp \;. \end{aligned}$$

This immediately implies (8.4.5).

The induction assumption on $a[z]$ is that

$$\left| \partial_{|p|}^{\beta} (a[z] \, + \, |p|) \right| \;\; \leq \;\; \sqrt{\epsilon}_0$$

(8.4.7)

uniformly in $|p|$, for $|p| \leq p_c$, and $\beta = 0, 1$. Due to the strong marginality of the operator $a[z]\mathbf{P}_f^{\|}$, our method to bound the left hand side of (8.4.7), is to 'integrate' the renormalization group flow equations up to the present scale in each renormalization iteration. This is done in Proposition 8.2, by which one ensures the validity of (8.4.7) in every iteration step. Since we prefer to solve the flow equations in the end, this approach via a 'nested induction argument' is convenient.

Furthermore, in the special case $|p| = 0$,

$$\lim_{|p| \to 0} T[\mathfrak{h}] \tag{8.4.8}$$

is fully $O(3)$-rotation and reflection invariant, and thus only depends on the operators H_f and $|\mathbf{P}_f|^2 = |\mathbf{P}_f^{\|}|^2 + |\mathbf{P}_f^{\perp}|^2$. Therefore,

$$\lim_{|p| \to 0} a[z] \;\; = \;\; 0 \; . \tag{8.4.9}$$

This, together with (8.4.7), implies that, by Taylor's theorem, $a[z]$ can be written in the form

$$a[z] \;\; = \;\; -|p|(1 + \delta_a[z]) \; ,$$

where $|\delta_a[z]| \leq \sqrt{\epsilon}_0$, uniformly in $|p|$, for $|p| \leq p_c$, and $p_c \in [\frac{1}{20}, 1)$ sufficiently small.

Furthermore, it is assumed that

$$\|\partial_z T[\mathfrak{h}]\| \;\; \leq \;\; C \; , \tag{8.4.10}$$

for a constant C that is independent of ϵ.

Under the induction hypotheses formulated above for $T[\mathfrak{h}]$, we obtain the upper and lower relative bounds

$$\frac{1}{3} \, H_f \;\; \leq \;\; T[\mathfrak{h}] \;\; \leq \;\; 3 \, H_f \; , \tag{8.4.11}$$

uniformly in $|p|$ for $|p| \leq p_c$. The upper bound is a consequence of

$$\begin{aligned} T[\mathfrak{h}] \;\; &\leq \;\; (1 \, + \, |a[z]|) H_f \, + \, \lambda_{n-l} H_f^2 \, + \, \frac{3}{2} \left(\frac{|p| + \lambda_{n-l}}{2} \, + \, O(\sqrt{\epsilon}_0) \right) H_f \\ &\leq \;\; \left(1 + \frac{7|p|}{4} + \frac{7}{8} \, + \, O(\sqrt{\epsilon}_0) \right) H_f \\ &\leq \;\; 3 \, H_f \; , \end{aligned} \tag{8.4.12}$$

using (8.4.5) and $\lambda_{n-l} \leq \frac{1}{2}$, for $|p| \leq p_c$ and $p_c \in [\frac{1}{20}, 1)$ sufficiently small.

The lower bound follows from

$$\begin{aligned} T[\mathfrak{h}] &\geq (1 - |a[z]|)H_f - \frac{3}{2}\left(\frac{|p| + \lambda_{n-l}}{2} + O(\sqrt{\epsilon}_0)\right) H_f \\ &\geq \left(1 - \frac{7|p|}{4} - \frac{3}{8} + O(\sqrt{\epsilon}_0)\right) H_f \\ &\geq \frac{1}{3} H_f \ , \end{aligned} \tag{8.4.13}$$

again using (8.4.5) and $\lambda_{n-l} \leq \frac{1}{2}$, for $|p| \leq p_c$ and $p_c \in [\frac{1}{20}, 1)$ sufficiently small. This completes the list of induction hypotheses for the non-interacting effective hamiltonian $T[\mathfrak{h}]$.

8.5. Induction hypotheses for interaction kernels of degree 1

It is, to begin with, assumed that the interaction kernels $w_{M,N}[z; \mathcal{O}; k, \lambda]$, $M + N = 1$, are **analytic** in $\mathbf{P}_f$.

They comprise the only marginal interaction kernels in the theory, namely

$$\begin{aligned} u_{M,N}[\mathfrak{h}] &= u_{M,N}[z; n_k, \lambda] \\ &= \lim_{\substack{|k| \to 0 \\ n_k \text{ fixed}}} w_{M,N}[\mathcal{O} \to 0; z; k, \lambda] \ . \end{aligned} \tag{8.5.1}$$

The induction hypothesis on $u_{M,N}[\mathfrak{h}]$ is that

$$u_{M,N}[\mathfrak{h}] = g\, a[z]\, \epsilon^{\|}_{M,N}[n_k, \lambda] \ , \tag{8.5.2}$$

where $a[z]$ is the coefficient of $\mathbf{P}^{\|}_f$ in $T_{lin}[\mathfrak{h}]$, and $\epsilon^{\|}_{M,N}[n_k, \lambda]$ is the projection of a photon polarization vector in the direction of the conserved momentum p.

Moreover, it is assumed that the lower bound

$$\epsilon \geq 2 \,|\!|\!|\, u_{M,N} \,|\!|\!| = 2\, g\, |a[z]| \ , \tag{8.5.3}$$

holds for ϵ, and that the auxiliary bound

$$\begin{aligned} |\!|\!|\, w_{M,N}[\mathfrak{h}] \,|\!|\!| &\leq \frac{16}{9}\, g\, |p|\, (1 + \sqrt{\epsilon}_0) + \frac{32}{9}\, g\, \lambda_{n-l} + O(\epsilon^{\frac{3}{2}} \xi) \\ &\leq \epsilon \ , \end{aligned}$$

holds. Furthermore, letting X denote the absolute value of a photon momentum $|k|$, the spectral parameter z, H_f, or $\mathbf{P}^{\perp}_f$, that

$$\begin{aligned} |\!|\!|\, \bar{P}^{\perp}_1[H_f] \left(\partial_{\mathbf{P}^{\|}_f} w_{M,N}\right) \bar{P}^{\perp}_1[H_f + |k|] \,|\!|\!| &\leq 2\, g\, \lambda_{n-l} + \epsilon^{\frac{3}{2}}\, \xi \ , \\ |\!|\!|\, \bar{P}^{\perp}_1[H_f] \left(\partial_X w_{M,N}\right) \bar{P}^{\perp}_1[H_f + |k|] \,|\!|\!| &\leq \epsilon^{\frac{3}{2}}\, \xi \ , \end{aligned} \tag{8.5.4}$$

for $\bar{P}^{\perp}_1[|k|] = \bar{P}^{\perp}_1[S_k H_f]|_{H_f \to 0}$. These induction assumptions are supplemented by additional requirements below, which can be formulated in a more general manner together with the conditions imposed on the cases $M + N \geq 2$.

8.5.1. Ward-Takahashi identities. The fact that $u_{M,N}[\mathfrak{h}]$ is determined by $a[z]$, the coefficient of $\mathbf{P}_f^{\|}$ in $T[\mathfrak{h}]$, is a consequence of the Ward-Takahashi identities. Formulated in the form of an induction hypothesis, they express that $H[\mathfrak{h}]$ satisfies

$$\mathcal{D}^{\sharp}_{WT,n_k,\lambda,\sigma}[H[\mathfrak{h}]] \;=\; 0 \;,$$

in the manner explained in Section 6.8.

8.6. General induction hypotheses for $w_{M,N}[\mathfrak{h}]$

All $w_{M,N}$ are assumed to be **analytic** in $\mathbf{P}_f$. Let X denote photon momenta $|k_i|, |\tilde{k}_j|$, for $i = 1, \ldots, M$, and $j = 1, \ldots, N$, or the spectral parameter z. Then, the induction hypotheses are that for $0 \leq |\alpha| \leq 2$, $0 \leq \beta \leq 1$,

$$\begin{aligned} \| \, \partial^{\beta}_{|p|} \, \partial^{\alpha}_{\mathcal{O}} \, w_{M,N}[\mathfrak{h}] \, \| \;&\leq\; (2\epsilon_0)^{\beta} \, \epsilon^{1-\beta} \xi \quad , \\ \| \, \partial^{\beta}_{|p|} \, \partial_X \, w_{M,N}[\mathfrak{h}] \, \| \;&\leq\; (2\epsilon_0)^{\beta} \, \epsilon^{1-\beta} \, \xi \;, \quad M+N = 1 \;, \end{aligned} \tag{8.6.1}$$

while for $M + N \geq 2$,

$$\begin{aligned} \| \, \partial^{\beta}_{|p|} \, \partial^{\alpha}_{\mathcal{O}} \, w_{M,N}[\mathfrak{h}] \, \| \;&\leq\; (2\epsilon_0)^{\beta} \, \epsilon^{\frac{7}{4} - \frac{|\alpha|}{4} - \beta} \, \xi^{M+N} \;, \\ \| \, \partial^{\beta}_{|p|} \, \partial_X \, w_{M,N}[\mathfrak{h}] \, \| \;&\leq\; (2\epsilon_0)^{\beta} \, \epsilon^{\frac{3}{2} - \beta} \, \xi^{M+N} \;. \end{aligned}$$

All bounds are assumed to be uniform in $|p|$, for $|p| \leq p_c$, with $p_c \in [\frac{1}{20}, 1)$ sufficiently small.

8.7. The main Theorem

The following Theorem is the main result of this chapter, and fully controls the renormalization group iteration.

THEOREM 8.1. *Let $\rho = \frac{1}{2}$, and let X denote photon momenta $|k_i|, |\tilde{k}_j|$, or the renormalized spectral parameter $\hat{z} = Z[z]$. There is a constant $p_c \in [\frac{1}{20}, 1)$, such that for all p with $|p| \leq p_c$, the following results hold. Assuming that ϵ_0, ξ are sufficiently small, the renormalized values of ϵ and κ are defined by*

$$\hat{\epsilon} \;=\; \begin{cases} \max\left\{ \dfrac{17}{18}\,\epsilon \;,\; 2\,(1+\sqrt{\epsilon_0})\,|p|\,\epsilon_0 \right\} & \text{if } \sigma \leq 1 \\ \dfrac{17}{18}\,\epsilon & \text{if } \sigma > 1 \quad , \end{cases}$$

with $\epsilon \leq 2\epsilon_0$, and

$$\sigma \;\longrightarrow\; \hat{\sigma} \;=\; \frac{\sigma}{\rho} \;=\; 2\,\sigma \;.$$

accounts for the renormalization of the running infrared cutoff. The renormalized effective hamiltonian $H[\hat{\mathfrak{h}}]$ is analytic in $\mathcal{O}$, and $\lim_{|p|\to} H[\hat{\mathfrak{h}}]$ is fully $O(3)$-symmetric.

The function $\eta[\mathfrak{h}]$ (cf. Proposition 6.2 for its definition) obeys the bound

$$\left| \partial^{\beta}_{|p|} \, (\eta[\mathfrak{h}] - 1) \right| \;\leq\; O(\epsilon_0^{\beta} \hat{\epsilon}^{2-\beta}) \;,$$

with $\beta = 0, 1$, and the renormalized coefficient $\hat{a}[Z[z]]$ of $\mathbf{P}_f^{\|}$ in $T_{lin}[\hat{\mathfrak{h}}]$ satisfies

$$\begin{aligned} \lim_{|p|\to 0} \hat{a}[Z[z]] &= 0 \ , \\ \left| \partial_{|p|}^{\beta} (\hat{a}[Z[z]] + |p|) \right| &\leq \sqrt{\epsilon}_0 \ . \end{aligned}$$

For the renormalized operator $T_{n-l}[\hat{\mathfrak{h}}]$,

$$\left| \partial_{|p|}^{\beta} \partial_{\mathcal{O}}^{\alpha} \left(\chi_1 \left(T_{n-l}[\hat{\mathfrak{h}}] \, - \, \hat{\lambda}_{n-l} \, |\mathbf{P}_f|^2 \right) \chi_1 \right) \right| \leq \epsilon \, + \, \hat{\kappa}_{n-l} \, \bar{P}_1 \ ,$$

for $0 \leq |\alpha| \leq 2$, $\beta = 0, 1$, and

$$\begin{aligned} 0 \, \leq \, \hat{\lambda}_{n-l} &= \rho \, \lambda_{n-l} \ , \\ |\partial_{|p|} \hat{\lambda}_{n-l}| \, , \, |\partial_z \hat{\lambda}_{n-l}| &= 0 \ . \end{aligned}$$

The constant $\hat{\kappa}_{n-l} < \infty$ is independent of $\hat{\epsilon}$. In particular,

$$\left| \left(T_{n-l}[\mathfrak{h}] \, - \, \lambda_{n-l} |\mathbf{P}_f|^2 \right) \right| \leq \epsilon \, + \, \left(\frac{|p| + \lambda_{n-l}}{2} \, + \, O(\sqrt{\epsilon}_0) \right) \bar{P}_1$$

and

$$\begin{aligned} \left| \partial_{H_f} T_{n-l}[\mathfrak{h}] \right| \leq \; & O(\sqrt{\epsilon}_0) \, + \, (|p| \, + \, \lambda_{n-l}) \, \times \\ & \times \left(2 \, + \, 3 \, |\partial_{H_f} (\chi_1^2 \bar{\chi}_1^2)| \, + \, 2(|p| + \lambda_{n-l}) \, |\partial_{H_f} \bar{\chi}_1^2| \right) \ . \end{aligned}$$

For $M + N = 1$,

$$\begin{aligned} u_{M,N}[\hat{\mathfrak{h}}] &= \lim_{\substack{|k|\to 0 \\ n_k \text{ fixed}}} w_{M,N}[\hat{\mathfrak{h}}] \\ &= g \, \hat{a}[Z[z]] \, \epsilon_{M,N}^{\|}[n_k, \lambda] \ , \end{aligned}$$

and in particular, for $|\alpha| = 1$,

$$\begin{aligned} \| \, \bar{P}_1^{\perp}[H_f] \left(\partial_{\mathcal{O}}^{\alpha} w_{M,N}[\hat{\mathfrak{h}}] \right) \bar{P}_1^{\perp}[H_f + |k|] \, \| &\leq 2 \, g \, \lambda_{n-l} \, + \, \hat{\epsilon}^{\frac{3}{2}} \, \xi \ , \\ \| \, \bar{P}_1^{\perp}[H_f] \left(\partial_{Y} w_{M,N}[\hat{\mathfrak{h}}] \right) \bar{P}_1^{\perp}[H_f + |k|] \, \| &\leq \hat{\epsilon}^{\frac{3}{2}} \, \xi \ , \end{aligned}$$

where Y denotes the absolute value of a photon momentum $|k|$, z, H_f or $\mathbf{P}_f^{\perp}$. Furthermore, for $0 \leq |\alpha| \leq 2$ and $0 \leq \beta \leq 1$,

$$\| \, \partial_{|p|}^{\beta} \, \partial_X \, w_{M,N}[\hat{\mathfrak{h}}] \, \| \ , \ \| \, \partial_{|p|}^{\beta} \, \partial_{\mathcal{O}}^{\alpha} \, w_{M,N}[\hat{\mathfrak{h}}] \, \| \leq (2\epsilon_0)^{\beta} \, \hat{\epsilon}^{1-\beta} \xi \quad , \text{ if } M + N = 1 \ ,$$

while for $M + N \geq 2$,

$$\begin{aligned} \| \, \partial_{|p|}^{\beta} \, \partial_{\mathcal{O}}^{\alpha} \, \hat{w}_{M,N} \, \| &\leq (2\epsilon_0)^{\beta} \, \hat{\epsilon}^{\frac{7}{4} - \frac{|\alpha|}{4} - \beta} \, \xi^{M+N} \ , \\ \| \, \partial_{|p|}^{\beta} \, \partial_X \, \hat{w}_{M,N} \, \| &\leq (2\epsilon_0)^{\beta} \, \hat{\epsilon}^{\frac{3}{2} - \beta} \, \xi^{M+N} \ . \end{aligned}$$

All bounds are uniform in $|p|$, for $|p| \leq p_c$.

8.8. Two key technical Lemmata

The auxiliary results presented in this section will be very helpful for the rest of the discussion.

LEMMA 8.1. *Let*

$$K[\mathfrak{h}] \;=\; a \;+\; \sum_{M+N\geq 0} K_{M,N}[\mathfrak{h}]$$

denote a Wick ordered operator on $P_1\mathcal{H}_p$ (for instance $\partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}}W[\mathfrak{h}]$ or $\partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}}H[\mathfrak{h}]$), where

$$a \;:=\; \langle K[\mathfrak{h}]\rangle_{\Omega_f} \;.$$

Furthermore, assume that the integral kernels are subject to bounds

$$\begin{aligned} \|\, k_{0,0}\,\| &\leq b \\ \|\!|\, k_{M,N}\,|\!\| &\leq c\,\xi^{M+N} \;. \end{aligned}$$

Then,

$$\begin{aligned} \|\, K[\mathfrak{h}]\,\Omega_f\,\| &\leq a \;+\; 4\,\sqrt{\pi}\,c\,\xi \\ \|\, \bar{\chi}_\rho\, K[\mathfrak{h}]\,\Omega_f\,\| &\leq 4\,\sqrt{\pi}\,c\,\xi \end{aligned}$$

and

$$\|\, K[\mathfrak{h}]\,\| \;\leq\; a \;+\; b \;+\; 4\,\sqrt{\pi}\,c\,\xi \;.$$

Proof.
We have

$$\|K_{M,N}[\mathfrak{h}]\| \;\leq\; (2\sqrt{\pi})^{M+N}\,|\!|\!|\, k_{M,N}\,|\!|\!| \;,$$

cf. Lemma 5.1, thus

$$\begin{aligned} \|\sum_{M+N\geq 1} K_{M,N}[\mathfrak{h}]\| &\leq c\sum_{M+N\geq 1}(2\sqrt{\pi}\xi)^{M+N} \\ &\leq 4\sqrt{\pi}\,c\,\xi \;. \end{aligned}$$

The asserted bounds follow immediately from $\|K_{0,0}\| \leq b$ and $K_{0,0}\Omega_f = 0$. ■

The following **main lemma** provides analytical control of the corrections of the interaction kernels and their derivatives under Feshbach decimation.

LEMMA 8.2. *Let X denote z or $|k_r|, |\tilde{k}_s|$. We assume that $M+N \geq 0$, and set $\rho = \frac{1}{2}$. Then, there are constants $C_{\alpha,\beta}$, $C_{X,\beta} < \infty$ independent of ϵ, ξ, such that*

$$|\!|\!|\, \partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}}\Delta w_{M,N}\,|\!|\!| \;\leq\; \epsilon_0^\beta\, C_{\alpha,\beta}\,\epsilon^{2-\beta}\,2^{M+N}\,\xi^{M+N} \;,$$

for $\beta = 0, 1$ and $0 \leq |\alpha| \leq 2$, and

$$|\!|\!|\, \partial^\beta_{|p|}\partial_X\Delta w_{M,N}\,|\!|\!| \;\leq\; \epsilon_0^\beta\, C_{X,\beta}\,\epsilon^{2-\beta}\,2^{M+N}\,\xi^{M+N} \;,$$

for $|p| \leq p_c$. In particular, the kernels $\Delta w_{M,N}$ are **analytic** *in $\mathbf{P}_f$ for all $M+N \geq 0$.*

Proof. The proof is given in the Appendix of Chapter 11. ■

8.9. Well-definedness of the Feshbach triple

A preliminary task in the discussion of the renormalization group iteration step is to verify that the Feshbach triple $(Q^\sharp_{P_\rho}[\mathfrak{h}], Q_{P_\rho}[\mathfrak{h}], F_{\chi_\rho,\tau}[H[\mathfrak{h}]])$ associated to $(P_\rho, P_1\mathcal{H}_p)$ is well-defined. In addition, it must be demonstrated that $F_{\chi_\rho,\tau}[H[\mathfrak{h}]]$ can be expanded into an absolutely convergent Neumann series for the operation of Feshbach decimation to be applicable.

Let

$$\begin{aligned} \tau[H[\mathfrak{h}]] &= (1+\Delta_{H_f}[z])H_f \,-\, (z-\Delta E_0[z]) \\ \omega[H[\mathfrak{h}]] &= H[\mathfrak{h}] \,-\, \tau[H[\mathfrak{h}]] \\ &= -\Delta_{H_f}[z]H_f \,-\, \Delta E_0[z] \,+\, \chi_1\, W[\mathfrak{h}]\,\chi_1 \;, \end{aligned} \tag{8.9.1}$$

where the definitions of $\Delta_{H_f}[z]$ and $\Delta E_0[z]$ are given in Section 4.5. Then,

$$\bar{R}_0[\mathfrak{h}] \;=\; (\tau[H[\mathfrak{h}]] \,+\, \bar{\chi}_\rho\, T'[\mathfrak{h}]\,\bar{\chi}_\rho)^{-1} \quad ,$$

denotes the 'free resolvent' on $\bar{P}_\rho\mathcal{H}_p$, with

$$\begin{aligned} T'[\mathfrak{h}] &= T[\mathfrak{h}] \,-\, \tau[H[\mathfrak{h}]] \\ &= -\Delta_{H_f}[z]H_f \,+\, \Delta E_0[z] \,+\, \chi_1 \left(a[z]\mathbf{P}_f^{\|} \,+\, T_{n-l}[\mathfrak{h}]\right)\chi_1 \;, \end{aligned}$$

and writing

$$\bar{\chi} \;\equiv\; \bar{\chi}_\rho\chi_1 \;,$$

the operator

$$\bar{R}[\mathfrak{h}] \;=\; (\tau[H[\mathfrak{h}]] \,+\, \bar{\chi}\,\omega[H[\mathfrak{h}]]\,\bar{\chi})^{-1} \quad ,$$

denotes the 'full resolvent'.

¿From the general discussion of the Feshbach map, we recall that it must be verified that

$$\begin{gathered} \bar{R}[\mathfrak{h}] \;,\; \bar{\chi}_\rho\,\bar{R}\,\bar{\chi}_\rho\,\omega[H[\mathfrak{h}]]\,\chi_\rho \;, \\ \chi_\rho\,\omega[H[\mathfrak{h}]]\,\bar{\chi}_\rho\bar{R}[\mathfrak{h}]\,\bar{\chi}_\rho \;, \\ \chi_\rho\,\omega[H[\mathfrak{h}]]\,\bar{\chi}_\rho\,\bar{R}[\mathfrak{h}]\,\bar{\chi}_\rho\,\omega[H[\mathfrak{h}]]\,\bar{\chi}_\rho \;, \\ \chi_\rho\,\omega[H[\mathfrak{h}]]\,\chi_\rho \end{gathered} \tag{8.9.2}$$

extend to bounded operators on $P_1\mathcal{H}_p$ in order for

$$\begin{aligned} Q^\sharp_{\chi_\rho,\tau}[\mathfrak{h}] &= \chi_\rho \,-\, \chi_\rho\,\omega[H[\mathfrak{h}]]\,\bar{\chi}_\rho\,\bar{R}\,\bar{\chi}_\rho \;, \\ Q_{\chi_\rho,\tau}[\mathfrak{h}] &= \chi_\rho \,-\, \bar{\chi}_\rho\,\bar{R}\,\bar{\chi}_\rho\,\omega[H[\mathfrak{h}]]\,\chi_\rho \;, \\ F_{\chi_\rho,\tau}[H[\mathfrak{h}]] &= \tau[H[\mathfrak{h}]] \,+\, \chi_\rho\,\omega[H[\mathfrak{h}]]\,\chi_\rho \\ &\quad -\chi_\rho\,\omega[H[\mathfrak{h}]]\,\bar{\chi}_\rho\,\bar{R}[\mathfrak{h}]\,\bar{\chi}_\rho\,\omega[H[\mathfrak{h}]]\,\bar{\chi}_\rho \end{aligned}$$

to be well-defined.

¿From Lemma 4.1, Neumann series expansion of the resolvent in the Feshbach map yields

$$\begin{aligned} F_{\chi_\rho,\tau}[H[\mathfrak{h}]] &= \tau[H[\mathfrak{h}]] + \chi_\rho T'[\mathfrak{h}]\,\Pi[\mathfrak{h}]\,\chi_\rho + \chi_\rho\,\Pi[\mathfrak{h}]\,W[\mathfrak{h}]\,\Pi[\mathfrak{h}]\,\chi_\rho \\ &- \sum_{L=2}^{\infty}(-1)^L \sum_{\substack{1\le M_i+N_i \\ i=2,\dots,L}} \chi_\rho\,\Pi[\mathfrak{h}] \left(\sum_{1\le M_1+N_1} W_{M_1,N_1}[\mathfrak{h}]\right) \bar\chi\,\bar R_0[\mathfrak{h}]\,\bar\chi\,\cdots \end{aligned}$$

$$\cdots\,\bar\chi\,W_{M_{L-1},N_{L-1}}[\mathfrak{h}]\,\bar\chi\,\bar R_0[\mathfrak{h}]\,\bar\chi \left(\sum_{1\le M_L+N_L} W_{M_L,N_L}[\mathfrak{h}]\right)\Pi[\mathfrak{h}]\,\chi_\rho\ , \tag{8.9.3}$$

where

$$\begin{aligned} \Pi[\mathfrak{h}] &= \mathbf{1} - \bar\chi_\rho T'[\mathfrak{h}]\bar\chi_\rho \bar R_0[\mathfrak{h}] \\ &= \bar P_\rho^\perp + \bar P_\rho\,\tau[H[\mathfrak{h}]]\,\bar R_0[\mathfrak{h}]\ . \end{aligned} \tag{8.9.4}$$

Wick ordering each of the summands in the Neumann series, and collecting all Wick monomials that are equal in the number of creation and annihilation operators, we find the image of $H[\mathfrak{h}]$ under smooth Feshbach decimation,

$$\begin{aligned} \mathcal{D}_\rho[H[\mathfrak{h}]] &= (1+\Delta_{H_f}[z])H_f - (z - \Delta E_0[z]) \\ &+ \chi_\rho(T'[\mathfrak{h}]\,\Pi[\mathfrak{h}] + \Delta T'[z';\mathcal{O}_0])\,\chi_\rho \\ &+ \sum_{M+N\ge 1} \chi_\rho \left(\Pi[\mathfrak{h}]\,W_{M,N}[\mathfrak{h}]\,\Pi[\mathfrak{h}] + \Delta W_{M,N}[z;\mathcal{O}]\right)\,\chi_\rho\ , \end{aligned} \tag{8.9.5}$$

in full analogy to the expression obtained in the first decimation step.

LEMMA 8.3. *For $z\in[-\epsilon,0]$ and $|p|\le p_c$ as required in the induction hypothesis,*

$$\frac{1}{4} \le \bar R_0[\mathfrak{h}]\,,\,\bar R[\mathfrak{h}] \le 7\,,$$

and in particular,

$$\|P_\rho \bar R_0[\mathfrak{h}] P_\rho\| \le \rho^{-1}\left(1-|p|+O(\sqrt{\epsilon_0})\right)^{-1}$$

with $\rho=\frac{1}{2}$. Furthermore,

$$\begin{aligned} \|\,\partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}}\,\bar R_0[\mathfrak{h}]\,\|\,,\ \|\,\partial_z\,\bar R_0[\mathfrak{h}]\,\| &\le C \\ \|\,\partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}}\,\bar R[\mathfrak{h}]\,\|\,,\ \|\,\partial_z\,\bar R[\mathfrak{h}]\,\| &\le C \end{aligned}$$

and

$$\begin{aligned} \|\partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}} W[\mathfrak{h}]\| &\le C'\,\epsilon_0^\beta \epsilon^{1-\beta}\,\xi \\ \|\partial_z W[\mathfrak{h}]\| &\le C\,\epsilon\,\xi \end{aligned}$$

for some constants C, C' that are independent of ϵ.

Proof. From $\frac{1}{3}H_f \le T \le 3H_f$ in (8.4.11) of the induction hypothesis immediately follows that

$$\frac{1}{3} \le \bar R_0 \le \frac{3}{\rho}\,.$$

Expanding $\bar{R}$ into a Neumann series, we have

$$\bar{R} \;=\; \bar{R}_0 \;+\; \sum_{j\geq 1} \bar{R}_0 \left(\bar{\chi} W \bar{\chi} \bar{R}_0\right)^j \;,$$

where the sum can be estimated by

$$\begin{aligned} \sum_{j\geq 1} \|\bar{R}_0\| \,\| \left(\bar{\chi} W \bar{\chi} \bar{R}_0\right) \|^j \;&\leq\; \|\bar{R}_0\| \sum_{j\geq 1} O(\epsilon^j\, \xi^j) \\ &=\; \|\bar{R}_0\|\, O(\epsilon\, \xi) \;, \end{aligned}$$

using $\|W[\mathfrak{h}]\| \leq O(\epsilon\, \xi)$ on $P_1 \mathcal{H}_p$ from choosing $K[\mathfrak{h}] = W[\mathfrak{h}]$ in Lemma 8.1. We thus obtain the asserted bounds, for ϵ sufficiently small.

Next, we note that

$$\begin{aligned} \left\| P_\rho\, \bar{R}_0[\mathfrak{h}]\, P_\rho \right\| \;&\leq\; \rho^{-1} \left(1 - |a[z]| - \|P_\rho \left(T_{n-l}[\mathfrak{h}] - \lambda_{n-l} |\mathbf{P}_f|^2\right) P_\rho\|\right)^{-1} \\ &\leq\; \rho^{-1} \left(1 - |p| + O(\sqrt{\epsilon}_0)\right)^{-1} \;, \end{aligned}$$

using (8.4.2) of the induction hypothesis.

The inequalities asserted for the derivatives of $\bar{R}_0[\mathfrak{h}]$ and $\bar{R}[\mathfrak{h}]$ and $W[\mathfrak{h}]$ are straightforward consequences of Lemma 4.2, and the induction assumptions on derivatives of $T[\mathfrak{h}]$ and $W[\mathfrak{h}]$, in addition to the fact that $\|\partial_{\mathcal{O}}^\alpha \chi\|, \|\partial_{\mathcal{O}}^\alpha \bar{\chi}\| \leq O(1)$. ■

Hence, all operators in (8.9.2) are indeed bounded on $P_1 \mathcal{H}_p$. This result also implies that $F_{\chi_\rho,\tau}[H[\mathfrak{h}]]$ can be expanded into an absolutely convergent Neumann series, cf. the detailed discussion in Section 7.2.

LEMMA 8.4. *Let $\rho = \frac{1}{2}$. Then, there is a constant C that is independent of ϵ, such that*

$$\|\partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha \Pi\| \;\leq\; C \tag{8.9.6}$$

holds on $P_1 \mathcal{H}_p$, for $0 \leq |\alpha| \leq 2$ and $\beta = 0, 1$. In particular, with

$$\|\partial_{H_f} (\chi_\rho^2 \bar{\chi}_\rho^2)\| \;,\; \|\partial_{H_f} \bar{\chi}_\rho^2\| \;,\; \|\partial_{H_f} \bar{\chi}_1^2\| \;\leq\; 10 \;,$$

the estimates

$$\begin{aligned} \|\Pi[\mathfrak{h}]\| \;&\leq\; \frac{4}{3} \\ \|\chi_\rho \left(\Pi \,-\, \mathbf{1}\right) \chi_\rho\| \;&\leq\; \frac{1}{3} \qquad (8.9.7) \\ \|\partial_{|p|} \Pi[\mathfrak{h}]\| \;&\leq\; 2 \qquad (8.9.8) \\ |\partial_{H_f} (\chi_\rho \Pi \chi_\rho)| \;&\leq\; 2 \,+\, 3\, |\partial_{H_f} (\chi_\rho^2 \bar{\chi}_\rho^2)| \\ &\qquad +\, 2(|p| + \lambda_{n-l})\, |\partial_{H_f} \bar{\chi}_\rho^2| \qquad (8.9.9) \end{aligned}$$

hold for $|p| \leq p_c$ with $p_c \in [\frac{1}{20}, 1)$ sufficiently small.

Proof. From

$$\left\| \partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha X \right\| \;\leq\; c$$

for

$$X \;=\; \chi_\rho^2 \bar{\chi}_\rho^2 \;,\; \bar{R}_0 \text{ and } T'[\mathfrak{h}]$$

(since $|\partial_{|p|}\Delta_{H_f}[z]|, |\partial_{|p|}\Delta E_0[z]| \leq O(\epsilon_0)$, as will be proved below, and using Lemma 8.3) follows immediately that there exists a constant $C < \infty$ such that (8.9.6) holds.

To prove (8.9.7), where (on $P_\rho \mathcal{H}_p$)

$$\Pi[\mathfrak{h}] \,-\, \mathbf{1} \;=\; -\,\bar{\chi}_\rho T'[\mathfrak{h}]\bar{\chi}_\rho \bar{R}_0[\mathfrak{h}] \;,$$

we first recall that

$$\left\| P_\rho \, \bar{R}_0[\mathfrak{h}] \, P_\rho \right\| \;\leq\; \rho^{-1} \left(1 - |p| + O(\sqrt{\epsilon}_0)\right)^{-1} \;,$$

from Lemma 8.3. Furthermore,

$$\|\chi_\rho^2 \bar{\chi}_\rho^2\| \;\leq\; \frac{1}{4} \;, \tag{8.9.10}$$

because of $\max_{x\in[0,1]}\{x^2(1-x^2)\} = \frac{1}{4}$. Finally, by definition of $T'[\mathfrak{h}]$ and $|\Delta_{H_f}[z]|, |\Delta E_0[z]| \leq O(\epsilon^2)$,

$$\begin{aligned} \|T'[\mathfrak{h}]\| \;&\leq\; |a[z]| \,+\, \lambda_{n-l} \,+\, O(\epsilon^2) \\ &\leq\; |p| \,+\, \lambda_{n-l} \,+\, O(\sqrt{\epsilon}_0) \;. \end{aligned}$$

Combining these estimates with $\rho = \frac{1}{2}$ and $\lambda_{n-l} \leq \frac{1}{2}$, we find that

$$\begin{aligned} \|\chi_\rho \, (\Pi[\mathfrak{h}] \,-\, \mathbf{1}) \, \chi_\rho\| \;&\leq\; \frac{\rho^{-1}}{4} \frac{|p| + \frac{1}{2} + O(\sqrt{\epsilon}_0)}{1 - |p| + O(\sqrt{\epsilon}_0)} \\ &\leq\; \frac{1}{3} \;, \end{aligned}$$

and from

$$\Pi[\mathfrak{h}] \;=\; \bar{P}_\rho^\perp \,+\, \bar{P}_\rho \, \tau[H[\mathfrak{h}]] \, \bar{R}_0[\mathfrak{h}]$$

we have

$$\begin{aligned} \|\Pi[\mathfrak{h}]\| \;&\leq\; \max\left\{1 \,,\; \|\tau[H[\mathfrak{h}]] \, \bar{P}_\rho \bar{R}_0[\mathfrak{h}]\|\right\} \\ &\leq\; (1 + O(\epsilon^2)) \max_{x\in[\frac{1}{2},1]} \frac{x}{(1 - |p|)x + O(\sqrt{\epsilon}_0)} \\ &\leq\; \frac{4}{3} \;, \end{aligned}$$

for sufficiently small $|p| \leq p_c$ with $p_c \in [\frac{1}{20}, 1)$, as claimed.

As to (8.9.8), we have

$$\begin{aligned} \|\partial_{|p|}\Pi[\mathfrak{h}]\| \;&\leq\; \|(\partial_{|p|}\tau[H[\mathfrak{h}]]) \, \bar{R}_0[\mathfrak{h}]\| \\ &\quad + \|\tau[H[\mathfrak{h}]]\bar{R}_0^2[\mathfrak{h}]\partial_{|p|}(\tau[H[\mathfrak{h}]] + \bar{\chi}_\rho T'\bar{\chi}_\rho)\| \\ &\leq\; \max_{x\in[\frac{1}{2},1]} \frac{(1 + O(\epsilon^2)) \, |\partial_{|p|}a[z]| \, x^2}{((1 - |p|)x + O(\sqrt{\epsilon}_0))^2} \,+\, O(\epsilon_0) \\ &\leq\; 2 \end{aligned}$$

on $P_\rho \mathcal{H}_p$. This establishes the claim.

To prove (8.9.9), we use

$$\left\| P_\rho \left(\partial_{H_f} T'[\mathfrak{h}]\right) P_\rho \right\| \;\leq\; O(\epsilon^2) \;,$$

using the fact that $P_\rho\,(\partial_{H_f}\bar\chi_1)\,P_\rho = 0$ since the spectral supports of $\partial_{H_f}\bar\chi_1$ and P_ρ are mutually disjoint, as well as

$$\begin{aligned}
\left|P_\rho\left(\partial_{H_f}\bar R_0[\mathfrak{h}]\right)P_\rho\right| &\leq \left\|P_\rho\bar R_0[\mathfrak{h}]P_\rho\right\|^2\left|P_\rho\left(\partial_{H_f}(\tau[H[\mathfrak{h}]]+\bar\chi_\rho T'[\mathfrak{h}]\bar\chi_\rho)\right)P_\rho\right| \\
&\leq \frac{\rho^{-2}\left(1+O(\epsilon^2)+\|P_\rho T'[\mathfrak{h}]P_\rho\|\,|\partial_{H_f}\bar\chi_\rho^2|\right)}{(1-|p|+O(\sqrt{\epsilon}_0))^2} \\
&\leq \frac{\rho^{-2}\left(1+(|p|+\lambda_{n-l})|\partial_{H_f}\bar\chi_\rho^2|+O(\sqrt{\epsilon}_0)\right)}{(1-|p|+O(\sqrt{\epsilon}_0))^2}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\partial(\chi_\rho\,\Pi[\mathfrak{h}]\,\chi_\rho)| &\leq |\partial_{H_f}(\chi_\rho^2\bar\chi_\rho^2)|\,\|P_\rho T'[\mathfrak{h}]P_\rho\|\,\|P_\rho\,\bar R_0[\mathfrak{h}]\,P_\rho\| \\
&\quad + \|\chi_\rho^2\bar\chi_\rho^2\|\,\|P_\rho(\partial_{H_f}T'[\mathfrak{h}])P_\rho\|\,\|P_\rho\,\bar R_0[\mathfrak{h}]\,P_\rho\| \\
&\quad + \|\chi_\rho^2\bar\chi_\rho^2\|\,\|P_\rho T'[\mathfrak{h}]P_\rho\|\,|P_\rho\left(\partial_{H_f}\bar R_0[\mathfrak{h}]\right)P_\rho| \\
&\leq \rho^{-1}\,\frac{(1+|p|+\lambda_{n-l}+O(\sqrt{\epsilon}_0))}{1-|p|+O(\sqrt{\epsilon}_0)}|\partial_{H_f}(\chi_\rho^2\bar\chi_\rho^2)| \;+\; O(\epsilon) \\
&\quad + \frac{\rho^{-2}}{4\,(1-|p|+O(\sqrt{\epsilon}_0))^2}(1+O(\sqrt{\epsilon}_0)) \\
&\quad + \frac{\rho^{-2}}{4}\frac{|p|+\lambda_{n-l}}{(1-|p|+O(\sqrt{\epsilon}_0))^2}\,|\partial_{H_f}\bar\chi_\rho^2| \\
&\leq 2\;+\;3\,|\partial_{H_f}(\chi_\rho^2\bar\chi_\rho^2)|\;+\;2(|p|+\lambda_{n-l})\,|\partial_{H_f}\bar\chi_\rho^2|
\end{aligned}$$

for $|p|\leq p_c$ with $p_c\in[\frac{1}{20},1)$ sufficiently small. ■

8.10. Renormalization of the spectral parameter

We absorb the quantity

$$\Delta E_0[z] \;:=\; \left\langle F_{\chi_\rho,\tau}\left[H[\mathfrak{h}]\right]+z\right\rangle_{\Omega_f} \tag{8.10.1}$$

into a shift of the spectral parameter,

$$z \;\longrightarrow\; z \;-\; \Delta E_0[z]\;. \tag{8.10.2}$$

Application of the rescaling transformation yields

$$\hat z \;=\; Z[z] \;:=\; (1+\Delta_{H_f}[z])^{-1}\;\rho^{-1}\;(z\;-\;\Delta E_0[z])\;, \tag{8.10.3}$$

the renormalized spectral parameter.

PROPOSITION 8.1. *The map $Z: z\to\hat z$ is of class C^1, and its first derivative is given by*

$$\partial_z Z[z] \;=\; \rho^{-1}(1+\Delta_{H_f}[z])^{-1}\;+\;O(\epsilon^2\,\xi^2)\;,$$

for z as required in the induction hypothesis. Thus, for any function f of class C^1 in z,

$$\left|\partial_{\hat z}f[Z^{-1}[\hat z]]\right|\;\leq\;(\rho\;+\;O(\epsilon))\;|\partial_z f[z]|\;. \tag{8.10.4}$$

In addition,

$$|\;\partial_z^\alpha\Delta E_0[z]\;|\;\leq\;O(\epsilon^2\,\xi^2)$$

for $\alpha=0,1$.

Proof. From straightforward calculation,

$$\partial_z Z[z] \;=\; \rho^{-1}(1+\Delta_{H_f}[z])^{-1}\left(1 \;-\; \partial_z \Delta E_0[z]\right) - \frac{\partial_z \Delta_{H_f}[z]}{1+\Delta_{H_f}[z]}\, Z[z] \quad .$$

We have

$$\begin{aligned} |\partial_z \Delta E_0[z]| \;&=\; \left| \partial_z \left\langle W[\mathfrak{h}]\, \bar{R}[\mathfrak{h}]\, W[\mathfrak{h}] \right\rangle_{\Omega_f} \right| \\ &\leq\; O(\epsilon^2\, \xi^2)\;, \end{aligned}$$

using

$$\|\partial_z^\beta W[\mathfrak{h}]\| \;\leq\; O(\epsilon\, \xi)$$

and

$$\|\partial_z^\beta \bar{R}\| \;\leq\; O(1)\;,$$

for $\beta = 0, 1$, from Lemma 8.3, and

$$|\partial_z^\beta \Delta_{H_f}[z]| \;\leq\; O(\epsilon^2\, \xi^2)$$

from Lemma 8.5 below. In the same manner, one shows that $|\Delta E_0[z]| \leq O(\epsilon^2\, \xi^2)$.

Therefore,

$$\partial_z Z[z] \;=\; \rho^{-1}(1+\Delta_{H_f}[z])^{-1} \;+\; O(\epsilon^2\, \xi^2)\;,$$

and (8.10.4) follows immediately. ■

The renormalized spectral parameter in $\hat{\mathfrak{h}} = \mathcal{R}[\mathfrak{h}] \in \mathfrak{H}$ is obtained by the substitution $z \to Z^{-1}[\hat{z}]$.

8.11. Renormalization of $T[\mathfrak{h}]$

Let us discuss the renormalization of the operators H_f, $a[z]\mathbf{P}_f^{\|}$ and $T_{n-l}[\mathfrak{h}]$ in $T[\mathfrak{h}]$.

8.11.1. The coefficient of H_f. Under Feshbach decimation, the coefficient 1 of H_f in $T_{lin}[\mathfrak{h}]$ is modified into

$$\begin{aligned} 1+\Delta_{H_f}[z] \;&=\; \left\langle \partial_{H_f} F_{\chi_\rho,\tau}\left[H[\mathfrak{h}]\right]\right\rangle_{\Omega_f} \\ &=\; \left\langle Q^\sharp_{\chi_\rho,\tau_z}[\mathfrak{h}]\; \left(1+\partial_{H_f}(T_{n-l}[\mathfrak{h}]+W[\mathfrak{h}])\right) Q_{\chi_\rho,\tau_z}[\mathfrak{h}]\right\rangle_{\Omega_f} \;\times \\ &\quad \times \Big(1 - \left\langle W[\mathfrak{h}]\bar{\chi}\bar{R}[\mathfrak{h}]\chi_\rho^2 \bar{R}[\mathfrak{h}]\bar{\chi} W[\mathfrak{h}]\right\rangle_{\Omega_f} \\ &\qquad + \left\langle Q^\sharp_{\chi_\rho,\tau_z}[\mathfrak{h}]\; \frac{\partial_{H_f}\bar{\chi}_\rho}{\bar{\chi}_\rho^3+\delta}\, \tau[H[\mathfrak{h}]]\; Q_{\chi_\rho,\tau_z}[\mathfrak{h}]\right\rangle_{\Omega_f}\Big)^{-1} \quad . \end{aligned} \tag{8.11.1}$$

Since we have decided to keep the coefficient of H_f fixed at the constant value 1 in the renormalization group iteration, the rescaling transformation must act by

$$\mathcal{S}_\rho[A] \;=\; (1+\Delta_{H_f}[z])^{-1}\; \rho^{-1}\; Ad_{U_\rho}[A] \tag{8.11.2}$$

on operators $A : P_\rho \mathcal{H}_p \to P_\rho \mathcal{H}_p$.

LEMMA 8.5. *For $|p| \leq p_c$, with $p_c \in [\frac{1}{20}, 1)$ sufficiently small, and under the assumptions of the induction hypothesis,*

$$\frac{1}{10}\, g^2\, |p|^2 \;+\; \lambda_{n-l}\, O(\epsilon^2 \xi^2) \;+\; O(\epsilon^{\frac{5}{2}} \xi^2) \;\leq\; \Delta_{H_f}[z] \;\leq\; O(\epsilon^2 \xi^2)\;.$$

Furthermore,

$$\left|\partial_{|p|} \Delta_{H_f}[z]\right| \;\leq\; C\, \epsilon_0^2$$

holds for a constant C that is independent of ϵ and g.

Proof. First of all, the denominator in (8.11.1) can be estimated by use of

$$\begin{aligned} \lim_{\delta \to 0} \left| \left\langle Q^{\sharp}_{\chi_\rho, \tau_z}[\mathfrak{h}] \bar{P}_\rho \frac{\chi_\rho^2}{\bar{\chi}_\rho^2 + \delta} \bar{P}_\rho Q_{\chi_\rho, \tau_z}[\mathfrak{h}] \right\rangle_{\Omega_f} \right| \;&\leq\; \left\| \chi_\rho \bar{R}[\mathfrak{h}] \bar{\chi}_\rho \right\|^2 \, \|W[\mathfrak{h}]\|^2 \\ &\leq\; O(\epsilon^2 \xi^2)\;. \end{aligned}$$

and

$$\begin{aligned} &\lim_{\delta \to 0} \left| \left\langle Q^{\sharp}_{\chi_\rho, \tau_z}[\mathfrak{h}] \bar{P}_\rho \frac{\partial_{H_f} \chi_\rho}{\bar{\chi}_\rho^3 + \delta} (H_f - \hat{z}) \bar{P}_\rho Q_{\chi_\rho, \tau_z}[\mathfrak{h}] \right\rangle_{\Omega_f} \right| \\ &\quad = \left| \left\langle W[\mathfrak{h}] \bar{\chi} \bar{R} (\partial_{H_f} \chi_\rho)(H_f - \hat{z}) \bar{R}_0 W[\mathfrak{h}] \right\rangle_{\Omega_f} \right| \\ &\qquad + \left| \left\langle W[\mathfrak{h}] \bar{\chi} \bar{R} (\partial_{H_f} \chi_\rho)(H_f - \hat{z}) \bar{R}_0 W[\mathfrak{h}] \bar{\chi} \bar{R} \bar{\chi} W[\mathfrak{h}] \right\rangle_{\Omega_f} \right| \\ &\quad \leq\; O(\epsilon^2 \xi^2)\;, \end{aligned}$$

by the second resolvent identity for $\bar{R}[\mathfrak{h}]$.

(I) The lower bound. Using the abbreviated notation

$$Q^{(\sharp)}[\mathfrak{h}] \;\equiv\; Q^{(\sharp)}_{\chi_\rho, \tau_z}[\mathfrak{h}]$$

it is clear that

$$\begin{aligned} &\left\langle Q^{\sharp}[\mathfrak{h}]\; (1 + \partial_{H_f}(T_{n-l}[\mathfrak{h}] + W[\mathfrak{h}]))\; Q[\mathfrak{h}] \right\rangle_{\Omega_f} \\ &\quad =\; 1 \;+\; \left\langle Q^{\sharp}[\mathfrak{h}]\; \bar{P}_\rho\; Q[\mathfrak{h}] \right\rangle_{\Omega_f} \\ &\qquad +\; \left\langle Q^{\sharp}[\mathfrak{h}] \bar{P}_\rho \left(\partial_{H_f} T_{n-l}[\mathfrak{h}]\right) \bar{P}_\rho Q[\mathfrak{h}] \right\rangle_{\Omega_f} \\ &\qquad +\; \left\langle Q^{\sharp}[\mathfrak{h}] \left(\partial_{H_f} W[\mathfrak{h}]\right) Q[\mathfrak{h}] \right\rangle_{\Omega_f}\;. \end{aligned} \tag{8.11.3}$$

The leading term in (8.11.3) is given by

$$\begin{aligned} \left\langle Q^{\sharp}[\mathfrak{h}]\; \bar{P}_\rho\; Q[\mathfrak{h}] \right\rangle_{\Omega_f} \;&=\; \left\langle\, W\, \bar{\chi}_\rho\, \bar{R}\, \bar{\chi}_\rho^2\; \bar{R}\, \bar{\chi}_\rho\, W \,\right\rangle_{\Omega_f} \\ &=\; \left\langle W_1\, \bar{\chi}_\rho\, \bar{R}\, \bar{\chi}_\rho^2\; \bar{R}\, \bar{\chi}_\rho\, W_1 \right\rangle_{\Omega_f} \qquad (8.11.4) \\ &\quad +\; \Big(\left\langle\, W_1\, \bar{\chi}_\rho\, \bar{R}\, \bar{\chi}_\rho^2\; \bar{R}\, \bar{\chi}_\rho\, W_{\geq 2} \,\right\rangle_{\Omega_f} \;+\; \text{h.c.} \Big) \qquad (8.11.5) \\ &\quad +\; \left\langle\, W_{\geq 2}\, \bar{\chi}_\rho\, \bar{R}\, \bar{\chi}_\rho^2\; \bar{R}\, \bar{\chi}_\rho\, W_{\geq 2} \,\right\rangle_{\Omega_f}\;, \qquad (8.11.6) \end{aligned}$$

where $W_1 = W_{0,1} + W_{1,0}$, and $W_{\geq 2} = W - W_1$.

(I.1) We claim that the leading term satisfies

$$\left\langle W_1\,\bar\chi_\rho\,\bar R\,\bar\chi_\rho^2\;\bar R\,\bar\chi_\rho\,W_1\,\right\rangle_{\Omega_f}\;\;\geq\;\;\mathfrak{I}[z]\;+\;O(\epsilon^{\frac{5}{2}}\xi^2)\,, \tag{8.11.7}$$

where

$$\mathfrak{I}[z]\;:=\;|a[z]|^2\,\sum_\lambda\int_{\rho\leq|k|\leq\frac{2}{3}}\frac{d^3k}{|k|}\;\left|\,\epsilon_\lambda^{\parallel}(n_k)\,\right|^2\;\bar\chi_\rho^2[|k|]\left\langle\,\bar R_0^2[z;S_k\mathcal{O}_0]\right\rangle_{\Omega_f}\,. \tag{8.11.8}$$

This can be seen as follows.

Application of the second resolvent identity to $\bar R$ yields

$$\bar R\;=\;\bar R_0\;-\;\bar R_0\bar\chi W\bar\chi\bar R\,,$$

where $\|W\|=O(\epsilon\xi)$ and $\|\bar R\|,\|\bar R_0\|=O(1)$. Therefore,

$$\left\langle W_1\,\bar\chi\,\bar R\,\bar\chi^2\;\bar R\,\bar\chi_\rho\,W_1\,\right\rangle_{\Omega_f}\;\;=\;\;\left\langle W_1\,\bar\chi\,\bar R_0\,\bar\chi^2\;\bar R_0\,\bar\chi\,W_1\,\right\rangle_{\Omega_f}\;+\;O(\epsilon^3\xi^3)$$

is immediate. Furthermore, it is clear that

$$\begin{aligned}\left\langle W_1\,\bar\chi\,\bar R_0\,\bar\chi_\rho^2\;\bar R_0\,\bar\chi\,W_1\,\right\rangle_{\Omega_f}\;\;&=\;\;\left\langle W_1\,\bar\chi^2\,\bar P_1^\perp\,\bar R_0^2\,\bar P_1^\perp\,\bar\chi^2\,W_1\,\right\rangle_{\Omega_f}\\&\quad+\;\left\langle W_1\,\bar\chi^2\,\bar P_1\,\bar R_0^2\,\bar P_1\,\bar\chi^2\,W_1\,\right\rangle_{\Omega_f}\\&\geq\;\;\left\langle W_1\,\bar\chi^2\,\bar P_1^\perp\,\bar R_0^2\,\bar P_1^\perp\,\bar\chi^2\,W_1\,\right\rangle_{\Omega_f}\,,\end{aligned}$$

and that

$$\mathfrak{I}[z]\;\;=\;\;\left\langle\,U_1\,\bar P_1^\perp\,\bar\chi^2\,\bar R_0^2\,\bar\chi^2\,\bar P_1^\perp\,U_1\,\right\rangle_{\Omega_f}\,,$$

where U_1 is defined in the same manner as W_1, but with the interaction kernels $w_{M,N}$ replaced by $u_{M,N}$ $(M+N=1)$. Defining the error term

$$\begin{aligned}\mathfrak{R}\;\;:=\;\;&\left\langle\,W_1\,\bar\chi^2\,\bar P_1^\perp\,\bar R_0^2\,\bar P_1^\perp\,\bar\chi^2\,W_1\,\right\rangle_{\Omega_f}\\&-\;\left\langle\,U_1\,\bar\chi^2\,\bar P_1^\perp\,\bar R_0^2\,\bar P_1^\perp\,\bar\chi^2\,U_1\,\right\rangle_{\Omega_f}\,,\end{aligned} \tag{8.11.9}$$

we have that, due to $W_1\Omega_f=W_1|_{\mathcal{O}\to0}\Omega_f$ and

$$\|\bar P_1^\perp(W_1|_{\mathcal{O}\to0}\;-\;U_1)\bar P_1^\perp\|\;\leq\;O(\epsilon^{\frac{3}{2}})\,\xi\,,$$

which is a consequence of (8.5.4) of the induction hypothesis,

$$\begin{aligned}\mathfrak{R}\;\;=\;\;&\left\langle\,(W_1-U_1)\bar P_1^\perp\,\bar\chi_\rho^2\,\bar R_0^2\,\bar\chi_\rho^2\,W_1\,\right\rangle_{\Omega_f}\\&-\;\left\langle\,W_1\,\bar\chi_\rho^2\,\bar R_0^2\,\bar\chi_\rho^2\,\bar P_1^\perp(W_1-U_1)\,\right\rangle_{\Omega_f}\\&+\;\left\langle\,(W_1-U_1)\bar P_1^\perp\,\bar\chi_\rho^2\,\bar R_0^2\,\bar\chi_\rho^2\,\bar P_1^\perp(W_1+U_1)\,\right\rangle_{\Omega_f}\end{aligned}$$

is bounded by

$$|\mathfrak{R}|\;\leq\;O(\epsilon^{\frac{5}{2}}\,\xi^2)\,.$$

Finally, an explicit calculation shows that for $p_c\in[\frac{1}{20},1)$ sufficiently small, and $\rho=\frac{1}{2}$, there is an estimate

$$\mathfrak{I}[z]\;\geq\;\frac{2}{5}\,g^2\,|p|^2\,,$$

for all $|p|\leq p_c$ with $p_c\in[\frac{1}{20},1)$ sufficiently small, and for z as required in the induction hypothesis. Furthermore, we have used the fact that $|a[z]|\leq|p|(1+O(\sqrt{\epsilon_0}))$, which is a

direct consequence of the induction assumption on $a[z]$.

(I.2) The error term (8.11.5) can be estimated by

$$\left| \left\langle \, W_1 \, \bar{\chi}^2 \, \bar{R}^2 \, \bar{\chi}^2 \, W_{\geq 2} \, \right\rangle_{\Omega_f} \, + \, \text{h.c.} \right| \, \leq \, O(\epsilon^{\frac{5}{2}} \, \xi^3) \, ,$$

since $\|W_{\geq 2}\| \leq O(\epsilon^{\frac{3}{2}} \, \xi^2)$, and using Lemma 8.3.

(I.3) The error term (8.11.6) can be bounded by

$$\left| \left\langle \, W_{\geq 2} \, \bar{\chi}^2 \, \bar{R}^2 \, \bar{\chi}^2 \, W_{\geq 2} \, \right\rangle_{\Omega_f} \right| \, \leq \, O(\epsilon^3 \, \xi^4) \, ,$$

for the same reasons as in the previous case under point (2).

(I.4) There are two additional error terms in (8.11.3) which have to be considered. The second vacuum expectation value appearing there can be estimated by

$$\begin{aligned}
&\left| \left\langle Q^\sharp[\mathfrak{h}] \bar{P}_\rho \left(\partial_{H_f} T_{n-l}[\mathfrak{h}] \right) \bar{P}_\rho Q[\mathfrak{h}] \right\rangle_{\Omega_f} \right| \\
&\quad = \, \left| \left\langle W \bar{\chi} \bar{R} \bar{\chi} \left(\partial_{H_f} T_{n-l} \right) \bar{\chi} \bar{R} \bar{\chi} \, W \, \right\rangle_{\Omega_f} \right| \\
&\quad \leq \, (2(|p| + \lambda_{n-l}) + O(\sqrt{\epsilon_0})) \left\langle W \bar{\chi} \bar{R} \bar{\chi}^2 \bar{R} \bar{\chi} W \, \right\rangle_{\Omega_f} \\
&\qquad + 3 \, (|p| \, + \, \lambda_{n-l}) \left\langle W \bar{\chi} \bar{R} \bar{\chi} | \partial_{H_f} (\chi_1^2 \bar{\chi}_1^2) | \bar{\chi} \bar{R} \bar{\chi} W \, \right\rangle_{\Omega_f} \\
&\qquad + 2(|p| + \lambda_{n-l})^2 \left\langle W \bar{\chi} \bar{R} \bar{\chi} | \partial_{H_f} \bar{\chi}_1^2 | \bar{\chi} \bar{R} \bar{\chi} W \, \right\rangle_{\Omega_f} \\
&\quad = \, 2 \, (|p| + O(\sqrt{\epsilon_0})) \left\langle W \bar{\chi} \bar{R} \bar{\chi}^2 \bar{R} \bar{\chi} W \, \right\rangle_{\Omega_f} \\
&\qquad + 3 \, |p| \, \left\langle W \bar{\chi} \bar{R} \bar{\chi} | \partial_{H_f} (\chi_1^2 \bar{\chi}_1^2) | \bar{\chi} \bar{R} \bar{\chi} W \, \right\rangle_{\Omega_f} \\
&\qquad + 2|p|^2 \left\langle W \bar{\chi} \bar{R} \bar{\chi} | \partial_{H_f} \bar{\chi}_1^2 | \bar{\chi} \bar{R} \bar{\chi} W \, \right\rangle_{\Omega_f} \\
&\qquad + \lambda_{n-l} \, O(\epsilon^2 \xi^2) \, , \qquad (8.11.10)
\end{aligned}$$

using (8.4.4) of the induction hypothesis, and the fact that $\left(\partial_{H_f} T_{n-l} \right) \Omega_f = 0$. Thus, the bounds (8.2.3) on the derivatives of χ_ρ, χ_1 imply that

$$\begin{aligned}
\left| \left\langle Q^\sharp[\mathfrak{h}] \bar{P}_\rho \left(\partial_{H_f} T_{n-l}[\mathfrak{h}] \right) \bar{P}_\rho Q[\mathfrak{h}] \right\rangle_{\Omega_f} \right| \, &\leq \, (14 + 8|p|) \, |p| \, \left\langle W \bar{\chi} \bar{R} \bar{\chi}^2 \bar{R} \bar{\chi} W \, \right\rangle_{\Omega_f} \\
&\quad + \lambda_{n-l} \, O(\epsilon^2 \xi^2) \\
&\leq \, \frac{3}{4} \left\langle W \bar{\chi} \bar{R} \bar{\chi}^2 \bar{R} \bar{\chi} W \, \right\rangle_{\Omega_f} \\
&\quad + \lambda_{n-l} \, O(\epsilon^2 \xi^2) \qquad (8.11.11)
\end{aligned}$$

for $|p| \leq p_c$ with $p_c \in [\frac{1}{20}, 1)$ small enough.

(I.5) The last error term in (8.11.3) can be estimated by

$$\begin{aligned}
\left| \, \left\langle Q^\sharp[\mathfrak{h}] \left(\partial_{H_f} W[\mathfrak{h}] \right) Q[\mathfrak{h}] \right\rangle_{\Omega_f} \, \right| \, &\leq \, \|\partial_{H_f} W[\mathfrak{h}]\| \left\langle W \bar{\chi} \bar{R} \bar{\chi}^2 \bar{R} \bar{\chi} W \, \right\rangle_{\Omega_f} \\
&\quad + 2 \left| \, \left\langle Q^\sharp[\mathfrak{h}] \, \bar{P}_\rho \, \left(\partial_{H_f} W[\mathfrak{h}] \right) \right\rangle_{\Omega_f} \, \right| \\
&\leq \, O(\epsilon^{\frac{5}{2}} \xi^2) \, , \qquad (8.11.12)
\end{aligned}$$

since

$$\|\bar{P}_1^{\perp}\partial_{H_f}W\Omega_f\| \;\le\; O(\epsilon^{\frac{3}{2}}\xi)\;,$$

due to (8.5.4) of the induction hypothesis, and

$$\|Q\,\Omega_f\| \;\le\; O(\epsilon\,\xi)\;.$$

(I.6) In total, we have, in view of (8.11.7) and the estimates derived in (I.1) $\sim$ (I.5), the lower bound

$$\begin{aligned}\Delta_{H_f}[z] \;&\ge\; (1-\frac{3}{4})\,\langle U_1\bar{\chi}^2\bar{R}_0^2\bar{\chi}^2U_1\,\rangle_{\Omega_f}\\ &\quad+\lambda_{n-l}\,O(\epsilon^2\xi^2)\;+\;O(\epsilon^{\frac{5}{3}}\xi^2)\;,\end{aligned}$$

which implies the claim.

(II) The upper bound. The estimate

$$|\Delta_{H_f}[z]| \;\le\; O(\epsilon^2\,\xi^2)$$

follows from

$$\langle W\bar{\chi}\bar{R}\bar{\chi}^2\bar{R}\bar{\chi}W\,\rangle_{\Omega_f} \;\le\; O(\epsilon^2\xi^2)\;,$$

which is evident from $\|\bar{R}[\mathfrak{h}]\|\le O(1)$, $\|W[\mathfrak{h}]\|\le O(\epsilon\xi)$.

(III) The estimate

$$|\partial_{|p|}\Delta_{H_f}[z]| \;\le\; O(\epsilon_0^2)$$

follows from $\|\partial_{|p|}\bar{R}[\mathfrak{h}]\|\le O(1)$, $\|\partial_{|p|}W[\mathfrak{h}]\|\le O(g)$, $g\ll\epsilon_0$, $\epsilon\le 2\epsilon_0$, and straightforwardly taking the derivative of (8.11.1) with respect to $|p|$. ■

8.11.2. Renormalization of the coefficient of $\mathbf{P}_f^{\parallel}$. The strategy in renormalizing $a[z]$ is to employ a nested induction argument. Assuming that Theorem 8.1 and the bound (8.4.7) hold for all $\mathfrak{h}^{(k)}$ with $0\le k<n$, it is proved that the assumptions of Propositions 6.3 and 6.2 are fulfilled. Consequently, the bound (8.4.7) also holds for $k=n$.

Proposition 8.2. *Let $\mathfrak{h}^{(n)}$ denote the point in $\mathfrak{V}_{2\epsilon_0,\xi}$ obtained from $\mathfrak{h}^{(0)}$ by n-fold iterating $\mathcal{R}$, and let $\rho=\frac{1}{2}$. Assume that both Theorem 8.1 and the bound (8.4.7) hold for $\mathfrak{h}^{(j-1)}\to\mathfrak{h}^{(j)}$, for all $j<n$. Then,*

$$\left|\partial_{|p|}^{\beta}(a_n[z_n]\;+\;|p|)\right| \;\le\; \sqrt{\epsilon}_0 \tag{8.11.13}$$

for $|p|\le p_c$.

Proof.
Lemmata 8.1 and 8.2 imply that

$$\begin{aligned}\|\partial_{|p|}^{\beta}\bar{R}[\mathfrak{h}^{(k)}]\| \;&\le\; C\\ \|\partial_{|p|}^{\beta}W[\mathfrak{h}^{(k)}]\| \;&\le\; \epsilon_0^{\beta}\,C'\,\epsilon^{1-\beta}\,\xi\end{aligned}$$

for constants C, C' that are independent of ϵ, $\beta = 0, 1$, and all $0 \leq k < n$. Furthermore,

$$\left| \partial_{|p|}^{\beta} \left(\eta[\mathfrak{h}^{(n-1)}] - 1 \right) \right| \;\leq\; C\, \epsilon_0^{\beta} \epsilon^{2-\beta} \;,$$

for $\beta = 0, 1$, where $\eta[\mathfrak{h}]$ is defined in Propositions 6.1 and 6.2.

(I) The assertion for the case $\beta = 0$ is a consequence of Proposition 6.3 under the condition that the main theorem holds for all $\mathfrak{h}^{(k)}$ with $0 \leq k < n$.

Lemma 6.3 implies that for Proposition 6.3 to hold, one must show

$$\left| \frac{\left\langle Q^{\sharp}\, Q \right\rangle_{(k;n)}}{\left\langle Q_n^{\sharp}[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right| \;\leq\; 2 \;, \tag{8.11.14}$$

for all $0 \leq k \leq n$, and that there exists a constant $B < \infty$ independent of n and ϵ_0, such that

$$\left\| Q^{\sharp}[\mathfrak{h}^{(0)}] \, \left(Q_0^{\sharp}[z]\, H_f\, Q_0[z] \,-\, H_f \right) \, Q[\mathfrak{h}^{(0)}] \right\| \;\leq\; B\, \epsilon_0^2 \tag{8.11.15}$$

and

$$\left\| Q^{\sharp}[\mathfrak{h}^{(k+1)}]\, U_{\rho} \, \left(Q^{\sharp}[\mathfrak{h}^{(n)}]\, H_f\, Q[\mathfrak{h}^{(k)}] \,-\, H_f \right) \, U_{\rho}^{*}\, Q[\mathfrak{h}^{(k+1)}] \right\| \;\leq\; B\, \epsilon_0^2 \tag{8.11.16}$$

for all $0 \leq k \leq n$.

(I.1) In order to prove (8.11.14), let us introduce the notation

$$F_{(k;n)}[H[\mathfrak{h}^{(k)}]] \;:=\; F_{\chi_{\rho^{n-k}}, \tau_n}[H[\mathfrak{h}^{(k)}]] \;, \tag{8.11.17}$$

where

$$H[\mathfrak{h}^{(n)}] \;=\; \frac{\rho^{-(n-k)}}{\langle \partial_{H_f} F_{(k;n)}[H[\mathfrak{h}^{(k)}]] \rangle_{\Omega_f}} Ad_{U_{\rho^{n-k}}} \left[F_{(k;n)}[H[\mathfrak{h}^{(k)}]] \right]$$

Next, we observe that according to (8.11.1),

$$\left\langle \partial_{H_f} F_{(k;n)}[H[\mathfrak{h}]] \right\rangle_{\Omega_f} \;=\; \left\langle Q^{\sharp}(1 + \partial_{H_f}(\chi_1(T_{n-l} + W)[\mathfrak{h}^{(k)}])\chi_1) Q \right\rangle_{(k;n)} \, \kappa[\mathfrak{h}^{(n-1)}] \;,$$

with

$$\begin{aligned} \kappa^{-1}[\mathfrak{h}^{(n-1)}] \;&=\; 1 - \left\langle Q^{\sharp}\, \frac{\chi_{\rho^{n-k}}^2}{\bar{\chi}_{\rho^{n-k}}^2}\, Q \right\rangle_{(k;n)} \\ &\qquad + \left\langle Q^{\sharp}\, \frac{\partial_{H_f} \bar{\chi}_{\rho^{n-k}}}{\bar{\chi}_{\rho^{n-k}}^3} (H_f - \rho z_n)\, Q \right\rangle_{(k;n)} \\ &=\; \eta_n^{-1}[z[z_n]] \;. \end{aligned}$$

Thus,

$$\kappa[\mathfrak{h}^{(n-1)}] \;=\; 1 \,+\, O(\epsilon_{n-1}^2) \;. \tag{8.11.18}$$

It is clear that

$$\begin{aligned}
&\left|\left\langle Q^\sharp(\partial_{H_f}(\chi_1(T_{n-l}+W)\chi_1)[\mathfrak{h}^{(k)}])Q\right\rangle_{(k;n)}\right| \\
&\quad\le \left(\|\partial_{H_f}(\chi_1 T_{n-l}\chi_1)\| + \|\partial_{H_f}(\chi_1 W[\mathfrak{h}^{(k)}]\chi_1)\|\right)\left\langle Q^\sharp Q\right\rangle_{(k;n)} \\
&\quad\le \left(\kappa_{n-l}+\sqrt{\epsilon}_0+O(\epsilon_k\xi)\right)\left\langle Q^\sharp Q\right\rangle_{(k;n)} \\
&\quad\le \frac{1}{3}\left\langle Q^\sharp Q\right\rangle_{(k;n)} \ , \qquad (8.11.19)
\end{aligned}$$

for $|p|\le p_c$ if $p_c\in[\frac{1}{20},1)$ is sufficiently small.

Hence, we find that

$$\begin{aligned}
\left\langle Q^\sharp Q\right\rangle_{(k;n)} &\le \left(\frac{3}{2}+O(\epsilon_{n-1}^2)\right)\left\langle \partial_{H_f}F_{(k;n)}[H[\mathfrak{h}]]\right\rangle_{\Omega_f} \\
&= \left(\frac{3}{2}+O(\epsilon_{n-1}^2)\right)\prod_{i=k}^{n}(1+\Delta_{H_f}^{(i)}[z_i])\ . \qquad (8.11.20)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{\left\langle Q^\sharp Q\right\rangle_{(k;n)}}{\left\langle Q_n^\sharp Q_n\right\rangle_{\Omega_f}} &= \eta_n[z[z_n]]\,\frac{\left\langle Q^\sharp Q\right\rangle_{(k;n)}}{\left\langle \partial_{H_f}F_n[z]\right\rangle_{\Omega_f}} \\
&= \eta_n[z[z_n]]\,\frac{\left\langle Q^\sharp Q\right\rangle_{(k;n)}}{\left\langle \partial_{H_f}F_{(k;n)}[H[\mathfrak{h}]]\right\rangle_{\Omega_f}}\,\frac{\left\langle \partial_{H_f}F_{(k;n)}[H[\mathfrak{h}]]\right\rangle_{\Omega_f}}{\left\langle \partial_{H_f}F_n[z]\right\rangle_{\Omega_f}} \\
&\le \left(\frac{3}{2}+O(\epsilon_{n-1}^2)\right)\prod_{i=0}^{k-1}(1+\Delta_{H_f}^{(i)}[z_i])^{-1} \\
&\le 2\ , \qquad (8.11.21)
\end{aligned}$$

as claimed, since $0<\Delta_{H_f}^{(i)}[z_i]\le C\epsilon_i^2$, and $\epsilon_k\le 2\epsilon_0$ for $0\le k\le n$.

(I.2) Next, let us prove (8.11.15). A straightforward calculation leads to

$$\begin{aligned}
Q_0^\sharp[z]\,H_f\,Q_0[z]\,-\,H_fP_1 \;=\;& \chi_1\omega[H_p(\sigma_0)]\bar\chi_1\bar R H_f\bar R\bar\chi_1\omega[H_p(\sigma_0)]\chi_1 \\
&-(\bar\chi_1+\chi_1\omega[H_p(\sigma_0)]\bar\chi_1\bar R\chi_1)H_f(\bar\chi_1+\chi_1\bar R\bar\chi_1\omega[H_p(\sigma_0)]\chi_1)\ ,
\end{aligned}$$

where we recall that

$$\omega[H_p(\sigma_0)] \;=\; \underbrace{-\Delta E_0[z]\,-\,\Delta_{H_f}[z]H_f\,+\,|p|\,\mathbf{P}_f^{\|}\,+\,\frac{1}{2}|\mathbf{P}_f|^2}_{=T'}+W\ ,$$

with $|\Delta E_0[z]|,|\Delta_{H_f}[z]|\le O(g^2)$. We claim that

$$\left\|H_f^{\frac{1}{2}}\bar R\bar\chi_1\omega[H_p(\sigma_0)]\chi_1\,Q[\mathfrak{h}^{(0)}]\right\| \;\le\; C\,\epsilon_0 \qquad (8.11.22)$$

$$\left\|H_f^{\frac{1}{2}}\bar\chi_1\,Q[\mathfrak{h}^{(0)}]\right\| \;\le\; C\,\epsilon_0 \qquad (8.11.23)$$

which implies (8.11.15). To prove (8.11.22), we note that since

$$Q[\mathfrak{h}^{(0)}] \;=\; \chi_\rho\,-\,\bar\chi_\rho\bar R[\mathfrak{h}^{(0)}]\bar\chi_\rho\omega[H[\mathfrak{h}^{(0)}]]\chi_\rho\ , \qquad (8.11.24)$$

we have

$$\begin{aligned}\left\| H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 \omega[H_p(\sigma_0)] \chi_1 \, Q[\mathfrak{h}^{(0)}] \right\| \;&\leq\; \left\| H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 W \chi_\rho \right\| \\ &+ \left\| H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 \omega[H_p(\sigma_0)] \chi_1 \bar{\chi}_\rho \bar{R}[\mathfrak{h}^{(0)}] \bar{\chi}_\rho \omega[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\| \;,\end{aligned}$$

since $\bar{\chi}_1 T'[H_p(\sigma_0)] \bar{\chi}_\rho = 0$. As a consequence of Lemma 7.3,

$$\begin{aligned}\left\| H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 W \chi_\rho \right\| \;&\leq\; \left\| H_f^{\frac{1}{2}} \bar{R}^{\frac{1}{2}} \right\| \, \left\| \bar{R}^{\frac{1}{2}} \bar{\chi}_1 W \chi_\rho \right\| \\ &\leq\; O(g) \;.\end{aligned}$$

Next, we apply the second resolvent formula to $\bar{R}[\mathfrak{h}^{(0)}]$, which yields

$$\begin{aligned}&\left\| H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 \omega[H_p(\sigma_0)] \chi_1 \, \bar{\chi}_\rho \bar{R}[\mathfrak{h}^{(0)}] \bar{\chi}_\rho \omega[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\| \\ &\quad\leq\; \left\| H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 \omega[H_p(\sigma_0)] \chi_1 \, \bar{\chi}_\rho \bar{R}_0[\mathfrak{h}^{(0)}] \bar{\chi}_\rho \omega[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\| \\ &\qquad+\; \left\| H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 \omega[H_p(\sigma_0)] \chi_1 \, \bar{\chi}_\rho \bar{R}_0[\mathfrak{h}^{(0)}] \bar{\chi} W[\mathfrak{h}^{(0)}] \bar{\chi} \bar{R}[\mathfrak{h}^{(0)}] \bar{\chi}_\rho \omega[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\| \;.\end{aligned}$$

The second term on the right hand side of the inequality sign is bounded by

$$\begin{aligned}&\underbrace{\left\| H_f^{\frac{1}{2}} \bar{R}^{\frac{1}{2}} \right\|}_{\leq O(1)} \underbrace{\left\| \bar{R}^{\frac{1}{2}} \bar{\chi}_1 \omega[H_p(\sigma_0)] \chi_1 \right\|}_{\leq O(1)} \underbrace{\left\| \bar{\chi}_\rho \bar{R}_0^{\frac{1}{2}}[\mathfrak{h}^{(0)}] \right\|}_{\leq O(1)} \underbrace{\left\| \bar{R}_0^{\frac{1}{2}}[\mathfrak{h}^{(0)}] \bar{\chi} W[\mathfrak{h}^{(0)}] \bar{\chi} \bar{R}^{\frac{1}{2}}[\mathfrak{h}^{(0)}] \right\|}_{\leq O(\epsilon_0)} \times \\ &\qquad\qquad \times \underbrace{\left\| \bar{R}^{\frac{1}{2}}[\mathfrak{h}^{(0)}] \bar{\chi}_\rho \omega[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\|}_{\leq O(1)} \;\leq\; O(\epsilon_0) \;,\end{aligned}$$

as a consequence of Lemmata 7.2, 7.3 and 8.3. The first term is bounded by

$$\begin{aligned}&\left\| \underbrace{H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 T'[H_p(\sigma_0)] \chi_1 \, \bar{\chi}_\rho \bar{R}_0[\mathfrak{h}^{(0)}] \bar{\chi}_\rho T'[H[\mathfrak{h}^{(0)}]] \chi_\rho}_{=\,0} \right\| \\ &+ \underbrace{\left\| H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 W \chi_1 \right\|}_{\leq O(g)} \underbrace{\left\| \bar{\chi}_\rho \bar{R}_0[\mathfrak{h}^{(0)}] \bar{\chi}_\rho T'[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\|}_{\leq O(1)} \\ &+ \underbrace{\left\| H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 T'[H_p(\sigma_0)] \chi_1 \right\|}_{\leq O(1)} \underbrace{\left\| \bar{\chi}_\rho \bar{R}_0[\mathfrak{h}^{(0)}] \bar{\chi}_\rho W[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\|}_{\leq O(\epsilon_0)} \\ &+ \underbrace{\left\| H_f^{\frac{1}{2}} \bar{R} \bar{\chi}_1 W[H_p(\sigma_0)] \chi_1 \right\|}_{\leq O(g)} \underbrace{\left\| \bar{\chi}_\rho \bar{R}_0[\mathfrak{h}^{(0)}] \bar{\chi}_\rho W[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\|}_{\leq O(\epsilon_0)} \;\leq\; O(\epsilon_0) \;.\end{aligned}$$

The first term here vanishes because $H_f, T'[H_p(\sigma_0)], \bar{R}_0[\mathfrak{h}^{(0)}]$ and $T'[H[\mathfrak{h}^{(0)}]]$ mutually commute, and $\bar{\chi}_1 \chi_\rho = 0$.

Likewise, (8.11.23) follows from

$$\begin{aligned}\left\| H_f^{\frac{1}{2}} \bar{\chi}_1 \, Q[\mathfrak{h}^{(0)}] \right\| &= \left\| H_f^{\frac{1}{2}} \bar{\chi}_1 \bar{R}[\mathfrak{h}^{(0)}] \bar{\chi}_\rho \omega[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\| \\ &\leq \underbrace{\left\| H_f^{\frac{1}{2}} \bar{\chi}_1 \bar{R}_0^{\frac{1}{2}}[\mathfrak{h}^{(0)}] \right\|}_{\leq O(1)} \underbrace{\left\| \bar{R}_0^{\frac{1}{2}}[\mathfrak{h}^{(0)}] \bar{\chi}_\rho W[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\|}_{\leq O(\epsilon_0)} \\ &\quad + \underbrace{\left\| H_f^{\frac{1}{2}} \bar{\chi}_1 \bar{R}_0^{\frac{1}{2}}[\mathfrak{h}^{(0)}] \right\|}_{\leq O(1)} \underbrace{\left\| \bar{R}_0^{\frac{1}{2}}[\mathfrak{h}^{(0)}] \bar{\chi} W[\mathfrak{h}^{(0)}] \bar{\chi} \bar{R}^{\frac{1}{2}}[\mathfrak{h}^{(0)}] \right\|}_{\leq O(\epsilon_0)} \times \\ &\qquad \times \underbrace{\left\| \bar{R}^{\frac{1}{2}}[\mathfrak{h}^{(0)}] \bar{\chi}_\rho \omega[H[\mathfrak{h}^{(0)}]] \chi_\rho \right\|}_{\leq O(1)} \\ &\leq O(\epsilon_0)\end{aligned}$$

by the second resolvent identity, applied $\bar{R}[\mathfrak{h}^{(0)}]$. This proves (8.11.15).

(I.3) The proof of (8.11.16) is completely analogous to the case (I.2). This is because the operator

$$U_\rho \left(Q^\sharp[\mathfrak{h}^{(n)}] \, H_f \, Q[\mathfrak{h}^{(k)}] \, - \, H_f \right) U_\rho^*$$

has (up to a numerical factor) precisely the same structure as

$$Q_0^\sharp[z] \, H_f \, Q_0[z] \, - \, H_f \ ,$$

which has been studied above. We note that in the present case, $H[\mathfrak{h}^{(k)}]$ is, in contrast to $H_p(\sigma_0)$, a bounded operator, and in comparison to the above case concerning the first decimation map, one must substitute $H_p(\sigma_0) \to H[\mathfrak{h}^{(k)}]$, $H[\mathfrak{h}^{(0)}] \to H[\mathfrak{h}^{(n+1)}]$ in the expressions appearing under (I.2). (8.11.16) follows straightforwardly by applying Lemmata 8.1 and 8.2. Due to the similarity to the case (I.2), this will not be carried out explicitly here.

This proves the claim for the case $\beta = 0$.

(II) The case $\beta = 1$. Here, we need to verify the assumptions of Proposition 6.2 under the condition that both Theorem 8.1 and the main hypothesis are correct for all scales $0 \leq k < n$.

The statement that

$$\epsilon_k \ \leq \ 2 \, \epsilon_0 \tag{8.11.25}$$

and

$$\| \partial_{|p|}^\beta T[\mathfrak{h}^{(k)}] \| \ \leq \ C \ , \ \| \partial_{|p|}^\beta W[\mathfrak{h}^{(k)}] \| \ \leq \ C \, \epsilon_0^\beta \epsilon_k^{1-\beta} \xi \tag{8.11.26}$$

straightforwardly follow from Theorem 8.1 and Lemma 8.1, and the induction hypothesis that

$$\left| \partial_{|p|} a_k[z_k] + 1 \right| \ \leq \ \sqrt{\epsilon}_0 \tag{8.11.27}$$

holds for all $0 \leq k < n$.

For Proposition 6.2 to hold, we must show that

$$\left| \frac{\langle \partial_{|p|} \partial_{H_f} F_n[z[z_n]] \rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right| \leq \epsilon_0 \, n \tag{8.11.28}$$

$$\left| \frac{\partial_{|p|} \left\langle Q_n^\sharp[z[z_n]] \, Q_n[z[z_n]] \right\rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right| \leq \epsilon_0 \, n \tag{8.11.29}$$

$$\left| \frac{\left\langle Q^\sharp \, Q \right\rangle_{(k;n)}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right| \leq 2 \tag{8.11.30}$$

$$\left\| (\partial_{|p|} Q[\mathfrak{h}^{(k)}]) U_\rho^* Q[\mathfrak{h}^{(k+1)}] \right\| \leq C \, \epsilon_0 \tag{8.11.31}$$

$$Q^\sharp[\mathfrak{h}^{(k)}] H[\mathfrak{h}^{(k)}] Q[\mathfrak{h}^{(k)}] \leq (1+\mu) F_{\chi_\rho,\tau}[H[\mathfrak{h}^{(k)}]] \tag{8.11.32}$$

for $\beta = 0, 1$, and all $0 \leq k \leq n$, and some constant $\mu \leq \frac{3}{4}$.

(II.1) To verify (8.11.28), we observe that

$$\begin{aligned} \left| \langle \partial_{|p|} \partial_{H_f} F_n[z[z_n]] \rangle_{\Omega_f} \right| &= \left| \partial_{|p|} \prod_{i=0}^{n} (1 + \Delta_{H_f}^{(i)}[z_i]) \right| \\ &\leq \left(\sum_{i=0}^{n} \left| \partial_{|p|} \Delta_{H_f}^{(i)}[z_i] \right| \right) \left| \prod_{i=0}^{n} (1 + \Delta_{H_f}^{(i)}[z_i]) \right| \\ &= \left(\sum_{i=0}^{n} \left| \partial_{|p|} \Delta_{H_f}^{(i)}[z_i] \right| \right) \left| \langle \partial_{H_f} F_n[z[z_n]] \rangle_{\Omega_f} \right| \ , \end{aligned}$$

since $\Delta_{H_f}^{(i)}[z_i] \geq 0$. Therefore,

$$\begin{aligned} \text{l.h.s. of (8.11.28)} &\leq \left(\sum_{i=0}^{n} \left| \partial_{|p|} \Delta_{H_f}^{(i)}[z_i] \right| \right) |\eta_n[z[z_n]]| \\ &\leq C \, g \, \epsilon_0 \, n \, (1 + O(\epsilon_0^2)) \ , \end{aligned}$$

since by Lemma 8.5,

$$\left| \partial_{|p|} \Delta_{H_f}^{(i)}[z_i] \right| \leq O(g \, \epsilon_i) \ ,$$

which implies the claim.

(II.2) In the case of (8.11.29), we have

$$\begin{aligned} \partial_{|p|} \left\langle Q_n^\sharp[z[z_n]] \, Q_n[z[z_n]] \right\rangle_{\Omega_f} &= \partial_{|p|} \left(\langle \partial_{H_f} F_n[z[z_n]] \rangle_{\Omega_f} \eta_n^{-1}[z[z_n]] \right) \\ &= \eta_n^{-1}[z[z_n]] \, \partial_{|p|} \langle \partial_{H_f} F_n[z[z_n]] \rangle_{\Omega_f} \\ &\qquad - \langle \partial_{H_f} F_n[z[z_n]] \rangle_{\Omega_f} \, \eta_n^{-2}[z[z_n]] \, \partial_{|p|} \eta_n[z_n] \ , \end{aligned}$$

by definition of $\eta_n[z[z_n]]$. Therefore,

$$\begin{aligned} |\text{l.h.s. of (8.11.29)}| &= \left| \eta_n^{-1}[z[z_n]] \, \frac{\langle \partial_{|p|} \partial_{H_f} F_n[z[z_n]] \rangle_{\Omega_f}}{\left\langle Q_n^\sharp[z[z_n]] Q_n[z[z_n]] \right\rangle_{\Omega_f}} \right| \\ &\quad + \left| \eta_n^{-1}[z[z_n]] \, \partial_{|p|} \eta_n[z[z_n]] \right| \\ &\leq (1 + O(\epsilon_0^2)) C \, \epsilon_0^2 \, n \\ &\quad + (1 + O(\epsilon_0^2)) C' \, \epsilon_0^2 \, , \end{aligned}$$

using the above estimate on the l.h.s. of (8.11.28).

(II.3) Inequality (8.11.30) has already been proved in the case $\beta = 0$.

(II.4) The bound (8.11.31) follows from the following argument. Since

$$\omega[\mathfrak{h}] \;=\; a[z] \mathbf{P}_f^{\shortparallel} \;+\; O(\epsilon) \, ,$$

(where $O(\epsilon)$ is with respect to the operator norm, of course) we have, for $0 \leq k < n$,

$$\begin{aligned} \left\| (\partial_{|p|} Q[\mathfrak{h}^{(k)}]) U_\rho^* Q[\mathfrak{h}^{(k+1)}] \right\| \;&\leq\; O(\epsilon_0) \\ &+ |a^{(k)}[z_k]| \, |a^{(k+1)}[z_{k+1}]| \left\| \bar{\chi}_\rho (\partial_{|p|} \bar{R}[\mathfrak{h}^{(k)}]) \bar{\chi}_\rho \mathbf{P}_f^{\shortparallel} \chi_\rho U_\rho^* \bar{\chi}_\rho \bar{R}[\mathfrak{h}^{(k+1)}] \bar{\chi}_\rho \mathbf{P}_f^{\shortparallel} \chi_\rho \right\| \\ &+ |\partial_{|p|} a^{(k)}[z_k]| \, |a^{(k+1)}[z_{k+1}]| \, \left\| \bar{\chi}_\rho \bar{R}[\mathfrak{h}^{(k)}] \bar{\chi}_\rho \mathbf{P}_f^{\shortparallel} \chi_\rho U_\rho^* \bar{\chi}_\rho \bar{R}[\mathfrak{h}^{(k+1)}] \bar{\chi}_\rho \mathbf{P}_f^{\shortparallel} \chi_\rho \right\| \, . \end{aligned}$$

Expanding $\bar{R} = \bar{R}_0 + O(\epsilon)$ according to the second resolvent identity (4.1.10), and using that $\chi_\rho U_\rho \chi_\rho = 0$, we arrive at the claim.

(II.5) To prove (8.11.32), we first of all observe that for $\mathfrak{h} \equiv \mathfrak{h}^{(k)}$ and any $0 \leq k < n$,

$$Q^\sharp[\mathfrak{h}] \, H[\mathfrak{h}] \, Q[\mathfrak{h}] \;=\; \chi_\rho^2 \, F_{\chi_\rho, \tau}[H[\mathfrak{h}]] \, \chi_\rho^2 \;+\; \tau[H[\mathfrak{h}]] \bar{\chi}_\rho^2 \chi_\rho^2 \, ,$$

which is obtained from straightforward calculation. Let us then find a sufficiently small number μ such that an estimate

$$\chi_\rho^2 \, F_{\chi_\rho, \tau}[H[\mathfrak{h}]] \, \chi_\rho^2 \;+\; \tau[H[\mathfrak{h}]] \bar{\chi}_\rho^2 \chi_\rho^2 \;\leq\; (1 + \mu) F_{\chi_\rho, \tau}[H[\mathfrak{h}]] \tag{8.11.33}$$

holds, and show that one can pick $\mu \leq \frac{3}{4}$. For $|p| \leq p_c$, and $p_c \in [\frac{1}{20}, 1)$ sufficiently small, there is a bound

$$F_{\chi_\rho, \tau}[H[\mathfrak{h}]] \;\geq\; C_{|p|} \, \tau[H[\mathfrak{h}]] \;+ O(\sqrt{\epsilon}_0) \, ,$$

for a constant

$$C_{|p|} \;\geq\; \frac{4}{9} \, ,$$

since

$$(1 + \Delta_{H_f}[z]) H_f \;+\; (a[z] + \Delta a[z]) \mathbf{P}_f^{\shortparallel} \;\geq\; (1 - |p|) \tau[H[\mathfrak{h}]] \;+ O(\sqrt{\epsilon}_0) \, ,$$

and

$$\| T_{n-l}[\mathfrak{h}] \, \Pi[\mathfrak{h}] + \Delta T_{n-l} - \lambda_{n-l} |\mathbf{P}_f|^2 \| \;\leq\; \frac{1}{2} \, ,$$

cf. (4.5.7) and Proposition 8.3 below, while

$$\|\Pi[\mathfrak{h}]W[\mathfrak{h}]\Pi[\mathfrak{h}] + \Delta W\| \;\leq\; O(\epsilon)\;.$$

Thus,

$$\tau[H[\mathfrak{h}]]\bar{\chi}_\rho^2\chi_\rho^2 \;\leq\; C_{|p|}^{-1}\bar{\chi}_\rho\chi_\rho F_{\chi_\rho,\tau}[H[\mathfrak{h}]]\bar{\chi}_\rho\chi_\rho \,+\, O(\sqrt{\epsilon}_0)\bar{\chi}_\rho^2\chi_\rho^2\;,$$

and since

$$\bar{\chi}_\rho^2\chi_\rho^2 \;\geq\; \frac{1}{2}\tau[H[\mathfrak{h}]]\bar{\chi}_\rho^2\chi_\rho^2\;,$$

it follows that

$$\tau[H[\mathfrak{h}]]\bar{\chi}_\rho^2\chi_\rho^2 \;\;\leq\;\; (1+O(\sqrt{\epsilon}_0))C_{|p|}^{-1}\bar{\chi}_\rho\chi_\rho F_{\chi_\rho,\tau}[H[\mathfrak{h}]]\bar{\chi}_\rho\chi_\rho\;.$$

Since $\|\bar{\chi}_\rho\chi_\rho\| \leq \frac{1}{2}$, this implies that

$$\tau[H[\mathfrak{h}]]\bar{\chi}_\rho^2\chi_\rho^2 \;\;\leq\;\; \frac{3}{4}F_{\chi_\rho,\tau}[H[\mathfrak{h}]]\;,$$

thus (8.11.33) holds for $\mu \leq \frac{3}{4}$.

(III) By (I) and (II), Proposition 6.2 implies that (8.11.27) holds for $k = n$. This proves the claim of the proposition. ■

It is thus a trivial consequence of the above proposition that

$$\hat{a}[Z[z]] \;=\; -\,|p|(1\,+\,\hat{\delta}_a[Z[z]])\;, \tag{8.11.34}$$

with

$$|\hat{\delta}_a[Z[z]]| \;\leq\; \sqrt{\epsilon}_0\;,$$

uniformly in $|p|$, for $|p| \leq p_c$, with $p_c \in [\frac{1}{20}, 1)$ small enough.

8.12. Renormalization of T_{n-l}

The following proposition controls the behaviour of T_{n-l} under $\mathcal{R}$.

Proposition 8.3. *Let $\rho = \frac{1}{2}$, and $\beta = 0, 1$. The renormalized nonlinear part $\hat{T}_{n-l} \equiv T_{n-l}[\hat{\mathfrak{h}}]$ of $T[\hat{\mathfrak{h}}]$ is $O(H_f^2)$ in the limit $H_f \to 0$, and satisfies*

$$\left|\partial_{|p|}^\beta\partial_{\mathcal{O}}^\alpha\left(\chi_1\left(T_{n-l}[\hat{\mathfrak{h}}] \,-\, \hat{\lambda}_{n-l}\,|\mathbf{P}_f|^2\right)\chi_1\right)\right| \;\leq\; \hat{\epsilon} \,+\, \kappa_{n-l}\,\bar{P}_1\;, \tag{8.12.1}$$

for $0 \leq |\alpha| \leq 2$, $\beta = 0, 1$, where

$$\begin{aligned} 0 \;\leq\; \hat{\lambda}_{n-l} \;&\leq\; \rho\,\lambda_{n-l}\;,\\ |\partial_{|p|}\hat{\lambda}_{n-l}|\,,\,|\partial_z\hat{\lambda}_{n-l}| \;&=\; 0\;. \end{aligned}$$

The constant $\hat{\kappa}_{n-l} < \infty$ is independent of $\hat{\epsilon}$. In particular,

$$\left|\left(T_{n-l}[\hat{\mathfrak{h}}] \,-\, \hat{\lambda}_{n-l}|\mathbf{P}_f|^2\right)\right| \;\leq\; \hat{\epsilon} \,+\, \left(\frac{|p|+\hat{\lambda}_{n-l}}{2} \,+\, O(\sqrt{\epsilon}_0)\right)\bar{P}_1 \tag{8.12.2}$$

and

$$\begin{aligned}\left|\partial_{H_f} T_{n-l}[\hat{\mathfrak{h}}]\right| \;&\leq\; O(\sqrt{\epsilon}_0) \;+\; (|p| \;+\; \hat{\lambda}_{n-l}) \;\times \\ &\times \left(2 \;+\; 3\,|\partial_{H_f}(\chi_1^2 \bar{\chi}_1^2)| \;+\; 2(|p| + \hat{\lambda}_{n-l})\,|\partial_{H_f} \bar{\chi}_1^2|\right) \;. \end{aligned} \tag{8.12.3}$$

The quantity $\hat{\epsilon}$ is defined in Proposition 8.5 below.

Proof.

¿From smooth Feshbach decimation, $\mathcal{D}_\rho : T_{n-l} \to T_{n-l} + \Delta T_{n-l}$, where ΔT_{n-l} is given as follows. By Lemma 4.1,

$$\begin{aligned}\mathcal{D}_1 \;:\; T_{n-l}[\mathfrak{h}] \;&\longrightarrow\; P_\rho T_{n-l}[\mathfrak{h}] P_\rho \;+\; T'[\mathfrak{h}]\,(\Pi[\mathfrak{h}] - \mathbf{1}) \\ &\quad + (\Delta w_{0,0} - \Delta w_{0,0}|_{\mathcal{O}\to 0} - \Delta_{H_f}[z] H_f - \Delta a[z] \mathbf{P}_f^{\|}) \\ &=\; P_\rho T_{n-l}[\mathfrak{h}] P_\rho \;-\; \bar{\chi}_\rho^2 (T')^2 \bar{R}_0 \\ &\quad + (\Delta w_{0,0} - \Delta w_{0,0}|_{\mathcal{O}\to 0} - \Delta_{H_f}[z] H_f - \Delta a[z] \mathbf{P}_f^{\|}) \\ &\equiv\; P_\rho\,(T_{n-l}[\mathfrak{h}] \;+\; \Delta T_{n-l})\,P_\rho \end{aligned}$$

on $P_\rho \mathcal{H}_p$, since

$$\Pi[\mathfrak{h}] \;=\; \mathbf{1} \;-\; \bar{\chi}_\rho T'[\mathfrak{h}] \bar{\chi}_\rho \bar{R}_0[\mathfrak{h}] \;,$$

as we recall. The contribution to ΔT_{n-l} from $\Delta w_{0,0}$ can be estimated by

$$\frac{1}{2!} \sum_{|\alpha|=2} \| \,\partial_{\mathcal{O}_0^r}^\alpha \,\Delta w_{0,0} \| H_f^2 \;\leq\; O(\epsilon^2)\, H_f^2 \;,$$

which is immediate from Lemma 8.2 and Taylor's theorem.

It thus remains to bound the operator

$$T'[\mathfrak{h}](\Pi[\mathfrak{h}] - \mathbf{1}) \;=\; -\,\bar{\chi}_\rho^2 (T'[\mathfrak{h}])^2 \bar{R}_0[\mathfrak{h}] \;, \tag{8.12.4}$$

together with its derivatives with respect to $\mathcal{O}$.

By (8.4.3) of the induction hypothesis, it is clear that

$$\begin{aligned}\left|P_\rho \,\left(T_{n-l}[\mathfrak{h}] \;-\; \lambda_{n-l} |\mathbf{P}_f|^2 \;+\; \Delta T_{n-l}\right)\, P_\rho\right| \\ &\leq\; O(\epsilon) \;+\; \|P_\rho T'[\mathfrak{h}] P_\rho\|\;|P_\rho(\Pi[\mathfrak{h}] - \mathbf{1}) P_\rho| \\ &\leq\; O(\epsilon) \;+\; \rho\,\frac{|p| + \hat{\lambda}_{n-l} + O(\sqrt{\epsilon}_0)}{2} \bar{P}_\rho \;. \end{aligned}$$

This is obtained by using Lemma 8.4, and the fact that

$$\begin{aligned}\|P_\rho T'[\mathfrak{h}] P_\rho\| \;&\leq\; O(\epsilon) \;+\; |a[z]|\,\|P_\rho \mathbf{P}_f^{\|} P_\rho\| \;+\; \lambda_{n-l} \|P_\rho |\mathbf{P}_f|^2 P_\rho\| \\ &\leq\; O(\sqrt{\epsilon}_0) \;+\; |p|\,\rho \;+\; \hat{\lambda}_{n-l}\,\rho \;, \end{aligned}$$

where by definition, $\hat{\lambda}_{n-l} = \rho \lambda_{n-l}$. Hence, rescaling by $\mathcal{S}_\rho = \rho^{-1}(1 + \Delta_{H_f}[z])^{-1} Ad_{U_\rho}$ yields (8.12.2).

Next, let us consider the derivative with respect to H_f. By (8.4.4) of the induction hypothesis,

$$\begin{aligned}
&\left|\partial_{H_f}\left(\chi_\rho\left(T_{n-l}[\mathfrak{h}] \,-\, \lambda_{n-l}|\mathbf{P}_f|^2 \,+\, \Delta T_{n-l}\right)\chi_\rho\right)\right| \\
&\qquad \leq\; O(\sqrt{\epsilon}_0) \,+\, \|P_\rho(\partial_{H_f}T'[\mathfrak{h}])P_\rho\|\,\|\Pi[\mathfrak{h}]\| \\
&\qquad\quad +\|P_\rho T'[\mathfrak{h}]P_\rho\|\,|\partial_{H_f}(\chi_\rho\Pi[\mathfrak{h}]\chi_\rho)| \\
&\qquad \leq\; O(\sqrt{\epsilon}_0) \,+\, (|p|+\hat{\lambda}_{n-l})\,|\partial_{H_f}(\chi_\rho\Pi[\mathfrak{h}]\chi_\rho)|\;,
\end{aligned}$$

since $\|P_\rho(\partial_{H_f}T'[\mathfrak{h}])P_\rho\| \leq O(\epsilon^2)$. Applying Lemma 8.4, and rescaling, one obtains the asserted bound.

For general $|\alpha|+\beta \geq 1$, the part in $\partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}}\Delta T_{n-l}$ stemming $\Delta w_{0,0}$ has an operator norm bounded by $O(\epsilon^2)$, again by Lemma 8.2 and Taylor's theorem. Since

$$\begin{aligned}
\|\partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}}T[\mathfrak{h}]\| &\leq O(1)\;, \\
\|\partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}}W[\mathfrak{h}]\| &\leq O(\epsilon_0^\beta\epsilon^{1-\beta}\xi)\;, \\
\|\partial^\beta_{|p|}\partial^\alpha_{\mathcal{O}}\chi_1^2\| &\leq O(1)\;,
\end{aligned}$$

where the implicit constants only depend on ρ, $|p|$, it follows straightforwardly that there exists a bound of the asserted form (8.12.1). ■

8.13. Renormalization of the marginal kernels of degree 1

The interaction kernels of degree $M+N=1$ are marginal, but their their renormalization group flow is determined by the marginal quantity $a[z]$ that has already been considered before. This is due to the operator-theoretic formulation of the Ward-Takahashi identities derived in Section 6.7.

We use the notation

$$\hat{w}_{M,N} \;=\; w_{M,N}[\hat{\mathfrak{h}}]$$

for brevity, and recall the shorthand

$$\hat{u}_{M,N} \;\equiv\; u_{M,N}[\hat{\mathfrak{h}}] \;=\; \lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} w_{M,N}[\hat{\mathfrak{h}}]\;,$$

from our previous discussion.

Proposition 8.4. *The renormalized expression for $u_{M,N}[\mathfrak{h}]$ is given by*

$$\hat{u}_{M,N}[Z[z];n_k,\lambda] \;=\; -\,g\,\hat{a}[Z[z]]\,\epsilon^{\|}_{M,N}[n_k;\lambda]\;, \tag{8.13.1}$$

where $\hat{a}[Z[z]]$ is the coefficient of $\mathbf{P}^{\|}_f$ in $T_{lin}[\hat{\mathfrak{h}}]$. Furthermore, picking $\rho=\frac{1}{2}$,

$$\begin{aligned}
\|\;\bar{P}_1^\perp[H_f]\left(\partial_{\mathbf{P}^{\|}_f}\hat{w}_{M,N}\right)\bar{P}_1^\perp[H_f+|k|]\;\| &\leq\; 2\,g\,\hat{\lambda}_{n-l} \,+\, \hat{\epsilon}^{\frac{3}{2}}\,\xi\;, \\
\|\;\bar{P}_1^\perp[H_f]\,(\partial_X\hat{w}_{M,N})\,\bar{P}_1^\perp[H_f+|k|]\;\| &\leq\; \hat{\epsilon}^{\frac{3}{2}}\,\xi\;. \qquad (8.13.2)
\end{aligned}$$

for X denoting either the absolute value $|k|$ of a photon momentum, H_f, $\mathbf{P}^\perp_f$ or z. The quantity $\hat{\epsilon}$ is defined in Proposition 8.5 below.

Proof.

(I) By induction hypothesis,

$$u_{M,N}[z;n_k,\lambda] \;=\; g\,a[z]\,\epsilon^{\|}_{M,N}[n_k;\lambda]\;. \tag{8.13.3}$$

We claim that the correction of $u_{M,N}[z;n_k,\lambda]$ under $\mathcal{D}_\rho$ is given by

$$\Delta u_{M,N}[z;n_k,\lambda] \;=\; -\,g\,\Delta a[z]\,\epsilon^{\|}_{M,N}[n_k,\lambda]\;, \tag{8.13.4}$$

where $\Delta a[z]$ denotes its correction under Feshbach decimation $\mathcal{D}_\rho$.

From

$$\begin{aligned} u_{0,1}[z;n_k,\lambda] &= \lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} \bar\kappa_\sigma^{-1}[|k|]\sqrt{|k|}\,[H[\mathfrak{h}],a^*_\lambda(k)] \\ &= -\,g\,\epsilon_{0,1}[n_k;\lambda]\cdot\partial_{\mathbf{P}_f}H[\mathfrak{h}]\;, \end{aligned} \tag{8.13.5}$$

and

$$\begin{aligned} u_{1,0}[z;n_k,\lambda] &= \lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} \bar\kappa_\sigma^{-1}[|k|]\sqrt{|k|}\,[a_\lambda(k),H[\mathfrak{h}]] \\ &= -\,g\,\epsilon_{1,0}[n_k;\lambda]\cdot\partial_{\mathbf{P}_f}H[\mathfrak{h}]\;, \end{aligned} \tag{8.13.6}$$

as required by the induction hypothesis, follows that

$$\begin{aligned} (u_{0,1}+\Delta u_{0,1})[z;n_k,\lambda] &= \lim_{\substack{|k|\to 0\\ n_k \text{ fixed}}} \bar\kappa_\sigma^{-1}[|k|]\sqrt{|k|}\,\left[F_{\chi_\rho,\tau}[H[\mathfrak{h}]],a^*_\lambda(k)\right] \\ &= -\,g\,\epsilon^{\|}_{0,1}[n_k;\lambda]\,\left\langle\partial_{\mathbf{P}^{\|}_f}F_{\chi_\rho,\tau}[H[\mathfrak{h}]]\right\rangle \\ &= -\,g\,(a[z]+\Delta a[z])\,\epsilon^{\|}_{0,1}[n_k;\lambda] \end{aligned} \tag{8.13.7}$$

due to Proposition 6.6, and likewise for $u_{1,0}+\Delta u_{1,0}$. Thus, by definition of $\hat a[Z[z]]$,

$$(u_{0,1}+\Delta u_{0,1})\,[z;n_k,\lambda] \;=\; -\,g\,\hat a[Z[z]]\,(1+\Delta_{H_f}[z])\,\epsilon^{\|}_{0,1}[n_k,\lambda]\;. \tag{8.13.8}$$

Hence, application of the rescaling transformation $\mathcal{S}_\rho$ yields

$$\hat u_{0,1}[Z[z];n_k,\lambda] \;=\; g\,\hat a[Z[z]]\,\epsilon^{\|}_{0,1}[n_k,\lambda] \tag{8.13.9}$$

for the renormalized expression.

(II) Next, for $|\alpha|=1$, and considering that $\bar P^\perp_\rho\Pi=\bar P^\perp_\rho$, we have

$$\begin{aligned} &\|\,\bar P^\perp_1[H_f]\,(\partial_Y\hat w_{M,N})\,\bar P^\perp_1[H_f+|k|]\,\| \\ &\quad\le\; \rho\,(1+\Delta_{H_f}[z])^{-1}\|\,\bar P^\perp_\rho[H_f]\,(\partial_Y\,(\Pi[z;\mathcal{O}]w_{M,N}\Pi[z;S_k\mathcal{O}]))\,\bar P^\perp_\rho[H_f+|k|]\| \\ &\qquad+(1+\Delta_{H_f}[z])^{-1}\|\,\partial_Y\Delta w_{M,N}\,\| \\ &\quad\le\; \rho\,(1+\Delta_{H_f}[z])^{-1}\|\,\bar P^\perp_\rho[H_f]\,(\partial_Y w_{M,N})\,\bar P^\perp_\rho[H_f+|k|]\| \\ &\qquad+(1+\Delta_{H_f}[z])^{-1}\|\,\partial_Y\Delta w_{M,N}\,\| \\ &\quad\le\; \rho\,\|\,\bar P^\perp_\rho[H_f]\,(\partial_Y w_{M,N})\,\bar P^\perp_\rho[H_f+|k|]\|\;+\;O(\epsilon^2\xi)\;, \end{aligned}$$

by unitarity of Ad_{U_ρ}, and for Y denoting $\mathbf{P}^{\|}_f$ or any of the quantities accounted for by X. Inserting (8.5.4) of the induction hypothesis, one immediately arrives at the claim. ■

8.14. Renormalization of kernels of degree 1 and higher

The following proposition controls general properties of the renormalized kernels

$$\hat{w}_{M,N} \;=\; w_{M,N}[\hat{\mathfrak{h}}] \;,$$

for all $M+N \geq 1$.

PROPOSITION 8.5. *Let X stand for $\hat{z} = Z[z]$, or $|k_r|, |\tilde{k}_s|$. Moreover, let $\rho = \frac{1}{2}$. Then, the renormalized integral kernels $\{\hat{w}_{M,N}\}$, for $M+N \geq 1$, satisfy*

$$\begin{aligned} \| \hat{w}_{M,N} \| &\leq \frac{16}{9}\, g\, |p|\, (1+\sqrt{\epsilon}_0) + \frac{32}{9}\, g\, \hat{\lambda}_{n-l} \,+\, O(\hat{\epsilon}^{\frac{3}{2}}\xi) \\ &\leq \hat{\epsilon}\, \xi \;, \end{aligned} \tag{8.14.1}$$

and generally, for $0 \leq |\alpha| \leq 2$, $0 \leq \beta \leq 1$,

$$\begin{aligned} \| \partial_{|p|}\, \partial_{\mathcal{O}}^{\alpha}\, \hat{w}_{M,N} \| &\leq (2\epsilon_0)^{\beta}\, \epsilon^{1-\beta}\, \xi \;, \\ \| \partial_{|p|}\, \partial_{X}\, \hat{w}_{M,N} \| &\leq (2\epsilon_0)^{\beta}\, \epsilon^{1-\beta}\, \xi \;, \quad M+N=1 \;, \end{aligned} \tag{8.14.2}$$

while for $M+N \geq 2$,

$$\| \partial_{|p|}^{\beta}\, \partial_{\mathcal{O}}^{\alpha}\, \hat{w}_{M,N} \| \leq (2\epsilon_0)^{\beta}\, \hat{\epsilon}^{\frac{7}{4}-\frac{|\alpha|}{4}-\beta}\, \xi^{M+N} \;, \tag{8.14.3}$$

$$\| \partial_{|p|}^{\beta}\, \partial_{X}\, \hat{w}_{M,N} \| \leq (2\epsilon_0)^{\beta}\, \hat{\epsilon}^{\frac{3}{2}-\beta}\, \xi^{M+N} \;. \tag{8.14.4}$$

All bounds are uniform in $|p|$, for $|p| \leq p_c$. The renormalized value of ϵ is given by

$$\hat{\epsilon} \;=\; \begin{cases} \max\left\{ \dfrac{17}{18}\,\epsilon \;,\; 2\,(1+\sqrt{\epsilon}_0)\,|p|\,\epsilon_0 \right\} & \text{if } \sigma \leq 1 \\ \dfrac{17}{18}\,\epsilon & \text{if } \sigma > 1 \;, \end{cases} \tag{8.14.5}$$

with $\hat{\epsilon} < 2\epsilon_0$. The running infrared cutoff σ is renormalized via

$$\sigma \;\longrightarrow\; \hat{\sigma} \;=\; \frac{\sigma}{\rho} \;. \tag{8.14.6}$$

In particular, $\hat{w}_{M,N}$ are, for all $M+N \geq 1$, **analytic** *in the components of $\mathbf{P}_f$.*

Proof. We recall, to begin with, that

$$\begin{aligned} \hat{w}_{M,N} \;=\; & \frac{\rho^{M+N-1}}{1+\Delta_{H_f}[z]} Ad_{U_\rho}\Big[\Pi[z; S^{(M)}\mathcal{O}]\, w_{M,N}[\mathfrak{h}]\, \Pi[z; S^{(N)}\mathcal{O}] \\ & + \Delta w_{M,N}[\mathcal{O}; z; K^{(M,N)}] \Big] \;, \end{aligned}$$

where $S^{(M)}$ shifts $\mathcal{O}$ by $(k_1, \ldots, k_M)$, that is,

$$S^{(M)} H_f \;=\; H_f \,+\, \sum_{i=1}^{M} |k_i| \quad , \quad S^{(M)} \mathbf{P}_f \;=\; \mathbf{P}_f \,+\, \sum_{i=1}^{M} k_i \;,$$

and likewise for $S^{(N)}$. Furhermore, we recall the induction hypotheses on $w_{M,N}$ of Section 8.6.

(I) Let us first consider (8.14.2). For the cases $M+N \geq 2$, we have

$$\begin{aligned} \| \hat{w}_{M,N} \| &\leq \rho \| w_{M,N} \| + O(\epsilon^2)\, \xi^{M+N} \\ &\leq \left(\frac{1}{2} + O(\epsilon^{\frac{1}{4}}) \right) \epsilon^{\frac{7}{4}}\, \xi^{M+N} \\ &\leq \hat{\epsilon}^{\frac{7}{4}}\, \xi^{M+N} \ , \end{aligned} \tag{8.14.7}$$

using the induction hypothesis $\| w_{M,N} \| \leq \epsilon^{\frac{7}{4}}\, \xi^{M+N}$, and for $\hat{\epsilon}$ as defined in (8.14.5).

For the case $M+N=1$,

$$\begin{aligned} \| \hat{w}_{M,N} \| &\leq (1+\Delta_{H_f}[z])^{-1} \left(\| Ad_{U_\rho}[\Pi[z;S_k\mathcal{O}] u_{M,N} \Pi[z;\mathcal{O}]] \| \right. \\ &\quad + \| Ad_{U_\rho}[\Pi[z;S_k\mathcal{O}](w_{M,N}-u_{M,N})\Pi[z;\mathcal{O}]] \| \\ &\quad \left. + \| \Delta w_{M,N} \| \right) \\ &\leq \|\Pi\|^2 \| u_{M,N} \| \\ &\quad + \rho\, \|\Pi\|^2 \left(\| P_\rho(\partial_{\mathbf{P}_f^\parallel} w_{M,N}) P_\rho \| + \sum_{Y=|k|,H_f,\mathbf{P}_f^\perp} \| P_\rho(\partial_Y w_{M,N}) P_\rho \| \right) \\ &\quad + O(\epsilon^2 \xi) \\ &\leq \frac{16}{9} (\| u_{M,N} \| + 2\, g\, \hat{\lambda}_{n-l}) + O(\epsilon^{\frac{3}{2}} \xi) \\ &= \frac{16}{9}\, g\, (|a[Z^{-1}[\hat{z}]]| + 2\, \hat{\lambda}_{n-l}) + O(\epsilon^{\frac{3}{2}} \xi) \ , \end{aligned} \tag{8.14.8}$$

where we have used Lemma 8.2 and (8.6.1) of the induction hypothesis, together with the fact that $\|\Pi\| \leq \frac{4}{3}$, by Lemma 8.4. This implies (8.14.1). In particular, since $\hat{\lambda}_{n-l} = \rho \lambda_{n-l} \leq \frac{1}{4}$, this is bounded by

$$\begin{aligned} \| \hat{w}_{M,N} \| &\leq \frac{16}{9}\, g\, |p|\, (1 + O(\sqrt{\epsilon_0})) + \frac{8}{9}\, g + O(\epsilon^{\frac{3}{2}} \xi) \\ &\leq 2\, g \\ &= \epsilon_0\, \xi \ , \end{aligned}$$

for $|p| \leq p_c$ with $p_c \in [\frac{1}{20}, 1)$.

Inequality (8.14.8) implies that if we choose

$$\hat{\epsilon} = \max \left\{ \frac{17}{18}\, \epsilon \, , \, 2(1+\sqrt{\epsilon_0})\, |p|\, g \right\} \ ,$$

it is always true that

$$\hat{\epsilon} \geq 2 \| \hat{u}_{M,N} \| \ .$$

Furthermore, it follows from a straightforward calculation that under the induction assumption

$$\frac{16}{9}\, g\, |p|\, (1+\sqrt{\epsilon_0}) + \frac{32}{9}\, g\, \lambda_{n-l} + O(\epsilon^{\frac{3}{2}} \xi) \leq \epsilon\, \xi \ ,$$

the estimate

$$\frac{16}{9}\, g\, |p|\, (1+\sqrt{\epsilon_0}) + \frac{32}{9}\, g\, \hat{\lambda}_{n-l} + O(\hat{\epsilon}^{\frac{3}{2}} \xi) \leq \hat{\epsilon}\, \xi$$

holds for this definition of $\hat{\epsilon}$. Therefore, we conclude that

$$\| \hat{w}_{M,N} \| \ \leq \ \hat{\epsilon}\, \xi \ ,$$

with $\hat{\epsilon} \leq \epsilon_0$.

(II) For (8.14.3), we first consider the case $M+N=1$, $|\alpha|=0$ and $\beta=1$. We have

$$\begin{aligned}
\| \partial_{|p|}\, \hat{w}_{M,N} \| \ &\leq \ (1+\Delta_{H_f}[z])^{-1} \left(\frac{|\partial_{|p|}\Delta_{H_f}[z]|}{1+\Delta_{H_f}[z]} \| \hat{w}_{M,N} \| \right. \\
&\quad + \underbrace{2\|\partial_{|p|}\Pi\|\,\|\Pi\|}_{\leq \frac{16}{3}\text{ , cf. Lemma 8.4.}} \| u_{M,N} \| \\
&\quad \left. + \|\Pi\|^2 \left(\underbrace{\| \partial_{|p|}\, u_{M,N} \|}_{\leq 1+\sqrt{\epsilon}_0} + \| P_\rho(\partial_{|p|}\, \partial_{|k|} w_{M,N}) P_\rho \| \right) \right. \\
&\quad \left. + \ \|\Pi\|^2 \sum_{|\alpha|=1} \| P_\rho(\partial_{|p|}\, \partial_{\mathcal{O}}^\alpha\, w_{M,N}) P_\rho \| \ + \ \| \partial_{|p|} \Delta w_{M,N} \| \right) \\
&\leq \ \frac{16}{3}\,|p|\, g \ + \ \frac{16}{9}\, g\,(1+O(\sqrt{\epsilon}_0)) \ + \ O(\epsilon_0 \epsilon^{\frac{1}{2}} \xi) \\
&\leq \ 2\,\epsilon_0\,\xi \ , \qquad (8.14.9)
\end{aligned}$$

for $|p| \leq p_c$, with $p_c \in [\frac{1}{20}, 1)$ sufficiently small, where we used $2g = \epsilon_0 \xi$, cf. (8.14.4), and Lemma 8.4.

Next, we consider the cases $M+N \geq 2$ and $0 \leq |\alpha| \leq 2$, $0 \leq \beta \leq 1$.
Here, we have

$$\begin{aligned}
\| \partial_{|p|}^\beta \partial_{\mathcal{O}}^\alpha \hat{w}_{M,N} \| \ &\leq \ \rho^{M+N-1+|\alpha|} \sum_{\alpha'+\alpha''=\alpha} \left(\begin{array}{c} |\alpha| \\ |\alpha'|, |\alpha''| \end{array} \right) \times \\
&\quad \times \left(2^\beta \|\partial_{|p|}^\beta \partial_{\mathcal{O}}^{\alpha'} \Pi\| \, \|\partial_{\mathcal{O}}^{\alpha''} \Pi\| \, \| \partial_{\mathcal{O}}^{\alpha-\alpha'-\alpha''}\, w_{M,N} \| \right. \\
&\quad \left. + \ \|\partial_{\mathcal{O}}^{\alpha'} \Pi\| \, \|\partial_{\mathcal{O}}^{\alpha''} \Pi\| \, \| \partial_{|p|}^\beta \partial_{\mathcal{O}}^{\alpha-\alpha'-\alpha''}\, w_{M,N} \| \right) \\
&\quad + \epsilon_0^\beta\, C_{\alpha,\beta}\, \epsilon^{2-\beta}\, (2\xi)^{M+N} \\
&\leq \ \rho\, \|\Pi\|^2 \, \| \partial_{|p|} \partial_{\mathcal{O}}^\alpha w_{M,N} \| \ + \ g^\beta\, C\, \epsilon^{\frac{7}{4}-\frac{|\alpha|-1}{4}-\beta}\, \xi^{M+N} \\
&\leq \ \epsilon_0^\beta \left(\frac{3}{4} \ + \ O(\epsilon^{\frac{1}{4}}) \right) \epsilon^{\frac{7}{4}-\frac{|\alpha|}{4}-\beta}\, \xi^{M+N} \\
&\leq \ \epsilon_0^\beta \left(\frac{4}{5} \right)^{\frac{7}{4}-\frac{|\alpha|}{4}-\beta} \epsilon^{\frac{7}{4}-\frac{|\alpha|}{4}-\beta}\, \xi^{M+N} \\
&\leq \ (2\epsilon_0)^\beta\, \hat{\epsilon}^{\frac{7}{4}-\frac{|\alpha|}{4}-\beta}\, \xi^{M+N} \ , \qquad (8.14.10)
\end{aligned}$$

This follows from the fact that for $|p| \leq p_c$,

$$\|\Pi\| \ \leq \ \frac{4}{3} \ ,$$

the induction hypotheses on $\partial_{|p|}\partial^{\alpha}_{\mathcal{O}} w_{M,N}$, and $|\partial_{|p|}\Delta_{H_f}[z]| \leq O(g\,\epsilon)$.

Likewise, for (8.14.4) and the same cases (i), (ii) as above,

$$\begin{aligned} \| \partial^{\beta}_{|p|}\partial_X \hat{w}_{M,N} \| &\leq \rho\,\|\Pi\|^2\,\| \partial^{\beta}_{|p|}\partial_X w_{M,N}\| + \epsilon_0^{\beta}\, C\,\epsilon^{\frac{3}{2}}\,\xi^{M+N} \\ &\leq (2\epsilon_0)^{\beta}\,\left(\frac{8}{9} + O(\epsilon^{\frac{1}{4}})\right)\,\epsilon^{\frac{5}{4}}\,\xi^{M+N} \\ &\leq (2\epsilon_0)^{\beta}\,\left(\frac{8}{9} + O(\epsilon^{\frac{1}{4}})\right)\,\epsilon^{\frac{5}{4}}\,\xi^{M+N} \\ &\leq (2\epsilon_0)^{\beta}\,\hat{\epsilon}^{\frac{5}{4}}\,\xi^{M+N} \qquad (8.14.11) \end{aligned}$$

is obtained in the same manner.

(VI) Finally, analyticity of $\hat{w}_{M,N}$ in $\mathbf{P}_f$ follows from the analyticity in $\mathbf{P}_f$ of $w_{M,N}$, cf. the induction hypothesis, and of $\Delta w_{M,N}$, cf. Lemma 8.2. ■

Proof of Theorem 8.1. Theorem 8.1 is a summary of the statements of the propositions of this chapters. ■

CHAPTER 9

THE RENORMALIZATION GROUP FLOW AND PROOF OF THE MAIN THEOREMS

The main insights gained in the two previous chapters can be summarized as follows. There is a point

$$\mathfrak{h}^{(0)} \in \mathfrak{V}_{2\epsilon_0,\xi} , \tag{9.0.12}$$

whose associated effective hamiltonian $H[\mathfrak{h}^{(0)}]$ in $\mathcal{B}(P_1\mathcal{H}_p)$ is, in the Feshbach sense, isospectral to the model hamiltonian $H_p(\sigma_0)$.

Iteration of $\mathcal{R}$ produces an orbit $\{\mathfrak{h}^{(n)}\}_{n\in\mathbf{N}_0}$ in $\mathfrak{V}_{2\epsilon_0,\xi}$ which emanates from $\mathfrak{h}^{(0)}$. Moreover, it establishes a map

$$\mathcal{R} \, : \, \mathfrak{V}_{\epsilon,\xi} \longrightarrow \mathfrak{V}_{\hat{\epsilon},\xi} ,$$

where

$$\epsilon \, , \, \hat{\epsilon} \leq 2\epsilon_0 ,$$

and where $\hat{\epsilon} \leq \epsilon$ is the renormalized value for ϵ. ξ is a fixed constant, while ϵ possesses the rôle of the running coupling constant.

In the systems considered in [**2, 3**], the interactions are irrelevant, such that the sequence $\{\epsilon_n\}_{n\in\mathbf{N}_0}$ (with $\hat{\epsilon}_n = \epsilon_{n+1}$) tends to zero. Hence, there is a sequence of nested polydiscs

$$\cdots \supseteq \mathfrak{V}_{\epsilon_n,\xi} \supseteq \mathfrak{V}_{\epsilon_{n+1},\xi} \supseteq \cdots$$

which converges, in the topology induced by the norm on $\mathfrak{V}_{2\epsilon_0,\xi}$ defined in Chapter 5, to a 1-dimensional polydisc that parametrizes non-interacting effective hamiltonians. In particular, this shows that $\mathcal{R}$ is a contraction on $\mathfrak{V}_{2\epsilon_0,\xi}$, and it is proved in [**2, 3**] by the Banach fixed point theorem that the 1-dimensional limiting element $\mathfrak{V}_{0,\xi}$ is the unique fixed point set.

The interactions within the model hamiltonian $H_p(\sigma_0)$ are also irrelevant, but only so due to the introduction of the artificial infrared regularization at $\sigma_0 > 0$. The limiting element $\mathfrak{V}_{0,\xi}$ in this case is 2-dimensional, and parametrizes operators of the form $\zeta_1 H_f + \zeta_2 \mathbf{P}_f^{||}$, with $\zeta_i \in \mathbb{R}$. In the limit $\sigma_0 \to 0$, where no regularization is included, the corresponding sequence $\{\epsilon_n\}_{n\in\mathbf{N}_0}$ does not tend to zero (unless $|p| = 0$, as we will see), since the interaction is strongly marginal. There are very strong reasons to expect that $\mathfrak{h}^{(\infty)} := \lim_{n\to\infty} \mathfrak{h}^{(n)}$ exists, and that the set of all such limiting points cover a 3-dimensional center manifold in the polydisc $\mathfrak{V}_{2\epsilon_0,\xi}$. However, the proof of these statements requires dynamical systems theory on Banach spaces, and can presumably not be established by a straightforward application of the Banach fixed point theorem. Clarification of these matters is beyond the scope of the present work.

9.1. Solution of the flow equations

The cumulative result of the previous two chapters is the system of flow equations generated by $\mathcal{R}$ on the polydisc $\mathfrak{V}_{\epsilon_0,\xi} \subset \mathfrak{H}$, given by Theorem 8.1, together with the initial condition $\mathfrak{h}^{(0)} \in \mathfrak{V}_{\epsilon_0,\xi}$, provided by Theorem 7.1. The orbit under $\mathcal{R}$ that emanates from $\mathfrak{h}^{(0)}$ is analytically controlled by the flow of the parameters ϵ and $a[z]$.

THEOREM 9.1. *Let $\rho = \frac{1}{2}$, and*

$$N_{\sigma_0} \;=\; \left\lceil \frac{\log \sigma_0}{\log \rho} \right\rceil \;.$$

Let $\{\mathfrak{h}^{(n)}\} \in \mathfrak{V}_{\epsilon_0,\xi}$ denote the orbit of $\mathfrak{h}^{(0)}$ obtained from iterating the renormalization map, that is,

$$\mathfrak{h}^{(n+1)} \;=\; \mathcal{R}[\mathfrak{h}^{(n)}] \;\;,\;\; \text{for } n \;\geq\; 0 \;,$$

and let the initial spectral parameter in $\mathfrak{h}^{(0)}$ satisfy

$$z_0 \;\in\; [E_{bare}[|p|] - g, E_{bare}[|p|]] \;.$$

Then, there is a constant $p_c \in [\frac{1}{20}, 1)$, such that for all p with $|p| \leq p_c$, the running coupling constant ϵ_n is given by

$$\epsilon_n \;=\; \begin{cases} \max\left\{ \left(\frac{17}{18}\right)^n \epsilon_0 \,,\; 2\,(1+\sqrt{\epsilon_0})\,|p|\,\epsilon_0 \right\} & n \leq N_{\sigma_0} \\ \left(\frac{17}{18}\right)^{n-N_{\sigma_0}} \epsilon_{N_{\sigma_0}} & n > N_{\sigma_0} \;, \end{cases} \tag{9.1.1}$$

such that $\epsilon_n \leq 2\epsilon_0$ for all n. The coefficient $a_n[z_n]$ of the operator $\mathbf{P}_f^{\parallel}$ in $T_{lin}[\mathfrak{h}^{(n)}]$ satisfies

$$\left| \partial_{|p|}^{\beta} \left(a_n[z_n] \,+\, |p| \right) \right| \;\leq\; \sqrt{\epsilon_0} \tag{9.1.2}$$

for $\beta = 0, 1$, hence

$$a_n[z_n] \;=\; -\,|p|(1 \,+\, O\left(\sqrt{\epsilon_0}\right)) \tag{9.1.3}$$

for all $n \in \mathbf{N}_0$.

Proof. We have

$$\epsilon_n \;\leq\; \max\left\{ \left(\frac{17}{18}\right)^n \epsilon_0 \,,\; 2\,(1+\sqrt{\epsilon_0})\,|p|\,\epsilon_0 \right\} \;, \tag{9.1.4}$$

which follows straightforwardly from

$$\hat{\epsilon} \;=\; \begin{cases} \max\left\{ \frac{17}{18}\,\epsilon \,,\; 2\,(1+\sqrt{\epsilon_0})\,|p|\,\epsilon_0 \right\} & \text{if } \sigma \leq 1 \\ \frac{17}{18}\,\epsilon & \text{if } \sigma > 1 \;, \end{cases}$$

as stated in Theorem 8.1.

The statements about $a_n[z_n]$ have also been proved there. ■

9.2. Proof of the main theorems

Using the results of the operator-theoretic renormalization group analysis for the orbit $\{\mathfrak{h}^{(n)}\}$, Theorems 3.1 and 3.2 from Chapter 3 can now be proved. Let us briefly recall their statements.

9.2.1. Theorem 3.1. It asserts that for every arbitrarily small, but fixed value of the infrared cutoff $\sigma_0 > 0$, there exists a constant $p_c \in [\frac{1}{20}, 1)$, such that for all p with $|p| \leq p_c$, the physical hamiltonian $H_p(\sigma_0)$ possesses an eigenvalue at the bottom of its spectrum, bordering to absolutely continuous spectrum, which corresponds to a unique eigenvector $\Omega[p;\sigma_0] \in \mathcal{H}_p$. Under the normalization condition

$$\langle\, \Omega_f \,,\, \Omega[p;\sigma_0] \,\rangle \;=\; 1 \;,$$

there are constants $c, C > 0$ and $c', C' > 0$, such that

$$c\, \exp\left[c'\, g^2\, |p|^2\, |\log \sigma_0|\right] \;\leq\; \|\, \Omega[p;\sigma_0]\, \|^2 \;\leq\; C\, \exp\left[C'\, g^2\, |p|^2\, |\log \sigma_0|\right] \;.$$

Thus, $\Omega[p;\sigma_0] \in \mathcal{H}_p$ for all $\sigma_0 \geq 0$ if and only if $|p| = 0$.

9.2.2. Theorem 3.2. It asserts that the model hamiltonian $H_p(\sigma_0)$ possesses a ground state eigenvalue $E_0[|p|,\sigma_0]$ at the bottom of its spectrum, bordering to absolutely continuous spectrum. It is at least of class C^2 with respect to $|p|$, and

$$\partial^{\beta}_{|p|} \left(E_0[|p|,\sigma_0] \;-\; \frac{|p|^2}{2} \right) \;\leq\; O(\epsilon_0)$$

holds for $\beta = 0, 1$, **uniformly** in $\sigma_0 \geq 0$. Furthermore,

$$\partial^{2}_{|p|} \left(E_0[|p|,\sigma_0] \;-\; \frac{|p|^2}{2} \right) \;\leq\; 0 \;,$$

that is, the renormalized mass of the infraparticle is bounded from below by the bare mass of the charged particle, for all $\sigma_0 \geq 0$. This theorem thus controls smoothness properties of the ground state energy under removal of the infrared regularization.

9.3. Proof of Theorem 3.1

We have seen in the previous chapters that the ratio

$$\eta_n[z_n] \;=\; \frac{\left\langle\, \partial_{H_f} F_n[z] \right\rangle_{\Omega_f}}{\left\langle\, Q^{\sharp}_n[z]\, Q_n[z] \right\rangle_{\Omega_f}}$$

is given by

$$\eta_n[z_n] \;=\; 1 \,+\, O(\epsilon_n^2) \;,$$

where $\lim_{n\to\infty} \epsilon_n = 0$. Since

$$\Omega[p;\sigma_0] \;=\; s - \lim_{n\to\infty} Q_n[z]\Omega_f \;,$$

it thus follows that

$$\begin{aligned} \|\Omega[p;\sigma_0]\|^2 &= \lim_{n\to\infty} \left\langle \partial_{H_f} F_n[z] \right\rangle_{\Omega_f} \\ &= \left(\prod_{i=1}^{N_{\sigma_0}} (1 + \Delta_{H_f}^{(i)}[E_{0,i}]) \right) \left(\prod_{i=N_{\sigma_0}+1}^{\infty} (1 + \Delta_{H_f}^{(i)}[E_{0,i}]) \right) \\ &\leq \exp\left[\sum_{i=1}^{N_{\sigma_0}} \Delta_{H_f}^{(i)}[E_{0,i}] \right] \exp\left[\sum_{i=N_{\sigma_0}+1}^{\infty} \Delta_{H_f}^{(i)}[E_{0,i}] \right] , \end{aligned} \tag{9.3.1}$$

while

$$\langle \Omega_f \, , \, \Omega[p;\sigma_0] \rangle_{\Omega_f} \;=\; 1 \; ,$$

because

$$\begin{aligned} \langle \Omega_f \, , \, Q_n[z]\Omega_f \rangle_{\Omega_f} &= \langle \Omega_f \, , \, \Omega_f \rangle_{\Omega_f} \\ &= 1 \end{aligned}$$

for all n.

The upper bound in the assertion is easy to obtain. It follows from

$$\| \, \Omega_0 \, \| \;\leq\; \exp\left(O\left(\epsilon_n^2 \, \xi^2\right) \sum_{j=0}^{N_{\sigma_0}} 1 \right) \exp\left[\sum_{j=N_{\sigma_0}+1}^{\infty} \left(\frac{17}{18}\right)^j \, \epsilon_0 \right] \; , \tag{9.3.2}$$

since

$$\epsilon_n \;\leq\; \max\left\{ \left(\frac{17}{18}\right)^n \, \epsilon_0 \, , \, 2\,(1+\sqrt{\epsilon_0})\,|p|\,\epsilon_0 \right\} \; , \tag{9.3.3}$$

as has been proved in the renormalization group analysis.

To prove the lower bound, we observe that

$$\begin{aligned} \prod_{i=1}^{N_{\sigma_0}} (1 + \Delta_{H_f}^{(i)}[E_{0,i}]) \;\geq\; & \exp\left[\frac{1}{20} \, g^2 \, |p|^2 \, N_{\sigma_0} \right] \times \\ & \times \exp\left[\sum_{j=0}^{N_{\sigma_0}} \left(\lambda_{n-l}^{(j)} \, O(\epsilon_j^2 \xi^2) \, + \, O\left(\epsilon_j^{\frac{5}{2}} \, \xi^2 \right) \right) \right] \; . \end{aligned}$$

Since

$$\epsilon_n \;\leq\; \max\left\{ \left(\frac{17}{18}\right)^n \, \epsilon_0 \, , \, 2\,(1+\sqrt{\epsilon_0})\,g\,|p| \right\} \; ,$$

for $n \leq N_{\sigma_0}$, we have

$$\epsilon_n^\gamma \;\leq\; \left(\frac{17}{18}\right)^{n\gamma} \epsilon_0^\gamma \, + \, \left(2\,(1+\sqrt{\epsilon_0})\,g\,|p|\right)^\gamma$$

for all $\gamma > 0$, and

$$\frac{1}{20} \, |p|^2 \, g^2 \, - \, O\left((g\,|p|)^{\frac{5}{2}} \right) \;\geq\; \frac{|p|^2 \, g^2}{22} \tag{9.3.4}$$

for g sufficiently small. Furthermore,

$$\begin{aligned}\prod_{i=N_{\sigma_0}+1}^{\infty}\left(1+\Delta_{H_f}^{(i)}[E_{0,i}]\right) &\geq \exp\left[-\sum_{j=N_{\sigma_0}+1}^{\infty}\left(\frac{17}{18}\right)^j\epsilon_0\right]\\ &= 1+O(\epsilon_0)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\,\Omega_0\,\| &\geq (1+O(\epsilon_0))\,\exp\left[\frac{1}{22}\,g^2\,|p|^2\,N_{\sigma_0}\right]\,\times\\ &\quad\times\exp\left[O\left(\epsilon_0^{\frac{5}{2}}\,\xi^2\right)\sum_{j=0}^{N_{\sigma_0}}\left(\frac{17}{18}\right)^j\right]\,\times\\ &\quad\times\exp\left[O\left(\epsilon_0^2\,\xi^2\right)\sum_{j=0}^{N_{\sigma_0}}\left(\frac{1}{2}\right)^j\lambda_{n-l}^{(0)}\right]\\ &\geq \frac{1}{2}\,\exp\left[\frac{1}{22}\,g^2\,|p|^2\,N_{\sigma_0}\right]\,, \qquad (9.3.5)\end{aligned}$$

since $\lambda_{n-l}^{(j)}=\lambda_{n-l}^{(0)}\rho^j$, where $\rho=\frac{1}{2}$. With

$$\begin{aligned}N_{\sigma_0} &= \left\lceil\frac{\log\sigma_0}{\log\rho}\right\rceil\\ &\geq \frac{|\log\sigma_0|}{2}\,,\end{aligned}$$

the assertion of Theorem 3.1 follows. ■

9.4. Proof of Theorem 3.2

Let us consider once more the eigenvalue equation for the ground state of $H_p(\sigma_0)$,

$$(H_p(\sigma_0)-E_0[|p|,\sigma_0])\,\Omega[p;\sigma_0] \;=\; 0\,, \qquad (9.4.1)$$

where

$$\Omega[p;\sigma_0] \;=\; s-\lim_{n\to\infty}Q_{\chi_n,\tau_n}[H_p(\sigma_0)-E_0]\,\Omega_f\,. \qquad (9.4.2)$$

Thus, taking two derivatives with respect to $|p|$,

$$\begin{aligned}
\partial^2_{|p|} E_0[|p|,\sigma_0] &= \lim_{n\to\infty} \frac{\left\langle Q^\sharp_n[E_0]\,,\,\left(\partial^2_{|p|} H_p(\sigma_0)\right)\,Q_n[E_0]\,\right\rangle_{\Omega_f}}{\left\langle Q^\sharp_n[E_0]\,Q_n[E_0]\right\rangle_{\Omega_f}} \\
&\quad - 2\,\lim_{n\to\infty} \frac{\left\langle\,(\partial_{|p|}Q^\sharp_n[E_0]\;(H_p(\sigma_0)-E_0)\;\partial_{|p|}Q_n[E_0]\right\rangle_{\Omega_f}}{\left\langle Q^\sharp_n[E_0]\,Q_n[E_0]\right\rangle_{\Omega_f}} \\
&= 1 \;-\; 2\,\lim_{n\to\infty} \frac{\left\langle\,(\partial_{|p|}Q^\sharp_n[E_0]\;(H_p(\sigma_0)-E_0)\;\partial_{|p|}Q_n[E_0]\right\rangle_{\Omega_f}}{\left\langle Q^\sharp_n[E_0]\,Q_n[E_0]\right\rangle_{\Omega_f}}\,, \qquad (9.4.3)
\end{aligned}$$

where

$$Q^{(\sharp)}_n[E_0] \;\equiv\; Q^{(\sharp)}_{\chi_{\rho^n},\tau_n}[H_p(\sigma_0)-E_0]$$

are maps between $P_{\rho^n}\mathcal{H}_p$ and $\mathcal{H}_p$. We have here used that $\partial^2_{|p|}H_p(\sigma_0)=1$.

¿From Propositions 6.4 and 8.2, we know that

$$\begin{aligned}
\partial_{|p|}a_n[z_n] &= -\frac{\partial_{|p|}\eta_n[z[z_n]]}{\eta_n[z[z_n]]}\,a_n[z_n] \;+\; \eta_n[z[z_n]]^{-1} \\
&\quad - 2\,\eta_n[z[z_n]]^{-1}\,\frac{\left\langle(\partial_{|p|}Q^\sharp_n[z[z_n]])\left(\partial_{\mathbf{P}^\parallel_f}H_p(\sigma_0)\right)\partial_{|p|}Q_n[z[z_n]]\right\rangle_{\Omega_f}}{\left\langle Q^\sharp_n[z[z_n]]Q_n[z[z_n]]\right\rangle_{\Omega_f}} \\
&\quad + O\left(\epsilon_0^2\,\rho^n\,n^2\right)\,. \qquad (9.4.4)
\end{aligned}$$

Thus,

$$\partial^2_{|p|}E_0 \;=\; \lim_{n\to\infty}\left(\eta_n[z[z_n]]\partial_{|p|}a_n[z_n] \;+\; (\partial_{|p|}\eta_n[z[z_n]])a_n[z_n]\right) \qquad (9.4.5)$$

(since $O\left(\epsilon_0^2\,\rho^n\,n^2\right)=o(n)$, with $\rho=\frac{1}{2}$).

Theorem 8.1 implies that

$$\left|\partial^\beta_{|p|}(a_n[z_n]+|p|)\right|\,,\,\left|\partial_{|p|}\eta_n[z[z_n]]\right| \;\leq\; \sqrt{\epsilon_0} \qquad (9.4.6)$$

holds for all n. Thus,

$$\begin{aligned}
\partial^2_{|p|}E_0 &= 1 \;+\; O(\sqrt{\epsilon}_0) \\
&= 1 \;+\; O(g^{\frac{1}{6}})\,,
\end{aligned}$$

as claimed. ■

CHAPTER 10

APPENDIX: PROOFS RELATED TO THE FIRST DECIMATION STEP

In this appendix, we summarize proofs of lemmata used for the analytical control of the first Feshbach decimation step in Chapter 7.

10.1. Proof of Lemma 7.2

Clearly, H_f is invertible on $\bar{P}_1\ \mathcal{H}_p$. Let us first consider the case of $W_1[\mathcal{O}]$. We recall that

$$| \ w_{M,N}[\mathcal{O}; k, \lambda] \ | \ \leq \ c\, g\, (|p| + |\mathbf{P}_f|)\ \chi_\Lambda[|k|] \tag{10.1.1}$$

if $M + N = 1$. Thus, picking any $\phi \in \mathcal{H}_p$, Cauchy-Schwarz gives

$$\begin{aligned}
\left\| \bar{R}_0^{\frac{1}{2}} \bar{\chi}_1 W_{0,1} \bar{\chi}_1 \bar{R}_0^{\frac{1}{2}}\ \phi \right\| \ &\leq \ c\, g \left\| \bar{\chi}_1 \frac{|p| + |\mathbf{P}_f|}{\sqrt{(1-|p|)H_f + \frac{1}{2}|\mathbf{P}_f|^2}} \bar{\chi}_1 \right\| \times \\
&\qquad \times \sum_\lambda \int d^3k\, \frac{\chi_\Lambda[|k|]}{\sqrt{|k|}}\ \| \, a_\lambda(k)\, \phi' \, \| \\
&\leq \ c\, g \left(\frac{|p|}{1-|p|} + 2 \right) \sum_\lambda \int d^3k\, \frac{\chi_\Lambda[|k|]}{\sqrt{|k|}}\ \| \, a_\lambda(k)\, \phi' \, \| \\
&\leq \ c\, g \left(\frac{|p|}{1-|p|} + 2 \right) \left(\sup_\lambda \int d^3k\ |k|^{-2}\ \chi_\Lambda[|k|] \right)^{\frac{1}{2}} \times \\
&\qquad \times \left(\sum_\lambda \int d^3k\ \langle\, \phi' \,,\, a_\lambda^*(k)\, |k|\, a_\lambda(k)\, \phi' \,\rangle \right)^{\frac{1}{2}} ,
\end{aligned} \tag{10.1.2}$$

with $\phi' \equiv \bar{R}_0^{\frac{1}{2}}\ \phi$, and where we have used

$$|\bar{R}_0| \ \leq \ \left((1-|p|)H_f + \frac{1}{2}|\mathbf{P}_f|^2 \right)^{-1}$$

on $\bar{P}_1 \mathcal{H}_p$. Thus, we obtain the bound

$$\begin{aligned}
\left\| \bar{R}_0^{\frac{1}{2}} \bar{\chi}_1 W_{0,1} \bar{\chi}_1 \bar{R}_0^{\frac{1}{2}}\ \phi \right\| \ &\leq \ c'\, g \left(\frac{|p|}{1-|p|} + 2 \right) \left\| \ H_f^{\frac{1}{2}}\ ((1-|p|)H_f)^{-\frac{1}{2}}\ \phi \right\| \\
&\leq \ c'\, g \left(\frac{|p|}{1-|p|} + 2 \right) \ \frac{1}{\sqrt{1-|p|}}\ \| \, \phi \, \| \\
&\leq \ c'\, g \, \|\phi\| \, ,
\end{aligned} \tag{10.1.3}$$

since $|p| \leq p_c$. The case for $W_{1,0}$ is identical.

Next, let us consider $W_2[\mathcal{O}]$. We recall that if $M+N=2$,

$$\| \; w_{M,N}[\mathcal{O};k,\lambda,\tilde{k},\tilde{\lambda}] \; \| \;\leq\; c\,g^2\,\chi_\Lambda[|k|]\;\chi_\Lambda[|\tilde{k}|]\;. \tag{10.1.4}$$

Picking any $\phi,\psi\in\mathcal{H}_p$, it is clear that for $M=N=1$,

$$\begin{aligned} |\,\langle\;\psi\;,\;W_{1,1}[\mathcal{O}]\;\phi\;\rangle\;| \;&\leq\; c\,g^2\sum_{\lambda,\tilde{\lambda}}\int\frac{d^3k}{\sqrt{|k|}}\,\frac{d^3\tilde{k}}{\sqrt{|\tilde{k}|}}\,\chi_\Lambda[|k|]\;\chi_\Lambda[|\tilde{k}|] \\ &\qquad\times\;\Big\|\,a_\lambda(k)\,\psi\,\Big\|\,\Big\|\,a_{\tilde{\lambda}}(\tilde{k})\,\phi\,\Big\| \\ &\leq\; c'\,g^2\,\Big\|\;H_f^{\frac{1}{2}}\;\psi\;\Big\|\;\Big\|\;H_f^{\frac{1}{2}}\;\phi\;\Big\| \end{aligned} \tag{10.1.5}$$

by the Cauchy-Schwarz inequality. Using

$$\|\bar{R}_0\,H_f\|\;\leq\;C\;, \tag{10.1.6}$$

this implies that

$$\Big\|\;\bar{R}_0^{\frac{1}{2}}\;\bar{\chi}_1 W_{1,1}[\mathcal{O}]\bar{\chi}_1\,\bar{R}_0^{\frac{1}{2}}\;\Big\| \;\leq\; c'\,C\,g^2\;. \tag{10.1.7}$$

For $M=0$ and $N=2$, we find

$$\begin{aligned} &\Big\|\;(H_f+\Lambda+1)^{-\frac{1}{2}}\;W_{0,2}[\mathcal{O}]\;\phi\;\Big\|^2 \\ \leq\;&\left(\sum_{\lambda,\tilde{\lambda}}\int d^3k\;d^3\tilde{k}\;|k|^{-\frac{1}{2}}\;|\tilde{k}|^{-\frac{1}{2}}\;\Big\|\;w_{0,2}[\mathcal{O};k,\lambda,\tilde{k},\tilde{\lambda}]\;\Big\|\;\times\right. \\ &\qquad\left.\times\;\Big\|\;(H_f+\Lambda+1)^{-\frac{1}{2}}\;a_\lambda(k)\;\;a_{\tilde{\lambda}}(\tilde{k})\;\phi\;\Big\|\right)^2 \\ \leq\;& c\,g^2\left(\int d^3k\;d^3\tilde{k}\;|k|^{-\frac{1}{2}}\;|\tilde{k}|^{-\frac{1}{2}}\;\chi_\Lambda[|k|]\;\chi_\Lambda[|\tilde{k}|]\right)\;\times \\ &\qquad\times\sum_{\lambda,\tilde{\lambda}}\int d^3k\;d^3\tilde{k}\;|k|\;|\tilde{k}|\;\chi_\Lambda[|k|]\;\times \\ &\qquad\qquad\times\;\Big\langle\;\phi\;,\;a_{\tilde{\lambda}}^*(\tilde{k})\;a_\lambda^*(k)\;(H_f+\Lambda+1)^{-1}\;a_\lambda(k)\;\;a_{\tilde{\lambda}}(\tilde{k})\;\phi\;\Big\rangle \\ \leq\;& c'\,g^2\;\Big\|\;H_f^{\frac{1}{2}}\;\phi\;\Big\|^2\;, \end{aligned} \tag{10.1.8}$$

again by using the Cauchy-Schwarz inequality, and

$$\begin{aligned} &\sum_\lambda\int d^3k\;|k|\;\chi_\Lambda[|k|]\;a_\lambda^*(k)\;(H_f+\Lambda+1)^{-1}\;a_\lambda(k) \\ =\;&\int d^3k\;|k|\;\chi_\Lambda[|k|]\;a_\lambda^*(k)\;a_\lambda(k)\;(H_f+\Lambda-|k|+1)^{-1} \\ \leq\;& H_f\;(H_f+1)^{-1} \\ \leq\;& 1\;. \end{aligned} \tag{10.1.9}$$

¿From

$$\Big\|\;H_f^{-\frac{1}{2}}\;\bar{\chi}_1\;(H_f+\Lambda+1)^{\frac{1}{2}}\;\Big\| \;\leq\; \sqrt{\Lambda+2} \tag{10.1.10}$$

and (10.1.6), we arrive at the desired bound

$$\left\| \bar{R}_0^{\frac{1}{2}}\, \bar{\chi}_1 W_{0,2} \bar{\chi}_1\, \bar{R}_0^{\frac{1}{2}} \right\| \;\leq\; c'\, g^2 \;. \tag{10.1.11}$$

The case of $W_{2,0}[\mathcal{O}]$ is treated in the same fashion.

Collecting the above bounds, this proves Lemma 7.2. ■

10.2. Proof of Lemma 7.3

Again using the bound

$$\| \! | \, w_{M,N} \, | \! \| \;\leq\; c\, g\, \chi_\Lambda[|k|] \tag{10.2.1}$$

for $M+N=1$, and picking any $\phi \in \mathcal{H}_p$, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\left\| \bar{R}_0^{\frac{1}{2}} \bar{\chi}_1 W_{0,1}[\mathcal{O}]\, P_1\, \phi \right\| \;&\leq\; c\, g \left\| \bar{\chi}_1 \frac{|p| + |\mathbf{P}_f|}{\sqrt{(1-|p|)H_f + \frac{1}{2}|\mathbf{P}_f|^2}} \bar{\chi}_1 \right\| \times \\
&\qquad \times \sum_\lambda \int d^3k\, \frac{\chi_\Lambda[|k|]}{\sqrt{|k|}} \, \| \, a_\lambda(k)\, P_1\, \phi \, \| \\
&\leq\; c\, g \left(\frac{|p|}{1-|p|} + 2 \right) \sum_\lambda \int d^3k\, \frac{\chi_\Lambda[|k|]}{\sqrt{|k|}} \, \| \, a_\lambda(k)\, P_1\, \phi \, \| \\
&\leq\; c\, g \left(\frac{|p|}{1-|p|} + 2 \right) \left(\sup_\lambda \int d^3k \; |k|^{-2} \; \chi_\Lambda[|k|] \right)^{\frac{1}{2}} \times \\
&\qquad \times \left(\sum_\lambda \int d^3k \; \langle \, P_1\, \phi \, , \, a_\lambda^*(k)\, |k|\, a_\lambda(k)\, P_1\, \phi \, \rangle \right)^{\frac{1}{2}} \;,
\end{aligned} \tag{10.2.2}$$

which is bounded by

$$c'\, g\, \|\phi\| \;,$$

as asserted, since $\|H_f^{\frac{1}{2}} P_1 \phi\| \leq \|\phi\|$. The case for $W_{1,0}$ is identical.

Next, let us consider $W_2[\mathcal{O}]$. Recalling (10.1.5), that is,

$$|\, \langle \, \psi \, , \, W_{1,1}[\mathcal{O}]\, \phi \, \rangle \, | \;\leq\; c'\, g^2\, \|H_f^{\frac{1}{2}} \psi\| \, \|H_f^{\frac{1}{2}} \chi\| \;,$$

we have

$$\left| \left\langle \, \psi \, , \, \bar{R}_0^{\frac{1}{2}}\, \bar{\chi}_1 W_{1,1}\, P_1 \phi \right\rangle \right| \;\leq\; c'\, g^2\, \|\psi\| \, \|\phi\| \;, \tag{10.2.3}$$

using (10.1.6).

For $M=0$ and $N=2$, we recall

$$\left\| \, (H_f + \Lambda + 1)^{-\frac{1}{2}} \, W_{0,2}[\mathcal{O}] \, \phi \, \right\| \;\leq\; c'\, g^2 \left\| \, H_f^{\frac{1}{2}} \, \phi \, \right\| \;,$$

and find

$$\left\| \bar{R}_0^{\frac{1}{2}}\, \bar{\chi}_1 W_{0,2}[\mathcal{O}]\, P_1\, \phi \right\| \;\leq\; c'\, g^2 \|\phi\| \;, \tag{10.2.4}$$

again using (10.1.6). The case of $W_{2,0}[\mathcal{O}]$ is treated in the same fashion.

This proves Lemma 7.3. ■

10.3. Proof of Lemma 7.6

We first prove the following bounds on the Hilbert space $P_1\mathcal{H}_p$.

LEMMA 10.1. *Under the assumptions of Lemma 7.6, there exists a constant $C < \infty$, such that*

$$\begin{aligned} \| \bar{R}_0^{\frac{1}{2}} \partial_{|p|}^{\beta} \bar{\chi}_1 (\partial_{\mathcal{O}}^{\alpha} W_{p,q}^{m,n}) \bar{\chi}_1 \bar{R}_0^{\frac{1}{2}} \| &\leq C \epsilon_0^{m+n+p+q} \xi^{m+n} (2\sqrt{\pi}\xi)^{p+q} \\ \| \bar{R}_0^{\frac{1}{2}} \bar{\chi}_1 (\partial_{|p|}^{\beta} \partial_{\mathcal{O}}^{\alpha} W_{p,q}^{m,n}) \chi_1 \| &\leq C \epsilon_0^{m+n+p+q} \xi^{m+n} (2\sqrt{\pi}\xi)^{p+q} \end{aligned}$$

for $|\alpha| \leq 2$, $\beta = 0, 1$, and

$$\begin{aligned} \| \bar{R}_0^{\frac{1}{2}} \bar{\chi}_1 (\partial_{|p|}^{\beta} \partial_X W_{p,q}^{m,n}) \bar{\chi}_1 \bar{R}_0^{\frac{1}{2}} \| &\leq C \epsilon_0^{m+n+p+q} \xi^{m+n} (2\sqrt{\pi}\xi)^{p+q} \\ \| \bar{R}_0^{\frac{1}{2}} (\partial_{|p|}^{\beta} \partial_X W_{p,q}^{m,n}) \chi_1 \| &\leq C \epsilon_0^{m+n+p+q} \xi^{m+n} (2\sqrt{\pi}\xi)^{p+q} \end{aligned}$$

on the Hilbert space $P_1\mathcal{H}_p$, for $m+n+p+q \leq 2$, and $X = |k_i|$, $|\tilde{k}_j|$ or z.

Proof. The asserted bounds are obtained in the precise same manner as in the cases of Lemma 7.2 and Lemma 7.3. The only difference now are the bounds

$$| \partial_{|p|}^{\beta} \partial_Y^{\alpha} w_{M,N} | \leq c\, g^{M+N} (1 + |\mathbf{P}_f|) \prod_{r=1}^{M} \chi_{[0,\Lambda]}(|k_r|) \prod_{s=1}^{N} \chi_{[0,\Lambda]}(|\tilde{k}_s|)$$

for $M + N \leq 2$, and Y denoting either $\mathcal{O}$ (with $|\alpha| \leq 2$) or X (with $|\alpha| \leq 1$), instead of those on $\| w_{M,N} \|$. Furthermore, we recall that $\epsilon_0\, \xi = 2g$. ■

LEMMA 10.2. *Let us assume that there is a constant C independent of ϵ, such that*

$$\| \partial_{|p|}^{\beta'} \partial_Y^{\alpha'} (\bar{\chi}_1 \bar{R}_0 \bar{\chi}_1) \| \,,\ \| \partial_{|p|}^{\beta'} \partial_Y^{\alpha'} \Pi \| \leq C$$

for any $0 \leq \beta' \leq 1$, and $0 \leq |\alpha'| \leq 2$. The operator

$$\mathcal{E}_L \left[\{ \underline{m}, \underline{n}, \underline{p}, \underline{q} \}; z; \mathcal{O}; K^{(M,N)} \right] = \left\langle \Pi W_{p_1,q_1}^{m_1,n_1} \bar{\chi}_1 \bar{R}_0 \bar{\chi}_1 \cdots W_{p_L,q_L}^{m_l,n_l} \Pi \right\rangle_{\mathcal{O},sym} ,$$

with $\underline{m} = (m_1, \ldots, m_L)$, etc., and $M = \sum m_i$, $N = \sum n_i$, satisfies

$$\| \partial_{|p|}^{\beta} \partial_{\mathcal{O}}^{\alpha} \mathcal{E}_L \| \leq c_{\alpha,\beta}\, L^{|\alpha|+\beta}\, \epsilon_0^L\, C^L\, \xi^{M+N}\, (2\sqrt{\pi}\xi)^{\sum (p_i+q_i)} .$$

for $|\alpha| \leq 2$, $\beta = 0, 1$, and

$$\| \partial_{|p|}^{\beta} \partial_X \mathcal{E}_L \| \leq c_{X,\beta}\, L^{1+\beta}\, \epsilon_0^L\, C^L\, \xi^{M+N}\, (2\sqrt{\pi}\xi)^{\sum (p_i+q_i)} .$$

The constants $c_{X,\beta}$ and $c_{\alpha,\beta}$ do not depend on ϵ, ξ. Furthermore, $\mathcal{E}_L$ is **analytic** *in $\mathcal{O}$ for all $M + N \geq 0$.*

Proof. $\mathcal{E}_L$ and the symbol $\langle \cdot \rangle_{\mathcal{O},sym}$ were defined below (4.5.12).

$\mathcal{E}_L$ is the $\langle \cdot \rangle_{\mathcal{O},sym}$-expectation value of $L + (L-1) < 2L$ operators (L interaction operators $W_{p_i,q_i}^{m_i,n_i}$ and $L-1$ free resolvents $\bar{R}_0$). Thus, by the product rule, $\partial_{|p|}^{\beta} \partial_{\mathcal{O}}^{\alpha} \mathcal{E}_L$ is the sum of $O(L^{|\alpha|+\beta})$ vacuum expectation values.

Derivatives of $\bar{\chi}_1 \bar{R}_0 \bar{\chi}_1$, Π can be estimated by

$$\| \partial_{|p|}^{\beta'} \partial_Y^{\alpha'} (\bar{\chi}_1 \bar{R}_0 \bar{\chi}_1) \| \,,\ \| \partial_{|p|}^{\beta'} \partial_Y^{\alpha'} \Pi \| \leq C ,$$

where the constant only depends on $|\alpha'| \leq |\alpha|, \beta' \leq \beta$. Derivatives of $W_{p_i,q_i}^{m_i,n_i}$ are controlled by Lemma 10.1.

In each of the $O(L^{|\alpha|+\beta})$ terms obtained from applying $\partial_{|p|}^{\beta}\partial_{\mathcal{O}}^{\alpha}$ to $\mathcal{E}_L$, there are at most $L-1+|\alpha|+\beta$ occurrences of $\bar{R}_0$, and two of Π, hence an overall factor

$$C^{L+1+|\alpha|+\beta}$$

in the bound. Furthermore, from each $W_{p_i,q_i}^{m_i,n_i}$ or its derivative, there emerges a factor

$$O(\epsilon_0)(2\sqrt{\pi}\xi)^{p_i+q_i}\xi^{m_i+n_i} \ .$$

Thus, there is an overall factor

$$O(\epsilon_0^L)(2\sqrt{\pi}\xi)^{\sum(p_i+q_i)}\xi^{\sum(m_i+n_i)}$$

stemming from the interaction operators.

Since all implicit constants in these estimates only depend on α and β, there is a constant $c_{\alpha,\beta}$, such that

$$\begin{aligned} \| \, \partial_{|p|}^{\beta}\partial_{\mathcal{O}}^{\alpha} \, \mathcal{E}_L \, \| \quad &\leq \quad c_{\alpha,\beta} \, L^{|\alpha|+\beta} \, \epsilon_0^L \, C^L \ \times \\ &\qquad \times \, \xi^{M+N} \, (2\sqrt{\pi}\xi)^{\sum(p_i+q_i)} \ . \end{aligned}$$

The asserted bound on $\| \, \partial_{|p|}^{\beta}\partial_X \, \mathcal{E}_L \, \|$ are proved in the same manner.

Because every $\bar{R}_0$ and every $W_{p_i,q_i}^{m_i,n_i}$ is analytic in $\mathbf{P}_f$, $\mathcal{E}_L$ also is.

For a more detailed exposition of the intermediate steps in the derivation of these bounds, cf. [**2, 3**]. ■

Proof of Lemma 7.6. Let X denote $|k|$ or any component $\mathcal{O}^r$ of $\mathcal{O} = (H_f, \mathbf{P}_f^{\|}, \mathbf{P}_f^{\perp})$. Then, using Lemma 10.2,

$$\begin{aligned} \left\| \, \partial_{|p|}^{\beta} \, \partial_{\mathcal{O}}^{\alpha} \, \Delta w_{M,N} \, \right\| \quad &\leq \quad c_{\alpha,\beta} \, \xi^{M+N} \, \sum_{L=2}^{\infty} \, L^{|\alpha|} \, \epsilon_0^L \, C^L \ \times \\ &\qquad \times \sum_{\substack{1 \leq M_i + N_i \leq 2 \\ i=1,\ldots,L}} \delta_{M,\sum_{i=1}^L m_i} \delta_{N,\sum_{i=1}^L n_i} \ \times \\ &\qquad \times \ \prod_{i=1}^{L} \left(\begin{array}{c} m_i + p_i \\ p_i \end{array} \right) \left(\begin{array}{c} n_i + q_i \\ q_i \end{array} \right) (2\sqrt{\pi}\xi)^{\sum(p_i+q_i)} \\ &\leq \quad c_{\alpha,\beta} \, \xi^{M+N} \, \sum_{L=2}^{\infty} \, 4^L \, L^{|\alpha|} \, \epsilon_0^L \, C^L \ \times \\ &\qquad \times \sum_{\substack{p_i+q_i \geq 0 \\ i=1,\ldots,L}} (2\sqrt{\pi}\xi)^{\sum(p_i+q_i)} \\ &\leq \quad C_{\alpha,\beta} \, \epsilon_0^2 \, \xi^{M+N} \end{aligned} \tag{10.3.1}$$

for a constant $C_{\alpha,\beta}$ that is independent of ϵ_0, ξ. The combinatorial factors have here been estimated by

$$\left(\begin{array}{c} m+p \\ p \end{array} \right) \leq 2 \ ,$$

since $m + p \leq 2$, and likewise for $m \leftrightarrow n$, $p \leftrightarrow q$.

The bound

$$\| \partial_{|p|}^{\beta} \partial_X \Delta w_{M,N} \| \ \leq \ C_{X,\beta} \, \epsilon_0^2 \, \xi^{M+N}$$

of the assertion follows in the same manner.

According to the previous lemma, $\mathcal{E}_L$ is analytic in $\mathcal{O}$ for all L. Thus, $\Delta w_{M,N}$ is given by a series of analytic functions of $\mathcal{O}$. In particular, the above results imply that the defining series for $\partial_{\mathcal{O}}^{\alpha} \Delta w_{M,N}$ are, for $|\alpha| \leq 1$, uniformly convergent with respect to $\| \cdot \|$. Thus, $\Delta w_{M,N}$ is analytic in $\mathcal{O}$.

This proves Lemma 7.6. ■

CHAPTER 11

APPENDIX: PROOFS RELATED TO THE INDUCTION STEP

In this appendix, we give the proofs that are missing in Chapter 8.

11.1. Proof of Proposition 8.5

LEMMA 11.1. *Let $0 \leq |\alpha| \leq 2$, $\beta = 0, 1$, and $X = |k_i|$, $|\tilde{k}_j|$ or z, corresponding to the assumptions of Lemma 8.5,*

$$\begin{aligned} \| \, \partial_{|p|}^{\beta} \, \partial_{\mathcal{O}}^{\alpha} \, W_{p,q}^{m,n} \, \| &\leq (2\epsilon_0)^{\beta} \, \epsilon^{\frac{7}{4} - \frac{|\alpha|}{4} - \beta} \, (2\sqrt{\pi}\xi)^{p+q} \, \xi^{m+n} \\ \| \, \partial_{|p|}^{\beta} \, \partial_{X} \, W_{p,q}^{m,n} \, \| &\leq (2\epsilon_0)^{\beta} \, \epsilon^{\frac{3}{2} - \beta} \, (2\sqrt{\pi}\xi)^{p+q} \, \xi^{m+n} \end{aligned} \tag{11.1.1}$$

on the Hilbert space $P_1 \mathcal{H}_p$.

Proof. This follows directly from

$$\| \, \partial_{|p|}^{\beta} \, \partial_{\mathcal{O}}^{\alpha} \, w_{M,N} \, \| \leq (2\epsilon_0)^{\beta} \, \epsilon^{\frac{7}{4} - \frac{|\alpha|}{4} - \beta} \, \xi^{M+N}$$

and

$$\| \, \partial_{|p|}^{\beta} \, \partial_{X} \, w_{M,N} \, \| \leq (2\epsilon_0)^{\beta} \, \epsilon^{\frac{3}{2} - \beta} \, \xi^{M+N}$$

implied by the induction hypothesis. ■

LEMMA 11.2. *Let $0 \leq |\alpha| \leq 2$, $\beta = 0, 1$, and $X = |k_i|$, $|\tilde{k}_j|$ or z, corresponding to the assumptions of Proposition 8.5,*

$$\mathcal{E}_L \left[\{\underline{m}, \underline{n}, \underline{p}, \underline{q}\}; z; \mathcal{O}; K^{(M,N)} \right] = \left\langle \, \Pi W_{p_1,q_1}^{m_1,n_1} \bar{\chi} \bar{R}_0 \bar{\chi} \cdots W_{p_L,q_L}^{m_l,n_l} \Pi \, \right\rangle_{\mathcal{O},sym} ,$$

with $\underline{m} = (m_1, \ldots, m_L)$, etc., and $M = \sum m_i$, $N = \sum n_i$, satisfies

$$\begin{aligned} \| \, \partial_{|p|}^{\beta} \partial_{\mathcal{O}}^{\alpha} \, \mathcal{E}_L \, \| \leq \epsilon_0^{\beta} \, c_{\alpha,\beta} \, L^{|\alpha|+\beta} \, \epsilon^{L-\beta} \, \rho^{1-L-|\alpha|-\beta} \, \times \\ \times \, \xi^{M+N} \, (2\sqrt{\pi}\xi)^{\sum (p_i+q_i)} \, . \end{aligned}$$

and

$$\begin{aligned} \| \, \partial_{|p|}^{\beta} \partial_{X} \, \mathcal{E}_L \, \| \leq \epsilon_0^{\beta} \, c_{X,\beta} \, L^{1+\beta} \, \epsilon^{L-\beta} \, \rho^{L-\beta} \, \times \\ \times \, \xi^{M+N} \, (2\sqrt{\pi}\xi)^{\sum (p_i+q_i)} \, . \end{aligned}$$

The constants $c_{\alpha,\beta}$ and $c_{X,\beta}$ are independent of ϵ, ξ. Furthermore, $\mathcal{E}_L$ is **analytic** *in $\mathcal{O}$ for all $M + N \geq 0$.*

Proof. The definition of the symbol $\langle \cdot \rangle_{\mathcal{O},sym}$ is given below (4.5.12). By induction hypothesis, T (and thus $\bar{R}_0[z]$) and $w_{M,N}$ are analytic in $\mathcal{O}$. Thus, $\mathcal{E}_L$ is analytic in $\mathcal{O}$.

¿From the product rule, it is first of all clear that there are $O(L^{|\alpha|+\beta})$ terms in total, because $\mathcal{E}_L$ is the $\langle \cdot \rangle_{\mathcal{O},sym}$-expectation value of a product of less than $2L$ operators.

Derivatives of $\bar\chi_\rho \bar R_0 \bar\chi_\rho$ and Π can be estimated by

$$\|\partial_{|p|}^{\beta'}\partial_X^{\alpha'}\,(\bar\chi_\rho \bar R_0 \bar\chi_\rho)\|\;,\;\|\partial_{|p|}^{\beta'}\partial_X^{\alpha'}\,\Pi\| \;\le\; C_\rho^{1+|\alpha|+\beta'}\;,$$

where the constant C_ρ only depend on ρ, while derivatives of $W_{p_i,q_i}^{m_i,n_i}$ can be estimated by Lemma 10.1.

In each of the $O(L^{|\alpha|+\beta})$ terms obtained from applying $\partial_{|p|}^{\beta}\partial_X^{\alpha}$ to $\mathcal{E}_L$, there are at most $L-1+|\alpha|+\beta$ occurrences of $\bar R_0$, and two of Π, hence an overall factor $C_\rho^{L+1+|\alpha|+\beta}$ in the bound. Furthermore, the induction hypotheses, together with Lemma 11.1, imply that

$$\|\!|\,\partial_{|p|}^{\beta_i}\partial_X^{\alpha'}\,W_{p_i,q_i}^{m_i,n_i}\,|\!\| \;\le\; (2\epsilon_0)^{\beta_i}\,\epsilon^{\frac{7}{4}-\frac{|\alpha'|}{4}-\beta}(2\sqrt{\pi}\xi)^{p_i+q_i}\xi^{m_i+n_i}\;,$$

where $\beta_i = 1$ if a derivative with respect to $|p|$ is taken, and $\beta_i = 0$ otherwise. Thus, there is an overall factor

$$O(\epsilon_0^\beta \epsilon^{L-\beta})(2\sqrt{\pi}\xi)^{\sum(p_i+q_i)}\xi^{\sum(m_i+n_i)}$$

due to the interaction operators.

Since all implicit constants in these estimates only depend on α and β, there is a constant $c_{\alpha,\beta}$, such that

$$\begin{aligned}\|\!|\,\partial_{|p|}^{\beta}\partial_{\mathcal{O}}^{\alpha}\,\mathcal{E}_L\,|\!\| \;&\le\; \epsilon_0^\beta\, c_{\alpha,\beta}\, L^{|\alpha|+\beta}\;\times\\ &\qquad\times\,\epsilon^{L-\beta}\,\xi^{M+N}\,(2\sqrt{\pi}\xi)^{\sum(p_i+q_i)}\;,\end{aligned}\tag{11.1.2}$$

which proves the claim.

The estimate

$$\begin{aligned}\|\!|\,\partial_{|p|}^{\beta}\partial_X\,\mathcal{E}_L\,|\!\| \;&\le\; \epsilon_0^\beta\, c_{X,\beta}\, L^{1+\beta}\,\epsilon^{L-\beta}\,C_\rho^L\;\times\\ &\qquad\times\,\xi^{M+N}\,(2\sqrt{\pi}\xi)^{\sum(p_i+q_i)}\end{aligned}$$

is proved in the same way.

Furthermore, for every $L\ge 2$, $\mathcal{E}_L$ is analytic in $\mathbf{P}_f$, since $\bar R_0$ and $W_{p_i,q_i}^{m_i,n_i}$ are.

We refer the interested reader to [**2, 3**] for a more explicit exposition of some of the intermediate steps. ■

Proof of Lemma 8.2. For $0\le|\alpha|\le 2$,

$$\begin{aligned}\|\!|\,\partial_{|p|}^{\beta}\,\partial_{\mathcal{O}}^{\alpha}\,\Delta w_{M,N}\,|\!\| \;&\le\; \sum_{L=2}^{\infty}\sum_{\substack{1\le M_i+N_i\\ i=1,\dots,L}}\delta_{M,\sum_{i=1}^L m_i}\delta_{N,\sum_{i=1}^L n_i}\;\times\\ &\qquad\times\prod_{i=1}^{L}\begin{pmatrix} m_i+p_i\\ p_i\end{pmatrix}\begin{pmatrix} n_i+q_i\\ q_i\end{pmatrix}\left\|\,\partial_{|p|}^{\beta}\,\partial_X^{\alpha}\,\mathcal{E}_L\,\right\|\\ &\le\; (2\xi)^{M+N}\,c_{|\alpha|,\beta}\sum_{L=2}^{\infty}L^{|\alpha|}\,\epsilon^{L-\beta}\,\rho^{1-L-|\alpha|-\beta}\;\times\\ &\qquad\times\,\epsilon_0^\beta\sum_{\substack{p_i+q_i\ge 0\\ i=1,\dots,L}}2^{\sum(p_i+q_i)}\,(2\sqrt{\pi}\xi)^{\sum(p_i+q_i)}\\ &\le\; \epsilon_0^\beta\,C_{|\alpha|,\beta}\,\epsilon^{2-\beta}\,(2\xi)^{M+N}\\ &=\; \epsilon_0^\beta\,C_{|\alpha|,\beta}\,\epsilon^{2-\beta}\,(2\xi)^{M+N}\;,\end{aligned}\tag{11.1.3}$$

using Lemma 11.2, and

$$\left(\begin{array}{c} m+p \\ p \end{array} \right) \leq 2^{m+p} ,$$

while setting $\rho = \frac{1}{2}$. This is the first assertion of Lemma 8.2.

The assertion

$$\| \partial^{\beta}_{|p|} \partial_X \Delta w_{M,N} \| \leq \epsilon_0^{\beta} C_{X,\beta} \epsilon^{2-\beta} (2\xi)^{M+N} , \tag{11.1.4}$$

for $X = |k_i|$, $|\tilde{k}_j|$ or z, is proved in the same way.

By the previous lemma, $\mathcal{E}_L$ is analytic in $\mathbf{P}_f$ for all L. Thus, $\Delta w_{M,N}$ is given by a series of analytic functions of $\mathbf{P}_f$. Since the first part of this proof showed that the defining series for $\partial^{\alpha}_{\mathbf{P}_f} \Delta w_{M,N}$ are, for $|\alpha| \leq 1$, uniformly convergent with respect to $\| \cdot \|$, $\Delta w_{M,N}$ is analytic in $\mathbf{P}_f$. ■

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Bibliography

[1] A. Arai, "Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation", preprint mp-arc (2000).

[2] V. Bach, J. Fröhlich, I.M. Sigal. "Quantum electrodynamics of confined non-relativistic particles", Adv. Math., **137 (2)**, 299-395, (1998).

[3] V. Bach, J. Fröhlich, I. M. Sigal, "Renormalization group analysis of spectral problems in quantum field theory", Adv. in Math. **137**, 205-298 (1998).

[4] V. Bach, J. Fröhlich, I. M. Sigal, "The smooth Feshbach map", unpublished preprint.

[5] F. Bloch, A. Nordsieck, Phys. Rev. **52**, 54, (1937). F. Bloch, A. Nordsieck, Phys. Rev. **52**, 59, (1937).

[6] V. Betz, F. Hiroshima, J. Lörinczi, R.A. Minlos, H. Spohn, "Ground state properties of the Nelson Hamiltonian - A Gibbs measure-based approach ", preprint math-ph/0106015 (2001).

[7] V. Chung, Phys. Rev. **B 140**, 1110, (1965).

[8] L.D. Faddeev, P.P. Kulish, (engl. transl.) Theor. and Math. Phys. **4**, 745, (1970).

[9] J. Fröhlich, "Ueber das Infrarot-Problem in einem Modell skalarer Elektronen und skalarer Bosonen der Ruhemasse Null", ETH-Diss 4920, (1973).

[10] J. Fröhlich, "On the infrared problem in a model of scalar electrons and massless scalar bosons", Ann. Inst. H. Poincaré **19**, 1 - 103 (1973).

[11] J. Fröhlich, M. Griesemer, B. Schlein, "Asymptotic Electromagnetic Fields in Models of Quantum-Mechanical Matter Interacting with the Quantized Radiation Field", preprint math-ph/0009033 (2000).

[12] J. Fröhlich, M. Griesemer, B. Schlein, "Asymptotic Completeness for Rayleigh Scattering", preprint math-ph/0103048 (2001).

[13] J. Fröhlich, G. Morchio, F. Strocchi, "Charged sectors and scattering states in quantum electrodynamics", Ann. Physics **119**, vol. 2, 241 - 284 (1979).

[14] C. Gérard, "On the existence of ground states in massless Pauli-Fierz hamiltonians", Ann. Henri Poincaré **1**, 443 - 459 (2000).

[15] C. Gérard, "On the scattering theory of massless Nelson models", preprint mp-arc 01-103 (2001).

[16] M. Griesemer, E. H. Lieb, M. Loss, "Ground States in Non-relativistic Quantum Electrodynamics", preprint math-ph/0007014 (2000).

[17] C. Itzykson, J.-B. Zuber, "Quantum field theory", McGraw-Hill, International edition (1985).

[18] T. Kibble, J. Math. Phys. **9**, 315, (1968).

[19] J. Lörinczi, R.A. Minlos, H. Spohn, "The infrared behaviour in Nelson's model of a quantum particle coupled to a massless scalar field", preprint math-ph/0011043 (2000).

[20] J. Lörinczi, R.A. Minlos, H. Spohn, "Infrared regular representation of the three dimensional massless Nelson model ", preprint math-ph/0107004 (2001).

[21] A. Pizzo, "One Particle (improper) states and Scattering States in Nelson's Massless Model", preprint math-ph/0010043 (2000).

[22] H. Spohn, "Asymptotic completeness in Rayleigh scattering", J. Math. Phys. **38**, 2281 - 2296 (1997).

[23] S. Teufel, H. Spohn, "Semiclassical limit for the Schrödinger equation with a short scale periodic potential", preprint math-ph/0010009 (2000).

[24] J. von Neumann, Composito Math. **6**, 1, (1938).

[25] D. Yennie, S. Frautschi, H. Suura, Ann. Phys. **13**, 379, (1961).