MIXING PROPERTIES AND EXPONENTIAL DECAY FOR LATTICE SYSTEMS IN FINITE VOLUMES

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Abstract. An infinite-volume mixing or exponential-decay property in a spin system or percolation model reflects the inability of the influence of the configuration in one region to propagate to distant regions, but in some circumstances where such properties hold, propagation can nonetheless occur in finite volumes endowed with boundary conditions. We establish the absense of such propagation, particularly in two dimensions in finite volumes which are simply connected, under a variety of conditions, mainly for the Potts model and the Fortuin-Kasteleyn (FK) random cluster model, allowing external fields. For example, for the FK model in two dimensions we show that exponential decay of connectivity in infinite volume implies exponential decay in simply connected finite volumes, uniformly over all such volumes and all boundary conditions, and implies a strong mixing property for such volumes with certain types of boundary conditions. For the Potts model in two dimensions we show that exponential decay of correlations in infinite volume implies a strong mixing property in simply connected finite volumes, which includes exponential decay of correlations in simply connected finite volumes, uniformly over all such volumes and all boundary conditions.

1. Introduction and Preliminaries

Many models encountered in statistical mechanics exhibit exponential decay of the two-point function at sufficiently high temperatures. Typical spin systems exhibit exponential decay of correlations, and many standard percolation models are known or believed to have exponential decay of connectivities for those non-critical parameter values at which there is no percolation. In particular this can be said for many random cluster models (graphical representations) corresponding to spin systems. Such exponential decay is by its nature an infinite-volume property, but it does have a finite-volume analog which has apparently been little studied. Let R be a bounded subset of $\mathbb{R}^n$. For a spin system on the lattice $\mathbb{Z}^n$, we consider the truncated correlation (covariance) $\langle \delta_{[\sigma_x=i]}; \delta_{[\sigma_y=j]} \rangle^{\beta}_{\Lambda,\eta}$ for the system on $\Lambda = R \cap \mathbb{Z}^n$ at inverse temperature β under boundary condition η. For a percolation model

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we consider the probability $P_{\mathcal{B},\rho}(x \leftrightarrow y)$ of the event $x \leftrightarrow y$ that there exists a path of open bonds from x to y, for the model on the set $\mathcal{B}$ of all bonds contained in R, under boundary condition ρ. We may ask, do there exists constants C and λ, *not* depending on the region R or on the boundary condition, such that for all $x, y \in \Lambda$ this finite-volume correlation or connectivity is bounded above by $C \exp(-\lambda d(x, y))$? There are two natural choices for the metric d here: the Euclidean metric d_2 and the *restricted-path metric*

$$d_R(x, y) = \min\{n \geq 0 : \text{ there exists a lattice path of length } n \text{ in } R \text{ from } x \text{ to } y\}.$$

When such C, λ exist we say there is *uniform exponential decay of finite-volume correlations* or *connectivities*. When the uniformity is only over some limited class $\mathfrak{C}$ of regions, or regions with boundary conditions, we refer to the uniform exponential decay as being *for the class* $\mathfrak{C}$. If the metric is not clear from the context, we refer to the decay as being *in the Euclidean metric* or *in the restricted-path metrics*. We will establish here sufficient conditions for such uniform exponential decay in two dimensions, in both metrics.

What makes the study of correlations, and analogously connectivities, under a boundary condition difficult is that the influence of one spin might in principle propagate (in part) along the boundary to affect distant spins much more than would occur in infinite volume. Such propagation was studied by Martinelli, Olivieri and Schonmann [24] in showing that in two dimensions, weak mixing implies a form of strong mixing. But these authors restricted attention to very regular regions Λ, specifically large squares or unions of large squares; we will consider more general regions here. We will not, however, consider spin systems as general as those in [24].

The region Λ cannot be completely arbitrary. The examples in ([23] pp. 458–459), one due to R. H. Schonmann, show that when Λ is so irregular that the boundary $\partial\Lambda$ permeates through the bulk of Λ, the influence of a single spin can propagate much more than is the case in infinite volume. The hypothesis we will make on Λ is a lattice version of simple connectedness, which we will see is enough to avoid this problem.

The main percolation model of interest to us is the Fortuin-Kasteleyn random cluster model (briefly, the *FK model*) of [13], [14], [15], including the version with external fields. The FK model is a graphical representation of the Potts model. When possible, however, we will state results for more general models.

In two dimensions one can also consider decay of dual connectivities in finite volumes, and here the effect of boundary conditions can be dramatic. For example, the FK model corresponding to an Ising model at subcritical temperature exhibits exponential decay of dual connectivity in infinite volume, but for a large square with Dobrushin boundary conditions (plus on the top half, minus on the bottom half), a long dual connection is forced to exist, meaning the exponential decay is destroyed.

For the Ising model in finite volume with boundary condition in which all boundary spins are plus or 0 (free), as observed by C. Pfister [26], the boundary condition may be viewed as an infinite external field applied to the boundary plus spins, together with the "turning off" of the couplings for all bonds with an endpoint at a boundary 0 spin. Standard symmetry inequalities then show that the truncated correlation in finite volume is bounded above by the infinite-volume truncated correlation, so exponential decay of correlations implies at least a weaker form of uniform exponential decay of finite-volume connectivities, in which we restrict the allowed boundary conditions. But symmetry inequalities tell us nothing here under Ising-model boundary conditions which mix plus and minus spins. And for other models of interest, such as the Potts model, symmetry inequalities aren't even available. Instead, our techniques involve first establishing uniform exponential decay of finite-volume connectivities in percolation models, then transferring these results to some spin systems using random cluster representations.

Beyond exponential decay of the two-point function, we consider mixing properties. Consider finite regions $\Delta \subset \Lambda$ with boundary condition on Λ^c. Roughly, weak mixing is the property that the maximum influence (measured additively) of the boundary condition on the probability of any event occuring in any fixed Δ decays to 0 exponentially in $d_2(\Delta, \Lambda)$ as $\partial\Lambda$ recedes to infinity, and ratio weak mixing is a similar but stronger property with influence measured multiplicatively. In strong mixing, the maximum (additive) influence of a change in the boundary condition on the probability of any event occurring in any subset Θ of Λ^c decays exponentially in $d_2(\Delta, \Theta)$. Thus weak mixing allows the influence of a region $\Theta \subset \Lambda^c$ to propagate along the boundary, but strong mixing does not allow this. In [5] the following facts are proved. For bond percolation models in two dimensions, under mild hypotheses (satisfied, for example, by the FK model), exponential decay of connectivities implies weak mixing. In any dimension, under additional hypotheses satisfied by the FK model, weak mixing and exponential decay of connectivity imply ratio weak mixing. For the Potts model without external fields above the critical temperature in two dimensions, exponential decay of correlations implies weak mixing, and in arbitrary dimension, weak mixing implies ratio weak mixing. Further, in [24] it is shown that in two dimensions, weak mixing is equivalent to a restricted version of strong mixing in which Λ is required to be a union of large squares. We consider here uniform versions of these results in finite volumes.

The condition in [24] that Λ be a union of large squares is more restrictive than it might first appear, for the following reason. It is often of interest to take two subsets $\Delta, \Sigma \subset \Lambda$ and, under a boundary condition specified on $\partial\Lambda$, consider the influence of an event or configuration on Σ on the probabilities of events occuring on Δ. One can hope to apply the results of [24] in this situation by treating Σ as part of the boundary, but this requires that $\Lambda\backslash\Sigma$ be a union of large squares, which is not generally natural.

Turning to formalities, a *bond*, denoted $\langle xy \rangle$, is an unordered pair of nearest neighbor sites of $\mathbb{Z}^n$. When convenient we view bonds as being closed line segments in the plane; this should be clear from the context. In particular for $R \subset \mathbb{R}^n$, $\mathcal{B}(R)$ denotes the set of all bonds for which the corresponding closed line segments are contained in R, and when we refer to distances between sets of bonds, we mean distances between the corresponding sets of line segments. The exception is for $\Lambda \subset \mathbb{Z}^n$, for which we set $\mathcal{B}(\Lambda) = \{\langle xy \rangle : x, y \in \Lambda\}$. (Again, this should be clear from the context.) For a set $\mathcal{D}$ of bonds we let $V(\mathcal{D})$ denote the set of all endpoints of bonds in $\mathcal{D}$, and

$$\partial\mathcal{D} = \{\langle xy \rangle : x \in V(\mathcal{D}), y \notin V(\mathcal{D})\}, \quad \overline{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}.$$

We write $\overline{\mathcal{B}}(\Lambda)$ for $\overline{\mathcal{B}(\Lambda)}$. A *bond configuration* is an element $\omega \in \{0,1\}^{\mathcal{B}(\mathbb{Z}^n)}$.

For $n = 2$, the *dual lattice* is the translation of the integer lattice by (1/2,1/2); we write x^* for $x + (1/2, 1/2)$. To each (regular) bond e of the lattice there corresponds a *dual bond* e^* which is its perpendicular bisector; the dual bond is defined to be open in a configuration ω precisely when the regular bond is closed, and the corresponding configuration of dual bonds is denoted ω^*. We write $(\mathbb{Z}^2)^*$ for $\{x^* : x \in \mathbb{Z}^2\}$.

A *cluster* in a given configuration is a connected component of the graph with site set $\mathbb{Z}^n$ and all open bonds; for $n = 2$, *dual clusters* are defined analogously for open dual bonds. (In contexts where there is a boundary condition consisting of a configuration on the complement $\mathcal{B}^c$ for some set $\mathcal{B}$ of bonds, a cluster may include bonds in $\mathcal{B}^c$.) For a configuration on $\overline{\mathcal{B}}(\Lambda)$ for some finite Λ, a *boundary cluster* is a cluster which intersects $\partial\Lambda$ and a *non-boundary cluster* is one which does not.

Given a set $\mathcal{D}$ of bonds, we write $\mathcal{D}^*$ for $\{e^* : e \in \mathcal{D}\}$. The set of all endpoints of bonds in $\mathcal{D}^*$ is denoted $V^*(\mathcal{D})$ or $V^*(\mathcal{D}^*)$.

For $\Lambda \subset \mathbb{Z}^n$ or $\Lambda \subset (\mathbb{Z}^2)^*$ we define

$$\partial\Lambda = \{x \notin \Lambda : x \text{ adjacent to } \Lambda\}, \quad \partial_{in}\Lambda = \{x \in \Lambda : x \text{ adjacent to } \Lambda^c\}$$

where adjacency is in the appropriate lattice $\mathbb{Z}^n$ or $(\mathbb{Z}^2)^*$.

A *(dual) path* is a sequence $\gamma = (x_0, \langle x_0 x_1 \rangle, x_1, \dots x_{n-1}, \langle x_{n-1} x_n \rangle, x_n)$ of alternating (dual) sites and bonds. γ is *self-avoiding* if all sites are distinct. We write $x \leftrightarrow y$ ($x \overset{*}{\leftrightarrow} y$) in ω if there is a path of open (dual) bonds from x to y in ω.

By a *bond percolation model* we mean a probability measure P on $\{0,1\}^{\mathcal{B}(\mathbb{Z}^n)}$. The finite-volume distribution for the model P under boundary condition $\rho \in \{0,1\}^{\mathcal{B}^c}$ is

$$P_{\mathcal{B},\rho} = P(\cdot \mid \omega_e = \rho_e \text{ for all } e \in \mathcal{B}^c),$$

where $\mathcal{B} \subset \mathcal{B}(\mathbb{Z}^n)$. We write ρ^i for the all-i boundary condition. We say a bond percolation model P has *bounded energy* if there exists $p_0 > 0$ such that

$$1 - p_0 > P(\omega_e = 1 \mid \omega_b, b \neq e) > p_0 \quad \text{for all } \{\omega_b, b \neq e\}. \tag{1.1}$$

Write $\omega_{\mathcal{D}}$ for $\{\omega_e : e \in \mathcal{D}\}$ and let $\mathcal{G}_{\mathcal{D}}$ denote the σ-algebra generated by $\omega_{\mathcal{D}}$. P has the *weak mixing property* if for some $C, \lambda > 0$, for all finite sets $\mathcal{D}, \mathcal{E}$ with $\mathcal{D} \subset \mathcal{E}$,

$$\begin{aligned}&\sup\{\mathrm{Var}(P_{\mathcal{E},\rho}(\omega_{\mathcal{D}} \in \cdot), P_{\mathcal{E},\rho'}(\omega_{\mathcal{D}} \in \cdot)) : \rho, \rho' \in \{0,1\}^{\mathcal{E}^c}\} \\ &\qquad \le C \sum_{x \in V(\mathcal{D}), y \in V(\mathcal{E}^c)} e^{-\lambda|x-y|},\end{aligned}$$

where $\mathrm{Var}(\cdot,\cdot)$ denotes total variation distance between measures. Roughly, the influence of the boundary condition on a finite region decays exponentially with distance from that region. Equivalently, for some $C, \lambda > 0$, for all sets $\mathcal{E}, \mathcal{F} \subset \mathcal{B}(\mathbb{Z}^2)$,

$$\begin{aligned}&\sup\{|P(E \mid F) - P(E)| : E \in \mathcal{G}_{\mathcal{E}}, F \in \mathcal{G}_{\mathcal{F}}, P(F) > 0\} \\ &\qquad \le C \sum_{x \in V(\mathcal{E}), y \in V(\mathcal{F})} e^{-\lambda|x-y|}.\end{aligned} \tag{1.2}$$

P has the *ratio weak mixing property* if for some $C, \lambda > 0$, for all sets $\mathcal{E}, \mathcal{F} \subset \mathcal{B}(\mathbb{Z}^2)$,

$$\begin{aligned}&\sup\left\{\left|\frac{P(E \cap F)}{P(E)P(F)} - 1\right| : E \in \mathcal{G}_{\mathcal{E}}, F \in \mathcal{G}_{\mathcal{F}}, P(E)P(F) > 0\right\} \\ &\qquad \le C \sum_{x \in V(\mathcal{E}), y \in V(\mathcal{F})} e^{-\lambda|x-y|},\end{aligned} \tag{1.3}$$

whenever the right side of (1.3) is less than 1. Note that in weak mixing the influence of the event F on the probability of E is measured additively, but in ratio weak mixing it is measured multiplicatively—F can alter the probability of E by at most a factor near 1. A multiplicative result is much stronger when dealing with events E of probability much smaller than the additive constant, i.e. the right side of (1.2).

The finite-volume analog of weak mixing is strong mixing, as studied for spin systems in [24] and elsewhere, sometimes under other names. We say P has the *strong mixing property* (for a class $\mathfrak{C}$ of regions and boundary conditions, in the metric d) if for some $C, \lambda > 0$, for all $(\mathcal{B}, \rho) \in \mathfrak{C}$ and all $\mathcal{E}, \mathcal{F} \subset \mathcal{B}$,

$$\begin{aligned}&\sup\{|P_{\mathcal{B},\rho}(E \mid F) - P_{\mathcal{B},\rho}(E)| : E \in \mathcal{G}_{\mathcal{E}}, F \in \mathcal{G}_{\mathcal{F}}, P_{\mathcal{B},\rho}(F) > 0\} \\ &\qquad \le C \sum_{x \in V(\mathcal{E}), y \in V(\mathcal{F})} e^{-\lambda d(x,y)}.\end{aligned} \tag{1.4}$$

The *ratio strong mixing property* is defined similarly with (1.4) replaced by

$$\sup\left\{\left|\frac{P_{\mathcal{B},\rho}(E\cap F)}{P_{\mathcal{B},\rho}(E)P_{\mathcal{B},\rho}(F)}-1\right| : E\in\mathcal{G}_{\mathcal{E}}, F\in\mathcal{G}_{\mathcal{F}}, P_{\mathcal{B},\rho}(E)P_{\mathcal{B},\rho}(F)>0\right\} \leq C\sum_{x\in V(\mathcal{E}),y\in V(\mathcal{F})} e^{-\lambda d(x,y)}. \tag{1.5}$$

Here d is either d_2 or the restricted-path metric $d_{\mathcal{B}}$. As we will see (Remark 1.4), in contrast to the situation for spin systems, for bond percolation models these properties are not always quite the right ones to consider, as influence may be transmitted, in effect, through the boundary configuration ρ. Hence later, for the FK model we will be restricting $\mathcal{E},\mathcal{F}$ in (1.4) and (1.5) in a manner depending on ρ.

Given $\rho\in\{0,1\}^{\overline{\mathcal{B}}(\Lambda)^c}$ we define (ω,ρ) to be the bond configuration on the full lattice which coincides with ω on $\overline{\mathcal{B}}(\Lambda)$ and with ρ on $\overline{\mathcal{B}}(\Lambda)^c$.

Let us use "$\leq$" to denote the coordinatewise partial ordering on $\{0,1\}^{\mathcal{B}}$. An event A is called *increasing* if $\omega\in A, \omega\leq\omega'$ imply $\omega'\in A$, and *decreasing* if its complement is increasing. A probability measure P on $\{0,1\}^{\mathcal{B}}$ is said to have the *FKG property* if

$$P(A\cap B)\geq P(A)P(B) \quad \text{for all increasing events } A,B.$$

P is said to satisfy the *FKG lattice condition* if

$$P(\omega\vee\omega')P(\omega\wedge\omega')\geq P(\omega)P(\omega') \quad \text{for all } \omega,\omega'. \tag{1.6}$$

As proved in [16], the FKG lattice condition implies the FKG property. For P_1 and P_2 probability measures on $\{0,1\}^{\mathcal{B}}$, we say P_1 *dominates* P_2 (in the FKG sense) if $P_1(A)\geq P_2(A)$ for all increasing events A.

We say that an (infinite-volume) bond percolation model P has *exponential decay of connectivities* if there exist $C,\lambda>0$ such that for all x and y,

$$P(x\leftrightarrow y)\leq Ce^{-\lambda|y-x|}.$$

If P has the FKG property, then $-\log P(0^*\leftrightarrow x^*)$ is a subadditive function of x, and therefore the limit

$$\tau(x)=\lim_{k\to\infty}-\frac{1}{k}\log P(0\leftrightarrow kx), \tag{1.7}$$

exists for $x\in\mathbb{Q}^2$, provided we take the limit through values of k for which $kx\in\mathbb{Z}^n$. This definition extends to $\mathbb{R}^n$ by continuity (see [3]); the resulting τ is a norm on $\mathbb{R}^n$. By standard subadditivity results, the limit in (1.7) is approached from above, so that, for $\theta=x/|x|$,

$$P(0\leftrightarrow x)\leq e^{-\tau(x)}=e^{-\tau(\theta)|x|} \quad \text{for all } x. \tag{1.8}$$

For $\Lambda\subset\mathbb{Z}^n$ finite, $\rho\in\{0,1\}^{\mathcal{B}(\Lambda^c)}$, and $\Gamma\subset\Lambda^c$ finite, we call $\mathcal{B}(\Gamma)$ a *controlling region* for $\overline{\mathcal{B}}(\Lambda)$ and ρ if for every $\rho'\in\{0,1\}^{\mathcal{B}(\Lambda^c)}$ such that $\rho=\rho'$ on $\mathcal{B}(\Gamma)$, we have $P_{\Lambda,\rho}=P_{\Lambda,\rho'}$. We say P has *exponentially bounded controlling regions* if there exist constants $C, \lambda > 0$ such that for every choice of disjoint finite sets Λ and Γ,

$$P(\{\rho \in \{0,1\}^{\mathcal{B}(\Lambda^c)} : \mathcal{B}(\Gamma) \text{ is not a controlling region for } \overline{\mathcal{B}}(\Lambda) \text{ and } \rho\})$$
$$\leq C \sum_{x \in \Lambda, y \in \Lambda^c \setminus \Gamma} e^{-\lambda|x-y|}.$$

Note that when $P(E)$ is much smaller than the right side of (1.2), the weak mixing condition (1.2) allows $P(E \mid F)$ to be many times larger than $P(E)$, but the ratio weak mixing condition (1.3) does not allow this. Nonetheless, it is proved in [5] that if P is translation invariant and has exponentially bounded controlling regions and the weak mixing property, then P has the ratio weak mixing property. (The hypothesis of translation invariance should have been included in the statement of this result in [5].)

We call $\mathcal{D} \subset \mathcal{B}(\mathbb{Z}^2)$ *simply lattice-connected* if $\mathcal{D}$ and $\mathcal{B}(\mathbb{Z}^2) \backslash \mathcal{D}$ are connected. We call $\Lambda \subset \mathbb{Z}^2$ simply lattice-connected if $\overline{\mathcal{B}}(\Lambda)$ has that property.

We can now state our first main result for percolation models, essentially that when open paths do not propagate far in infinite volume, neither can they propagate in finite volumes, along the boundary or through the bulk, provided the model has the weak mixing property in infinite volume. The statement is given for the square lattice for ease of exposition, but the result is valid for any planar lattice, as all cited results used in the proof similarly so extend. We defer all proofs to Section 2.

Theorem 1.1. *Let P be a translation-invariant bond percolation model on $\mathcal{B}(\mathbb{Z}^2)$ having exponential decay of connectivities and the weak mixing property. Then P has uniform exponential decay of finite-volume connectivities for the class of all simply lattice-connected subsets of $\mathcal{B}(\mathbb{Z}^2)$ with arbitrary bond boundary conditions, for both the Euclidean and restricted-path metrics.*

This result for the restricted-path metric clearly implies the result for the Euclidean metric, so the last phrase is really just for emphasis. The same applies to the other results of this paper.

We do not know whether the rate of exponential decay in infinite volume and the uniform rate of exponential decay in finite volumes are the same in Theorem 1.1.

For $p \in [0,1], q > 0$ and $\mathcal{B} \subset \mathcal{B}(\mathbb{Z}^n)$, the *FK model* $P^{p,q}_{\Lambda,w}$ on the graph $(V(\mathcal{B}), \mathcal{B})$ with parameters (p,q) and free boundary condition is defined by the weights

$$W(\omega) = p^{|\omega|}(1-p)^{|\mathcal{B}|-|\omega|} q^{K(\omega)} \tag{1.9}$$

Here $|\omega|$ means the number of open bonds in ω and $K(\omega)$ denotes the number of open clusters in ω. More generally, let $K(\omega \mid \rho)$ be the number of open clusters of (ω, ρ) which intersect $V(\mathcal{B})$. The FK model $P^{p,q}_{\Lambda,\rho}$ with bond boundary condition ρ is given by the weights in (1.9) with $K(\omega)$ replaced by $K(\omega \mid \rho)$. The boundary condition ρ^1 is also called the *wired boundary condition* and we write $K_w(\omega)$ for $K(\omega \mid \rho^1)$; the corresponding weights are

$$W_w(\omega) = p^{|\omega|}(1-p)^{|\mathcal{B}|-|\omega|}q^{K_w(\omega)} \tag{1.10}$$

Alternately, we consider site boundary conditions. For notational convenience we allow an additional spin value 0 at boundary sites, i.e. $\eta \in \{0,1,..,q\}^\Lambda$; taking $\eta_x = 0$ makes the boundary condition free at x. Specifically, suppose $\mathcal{B} = \overline{\mathcal{B}}(\Lambda)$ for some finite $\Lambda \subset \mathbb{Z}^n$; given $\eta \in \{0,1,..,q\}^{\partial\Lambda}$ define

$$\begin{aligned} J(\Lambda,\eta) = \Big\{\omega \in \{0,1\}^{\overline{\mathcal{B}}(\Lambda)} : \eta_x = \eta_y \text{ for every } x, y \in \partial\Lambda \text{ for which } x \leftrightarrow y \text{ in } \omega, \\ \omega_e = 0 \text{ for all } e \in \{\langle xy\rangle : x \in \Lambda, y \in \partial\Lambda, \eta_y = 0\}\Big\}. \end{aligned} \tag{1.11}$$

Here $x \leftrightarrow y$ means there is a path of open bonds connecting x to y. The FK model $P^{p,q}_{\overline{\mathcal{B}}(\Lambda),\eta}$ with site boundary condition η is given by the weights in (1.10), multiplied by $\delta_{J(\Lambda,\eta)}(\omega)$. We write η^i for the all-i site boundary condition. Taking $\eta = \eta^0$ gives the FK model with free boundary condition; we denote it $P^{p,q}_{\mathcal{B}(\Lambda),f}$. Let

$$\mathcal{B}^+(\Lambda,\eta) = \mathcal{B}(\Lambda) \cup \{\langle xy\rangle : x \in \Lambda, y \in \partial\Lambda, \eta_y \neq 0\}.$$

Generally η is understood from the context and we suppress it in the notation, writing $\mathcal{B}^+(\Lambda)$. The the model on $\overline{\mathcal{B}}(\Lambda)$ with boundary condition η is equivalent to the model on $\mathcal{B}^+(\Lambda)$ with site boundary condition defined only on $\{y \in \partial\Lambda : \eta_y \neq 0\}$. We call η a *single-species boundary condition* if for some i, $\eta_x \in \{0,i\}$ for all x. For a summary of basic properties of the FK model, see [17]. In particular, since we are in two dimensions, for $p \neq \sqrt{q}/(1+\sqrt{q})$ there is a unique infinite-volume FK measure on $\mathcal{B}(\mathbb{Z}^2)$, which can be obtained as the limit of $P^{p,q}_{\overline{\mathcal{B}}(\Lambda),w}$ as $\Lambda \nearrow \mathbb{Z}^2$; we denote this measure $P^{p,q}$ and we say that *random-cluster uniqueness* holds for the FK model at (p,q). For $q \geq 1$, the FK model satisfies the FKG lattice condition, under any bond or single-species site boundary condition.

For the FK model with external fields h_i, $i = 1,\dots,q$ and free boundary, the factor $q^{K(\omega)}$ in the weight $W(\omega)$ is replaced by

$$\prod_{C \in \mathcal{C}(\omega)} \left((1-p)^{h_1 s(C)} + (1-p)^{h_2 s(C)} + \ldots + (1-p)^{h_q s(C)}\right), \tag{1.12}$$

where $\mathcal{C}(\omega)$ is the set of clusters in $(\overline{\Lambda}, \overline{\mathcal{B}}(\Lambda))$ in the configuration ω and $s(C)$ denotes the number of sites in the cluster C. The parameters are then $(p,q,\{h_i\})$; q must be an integer, and we may omit $\{h_i\}$ when all external fields are 0. We need only consider $0 = h_1 \geq h_2 \geq \ldots \geq h_q$, so we henceforth assume this in our notation. Species i is called *stable* if h_i is maximal, i.e. $h_i = h_1 = 0$. For bond boundary conditions ρ we replace (1.12) with

$$\prod_{C \in \mathcal{C}(\omega|\rho)} \left((1-p)^{h_1 s(C)} + (1-p)^{h_2 s(C)} + \ldots + (1-p)^{h_q s(C)} \right), \tag{1.13}$$

where $\mathcal{C}(\omega \mid \rho)$ is the set of open clusters of $(\omega\rho)$ which intersect $V(\mathcal{B})$. If $\mathcal{B}(\mathbb{Z}^n)\backslash\mathcal{B}$ is connected then for stable i the wired boundary condition is equivalent to the site boundary condition η^i; we therefore refer to η^i as the *i-wired* boundary condition. For general site boundary conditions η for the model on $\overline{\mathcal{B}}(\Lambda)$ the factor (1.12) is multiplied by

$$\prod_{C \in \mathcal{C}_{int}(\omega)} \left((1-p)^{h_1 s(C)} + (1-p)^{h_2 s(C)} + \ldots + (1-p)^{h_q s(C)} \right) \tag{1.14}$$
$$\times \prod_{C \in \mathcal{C}_\partial(\omega)} (1-p)^{h_{i(C)} s(C)} \quad \times \quad \delta_{J(\Lambda,\eta)}(\omega),$$

where $\mathcal{C}_\partial(\omega)$ (respectively $\mathcal{C}_{int}(\omega)$) is the set of clusters in the configuration ω which do (respectively don't) intersect $\partial\Lambda$ and $i(C)$ is the species for which $\eta_x = i$ for all $i \in \partial\Lambda \cap C$. (The existence of such an i is forced by the event $J(\Lambda, \eta)$.) We call η a *single-stable-species boundary condition* if for some stable i, $\eta_x \in \{0, i\}$ for all x. For $q \geq 1$, the FK model with external fields satisfies the FKG lattice condition, under any bond or single-stable-species site boundary condition. It should be noted that a single-stable-species site boundary condition η is equivalent to a bond boundary condition in which all sites $x \in \partial\Lambda$ where $\eta_x \neq 0$ are part of a single infinite cluster in the boundary $\overline{\mathcal{B}}(\Lambda)^c$.

Let $p_c(q, n, \{h_i\})$ denote the percolation critical point of the FK model on $\mathbb{Z}^n$; we omit the $\{h_i\}$ when there are no external fields. The following facts are known for $n = 2$. For $q = 1, q = 2$, and $q \geq 25.72$, we have $p_c(q, 2) = \frac{\sqrt{q}}{1+\sqrt{q}}$ [22], and the connectivity decays exponentially for all $p < p_c(q, 2)$ [18]. This is believed to be true for all q; for $2 < q < 25.72$ the connectivity is known to decay exponentially at least for all $p < \frac{\sqrt{q-1}}{1+\sqrt{q-1}}$, and analogous results hold for other planar lattices [2]. For general $q \geq 1$, if the connectivity decays exponentially then the model has the ratio weak mixing property [5]. (This result is actually given assuming a nonnegative external field applied to at most one species, but the proof carries over without change to arbitrary external fields; the necessary FKG property is proved in [7].) From this and Theorem 1.1 we immediately get the following.

Theorem 1.2. *Let $P = P^{p,q,\{h_i\}}$ be an FK model on $\mathcal{B}(\mathbb{Z}^2)$ with $p < p_c(q, 2, \{h_i\})$, and suppose P has exponential decay of connectivities. Then P has uniform exponential decay of finite-volume connectivities for the class of all simply lattice-connected subsets of $\mathcal{B}(\mathbb{Z}^2)$ with arbitrary (site or bond) boundary condition, for both the Euclidean and restricted-path metrics.*

Theorem 1.2 is implicitly stated here, and later proved, for q an integer, but in the absense of external fields it is valid for noninteger q as well. The proof requires only minor modifications. Similar considerations hold for all our other results.

When long paths of open bonds exist for subcritical p, the standard heuristic picture is that the path has a "string of beads" structure, meaning that "blobs" of open bonds, containing many double connections between pairs of sites and having a linear scale on the order of the correlation length, are connected by short single paths of open bonds. Such a picture has been made rigorous in [9] and [10] for Bernoulli percolation. The next corollary establishes a small part of such a picture for the FK model in two dimensions, by showing that long double connections have an exponential cost over and above the cost of a single connection.

Corollary 1.3. *Let $P = P^{p,q,2,\{h_i\}}$ be an FK model on $\mathcal{B}(\mathbb{Z}^2)$ with $p < p_c(q, 2, \{h_i\})$, and suppose P has exponential decay of connectivities. Then there exists $\epsilon > 0$ such that for all $x \in \mathbb{Z}^2$, for $\theta = x/|x|$,*

$$P(\text{there exist two bond-disjoint open paths } 0 \leftrightarrow x) \leq e^{-(\tau(\theta)+\epsilon)|x|}.$$

Remark 1.4. For a bond percolation model on a finite set $\mathcal{B}$ of bonds under bond boundary condition ρ, influence can propagate not only along the boundary inside $\mathcal{B}$ but also through the exterior via ρ. For example, for the FK model, suppose $\langle uv \rangle, \langle xy \rangle \in \mathcal{B}$ and in ρ there are open paths from u to x and from v to y, and these two paths are in different clusters of the configuration ρ. That is, the two paths form a tunnel from $\langle uv \rangle$ to $\langle xy \rangle$. It is then straightforward to show that the events that $\langle uv \rangle$ and $\langle xy \rangle$ are open can have a correlation bounded away from 0 uniformly in the length of the tunnel. Tunnels of dual bonds can cause similar problems and effectively can exist even under site boundary conditions (see Example 1.5 below.) Nonetheless, restricting some of our results to site boundary conditions makes tunneling a manageable problem. As we have noted, single-stable-species site boundary conditions are equivalent to bond boundary conditions, and are thus effectively a natural class of bond boundary conditions which do not allow the tunneling phenomenon to be a problem. The results we state under such boundary conditions can be generalized to some other bond boundary conditions for which tunneling does not occur in such a way as to be a problem.

For $x \in \mathbb{R}^n$ and $l > 0$ let $Q_l(x)$ denote the closed cube of side $2l$ centered at x. Write $Q(x)$ for $Q_{1/2}(x)$, and for $\Lambda \subset \mathbb{Z}^n$ let $Q(\Lambda) = \cup_{x \in \Lambda} Q(x)$.

Example 1.5. Define bonds of the square lattice

$$b_n^+ = \langle (n, 0), (n + 1, 0) \rangle, \qquad b_n^- = \langle (-n - 1, 0), (-n, 0) \rangle$$

and consider the FK model on $\overline{\mathcal{B}}(Q_n(0))$, without external fields, at $(1 - \epsilon, q)$ where $\epsilon > 0$ and $q > 1$, with site boundary condition $\eta_x = 1$ if $x = (n + 1, 0)$ or $x = (-n - 1, 0)$, $\eta_x = 0$

otherwise. It is straightforward to verify the following statements:

$$P((-n,0) \leftrightarrow (n,0)) = 1 - O(\epsilon^3);$$
$$P(b_n^+ \text{ is closed} \mid (-n,0) \leftrightarrow (n,0)) = P(b_n^- \text{ is closed} \mid (-n,0) \leftrightarrow (n,0)) = \epsilon + O(\epsilon^2);$$
$$P(b_n^+ \text{ and } b_n^- \text{ are closed} \mid (-n,0) \leftrightarrow (n,0)) = q\epsilon^2 + O(\epsilon^3).$$

In all cases the $O(\cdot)$ is uniform in n. The first statement implies that the last two are true without the conditioning on $[(-n,0) \leftrightarrow (n,0)]$. It follows that when ϵ is sufficiently small, the correlation between the events $[b_n^+ \text{ is closed}]$ and $[b_n^- \text{ is closed}]$ does not decay to 0 as $n \to \infty$. The boundary condition η is equivalent to a bond boundary condition in which an open path connects $(-n-1,0)$ to $(n+1,0)$ outside $\overline{\mathcal{B}}(Q_n(0))$. Thus strong mixing fails for bond and for single-species-site boundary conditions, although exponential decay of dual connectivity and weak mixing both hold in infinite volume, and uniform exponential decay of finite-volume dual connectivities holds, by Theorem 1.2.

This same phenomenon persists in the presence of external fields, provided there is more than one stable species. In Theorem 1.6 we will see that when there is a unique stable species, we do obtain strong mixing.

Replacing regular bonds with dual bonds throughout this example (in the case of no external fields), including in the bond form of the boundary condition, we obtain an example in which there is exponential decay of connectivity and weak mixing in infinite volume, and uniform exponential decay of finite-volume connectivities, but strong mixing for general bond boundary conditions fails. This is an example of the tunneling phenomenon of Remark 1.4.

Given a metric d we call a class $\mathfrak{C}$ of subsets of $\mathcal{B}(\mathbb{Z}^n)$ *inheriting* with respect to d if for all $\mathcal{B} \in \mathfrak{C}, x \in V(\mathcal{B})$ and $r > 0$, the connected component of x in $\{e \in \mathcal{B} : d(e,x) \leq r\}$ is in $\mathfrak{C}$. The class of all simply lattice-connected subsets of $\mathcal{B}(\mathbb{Z}^2)$ is clearly inheriting, with respect to the Euclidean and restricted-path metrics.

We define a *closure subset* of $\mathcal{B}(\mathbb{Z}^n)$ to be a subset of form $\overline{\mathcal{B}}(\Lambda)$ for some finite Λ. These are the subsets which can take site boundary conditions. If $\mathcal{E} = \overline{\mathcal{B}}(\Lambda)$ is a simply lattice-connected closure subset of $\mathcal{B}(\mathbb{Z}^n)$, then $\mathcal{E}_{x,r} = \mathcal{E} \cap \{e \in \mathcal{B}(\mathbb{Z}^n) : d(e,x) \leq r\}$ is simply lattice-connected but is not typically a closure subset, so strictly speaking the class of all simply lattice-connected subsets of $\mathcal{B}(\mathbb{Z}^n)$ is not inheriting. However we can add a few bonds to $\mathcal{E}_{x,r}$—specifically, those in $\partial\mathcal{E}_{x,r} \backslash \partial\mathcal{B}(\Lambda)$—to create a closure subset. Classes which are "almost" inheriting in this sense work perfectly well in our proofs, so we tacitly treat them as if they were inheriting in what follows.

The next result, for the FK model in general dimension, says in effect that when paths cannot propagate in any of a class of finite regions, then neither can influence (of one event on another distant one) so propagate, under site boundary conditions, even when this influence is measured multiplicatively. This result is not restricted to 2 dimensions. The underlying idea for establishing mixing from uniform exponential decay occurs in [6] and [25], though without the "ratio" aspect, which appears in [5].

Theorem 1.6. *Let $P = P^{p,q,\{h_i\}}$ be the FK model at $(p, q, \{h_i\})$ on $\mathcal{B}(\mathbb{Z}^n)$ and suppose random-cluster uniqueness holds. Let d be either the Euclidean or restricted-path metric, and let $\mathfrak{C}$ be a class of closure subsets of $\mathcal{B}(\mathbb{Z}^n)$ which is inheriting with respect to d. Suppose P has uniform exponential decay of finite-volume connectivities for the class $\mathfrak{C}$ with wired boundary conditions, for the metric d. Then P has the ratio strong mixing property for the class $\mathfrak{C}$ and arbitrary site boundary conditions, for the metric d.*

For dimension $n = 2$, as in Theorem 1.2 we need not assume uniformity of the exponential decay to draw a similar conclusion. Specifically, we will prove the following.

Theorem 1.7. *Let $P = P^{p,q,\{h_i\}}$ be the FK model at $(p, q, \{h_i\})$ on $\mathbb{Z}^2$, and let $\mathfrak{C}$ be the class of all simply lattice-connected closure subsets of $\mathcal{B}(\mathbb{Z}^2)$. Suppose random-cluster uniqueness holds at $(p, q, \{h_i\})$. Let d be either the Euclidean or the restricted-path metric.*

(i) Suppose P has exponential decay of connectivities. Then P has the ratio strong mixing property for the class $\mathfrak{C}$ with arbitrary site boundary conditions.

(ii) Suppose P has exponential decay of dual connectivities. Then P has the ratio strong mixing property for the class $\mathfrak{C}$ with free and wired boundary conditions.

(iii) Suppose there is a unique stable species and P has exponential decay of dual connectivities. Then P has the ratio strong mixing property for the class $\mathfrak{C}$ with arbitrary site boundary conditions.

From Theorem 1.7(i) we see that for the FK model on $\mathcal{B}(\mathbb{Z}^2)$, exponential decay of connectivity in infinite volume is enough to ensure that in finite regions, influence cannot propagate along the boundary or through the bulk, under site boundary conditions.

Example 1.5 shows that significantly more general boundary conditions cannot be allowed in Theorem 1.7(ii), in contrast to Theorem 1.7(iii).

From [7], the assumption of random-cluster uniqueness in Theorem 1.7 is satisfied except possibly at the (unique) percolation critical point, which we denote $p_c(q, n, \{h_i\})$.

We turn our attention now to spin systems. We restrict attention mainly to the *q-state Potts model* (with possible external fields), which is the spin system with single-spin space $S = \{1, .., q\}$ and Hamiltonian

$$H_{\Lambda,\eta}(\sigma_\Lambda) = - \sum_{\langle xy \rangle \in \overline{\mathcal{B}}(\Lambda)} \delta_{[(\sigma\eta)_\Lambda(x) = (\sigma\eta)_\Lambda(y)]} - \sum_{i=1}^{q} \sum_{x \in \Lambda} h_i \delta_{[\sigma_x = i]}$$

for the model on Λ with boundary condition η; the corresponding finite-volume Gibbs distribution at inverse temperature β is given by

$$\mu_{\Lambda,\eta}^{\beta,\{h_i\}}(\sigma_\Lambda) = \frac{1}{Z_{\Lambda,\eta}^{\beta,\{h_i\}}} e^{-\beta H_{\Lambda,\eta}(\sigma_\Lambda)}, \quad \sigma_\Lambda \in S^\Lambda$$

where $Z_{\Lambda,\eta}^{\beta,\{h_i\}}$ is the partition function. We denote the critical inverse temperature of the model on $\mathbb{Z}^n$, without external fields, by $\beta_c(q,d)$.

As shown by Edwards and Sokal in [12], for β given by $p = 1 - e^{-\beta}$, a configuration of the q-state Potts model (without external fields) on Λ with boundary condition η at inverse temperature β can be obtained from a configuration ω of the FK model at (p,q) with site boundary condition η, by choosing a label for each non-boundary cluster of ω independently and uniformly from $\{1,...,q\}$. For a cluster intersecting $\{x \in \partial\Lambda : \sigma_x = i\}$ (which can happen for at most one i), all sites are labeled i. This *cluster-labeling construction* yields a joint site-bond configuration, which we call an *Edwards-Sokal configuration*, in which the sites are a Potts model and the bonds are an FK model. When the parameters are related in this way, we call the Potts and FK models *corresponding*. Alternatively, if one selects a Potts configuration σ_Λ and does independent percolation at density p on the set of bonds

$$\{\langle xy\rangle \in \overline{\mathcal{B}}(\Lambda) : (\sigma\eta)_x = (\sigma\eta)_y\},$$

the resulting bond configuration is a realization of the corresponding FK model. We call this the *percolation construction* of the FK model. For the q-state Potts model (without external fields) at inverse temperature $\beta < \beta_c(q,n)$, for $p = 1 - e^{-\beta}$ and the FK model at (p,q), the covariance in the Potts model and the connectivity in the FK model are related by

$$q^2 \operatorname{cov}(\delta_{[\sigma_0=i]}, \delta_{[\sigma_x=i]}) = (q-1)P(0 \leftrightarrow x), \quad i = 1,..,q; \tag{1.15}$$

see [1] or [17]. Thus exponential decay of connectivities in the FK model is equivalent to exponential decay of correlations in the corresponding Potts model. Further, we have

$$p_c(q,d) = 1 - e^{-\beta_c(q,n)}; \tag{1.16}$$

again see [1] or [17]. When external fields are present, we can take (1.16) as the definition of an inverse temperature $\beta_c(q,d,\{h_i\})$, that is,

$$p_c(q,n,\{h_i\}) = 1 - e^{-\beta_c(q,n,\{h_i\})}. \tag{1.17}$$

This $\beta_c(q,n,\{h_i\})$ is not necessarily a true critical point, however, if there is a unique stable species.

Under external fields $\{h_i\}$, the cluster-labeling construction is modified as follows. Let $s(C)$ denote the number of sites in a cluster C. For non-boundary clusters C, the spin i is chosen for the cluster C with probability proportional to $e^{\beta h_i s(C)}$.

The (ratio) weak mixing and (ratio) strong mixing properties extend straightforwardly to spin systems. We do not write out a separate definition explicitly; see [2].

The following is the Potts-model analog of Theorem 1.7.

Theorem 1.8. *Let $\mu^{\beta,q,\{h_i\}}$ be the q-state Potts model at $(\beta, \{h_i\})$ on $\mathbb{Z}^2$. Let $\mathfrak{C}$ be the class of all finite simply lattice-connected subsets of $\mathbb{Z}^2$ with arbitrary boundary condition, and let d be either the Euclidean or the restricted-path metric.*

(i) If there are multiple stable species, $\beta < \beta_c(q, 2, \{h_i\})$ and $\mu^{\beta,q,\{h_i\}}$ has exponential decay of correlations, then it has the ratio strong mixing property for the class $\mathfrak{C}$.

(ii) If $\beta < \beta_c(q, 2, \{h_i\})$ and the corresponding FK model has exponential decay of connectivities, then $\mu^{\beta,q,\{h_i\}}$ has the ratio strong mixing property for the class $\mathfrak{C}$.

(iii) If there is a unique stable species, $\beta > \beta_c(q, 2, \{h_i\})$ and the corresponding FK model has exponential decay of dual connectivities, then $\mu^{\beta,q,\{h_i\}}$ has the ratio strong mixing property for the class $\mathfrak{C}$.

Note that in Theorem 1.8 we are implicitly viewing $\mathfrak{C}$ as a class of pairs consisting of a set and a boundary condition on that set, rather than just as a class of sets. We will use each meaning of $\mathfrak{C}$ at various times; which one we are using should be clear from the context.

From the relation (1.16) and the remarks preceding Theorem 1.2, we see that the hypothesis in Theorem 1.8(i) of exponential decay of correlations is known to be satisfied whenever $e^\beta < e^{\beta_c} = 1 + \sqrt{q}$ if $q = 2$ or $q \geq 26$, and whenever $e^\beta < 1 + \sqrt{q-1}$ if $3 \leq q \leq 25$, in the absense of external fields.

The Potts-model analog of Theorem 1.6 is contained in the following theorem.

Theorem 1.9. *Consider the q-state Potts model $\mu^{\beta,q,\{h_i\}}$ on $\mathbb{Z}^n$ at inverse temperature β with external fields h_i. Let d be either the Euclidean or the restricted path metric and let $\mathfrak{D}$ be a class of subsets of $\mathbb{Z}^n$ such that the class $\mathfrak{C} = \{\overline{\mathcal{B}}(\Lambda) : \Lambda \in \mathfrak{D}\}$ is inheriting with respect to d. Suppose that the corresponding FK model has uniform exponential decay of finite-volume connectivities for the class $\mathfrak{C}$ and wired boundary condition, for the metric d. Then $\mu^{\beta,q,\{h_i\}}$ has the ratio strong mixing property for the class $\mathfrak{D}$ and arbitrary boundary conditions, for the same metric.*

Remark 1.10. The notion of a "corresponding random cluster model" makes sense in the context of any random cluster representation of any spin system. The analog of Theorem 1.9 will be valid for a nearest-neighbor spin system provided the following conditions are satisfied. (1) There exists a bond percolation model P on $\mathbb{Z}^n$ such that for every finite Λ, $P_{\overline{\mathcal{B}}(\Lambda),w}$ FKG-dominates the random cluster representation on $\overline{\mathcal{B}}(\Lambda)$ with arbitrary spin boundary condition, and P has uniform exponential decay of finite-volume connectivities. (In Theorem 1.9, P is the FK model itself.) (2) The joint site-bond configuration obtained in the cluster-labeling construction has the following Markov property for open dual surfaces: for every $\Delta \subset \Lambda$, every boundary condition η on $\partial\Lambda$, and every configuration σ_Δ on Δ, given σ_Δ and given that all bonds in $\overline{\mathcal{B}}(\Lambda) \cap \partial\mathcal{B}(\Delta)$ are closed, the configuration $\sigma_{\Lambda\backslash\Delta}$ has distribution corresponding to the boundary condition on $\partial(\Lambda\backslash\Delta)$ which is given by η_x for $x \in \partial(\Lambda\backslash\Delta) \cap \partial\Lambda$, free for $x \in \partial(\Lambda\backslash\Delta) \cap \Lambda$. Condition (2) is valid provided that there is no interaction between clusters in the random cluster representation, that is, the weight assigned to a bond configuration is a product of weights assigned to clusters. This is true for the FK model (with or without external fields) and for the random cluster representations of the models given studied in [6], which are given as follows: the single-spin space S is a compact group, $\mathcal{E} : S \to \mathbb{R}$ is a function, and the Hamiltonian is

$$H_{\Lambda,\eta}(\sigma_\Lambda) = -\frac{1}{2} \sum_{\langle xy\rangle \in \overline{\mathcal{B}}(\Lambda)} \mathcal{E}(\sigma_x^{-1}\sigma_y). \tag{1.18}$$

This includes the Potts model without external fields.

2. Proofs

Throughout the paper $c_1, c_2, ...$ and $\epsilon_1, \epsilon_2, ...$ represent constants which depend only on the infinite-volume model under discussion, with ϵ_i used for "sufficiently small" constants. d_p denotes the l^p metric.

We begin with a result needed for the proof of Theorem 1.1. For $\mathcal{B} \subset \mathcal{B}(\mathbb{Z}^n)$ and $r > 0$, let

$$\mathcal{B}_r = \{e \in \mathcal{B} : d(e, \mathcal{B}(\mathbb{Z}^n \backslash \mathcal{B})) \geq r\}.$$

By a *chain of l-cubes* (from x_1 to x_k) we mean a finite sequence $Q_l(x_1), .., Q_l(x_k)$ of cubes with disjoint interiors, with $x_i \in l\mathbb{Z}^n$ for all $i \leq k$ and $d_\infty(x_i, x_{i+1}) = 2l$ for all $i \leq k-1$.

Proposition 2.1. *Let P be a translation-invariant bond percolation model on $\mathbb{Z}^n$ having exponential decay of connectivities and the weak mixing property. Then for each sufficiently large r there exist $C_r, \lambda_r > 0$ such that for all $\mathcal{B} \subset \mathcal{B}(\mathbb{Z}^n)$, all $x, y \in V(\mathcal{B}_r)$, and every boundary condition $\rho \in \{0,1\}^{\mathcal{B}_r^c}$,*

$$P_{\mathcal{B},\rho}(x \leftrightarrow y \text{ via a path in } \mathcal{B}_r) \leq C_r e^{-\lambda_r d_2(x,y)}.$$

Proof. From the weak mixing property, there exist $c_1, c_2, \epsilon_1, \epsilon_2 > 0$ such that if $z \in \mathbb{Z}^n$ and $r \in \mathbb{Z}$ is sufficiently large, then for every boundary condition θ on $\big(\mathcal{B} \cap \mathcal{B}(Q_{2r}(z))\big)^c$,

$$\begin{aligned} &P_{\mathcal{B}\cap\mathcal{B}(Q_{2r}(z)),\theta}\big(\text{there is an open path in } \mathcal{B}_r \cap \mathcal{B}(Q_r(z)) \text{ from } u \text{ to } v \\ &\qquad\qquad \text{for some } u, v \text{ with } d_2(u,v) \geq r/2\big) \\ &\leq P\big(\text{there is an open path in } \mathcal{B}_r \cap \mathcal{B}(Q_r(z)) \text{ from } u \text{ to } v \\ &\qquad\qquad \text{for some } u, v \text{ with } d_2(u,v) \geq r/2\big) + c_1 e^{-\epsilon_1 r} \\ &\leq c_2 e^{-\epsilon_2 r}. \end{aligned} \tag{2.1}$$

For fixed r we call a cube $Q_{2r}(z)$ *bad* if the event on the left side of (2.1) occurs, and *good* otherwise. We define a site percolation process on $\mathbb{Z}^n$ by setting $\xi_q = 1$ if $Q_{2r}(2rq)$ is bad, and $\xi_q = 0$ otherwise. By (2.1), for every set of cubes $Q_{2r}(2rq_1), .., Q_{2r}(2rq_k)$ having disjoint interiors, with $q_i \in \mathbb{Z}^n$, the configuration $\{\xi_{q_i} : i \leq k\}$ is FKG-dominated by Bernoulli site percolation at density $c_2 e^{-\epsilon_2 r}$. In particular, the probability for all $2r$-cubes in a given chain of length k to be bad is at most $(c_2 e^{-\epsilon_2 r})^k$.

If $d_2(x, y) \geq r/2$ and $x \leftrightarrow y$ via a path in $\mathcal{B}_r$, it is easy to see that there exist $z, w \in 2r\mathbb{Z}^n$ such that $x \in Q_{2r}(w), y \in Q_{2r}(z)$, and there is a chain of bad $2r$-cubes from z to w. The number of chains of $2r$-cubes from z to w of length k is at most 5^{dk}, so the proposition now follows from a routine Peierls-type argument. □

The next lemma establishes a weaker analog of Theorem 1.1, which we will later bootstrap using renormalization.

Lemma 2.2. *Let P be a translation-invariant bond percolation model on $\mathcal{B}(\mathbb{Z}^2)$ having bounded energy, exponential decay of connectivities and the weak mixing property. Let $\mathfrak{C}$ be the class of all finite simply lattice-connected subsets of $\mathcal{B}(\mathbb{Z}^2)$. There exist constants c_i, ϵ_3 such that for all $\mathcal{B} \in \mathfrak{C}$, all boundary conditions ρ and all $x, y \in V(\mathcal{B})$ with $d_2(x, y) \geq c_3$,*

$$P_{\mathcal{B},\rho}(x \leftrightarrow y) \leq c_4 e^{-\epsilon_3 d_2(x,y)/\log d_2(x,y)}.$$

Proof. Let r, C_r, λ_r be as in Proposition 2.1, and fix $x, y, \mathcal{B}$ with $d_2(x, y) \geq 30r$ and $x, y \in V(\mathcal{B})$. Let γ be the lattice dual circuit which is the outer boundary of $Q(V(\mathcal{B}))$. Let x', y' be the dual sites in γ closest to x and y, respectively (breaking ties arbitrarily), and let γ_R and γ_L denote the segments of γ from x' to y' (counterclockwise) and from y' to x', respectively. Let $\epsilon > 0$.

Case 1: Either $d_2(x, \gamma) \geq \frac{1}{10} d_2(x, y)$ or $d_2(y, \gamma) \geq \frac{1}{10} d_2(x, y)$; we may assume the former. Then by Proposition 2.1, if $d_2(x, y)$ is sufficiently large (depending on r), then for every boundary condition ρ,

$$\begin{aligned}(2.2)\qquad & P_{\mathcal{B},\rho}(x \leftrightarrow y) \\ & \leq P_{\mathcal{B},\rho}\big(x \leftrightarrow z \text{ via a path in } \mathcal{B}_r \text{ for some } z \text{ with } d_2(x, z) \geq \frac{1}{20} d_2(x, y)\big) \\ & \leq c_5 d_2(x, y) C_r e^{-\lambda_r d_2(x,y)/20}.\end{aligned}$$

Case 2:

$$\max\big(d_2(x, \gamma), d_2(y, \gamma)\big) < \frac{1}{10} d_2(x, y).$$

Let $\mathcal{S}$ denote the set of bonds in $\mathcal{B}^c$ which are surrounded by γ. Since $\mathcal{B}$ is simply lattice-connected, every dual bond in $\mathcal{S}^*$ must be connected to exactly one of γ_L, γ_R, by a path in $\mathcal{S}^*$.

This partitions $\mathcal{S}^*$ into two subsets, call them $\mathcal{S}^*_L$ and $\mathcal{S}^*_R$, and we define $\tilde\gamma_L = \gamma_L \cup \mathcal{S}^*_L, \tilde\gamma_R = \gamma_R \cup \mathcal{S}^*_R$. For $k \geq 1$ let

$$\Omega_k = \{z \in V(\mathcal{B}) : (k-1)c_6 \log d_2(x,y) < d_2(z,x') \leq kc_6 \log d_2(x,y)\},$$

where $c_6 = c_6(r)$ is a (large) constant to be specified, and let

$$\mathcal{D}_k = \{\langle uv\rangle \in \mathcal{B} : u \in \Omega_k, v \in \Omega_{k+1}\}.$$

We may assume x, y satisfy

$$\log d_2(x,y) > \frac{4r}{c_6}.$$

Let $k_{\min}-1$ and $k_{\max}$ be the smallest and largest integers k, respectively, for which $d_2(x',x) < kc_6 \log d_2(x,y) < d_2(x',y)$. Then

$$k_{\max} - k_{\min} \geq \frac{d_2(x,y)}{2c_6 \log d_2(x,y)}. \tag{2.3}$$

For each $k_{\min} \leq k \leq k_{\max}$, every path from x to y must pass through $\mathcal{B}(\Omega_k)$, and $\overline{\mathcal{B}}(\Omega_k)^*$ must include bonds of both $\tilde\gamma_L$ and $\tilde\gamma_R$. Among the connected components of $\mathcal{D}^*_k$ there is at least one which has one endpoint in $\tilde\gamma_L$ and the other in $\tilde\gamma_R$; let us call such a component *k-crossing.* Among the k-crossing components there is one, which we denote $\mathcal{E}^*_k$, with the property that for every path φ from x to y, $\mathcal{E}^*_k$ is the last k-crossing crossed by φ. Now $\mathcal{E}^*_{k-1}$ and $\mathcal{E}^*_k$ divide $\mathcal{B}$ into 3 pieces, and we call the middle one $\mathcal{C}_k$; more precisely, $\mathcal{C}_k$ is the connected component of $\mathcal{B}\backslash(\mathcal{E}_{k-1} \cup \mathcal{E}_k)$ for which both $\mathcal{E}_{k-1} \subset \partial\mathcal{C}_k$ and $\mathcal{E}_k \subset \partial\mathcal{C}_k$. For $t > 0$ let

$$\Xi^{L,t}_k = \{z \in V(\mathcal{C}_k) : d_1(z, \tilde\gamma_L) \leq t\}, \quad \Xi^{R,t}_k = \{z \in V(\mathcal{C}_k) : d_1(z, \tilde\gamma_R) \leq t\},$$

$$\Delta^+_k = V(\mathcal{C}_k) \cap V(\mathcal{E}_k), \quad \Delta^-_k = V(\mathcal{C}_k) \cap V(\mathcal{E}_{k-1}),$$

and

$$\hat{\mathcal{C}}_k = \mathcal{C}_{k-1} \cup \mathcal{D}_{k-1} \cup \mathcal{C}_k \cup \mathcal{D}_k \cup \mathcal{C}_{k+1}.$$

Case 2a. $\Xi^{L,3r}_k \cap \Xi^{R,3r}_k \neq \phi$. Then there is a dual path of length at most $6r$ from $\tilde\gamma_L$ to $\tilde\gamma_R$ in $\hat{\mathcal{C}}^*_k$. If this dual path is open, then there can be no open path in $\hat{\mathcal{C}}_k$ from Δ^-_{k-1} to Δ^+_{k+1}. For p_0 as in (1.1) we therefore have for all boundary conditions ρ and all bond configurations $\theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}$:

$$P_{\mathcal{B},\rho}\big(\Delta^-_{k-1} \leftrightarrow \Delta^+_{k+1} \text{ via a path in } \hat{\mathcal{C}}_k \mid \omega_{\mathcal{B}\backslash\hat{\mathcal{C}}_k} = \theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}\big) \leq 1 - p_0^{6r}. \tag{2.4}$$

Case 2b. $\Xi^{L,3r}_k \cap \Xi^{R,3r}_k = \phi$. Let $\mathcal{A}_k = \mathcal{C}_k\backslash\overline{\mathcal{B}}(\Xi^{L,2r}_k \cup \Xi^{R,2r}_k)$. Then there exist paths from x to y in $\mathcal{B}$ outside $\overline{\mathcal{B}}(\Xi^{L,2r}_k \cup \Xi^{R,2r}_k)$, and there is a unique connected component $\mathcal{F}^*_k$ of $\mathcal{E}^*_{k-1} \cap \partial Q(V(\mathcal{A}_k))$ such that all such paths cross $\mathcal{F}^*_k$. Necessarily $\mathcal{F}^*_k$ has one endpoint in

$\partial Q(\Xi_k^L)$ and the other in $\partial Q(\Xi_k^R)$. From duality, there are two possibilities: either there is an open path from Δ_k^- to Δ_k^+ in $\mathcal{A}_k$, or there is an open dual path from $\partial Q(\Xi_k^L)$ to $\partial Q(\Xi_k^R)$ in $\mathcal{A}_k^*$. Since $\mathcal{A}_k \subset \mathcal{B}_r$, it follows from Proposition 2.1 that, provided c_6 is large, for all boundary conditions ρ and all bond configurations $\theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}$,

$$\begin{aligned} \text{(2.5)} \qquad & P_{\mathcal{B},\rho}\big(\partial Q(\Xi_k^{L,2r}) \overset{*}{\leftrightarrow} \partial Q(\Xi_k^{R,2r}) \text{ via a dual path in } \mathcal{A}_k^* \mid \omega_{\mathcal{B}\backslash\hat{\mathcal{C}}_k} = \theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}\big) \\ &\geq 1 - P_{\mathcal{B},\rho}\big(\Delta_k^- \leftrightarrow \Delta_k^+ \text{ via a path in } \mathcal{A}_k \mid \omega_{\mathcal{B}\backslash\hat{\mathcal{C}}_k} = \theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}\big) \\ &\geq 1 - |\Delta_k^-||\Delta_k^+|C_r e^{-\lambda_r c_6 (\log d_2(x,y))/2} \\ &\geq 1 - (c_4 d_2(x,y))^2 C_r e^{-\lambda_r c_6 (\log d_2(x,y))/2} \\ &\geq \frac{1}{2}. \end{aligned}$$

When there is an open dual path in $\mathcal{A}_k^*$ from $\partial Q(\Xi_k^L)$ to $\partial Q(\Xi_k^R)$, there is a unique such path, which we denote α_k, which is "closest to $\mathcal{F}_k^*$" (analogous to the "lowest occupied crossing" of e.g. [20].) Let Y_k^L and Y_k^R denote the (random) initial and final sites, respectively, of α_k.

For every dual site u in $\partial Q(\Xi_k^{L,2r}) \cap V(\mathcal{A}_k^*)$ there is a dual path of length at most $2r+2$ from u to $\tilde{\gamma}_L$ outside $\mathcal{A}_k^*$; we denote this path by β_u^L, making an arbitrary choice of β_u^L if more than one is possible. We define β_u^R analogously for u in $\partial Q(\Xi_k^R) \cap V(\mathcal{A}_k^*)$. Conditionally on the connection event on the left side of (2.5), the probability that $\beta_{Y_k^L}^L$ and $\beta_{Y_k^R}^R$ are both open dual paths is at least p_0^{4r+4}. Further, when these two dual paths are both open, there is an open dual path in $\hat{\mathcal{C}}_k^*$ from $\tilde{\gamma}_L$ to $\tilde{\gamma}_R$. From this and (2.5) we conclude that for all boundary conditions ρ and all bond configurations $\theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}$,

$$\begin{aligned} & P_{\mathcal{B},\rho}\big(\tilde{\gamma}_L \overset{*}{\leftrightarrow} \tilde{\gamma}_R \text{ via a dual path in } \hat{\mathcal{C}}_k^* \mid \omega_{\mathcal{B}\backslash\hat{\mathcal{C}}_k} = \theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}\big) \\ &\quad \geq p_0^{4r+4} P_{\mathcal{B},\rho}\big(\partial Q(\Xi_k^L) \overset{*}{\leftrightarrow} \partial Q(\Xi_k^R) \text{ via a dual path in } \mathcal{A}_k^* \mid \omega_{\mathcal{B}\backslash\hat{\mathcal{C}}_k} = \theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}\big) \\ &\quad \geq \frac{1}{2} p_0^{4r+4}. \end{aligned}$$

Using duality again, this shows that, again for all boundary conditions ρ and all bond configurations $\theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}$,

$$\text{(2.6)} \qquad P_{\mathcal{B},\rho}\big(\Delta_k^- \leftrightarrow \Delta_k^+ \text{ via a path in } \hat{\mathcal{C}}_k \mid \omega_{\mathcal{B}\backslash\hat{\mathcal{C}}_k} = \theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}\big) \leq 1 - \frac{1}{2} p_0^{4r+4}.$$

We have shown that provided r is large, in both Cases 2a and 2b,

$$\text{(2.7)} \qquad P_{\mathcal{B},\rho}\big(\Delta_{k-1}^- \leftrightarrow \Delta_{k+1}^+ \text{ via a path in } \hat{\mathcal{C}}_k \mid \omega_{\mathcal{B}\backslash\hat{\mathcal{C}}_k} = \theta_{\mathcal{B}\backslash\hat{\mathcal{C}}_k}\big) \leq 1 - p_0^{6r}.$$

Let G_j denote the event $[\Delta_{3j-1}^- \leftrightarrow \Delta_{3j+1}^+$ via a path in $\hat{\mathcal{C}}_{3j}]$ and let $j_{\min}$ and $j_{\max}$ be the smallest and largest integers, respectively, in $\{j \in \mathbb{Z} : k_{\min} < 3j < k_{\max}\}$. From (2.7) we have

$$P_{\mathcal{B},\rho}(G_j \mid G_{j_{\min}} \cap .. \cap G_{j-1}) \leq 1 - \frac{1}{2} p_0^{4r+4} \quad \text{for all } j_{\min} \leq j \leq j_{\max},$$

and hence

$$P_{\mathcal{B},\rho}(x \leftrightarrow y) \leq P_{\mathcal{B},\rho}(\cap_{j=j_{\min}}^{j_{\max}} G_j) \leq (1 - \frac{1}{2} p_0^{4r+4})^{j_{\max}-j_{\min}+1}.$$

From (2.3), provided $d_2(x,y)$ is sufficiently large,

$$j_{\max} - j_{\min} + 1 \geq \frac{1}{3}(k_{\max} - k_{\min}) - 2 \geq \frac{d_2(x,y)}{7c_6 \log d_2(x,y)},$$

and the lemma follows from this and (2.2). □

We call a subset Λ of $\mathbb{Z}^n$ *l^∞-connected* if for all $x, y \in \Lambda$ there is a sequence of sites $x = x_1, \dots, x_n = y$ in Λ with $d_\infty(x_i, x_{i+1}) \leq 1$ for all i.

Proof of Theorem 1.1. From Lemma 2.2 we see that given $\epsilon > 0$ there exists l such that for all $\mathcal{B} \in \mathfrak{C}$, all boundary conditions ρ, and all $z \in \mathbb{Z}^2$,

$$P_{\mathcal{B} \cap \mathcal{B}(Q_l(z) \cap \mathbb{Z}^2),\rho}\big(u \leftrightarrow v \text{ for some } u, v \in Q_l(z) \text{ with } d_2(u,v) \geq \frac{l}{4}\big) < \epsilon. \tag{2.8}$$

Fix $\mathcal{B} \in \mathfrak{C}$. For $z \in l\mathbb{Z}^2$ we call $Q_l(z)$ *bad* if the event on the left side of (2.8) occurs, and *good* otherwise. As in the proof of Proposition 2.1, for disjoint cubes the configuration of bad cubes is FKG-dominated by Bernoulli site percolation at density ϵ. Further, if $x \leftrightarrow y$ for some x, y with $d_2(x,y) \geq 8l$, then there is an chain of bad l-cubes from w to z, of length at least $d_2(x,y)/4l$, where w, z are sites in $l\mathbb{Z}^2$ with $x \in Q_l(w), y \in Q_l(z)$. Provided ϵ is sufficiently small, a routine Peierls argument therefore gives that for all boundary conditions θ and all x, y with $d_2(x,y) \geq 8l$,

$$P_{\mathcal{B},\theta}(x \leftrightarrow y) \leq c_7 e^{-d_2(x,y)/4l}.$$

This proves the theorem for the metric d_2.

Next we consider $d_{\mathcal{B}}$. Suppose $x \leftrightarrow y$ for some x, y with $d_{\mathcal{B}}(x,y) \geq 15l^2$. Fix $z \in l\mathbb{Z}^2$ with $x \in Q_{l/2}(z)$ and let A be the connected component of $Q_l(z)$ in $\cup\{Q_l(w) : w \in l\mathbb{Z}^2, Q_l(w) \text{ is bad}\}$. Let

$$\Theta = \{w \in \mathbb{Z}^2 : Q_l(lw) \subset A, Q_l(lw) \text{ is bad}\}.$$

Note that since any one bond is contained in at most 4 sets $Q_l(lw)$ with $w \in \mathbb{Z}^2$, there must be a subset $f(\Theta) \subset \Theta$ with $|f(\Theta)| \geq |\Theta|/4$ and $\{Q_l(lw), w \in f(\Theta)\}$ mutually disjoint. Let C_x be the open cluster of x and $A^t = \{u \in \mathbb{R}^2 : d_2(u, A) \leq t\}$; then

$$C_x \subset A^{l/4}, \qquad |\{\text{bonds in } C_x\}|\ geq d_{\mathcal{B}}(x,y).$$

There are at most $9l^2$ bonds in a cube $Q_l(w)$, and hence at most $9|\Theta|l^2$ in $\mathcal{B}(A)$ and at most $15|\Theta|l^2$ in $\mathcal{B}(A^{l/4})$. Therefore $|\Theta| \geq d_{\mathcal{B}}(x,y)/15l^2$. Now Θ must be an l^∞-connected subset of $\mathbb{Z}^2$ with $z \in \Theta$. The number of possible such lattice animals Θ with $|\Theta| = n$ is at most c_8^n for all n, by the argument of ([21], p. 85.) Thus provided ϵ is sufficiently small we have, again by a routine Peierls-type argument,

$$P_{\mathcal{B},\rho}(x \leftrightarrow y) \leq P_{\mathcal{B},\rho}\left(|\Theta| \geq \frac{d_{\mathcal{B}}(x,y)}{15l^2}\right) \leq P_{\mathcal{B},\rho}\left(|f(\Theta)| \geq \frac{d_{\mathcal{B}}(x,y)}{60l^2}\right) \leq c_9 e^{-\epsilon_4 d_{\mathcal{B}}(x,y)/60l^2},$$

which proves the theorem for the metric $d_{\mathcal{B}}$. □

Let $B_\tau(x,r)$ denote the τ-ball of radius r centered at x. Consider distinct points $x, y \in \mathbb{R}^2$ with $|y-x| \geq 4\sqrt{2}$. We let $S(x,y)$ denote the closed slab between the tangent line to $\partial B_\tau(x, \tau(y-x))$ at y and the parallel line through x; we call $S(x,y)$ the *natural slab* of x and y. (Note the tangent line is not necessarily unique; if it is not we make some arbitrary choice.) Due to the 8-fold symmetry of the lattice, $S(x,y)$ makes an angle of not more than 45° with $y-x$. Note that if u, v are on opposite sides of $S(x,y)$, then

$$\tau(v-u) \geq \tau(y-x). \tag{2.9}$$

Proof of Corollary 1.3. Fix x and let R be the closed parallelogram of which two of the sides are segments of $\partial S(0,x)$ of length $8|x|$ centered at 0 and x respectively. Note the other two sides are parallel to x. We denote the sides containing 0 and x by l_0 and l_x, respectively. We view l_0 as the left side of the parallelogram, which specifies which short side is the bottom. Let Γ_R denote the lowest open crossing of R from l_0 to l_x, when an open crossing exists. Given a path γ from l_0 to l_x, we let U_γ denote the portion of R which is strictly above γ, and let $V_{\gamma,0}$ (respectively $V_{\gamma,x}$) denote the set of sites in $V(\mathcal{B}(U_\gamma))$ which are endpoints of bonds intersecting l_0 (respectively l_x.) By Theorem 1.2 there exists $\epsilon > 0$ such that for each such γ we have

$$P_{\mathcal{B}(U_\gamma),w}(V_{\gamma,0} \leftrightarrow V_{\gamma,x}) \leq e^{-2\epsilon|x|}.$$

Therefore provided x is sufficiently large,

$$
\begin{aligned}
(2.10)\qquad & P(\text{there exist two bond-disjoint open paths } 0 \leftrightarrow x) \\
&\leq P(0 \leftrightarrow y \text{ for some } y \text{ with } |y| \geq 2|x|) \\
&\quad + P(\text{there exist two bond-disjoint open paths } l_0 \leftrightarrow l_x \text{ in } R) \\
&\leq c_{10} e^{-\frac{3}{2}\tau(\theta)|x|} + \sum_{\gamma} P(\Gamma_R = \gamma; l_0 \leftrightarrow l_x \text{ via an open path in } U_\gamma) \\
&\leq c_{10} e^{-\frac{3}{2}\tau(\theta)|x|} + \sum_{\gamma} P(\Gamma_R = \gamma) P_{\mathcal{B}(U_\gamma),w}(V_{\gamma,0} \leftrightarrow V_{\gamma,x}) \\
&\leq c_{10} e^{-\frac{3}{2}\tau(\theta)|x|} + e^{-2\epsilon|x|} P(l_0 \leftrightarrow l_x) \\
&\leq c_{10} e^{-\frac{3}{2}\tau(\theta)|x|} + e^{-(\tau(\theta)+\epsilon)|x|}
\end{aligned}
$$

where the third inequality uses the FKG property and the fifth uses (2.9), and the sums are over all self-avoiding paths γ from l_0 to l_x. This completes the proof. □

We turn next to the proof of Theorem 1.6. We first establish strong mixing, and later obtain ratio strong mixing as a consequence. We need to use certain variants of the Markov property. A *dual plaquette* is a face of a hypercube $Q(x)$ for some $x \in \mathbb{Z}^n$. As with dual bonds, a dual plaquette is defined to be open precisely when the bond which it perpendicularly bisects is closed. For a set A of bonds (or dual plaquettes), let Open(A) denote the event that all bonds (or dual plaquettes) in A are open. By a *dual surface* we mean a set of dual plaquettes which is the outer boundary of $Q(\Delta)$ for some finite $\Delta \subset \mathbb{Z}^n$ for which the interior of $Q(\Delta)$ is connected. We say a bond percolation model P has the *Markov property for open dual surfaces* if for every dual surface S, the bond configurations inside and outside S are independent given the event Open(S). The FK model with arbitrary external fields has this property. In two dimensions, the *Markov property for open circuits* (of regular bonds) is defined analogously; the FK model has this property if and only if there are no external fields (see [4].) However, we can come close to the Markov property even under external fields: if P is the infinite-volume k-wired FK model on $\mathcal{B}(\mathbb{Z}^2)$ for some stable k, then letting ω_{int} and ω_{ext} denote the bond configurations inside and outside γ, respectively, we have (see [4]) for some $C, a > 0$

$$
\begin{aligned}
(2.11)\qquad & \sup\left\{ \left| \frac{P(\omega_{int} \in A \mid \text{Open}(\gamma), \omega_{ext} \in B)}{P(\omega_{int} \in A \mid \text{Open}(\gamma))} - 1 \right| : A \in \mathcal{G}_{\mathcal{B}(\text{Int}(\gamma))}, B \in \mathcal{G}_{\mathcal{B}(\text{Ext}(\gamma))} \right\} \\
&\quad \leq Ce^{-a|\gamma|} \qquad \text{for all } \gamma
\end{aligned}
$$

where Int and Ext denote the interior and exterior, respectively. When (2.11) holds we say P has the *near-Markov property for open circuits.*

A *coupling* of two measures P_1 and P_2 on $\{0,1\}^{\mathcal{B}}$ is a measure $\mathbb{P}$ on $\{0,1\}^{\mathcal{B}} \times \{0,1\}^{\mathcal{B}}$ with marginals P_1 and P_2 (in order.) A standard way of constructing couplings (see [5], [6], [25]) is via what we call a *construction algorithm*, which is a rule specifying for each subset $\mathcal{E}$ of $\mathcal{B}$ and each pair $(\omega^1_{\mathcal{E}}, \omega^2_{\mathcal{E}})$ of configurations on $\mathcal{E}$ a choice of a bond $b = b(\mathcal{E}, \omega^1_{\mathcal{E}}, \omega^2_{\mathcal{E}})$ and a choice of a "single-bond" coupling of $P_1(\omega_b = \cdot \mid \omega^1_{\mathcal{E}})$ and $P_2(\omega_b = \cdot \mid \omega^2_{\mathcal{E}})$ on $\{0,1\}^2$. In particular there is an initial bond $b_1 = b(\phi)$ and an initial single-bond coupling on b_1. We construct a coupled pair of configurations by first choosing $(\omega^1_{b_1}, \omega^2_{b_1})$ under the initial single-bond coupling, then applying the rule to determine both the second bond $b_2 = b(\{b_1\}, \omega^1_{b_1}, \omega^2_{b_1})$ and the single-bond coupling on b_2, then choosing $(\omega^1_{b_2}, \omega^2_{b_2})$ using this single-bond coupling on b_2, and iterating in this manner until the entire configuration is constructed. We let $\mathcal{E}_n$ denote the (random) set consisting of the first n bonds chosen. We also consider *stopped construction algorithms* in which the construction stops after a random number τ of steps, with τ a stopping time relative to $\{\mathfrak{G}_n\}$, where $\mathfrak{G}_n$ is the σ-field generated by the first n steps of the construction. It is easy to see that, given a set A of bonds or sites, τ may be chosen so that S_τ is the closure of the cluster of A; one merely "builds the cluster outwards from A."

We will use two particular types of construction algorithms. The first type are *independent (stopped) construction algorithms*, in which ω^1_b and ω^2_b are chosen independently of each other in each iteration, i.e. the single-bond coupling is product measure. The second type are *FKG (stopped) construction algorithms* [25] which exist if P_1 or P_2 satisfies the FKG lattice condition and $P_1(\omega)/P_2(\omega)$ is an increasing function of ω. This ensures that $P_1(\omega_{\mathcal{E}^c} \in \cdot \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}})$ dominates $P_2(\omega_{\mathcal{E}^c} \in \cdot \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}})$ for all $\mathcal{E}$ and $\rho_{\mathcal{E}}$. In turn this means that at each step of the construction, the single-bond coupling can be chosen so $\omega^1_b \geq \omega^2_b$, and thus at the end, $\omega^1_{\mathcal{E}_\tau} \geq \omega^2_{\mathcal{E}_\tau}$.

Proposition 2.3. *Let $P = P^{p,q,\{h_i\}}$ be the FK model at $(p, q, \{h_i\})$ on $\mathbb{Z}^n$, let d be either the Euclidean or the restricted path metric and let $\mathfrak{C}$ be a class of subsets of $\mathbb{Z}^n$ which is inheriting with respect to d. Suppose P has uniform exponential decay of finite-volume connectivities for the class $\mathfrak{C}$ with wired boundary conditions. Then P has the strong mixing property for the class $\mathfrak{C}$ and arbitrary site boundary conditions, for the metric d.*

Proof. Fix a finite $\mathcal{B} = \overline{\mathcal{B}}(\Lambda) \in \mathfrak{C}$, a site boundry condition η and $\mathcal{E}, \mathcal{F} \subset \mathcal{B}$. For a bond configuration $\omega_{\mathcal{B}\setminus\mathcal{F}}$ let

$$C_{\mathcal{F}}(\omega_{\mathcal{B}\setminus\mathcal{F}}) = \cup_{x \in V(\mathcal{B}\setminus\mathcal{F}) \cap V(\mathcal{F})} \, C_x(\omega_{\mathcal{B}\setminus\mathcal{F}}),$$

where $C_x(\omega_{\mathcal{B}\backslash\mathcal{F}})$ is the cluster of x in $\omega_{\mathcal{B}\backslash\mathcal{F}}$. Note that all bonds in $\partial C_{\mathcal{F}}(\omega_{\mathcal{B}\backslash\mathcal{F}}) \cap (\mathcal{B}\backslash\mathcal{F})$ are closed. Also,

(2.12) conditionally on the event $[C_{\mathcal{F}} = \mathcal{A}]$, the bond configuration on $\mathcal{B}\backslash(\mathcal{A}\cup\mathcal{F})$ is the FK model with site boundary condition given by η on $V(\mathcal{B}\backslash(\mathcal{A}\cup\mathcal{F})) \cap \partial\Lambda$, 0 on $V(\mathcal{B}\backslash(\mathcal{A}\cup\mathcal{F})) \cap \Lambda$.

(This is a straightforward extension of the Markov property for open dual surfaces.) Further, since $P_{\mathcal{B},w}$ has the strong FKG property, the measure $P_{\mathcal{B}\backslash\mathcal{F},w}$ FKG-dominates $P_{\mathcal{B},\eta}(\omega_{\mathcal{B}\backslash\mathcal{F}} \in \cdot \mid \omega_{\mathcal{F}} = \theta_{\mathcal{F}})$ for all $\theta_{\mathcal{F}}$. From these observations, as in [25] and [6], we see that for each pair of configurations $\theta_{\mathcal{F}}, \theta'_{\mathcal{F}}$ it is possible, using an FKG construction algorithm, to construct a coupling measure $\mathbb{P}$ on $(\{0,1\}^{\mathcal{B}\backslash\mathcal{F}})^3$ such that

- (i) the marginals of $\mathbb{P}$ are (in order) $P_{\mathcal{B},\eta}(\omega_{\mathcal{B}\backslash\mathcal{F}} \in \cdot \mid \omega_{\mathcal{F}} = \theta_{\mathcal{F}}), P_{\mathcal{B},\eta}(\omega_{\mathcal{B}\backslash\mathcal{F}} \in \cdot \mid \omega_{\mathcal{F}} = \theta'_{\mathcal{F}})$, and $P_{\mathcal{B}\backslash\mathcal{F},w}$;
- (ii) with $\mathbb{P}$-probability 1, the configuration $(\omega, \omega', \omega'')$ satisfies

$$\omega \le \omega'', \quad \omega' \le \omega'', \quad \omega_{\mathcal{B}\backslash(\mathcal{F}\cup\mathcal{C}_{\mathcal{F}}(\omega''_{\mathcal{B}\backslash\mathcal{F}}))} = \omega'_{\mathcal{B}\backslash(\mathcal{F}\cup\mathcal{C}_{\mathcal{F}}(\omega''_{\mathcal{B}\backslash\mathcal{F}}))}.$$

That is, the first 2 configurations agree outside the cluster of $\mathcal{F}$ in the largest configuration ω''. Then for $E \in \mathcal{G}_{\mathcal{E}}$,

$$\begin{aligned} |P_{\mathcal{B},\eta}(E \mid \omega_{\mathcal{F}} = \theta_{\mathcal{F}}) &- P_{\mathcal{B},\eta}(E \mid \omega_{\mathcal{F}} = \theta'_{\mathcal{F}})| \\ &\le \mathbb{P}(\omega \ne \omega') \\ &\le P_{\mathcal{B}\backslash\mathcal{F},w}(C_{\mathcal{F}}(\omega_{\mathcal{B}\backslash\mathcal{F}}) \cap \mathcal{E} \ne \phi) \\ &= P_{\mathcal{B}\backslash\mathcal{F},w}(x \leftrightarrow y \text{ for some } x \in V(\mathcal{F}), y \in V(\mathcal{E})). \end{aligned} \tag{2.13}$$

We claim that for some c_{11}, ϵ_5 (not depending on $\mathcal{B}, \mathcal{E}, \mathcal{F}$), the right side of (2.13) is bounded above by

$$\sum_{x \in V(\mathcal{E}), y \in V(\mathcal{F})} c_{11} e^{-\epsilon_5 d(x,y)}. \tag{2.14}$$

(This is not immediate from the uniform exponential decay assumption because we do not assume $\mathcal{B}\backslash\mathcal{F} \in \mathfrak{C}$.) Let $\mathcal{F}^+ = \{x \in \mathbb{Z}^n : d(x,\mathcal{F}) \le d(x,\mathcal{E})\}$. For $z \in \partial\mathcal{F}^+ \cap \Lambda$ let U_z be the connected component of z in the "ball" $W_z = \{e \in \mathcal{B} : d(z,e) \le \frac{1}{2} d(z,\mathcal{F})\}$. Then from the uniform exponential decay of connectivities, the FKG property and the inheriting property of $\mathfrak{C}$, for some c_i and ϵ_6,

$$\begin{aligned}
(2.15)\qquad & P_{\mathcal{B}\backslash\mathcal{F},w}(x \leftrightarrow y \text{ for some } x \in V(\mathcal{E}), y \in V(\mathcal{F})) \\
&\leq P_{\mathcal{B}\backslash\mathcal{F},w}(z \leftrightarrow V(\mathcal{B}\backslash W_z) \text{ for some } z \in \partial\mathcal{F}^+) \\
&\leq \sum_{z\in\partial\mathcal{F}^+} P_{U_z,w}(z \leftrightarrow V(\mathcal{B}\backslash W_z)) \\
&\leq \sum_{z\in\partial\mathcal{F}^+} c_{12} d(z,\mathcal{F})^{d-1} e^{-\epsilon_6 d(z,\mathcal{F})} \\
&\leq \sum_{z\in\partial\mathcal{F}^+} c_{13} e^{-\epsilon_6 d(z,\mathcal{F})/2}.
\end{aligned}$$

For each $z \in \partial\mathcal{F}^+$ let $f(z)$ be the site in $V(\mathcal{F})$ which is closest to z (in the metric d, breaking ties arbitrarily.) Then for $x \in V(\mathcal{F})$ and $z \in f^{-1}(x)$, from simple geometry

$$d(x,\mathcal{E}) \geq d(z,\mathcal{E}) - d(z,x) \geq d(z,x) - 2$$

and hence

$$|f^{-1}(x)| \leq c_{14}(d(x,\mathcal{E}) + 2)^d.$$

Therefore

$$\begin{aligned}
(2.16)\qquad & \sum_{z\in\partial\mathcal{F}^+} e^{-\epsilon_6 d(z,\mathcal{F})/2} \\
&\leq \sum_{x\in V(\mathcal{F})} \sum_{z\in f^{-1}(x)} e^{-\epsilon_6 d(z,x)/2} \\
&\leq \sum_{x\in V(\mathcal{F})} c_{15}(d(x,\mathcal{E}) + 2)^d e^{-\epsilon_6 d(x,\mathcal{E})/2} \\
&\leq \sum_{x\in V(\mathcal{F})} c_{16} e^{-\epsilon_6 d(x,\mathcal{E})/4} \\
&\leq \sum_{x\in V(\mathcal{F}), y\in V(\mathcal{E})} c_{16} e^{-\epsilon_6 d(x,y)/4}.
\end{aligned}$$

This and (2.15) prove the claim of the bound (2.14). Combining the claim with (2.13), and averaging over $\theta_{\mathcal{F}}$ and then over $\theta'_{\mathcal{F}}$ shows that

$$|P_{\mathcal{B},\eta}(E \mid F) - P_{\mathcal{B},\eta}(E)| \leq \sum_{x\in V(\mathcal{F}), y\in V(\mathcal{E})} c_{17} e^{-\epsilon_6 d(x,y)/4},$$

which proves the proposition. $\square$

The next two results form the analog of the last proposition for dual connectivity. The proof does not carry over, due to the fact that, under external fields, the FK model has only the near-Markov property for open circuits, not the full Markov property for open circuits, and the extension (2.12) of the Markov property breaks down completely. We evade this difficulty by using the ARC model of [2]. For $q > 0, Q \geq 1$ and $p_r, p_g \in [0,1]$, the ARC model on $(\overline{\Lambda}, \overline{\mathcal{B}}(\Lambda))$ with parameters (p_r, p_g, q, Q) is given by the weights

$$W(\omega_r, \omega_g) = p_r^{|\omega_r|}(1-p_r)^{|\overline{\mathcal{B}}(\Lambda)|-|\omega_r|} p_g^{|\omega_g|}(1-p_g)^{|\overline{\mathcal{B}}(\Lambda)|-|\omega_g|} q^{K(\omega_r)} Q^{I(\omega_r \vee \omega_g)}, \tag{2.17}$$

assigned to configurations $(\omega_r, \omega_g) \in \{0,1\}^{\overline{\mathcal{B}}(\Lambda)} \times \{0,1\}^{\overline{\mathcal{B}}(\Lambda)}$. Here $I(\omega)$ denotes the number of isolated sites (singleton open clusters) in Λ in the configuration ω, and $\omega_r \vee \omega_g$ denotes the coordinatewise maximum. Bonds in ω_r are called *red bonds*, and bonds in ω_g are called *green bonds*. The *black configuration* is defined to be $\omega_b = \omega_r \vee \omega_g$, the bondwise maximum. The ARC model is a graphical representation of the q-state Potts lattice gas, and hence, as a special case, of the $(q+1)$-state Potts model on $\Lambda \subset \mathbb{Z}^n$ at inverse temperature β with an external field applied to one species only, say h_1 applied to species 1, when we take

$$p_r = p_g = 1 - e^{-\beta}, \qquad Q = 1 + \frac{e^{\beta(2d+h_1)}}{q}. \tag{2.18}$$

Under site boundary conditions η with the property that, for some $i \in \{2, .., q+1\}, \eta_x \in \{0, 1, i\}$ for all $x \in \partial\Lambda$, the ARC model satisfies the FKG lattice condition [2].

It should be noted here that in [2], the species for the $(q+1)$-state Potts model were given as $0, 1, .., q$ instead of $1, 2, .., q+1$, and the external field was applied to species 0. Here we instead retain the meaning of 0 as a free boundary condition.

As with the FK model, for integer q it is straightforward in the ARC model to allow external fields applied to all species, by replacing the factor $q^{K(\omega_r)}$ in the ARC weight (2.17) with the weight

$$\prod_{C \in \mathcal{C}(\omega_r)} \left((1-p)^{h_2 s(C)} + (1-p)^{h_3 s(C)} + \ldots + (1-p)^{h_{q+1} s(C)} \right)$$

as in (1.12) under free boundary conditions, with modifications analogous to (1.13), (1.14) for bond and site boundary conditions. It is readily checked that the FKG lattice condition still holds under all bond boundary conditions and under site boundary conditions with $\eta_x \in \{0, 1, i\}$ for all x, where $i \in \{2, .., q\}$. The parameters are $(p_r, p_g, q, Q, \{h_i\})$. In the ARC model with external fields we need not require that $h_1 \geq h_2 \geq .. \geq h_{q+1}$ but rather only that $h_2 \geq .. \geq h_{q+1}$; the position of h_1 relative to the other h_i's is arbitrary. To make the model a graphical representation of the Potts model with external fields, one should replace q in (2.18) with

$$e^{\beta h_2} + \ldots + e^{\beta h_{q+1}}.$$

For parameters as in (2.18) (modified as above for external fields, if necessary), the cluster-labeling construction of the Potts model from the ARC model works as follows [2]. Each non-boundary cluster C of ω_r which is not a single isolated site in ω_b is independently given label i $(2 \leq i \leq q+1)$ with probability proportional to $e^{\beta h_i s(C)}$; no such cluster is ever given label $i = 1$. In each boundary cluster of ω_r intersecting $\{x \in \partial\Lambda : \sigma_x = i\}$, all sites are labeled i. Each site in Λ which is isolated in ω_b is independently given label i with probability proportional to $e^{\beta h_i}$ for $2 \leq i \leq q+1$ and proportional to $e^{\beta(2d+h_i)}$ for $i = 1$. (Note the resulting probability for label 1 is $(Q-1)/Q$.) Thus in the Edwards-Sokal-type joint construction, sites with species 1 are always isolated in the black configuration ω_b.

In certain contexts it is useful to consider the model obtained from the ARC model when only a portion of the cluster labeling is done. Specifically, given an ARC-model configuration with parameters as in (2.18) we can independently label each isolated site as being of species 1 with probability $(Q-1)/Q$, leaving all other sites unlabeled. As in [2] we call the resulting measure on site/bond configurations the *particle-bond form* of the ARC model. Given a realization of the particle-bond ARC model on some set $\overline{\mathcal{B}}(\Lambda)$, with red-bond configuration ω_r, we can enlarge the set of open red bonds by doing independent percolation, with red bonds, at density p_r on the set of bonds $\{\langle xy\rangle : \sigma_x = \sigma_y = 1\}$ (noting that such x and y are necessarily isolated in ω_r). The resulting enlarged red configuration is a realization of the FK model on $\overline{\mathcal{B}}(\Lambda)$ at $(p_r, q, \{h_i\})$ [2]. We call this the *ARC-based percolation construction* of the FK model.

We say that a q -state Potts model μ (in infinite volume) has *exponential decay of non-1 connectivities* if there exists $C, \lambda > 0$ such that for all x, y,

$$\mu(\text{there exists a lattice path from } x \text{ to } y \text{ in which every site } z \text{ has } \sigma_z \neq 1) \leq Ce^{-\lambda|y-x|}.$$

We say that *Gibbs uniqueness holds* at $(\beta, \{h_i\})$ if there is a unique infinite-volume Gibbs distribution for the Potts model with those parameters. If Gibbs uniqueness holds and $\beta > \beta_c(q, n, \{h_i\})$, then there must be a unique stable species. Conversely, if there is a unique stable species and the dimension $n = 2$, then random-cluster uniqueness holds for the corresponding FK model except possibly at $p = p_c(q, 2, \{h_i\})$ [7], and it follows readily from the cluster-labeling construction that Gibbs uniqueness holds for the Potts model except possibly at $\beta = \beta_c(q, 2, \{h_i\})$. This exception is a real one, at least for large q and small fields—the existence of a unique stable species need not imply Gibbs uniqueness at $\beta_c(q, 2, \{h_i\})$ [11].

Proposition 2.4. *Let $\mu^{\beta,q,\{h_i\}}$ be the q-state Potts model at $(\beta, \{h_i\})$ on $\mathbb{Z}^2$. Suppose Gibbs uniqueness holds. Let d be either the Euclidean or the restricted path metric. Suppose one of the following holds:*

(i) *the corresponding FK model has exponential decay of connectivities;*
(ii) *the Potts model $\mu^{\beta,q,\{h_i\}}$ has exponential decay of non-1 connectivities;*

(iii) *the corresponding FK model has exponential decay of dual connectivities;*
(iv) *the corresponding ARC model black configuration has exponential decay of connectivities.*

Then $\mu^{\beta,q,\{h_i\}}$ *(respectively, the corresponding FK and ARC models) has the strong mixing property for the class of all simply lattice-connected subsets of* $\mathbb{Z}^2$ *(respectively, of* $\mathcal{B}(\mathbb{Z}^2)$*) and arbitrary site boundary conditions, for the metric* d.

By the remarks preceding Proposition 2.4, if (ii) or (iii) (and Gibbs uniqueness) hold, then there must be a unique stable species, but (i) and (iv) allow multiple stable species, if the temperature is supercritical. Also, (i) can only be valid at high temperatures ($\beta < \beta_c(q, 2, \{h_i\})$) and (iii) can only be valid at low temperatures ($\beta > \beta_c(q, 2, \{h_i\})$) but (ii) and (iv) may be valid at arbitrary temperatures.

Gibbs uniqueness in the Potts model implies random-cluster uniqueness in the corresponding FK and ARC models, as is apparent from the percolation construction, so (i), (iii) and (iv) are not ambiguous.

Proof of Proposition 2.4. Suppose (iii) holds. For the corresponding Potts model define

$$\Sigma_1 = \{x : \sigma_x = 1\}, \qquad \Sigma_u = \{x : \sigma_x \in \{2, .., q\}\}.$$

Consider a joint Potts-FK-ARC configuration. In infinite volume, every connected component of $\mathcal{B}(\Sigma_u)$ in the Potts configuration must be finite, and must be surrounded by an open dual circuit in the FK configuration. It follows that (ii) holds. Since all x with $\sigma_x = 1$ are isolated sites in the ARC model, (ii) implies (iv). Thus, except under (i), we may assume (iv).

The ARC model then has the weak mixing property, by Theorem 3.1 of [5]. (Strictly speaking, that theorem is stated for a single configuration whereas the ARC model has separate red and green configurations, but the extension is trivial.) Therefore by Theorem 1.1, the ARC model has uniform exponential decay for the class of all simply lattice-connected subsets of $\mathcal{B}(\mathbb{Z}^2)$, with arbitrary site or bond boundary conditions. The proof of Proposition 2.3 then goes through (with 0 replaced by 1 in (2.12)) to show that the ARC model has the strong mixing property for the class of all simply lattice-connected subsets of $\mathcal{B}(\mathbb{Z}^2)$ with arbitrary site boundary conditions.

In fact, the proof of Proposition 2.3 shows more. Consider the joint Potts-ARC configuration under a site boundary condition η on $\partial\Lambda$, and let $\omega = (\omega_r, \omega_g)$ denote a generic ARC model configuration. Let $\Delta, \Gamma \subset \Lambda$ and let ξ_Γ, ξ'_Γ be Potts-model configurations on Γ. Define the ARC-model cluster of Γ by

$$C_\Gamma(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{B}(\Gamma)}) = \cup_{x\in\Gamma}\, C_x(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{B}(\Gamma)}),$$

where $C_x(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{B}(\Gamma)})$ is the cluster of x in the ARC model black configuration $(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{B}(\Gamma)})_b$. Analogously to (2.12), conditionally on the event $[C_\Gamma = \mathcal{A}]$, the site configuration on $\Lambda\backslash V(\mathcal{A})$

is the Potts model with site boundary condition given by η on $\partial(\Lambda\backslash V(\mathcal{A})) \cap \partial\Lambda$, 0 on $\partial(\Lambda\backslash V(\mathcal{A})) \cap \Gamma$. Observe that the ARC model measures on $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{B}(\Gamma)$ with site boundary conditions η on $\partial\Lambda$ and ξ_Γ or ξ'_Γ on Γ are both FKG-dominated by the wired measure on $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{B}(\Gamma)$, and the latter has the FKG property. Therefore these three ARC-model measures can be coupled so that the configurations agree outside the cluster of Γ in the largest (wired) configuration. Then via the cluster-labeling construction, the corresponding three Potts configurations can be coupled so that they agree outside this same cluster of Γ. As in (2.13) – (2.16) this shows that for any event $E \in \mathcal{G}_\Delta$,

$$\begin{aligned}|\mu_{\Lambda,\eta}^{\beta,q,\{h_i\}}&(E \mid \sigma_\Gamma = \xi_\Gamma) - \mu_{\Lambda,\eta}^{\beta,q,\{h_i\}}(E \mid \sigma_\Gamma = \xi'_\Gamma)| \\ &\leq P_{\overline{\mathcal{B}}(\Lambda),w}(x \leftrightarrow y \text{ for some } x \in \Gamma, y \in \Delta) \\ &\leq \sum_{x\in\Gamma, y\in\Lambda} c_{18} e^{-\epsilon_7 d(x,y)},\end{aligned}$$

where $P_{\overline{\mathcal{B}}(\Lambda),w}$ is the wired ARC model. This proves that $\mu^{\beta,q,\{h_i\}}$ has the desired strong mixing property.

Under (i), essestially the same proof works, with the FK model substituted for the ARC model, to yield that $\mu^{\beta,q,\{h_i\}}$ has the desired strong mixing property.

Given $\Gamma \subset \Lambda$ and a joint Potts-FK configuration $(\xi_\Gamma, \rho_{\mathcal{B}(\Gamma)})$ on $(\Gamma, \mathcal{B}(\Gamma))$, it is easy to see that conditioning on both ξ_Γ and $\rho_{\mathcal{B}(\Gamma)}$ is the same as conditioning only on ξ_Γ. As in the proof of ([5], Theorem 6.1(i)), this observation and the percolation and cluster-labeling constructions of the FK model can be used to show that whenever the Potts model has the stated strong mixing property, so does the corresponding FK model. □

Proposition 2.4 does not cover the case in which there are multiple stable species and the FK model has exponential decay of dual connectivities. Of course, in that case the Potts model itself will not satisfy Gibbs uniqueness so will not have any reasonable sort of mixing property, but the corresponding FK model still may. The next proposition, like Theorem 1.6(iii), removes the assumption, in Proposition 2.4(iii), of Gibbs uniquesness in the corresponding Potts model, allowing multiple stable species, at the expense of great restriction on the boundary condition. The proof, which will follow some preliminary lemmas, is also significantly more complex.

Proposition 2.5. *Let $P = P^{p,q,\{h_i\}}$ be the FK model at $(p, q, \{h_i\})$ on $\mathbb{Z}^2$. Suppose random-cluster uniqueness holds and P has exponential decay of dual connectivities. Let d be either the Euclidean or restricted path metric. Then $P^{p,q,\{h_i\}}$ has the strong mixing property for the class of all simply lattice-connected closure subsets of $\mathcal{B}(\mathbb{Z}^2)$ with free and wired boundary conditions, for the metric d.*

Since, under the assumptions of Proposition 2.5, there is not necesssarily Gibbs uniqueness in the corresponding Potts and ARC models, (iii) and especially (iv) in Proposition 2.4

become potentially ambiguous statements and thus the proof of Proposition 2.4 does not carry over. Instead we will make use of the following. Consider a q-state Potts model at $(\beta, \{h_i\})$ on a finite Λ with stable species $1,..,k$ and unstable species $k+1,..,q$. Starting from the corresponding Edwards-Sokal joint Potts/FK model, with $p = 1 - e^{-\beta}$, as in [2] we color each site x yellow if σ_x is stable, white if σ_x is unstable, and designate each open bond to have the color of its (necessarily matching) endpoints. We call the corresponding configuration of colored bonds and sites (without the values σ_x) the *bicolored FK model*. When necessary for clarity, we call the usual FK model the *uncolored FK model*. Using the Edwards/Sokal joint Potts/FK model, we see that in either the bicolored or uncolored FK model, one can have bond boundary conditions $\rho \in \{0,1\}^{\overline{\mathcal{B}}(\Lambda)^c}$ for which each cluster is designated to be one of 3 types: yellow, white, or uncolored (see Remark 2.7 below.) Note that the yellow-wired boundary condition is the same as the uncolored wired boundary condition, since infinite clusters can only be yellow. The configuration of open yellow bonds $\{\delta_{[b \text{ open and yellow}]} : b \in \overline{\mathcal{B}}(\Lambda)\}$ (or its distribution, in a harmless abuse of terminology) is called the *stable partial FK model*, or briefly the *SPFK model*, with parameters $(p, q, k, \{h_i\})$, and its distribution is denoted $P^{(p,q,k,\{h_i\})}_{SPFK,\overline{\mathcal{B}}(\Lambda),\rho}$. For the SPFK model we can think of all open bonds, including those in the boundary condition, as being yellow. Define

$$\mathcal{I}(\omega,\Lambda) = \{x \in \Lambda : x \text{ is an isolated site of } \omega\}, \qquad I(\omega,\Lambda) = |\mathcal{I}(\omega,\Lambda)|,$$

$$r_n = r_n(p, h_{k+1}, .., h_q) = \sum_{i=k+1}^{q} (1-p)^{-nh_i}$$

It is easily calculated (see [2]) that the weight associated to an SPFK configuration ω under a bond boundary condition ρ is

$$W_{SPFK}(\omega) = p^{|\omega|}(1-p)^{|\overline{\mathcal{B}}(\Lambda)|-|\omega|} k^{K(\omega|\rho)} G(\omega)$$

where

$$G(\omega) = k^{-I((\omega,\rho),\overline{\Lambda})} \sum_{\omega_w \in \{0,1\}^{\mathcal{B}(\mathcal{I}((\omega,\rho),\overline{\Lambda}))\cap\overline{\mathcal{B}}(\Lambda)}} \left(\frac{p}{1-p}\right)^{|\omega_w|} \left(\prod_{C\in\mathcal{C}(\omega_w)} r_{|C|}\right) \left(\frac{k+r_1}{r_1}\right)^{I((\omega\vee\omega_w,\rho),\overline{\Lambda})}.$$

Here the configurations ω_w correspond to white configurations in the bicolored FK model. Given a bond configuration ω and a bond e, we let $\omega \vee e$ denote the configuation obtained by adding the bond e to ω (that is, by declaring e to be open.)

Remark 2.6. The reason for interest in the SPFK model is as follows. Consider a cicuit γ and let Λ be the set of sites surrounded by γ, with boundary condition in which all bonds of γ are open. The FK model with external fields lacks the Markov property for open dual circuits because the weight attached to the boundary cluster in $\overline{\mathcal{B}}(\Lambda)$ depends on the number of sites outside γ in that cluster, hence is affected by the bond configuration outside γ. In the SPFK model, the same boundary cluster gets weight k in all configurations, so it is easy to see that the Markov property for open dual circuits does hold.

Remark 2.7. A boundary condition ρ for the SPFK model can be viewed as a boundary condition for the full FK model, with all non-singleton clusters of ρ conditioned to be be labeled with stable species in the cluster-labeling construction, and with the label types (stable/unstable) for singleton clusters not specified. This idea can be extended to allow boundary conditions ρ for the SPFK model which are bond configurations for the full FK model, with stable/unstable labels specified for some clusters, with both singleton and non-singleton clusters of ρ allowed to have specified or unspecified labels. We call such a ρ a *partly labeled bond boundary condition.*

Consider an unlabeled nonsingleton boundary cluster C in a partly labeled bond boundary condition ρ, with $C \cap \partial\Lambda \neq \phi$. The effect of the presence of C is to identify the sites $C \cap \partial\Lambda$ into a single "condensed site" which should be treated as one site in determining the set $\mathcal{I}((\cdot,\rho),\overline{\Lambda})$ of isolated sites in the formula for $G(\omega)$. However, if the condensed site is isolated the weight associated to it in the FK model is $k + r_{s(C)}$, not $k + r_1$, so the factor $(k + r_1)/r_1)$ associated to such an isolated condensed site should be replaced by $(k+r_{s(C)})/r_{s(C)}$. similarly, if the condensed site C consistts of j original sites and is not isolated, all j must be counted in calculating the size $s(\cdot)$ of the cluster containing C.

The proofs of Lemmas 2.8 and 2.9 are both valid for partly labeled bond boundary conditions, with no significant modifications.

Lemma 2.8. *For every finite Λ, every $(p, q, k, \{h_i\})$ and every bond boundary condition ρ, the SPFK model $P^{p,q,k,\{h_i\}}_{SPFK,\Lambda,\rho}$satisfies the FKG lattice condition.*

Proof. As is standard, it is sufficient to show that, for every bond e, $W_{SPFK}(\omega\vee e)/W_{SPFK}(\omega)$ is an increasing function of ω. Thus fix $e = \langle xy\rangle \in \overline{\mathcal{B}}(\Lambda)$. Since $K(\omega \vee e \mid \rho) - K(\omega \mid \rho)$ is increasing, it is sufficient to show that $G(\omega\vee e)/G(\omega)$ is increasing. We proceed as in the proof of ([2], Proposition 4.16). Let $F(\omega) = k^{I((\omega,\rho),\overline{\Lambda})}G(\omega)$ denote the sum in $G(\omega)$ and let $\mathcal{D}(\omega) = \mathcal{B}(\mathcal{I}((\omega,\rho),\overline{\Lambda})) \cap \overline{\mathcal{B}}(\Lambda)$ denote the set of bonds on which the configurations ω_w exist. Note $F(\omega)$ is the partition function of the ARC model on $\mathcal{D}(\omega)$ at $(p, 0, q - k, \frac{k+r_1}{r_1}, \{h_{k+1}, .., h_q\})$ with free boundary; we denote this model by P^{ARC}_ω. Let

$$\Delta(\omega) = I((\omega,\rho),\overline{\Lambda}) - I((\omega \vee e,\rho),\overline{\Lambda}) = |\{x, y\} \cap \mathcal{I}((\omega,\rho),\overline{\Lambda})|.$$

Terms in the sum $F(\omega \vee e)$, each multiplied by $(k + r_1)^{\Delta(\omega)}$, correspond precisely to those configurations ω_w of this ARC model in which $\{x, y\} \cap \mathcal{I}((\omega,\rho),\overline{\Lambda}) \subset \mathcal{I}((\omega \vee \omega_w,\rho),\overline{\Lambda})$, or equivalently, in which all bonds in $\mathcal{D}(\omega) \cap \overline{\{e\}}$ are closed. It follows that

$$\frac{F(\omega \vee e)}{F(\omega)} = (k + r_1)^{-\Delta(\omega)}P^{ARC}_\omega\big(\{\omega_w : \text{ all bonds in } \mathcal{D}(\omega) \cap \overline{\{e\}} \text{ are closed in } \omega_w\}\big)$$

and then that
(2.19)
$$\frac{G(\omega \vee e)}{G(\omega)} = \left(\frac{k}{k+r_1}\right)^{\Delta(\omega)} P_{\omega}^{ARC}\big(\{\omega_w :\ \text{all bonds in } \mathcal{D}(\omega) \cap \overline{\{e\}} \text{ are closed in } \omega_w\}\big).$$

It is easy to see that $\left(\frac{k}{k+r_1}\right)^{\Delta(\omega)}$ is an increasing function of ω. It is proved in [2] that the ARC model without external fields satisfies the FKG lattice condition. With external fields the proof is similar except that one must establish the straightforward fact that $r_{n+m}/r_n r_m$ is an increasing function of m and n. Thus P_{ω}^{ARC} satisfies the FKG lattice condition. Let $b = \langle uv \rangle$ be a bond which is closed in $\omega \vee e$. Then

$$P_{\omega \vee b}^{ARC}(\cdot) = P_{\omega}^{ARC}\big((\omega_w)_{\mathcal{D}(\omega \vee b)} \in \cdot \mid \ \text{all bonds in } \mathcal{D}(\omega) \cap \overline{\{b\}} \text{ are closed in } \omega_w\big)$$

and the conditioning here is on a decreasing event. Therefore

$$\begin{aligned} (2.20) \qquad & P_{\omega \vee b}^{ARC}\big(\{\omega_w :\ \text{all bonds in } \mathcal{D}(\omega \vee b) \cap \ overline\{e\} \text{ are closed in } \omega_w\}\big) \\ &= P_{\omega}^{ARC}\big(\ \text{all bonds in } \mathcal{D}(\omega \vee b) \cap \overline{\{e\}} \text{ are closed in } \omega_w \\ &\qquad\qquad \mid \ \text{all bonds in } \mathcal{D}(\omega) \cap \overline{\{b\}} \text{ are closed in } \omega_w\big) \\ &\geq P_{\omega}^{ARC}\big(\ \text{all bonds in } \mathcal{D}(\omega) \cap \overline{\{e\}} \text{ are closed in } \omega_w\big), \end{aligned}$$

so $G(\omega \vee e)/G(\omega)$ is increasing, as desired. □

It follows from Lemma 2.8 that the free- and wired-boundary SPFK models have infinite-volume limits, and there is a unique infinite-volume limit if and only if the free and wired models are equal.

Lemma 2.9. *Let $P = P^{p,q,\{h_i\}}$ be the FK model at $(p, q, \{h_i\})$ on $\mathbb{Z}^2$. Suppose random-cluster uniqueness holds and P has exponential decay of dual connectivities. Then the corresponding SPFK model has the weak mixing property and has uniform exponential decay of finite-volume dual connectivities for the class of all simply lattice-connected subsets of $\mathcal{B}(\mathbb{Z}^2)$ with arbitrary bond boundary condition, for both the Euclidean and restricted-path metrics.*

Proof. It is clear that the SPFK model also satisfies random-cluster uniqueness. Further, in the infinite-volume FK model there is with probability 1 a unique infinite cluster with finite "holes" defined by exterior open dual circuits, with an exponentially decreasing tail for the distribution of hole sizes. In the Edwards-Sokal joint construction the infinite cluster is stable with probability 1, so the same infinite cluster with the same holes is present in the SPFK configuration. It follows that the SPFK model has exponential decay of dual connectivities.

Let γ_n be the dual circuit forming the boundary of $[-n-\frac{1}{2}, n+\frac{1}{2}]^2$ and let $\Lambda_n = [-n, n]^2 \cap \mathbb{Z}^2$. Let k be the number of stable species. Given a bond boundary condition ρ on $\overline{\mathcal{B}}(\Lambda_n)^c$, by

Lemma 2.8 and the Markov property for open circuits (see Remark 2.6), there exists a coupling of the measures $P^{p,q,k,\{h_i\}}_{SPFK,\overline{\mathcal{B}}(\Lambda_n),\rho}$ and $P^{p,q,k,\{h_i\}}_{SPFK,\overline{\mathcal{B}}(\Lambda_n),f}$ with the property that the respective configurations ω, ω^f on $\overline{\mathcal{B}}(\Lambda)$ agree outside the set

$$\{b \in \overline{\mathcal{B}}(\Lambda_n) : b^* \overset{*}{\leftrightarrow} \gamma_n \text{ in } (\omega^f)^*\}.$$

(Such couplings can be made using an FKG construction algorithm—see [5], [6] or [25].) Therefore by ([5], Theorem 3.1) the SPFK model has the weak mixing property. The uniform exponential decay now follows from Theorem 1.1. □

The next lemma controls the size of the portion of the FK-model boundary cluster which is attached to boundary sites occupied by unstable species.

Lemma 2.10. *Let $P = P^{p,q,\{h_i\}}$ be an FK model on $\mathcal{B}(\mathbb{Z}^2)$, let d be the Euclidean or restricted-path metric. There exist constants C, λ such that for every finite $\Gamma \subset \mathbb{Z}^2$, every site boundary condition η, every $x \in \Gamma$ and every unstable species j, for $(\partial\Gamma)_j = \{y \in \partial\Gamma : \sigma_y = j\}$,*

$$P_{\Gamma,\eta}(x \leftrightarrow (\partial\Gamma)_j) \leq Ce^{-\lambda d(x,(\partial\Gamma)_j)}.$$

Proof. We will do the proof for the Euclidean metric; the modifications for the restricted-path metric are minor. Fix Γ, η and $x \in \Gamma$. Let k be the number of stable species. Recall that $Q_r(x)$ denotes the closed square of side $2r$ centered at x and let $\mathcal{D}_m$ denote the set of all bonds which cross $\partial Q_{m+\frac{1}{2}}(x)$. Let

$$m_{\min} = \lfloor \frac{1}{3} d(x,(\partial\Gamma)_j) \rfloor, \qquad m_{\max} = \lfloor \frac{2}{3} d(x,(\partial\Gamma)_j) \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Let C_x denote the open cluster of x, fix $m_{\min} \leq m \leq m_{\max}$ and define events

$$B_\Theta = [x \leftrightarrow (\partial\Gamma)_j, C_x \cap \mathcal{D}_m = \Theta]$$

for $\Theta \subset \mathcal{D}_m$. We fix such a Θ and write bond configurations ω as $(\omega_{\overline{\mathcal{B}}(\Gamma)\backslash\Theta}, \omega_\Theta)$. Thus we define

$$\tilde{B}_\Theta = \{\omega_{\overline{\mathcal{B}}(\Gamma)\backslash\Theta} : (\omega_{\overline{\mathcal{B}}(\Gamma)\backslash\Theta}, \rho^1_\Theta) \in B_\Theta\},$$

where, as will be recalled, ρ^i_Θ denotes the all-i configuration on Θ.

Given a configuration $\omega \in B_\Theta$, closing all bonds in Θ breaks the cluster C_x into one or more clusters in $Q_m(x)$, and one or more clusters outside the interior of $Q_{m+1}(x)$. More precisely, the first group is

$$\hat{\mathcal{C}}_x(\omega_{\overline{\mathcal{B}}(\Gamma)\backslash\Theta}) = \{\text{all clusters of } (\omega_{\overline{\mathcal{B}}(\Gamma)\backslash\Theta}, \rho^0_\Theta) \text{ contained in } C_x((\omega_{\overline{\mathcal{B}}(\Gamma)\backslash\Theta}, \rho^1_\Theta)) \cap Q_m(x)\},$$

and the FK weight assigned to each cluster C in this group is $k + r_{|C|} \geq 1$. Let $N(\omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta})$ denote the total number of sites in the clusters in $\hat{\mathcal{C}}_x(\omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta})$, and let $W(\omega)$ denote the FK weight of ω. We then have, for $\omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta} \in \hat{B}_\Theta$, $N(\omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta}) \geq m$ and therefore

$$\begin{aligned} P\big((\omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta}, \rho_\Theta^1) \mid \omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta}\big) &\leq \frac{W((\omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta}, \rho_\Theta^1))}{W((\omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta}, \rho_\Theta^0))} \\ &\leq \left(\frac{p}{1-p}\right)^{|\Theta|} (1-p)^{-h_{k+1} N(\omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta})} \\ &\leq \left(\frac{p}{1-p} \vee 1\right)^{|\Theta|} (1-p)^{-h_{k+1} m}. \end{aligned} \tag{2.21}$$

Therefore

$$P(B_\Theta) \leq \max\left\{P\big((\omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta}, \rho_\Theta^1) \mid \omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta}\big) : \omega_{\overline{\mathcal{B}}(\Gamma)\setminus\Theta} \in \tilde{B}_\theta\right\} \leq \left(\frac{p}{1-p} \vee 1\right)^{|\Theta|} (1-p)^{-h_{k+1} m}.$$

Fix $M > 0$ to be specified. We have

$$\begin{aligned} &P(x \leftrightarrow (\partial\Gamma)_j, |C_x \cap \mathcal{D}_m| \leq M) \\ &\quad\leq \sum_{\Theta \subset \mathcal{D}_m : |\Theta| \leq M} P(B_\Theta) \\ &\quad\leq c_{19} |\mathcal{D}_m|^M \left(\frac{p}{1-p} \vee 1\right)^M (1-p)^{-h_{k+1} m} \end{aligned} \tag{2.22}$$

so that, provided $d(x, (\partial\Gamma)_j)$ is sufficiently large (depending on M),

$$\begin{aligned} &P(x \leftrightarrow (\partial\Gamma)_j, |C_x \cap \mathcal{D}_m| \leq M \text{ for some } m_{\min} \leq m \leq m_{\max}) \\ &\quad\leq c_{19} d(x, (\partial\Gamma)_j) |\mathcal{D}_m|^M \left(\frac{p}{1-p} \vee 1\right)^M (1-p)^{-h_{k+1} m_{\min}} \\ &\quad\leq c_{20} (1-p)^{-h_{k+1} d(x, (\partial\Gamma)_j)/6}. \end{aligned} \tag{2.23}$$

Next we consider configurations satisfying

$$x \leftrightarrow (\partial\Gamma)_j, \quad |C_x \cap \mathcal{D}_m| > M \text{ for all } m_{\min} \leq m \leq m_{\max}.$$

Taking $m = m_{\max}$ and Θ as above, in (2.21) we have for such configurations $N(\omega_{\overline{\mathcal{B}}(\Gamma)\backslash\Theta}) \geq Md(x, (\partial\Gamma)_j)/3$. Therefore

$$
\begin{aligned}
(2.24) \qquad & P(x \leftrightarrow (\partial\Gamma)_j, |C_x \cap \mathcal{D}_m| > M \text{ for all } m_{\min} \leq m \leq m_{\max}) \\
&\leq \sum_{\Theta \subset \mathcal{D}_{m_{\max}}} P\left(B_\Theta \cap \left[N(\omega_{\overline{\mathcal{B}}(\Gamma)\backslash\Theta}) \geq \frac{1}{3}Md(x, (\partial\Gamma)_j))\right]\right) \\
&\leq 2^{|\mathcal{D}_{m_{\max}}|}\left(\frac{p}{1-p} \vee 1\right)^{|\mathcal{D}_{m_{\max}}|}(1-p)^{-h_{k+1}Md(x,(\partial\Gamma)_j)/3} \\
&\leq c_{21}(1-p)^{-h_{k+1}Md(x,(\partial\Gamma)_j)/6}
\end{aligned}
$$

provided M is sufficiently large. With (2.23) this completes the proof. $\square$

Proof of Proposition 2.5. Let Λ be a finite subset of $\mathbb{Z}^2$ and $\mathcal{E}, \mathcal{F} \subset \overline{\mathcal{B}}(\Lambda)$, let η be the free or wired boundary condition on $\partial\Lambda$ and let $\rho_\mathcal{E}, \rho'_\mathcal{E}$ be bond configurations on $\mathcal{E}$. We wish to show roughly that $P_{\overline{\mathcal{B}}(\Lambda),\eta}(\omega_\mathcal{F} \in \cdot \mid \omega_\mathcal{E} = \rho_\mathcal{E})$ and $P_{\overline{\mathcal{B}}(\Lambda),\eta}(\omega_\mathcal{F} \in \cdot \mid \omega_\mathcal{E} = \rho'_\mathcal{E})$ differ by an exponentially small amount. Let $\mathbb{P}$ denote the distribution of the Edwards-Sokal joint Potts/FK configuration (σ, ω) on $\overline{\mathcal{B}}(\Lambda)$. Define

$$U(\sigma) = \{x \in V(\mathcal{E}) : \sigma_x \text{ is stable}\}$$

and consider the measures

$$\mathbb{P}(\cdot \mid \omega_\mathcal{E} = \rho_\mathcal{E}, U(\sigma) = U_0), \quad \mathbb{P}(\cdot \mid \omega_\mathcal{E} = \rho^1_\mathcal{E}, U(\sigma) = \phi)$$

where $U_0 \subset V(\mathcal{E})$ is *compatible with* $\rho_\mathcal{E}$, that is, U_0 contains either all or none of each cluster of $\rho_\mathcal{E}$. We call the first of these measures the *lower measure* and the second the *upper measure.* It is sufficient to show that for arbitrary compatible U_0, these two measures can be coupled so that the corresponding bond configurations agree on $\mathcal{F}$, except with an exponentially small probability. Note that if η is the wired boundary condition, the Potts model under $\mathbb{P}$ is conditioned on the event that all boundary spins are the same stable species. Also, we may assume $\mathcal{E} = \mathcal{B}(E) \cap \overline{\mathcal{B}}(\Lambda)$ for some $E \subset \overline{\Lambda}$. Then $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E} = \overline{\mathcal{B}}(\Lambda\backslash E)$.

The coupling is constructed as follows. The lower and upper bond configurations are denoted $\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}$ and $\omega'_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}$, respectively. First we use a stopped construction algorithm to create $\overline{C}_{U_0}(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}})$, the closure of the cluster of U_0, in the lower FK configuration $\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}$. Suppose the resulting value of the cluster closure is $\overline{C}_{U_0}(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}) = \overline{\mathcal{B}}(G)$ for some $G \subset \Lambda\backslash E$. (Note the cluster closure always has such a form for some G.) We then choose the stable part of the upper bond configuration on this same set $\overline{\mathcal{B}}(G)$, using the SPFK model, making this choice of upper SPFK configuration independently of the lower FK configuration we selected on $\overline{\mathcal{B}}(G)$. Let $\mathcal{E}_+ = \mathcal{E} \cup \overline{\mathcal{B}}(G)$. Note that in the lower configuration, all bonds in $\partial\mathcal{B}(G) \cap \overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}$ are closed. We now treat the upper and lower configurations on $\overline{\mathcal{B}}(G)$ as parts of extended boundary conditions for the remaining upper and lower configurations on $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}_+$. (Here "extended" refers to the fact that these boundary conditions are configurations on $\overline{\mathcal{B}}(\Lambda)^c \cup \mathcal{E}_+$, not just on $\overline{\mathcal{B}}(\Lambda)^c$.) Since all of the open bonds relevant to these extended boundary conditions have stable species at their endpoints, we can treat them as boundary conditions for the SPFK model, rather than for the full FK model. (Those with unstable species at their endpoints are irrelevant because they cannot have endpoints in $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}_+$.) Let $\tilde{\omega}_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}_+}$ and $\tilde{\omega}'_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}_+}$ respectively denote the lower and upper SPFK configurations. Note that the extended boundary condition for the upper configuration is always larger, in the usual ordering, than the extended boundary condition for the lower configuration. Therefore by Lemma 2.8, the upper SPFK configuration on $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}_+$ FKG-dominates the lower one.

Note in part of what follows, we will use the domain $\mathcal{B}^+(\Lambda)$ instead of $\overline{\mathcal{B}}(\Lambda)$, because for η free we do not want to allow the use of the boundary dual bonds $(\partial\mathcal{B}(\Lambda))^*$, which are always open, in forming our dual paths.

We next use another stopped construction algorithm to create $\overline{C^*_{\mathcal{E}^*_+}}(\tilde{\omega}_{\mathcal{B}^+(\Lambda)\backslash\mathcal{E}_+})$, the closure of the dual cluster of $\mathcal{E}^*_+$ in the lower SPFK configuration $\tilde{\omega}_{\mathcal{B}^+(\Lambda)\backslash\mathcal{E}_+}$, while simultaneously creating the upper SPFK configuration on the same (random) set of dual bonds. By the FKG domination, this can be done with the upper SPFK configuration larger than the lower one. Suppose the resulting values of this dual-cluster closure and the dual-cluster boundary (in $\mathcal{B}^+(\Lambda)\backslash\mathcal{E}_+$) are $\overline{C^*_{\mathcal{E}^*_+}}(\tilde{\omega}_{\mathcal{B}^+(\Lambda)\backslash\mathcal{E}_+}) = \mathcal{H}^*$ and $\partial C^*_{\mathcal{E}^*_+}(\tilde{\omega}_{\mathcal{B}^+(\Lambda)\backslash\mathcal{E}_+}) = \mathcal{D}^*$ for some $\mathcal{D}, \mathcal{H} \subset \mathcal{B}^+(\Lambda)\backslash\mathcal{E}_+$. Then all (regular) bonds in $\mathcal{D}$ are open, in both the upper and lower SPFK configurations. Let $\mathcal{E}_{++} = \mathcal{E}_+ \cup \mathcal{H}$. As before with $\mathcal{E}_+$, the upper and lower SPFK configurations on $\mathcal{E}_{++}$ can each be treated as further-extended boundary conditions for the corresponding configurations on $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}_{++}$. Because all bonds in $\mathcal{D}$ are open, these further-extended boundary conditions are both the same—wired on each component of $\mathcal{D}$ if η is free, and fully wired if η is wired. Here we are using an extension, analogous to (2.12), of the Markov property for open surfaces valid for the SPFK model—see Remark 2.6—and for η free we are using simple lattice-connectedness of $\overline{\mathcal{B}}(\Lambda)$ to ensure that no two components of $\mathcal{D}$ are connected to each other both by an open path in $\mathcal{E}_{++}$ and by a path in $\overline{\mathcal{B}}(\Lambda)$ outside $\mathcal{E}_{++}$. Even when the SPFK configurations we have constructed on portions of $\mathcal{E}_{++}$ are extended to full FK configurations on the same portions of $\mathcal{E}_{++}$, the boundary condition for configurations on $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}_{++}$ is not affected. (Here "portions of $\mathcal{E}_{++}$" means $\mathcal{H}$ in the lower configuration and all of $\mathcal{E}_{++}$ in the upper configuration.) Thus in completing the upper and lower FK configurations we can choose $\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}_{++}}$ and $\omega'_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}_{++}}$ to be equal. The set $\mathcal{E}_{++} = \mathcal{E}_{++}(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}})$ is of course random, and we have

$$\begin{aligned} &|\mathbb{P}(F \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}}, U(\sigma) = U_0) - \mathbb{P}(F \mid \omega_{\mathcal{E}} = \rho^1_{\mathcal{E}}, U(\sigma) = \phi)| \\ &\quad \le \mathbb{P}(\mathcal{E}_{++}(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}) \cap \mathcal{F} \ne \phi \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}}, U(\sigma) = U_0), \quad F \in \mathcal{G}_{\mathcal{F}}. \end{aligned} \tag{2.25}$$

For $0 < \alpha < 1$ let

$$\Gamma(\mathcal{E}, \alpha) = \left\{ x \in \Lambda : d(x, \mathcal{E}) \le \frac{\alpha}{1-\alpha} d(x, \mathcal{F}) \right\},$$

$$\Gamma^*(\mathcal{E}, \alpha) = \left\{ x \in V^*(\overline{\mathcal{B}}(\Lambda)) : d(x, \mathcal{E}) \le \frac{\alpha}{1-\alpha} d(x, \mathcal{F}) \right\}.$$

These are the points which are, roughly speaking, at most α fraction of the way from $\mathcal{E}$ to $\mathcal{F}$. Then

(2.26)

$$\begin{aligned} &\mathbb{P}(\mathcal{E}_{++}(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}) \cap \mathcal{F} \ne \phi \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}}, U(\sigma) = U_0) \\ &\quad \le \mathbb{P}\left(\mathcal{E}_{+}(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}) \cap \Gamma\left(\mathcal{E}, \frac{1}{3}\right)^c \ne \phi \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}}, U(\sigma) = U_0 \right) \\ &\qquad + \mathbb{P}\left(\mathcal{E}_{+}(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}) \cap \Gamma\left(\mathcal{E}, \frac{1}{3}\right)^c = \phi, \mathcal{E}_{++}(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}) \cap \mathcal{F} \ne \phi \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}}, U(\sigma) = U_0 \right) \end{aligned}$$

Since conditioning on $\omega_{\mathcal{E}} = \rho_{\mathcal{E}}, U(\sigma) = U_0$ is just an average of certain site boundary conditions, Lemma 2.10 yields that for some c_{22}, ϵ_8,

$$\mathbb{P}\left(\mathcal{E}_{+}(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}) \cap \Gamma\left(\mathcal{E}, \frac{1}{3}\right)^c \ne \phi \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}}, U(\sigma) = U_0 \right) \le c_{22} \sum_{x \in \mathcal{E}, y \in \Gamma(\mathcal{E}, \frac{1}{3})^c} e^{-\epsilon_8 d(x,y)} \tag{2.27}$$

for some constants c_i. As in the proof of ([5], Theorem 3.3) we have for some c_{23}, ϵ_9,

$$\sum_{x \in \mathcal{E}, y \in \Gamma(\mathcal{E}, \frac{1}{3})^c} e^{-\epsilon_8 d(x,y)} \le c_{23} \sum_{x \in \mathcal{E}, z \in \mathcal{F}} e^{-\epsilon_9 d(x,z)}. \tag{2.28}$$

Define for $x \notin \mathcal{E}$

$$B_x = \{y \in \mathbb{R}^2 : d(x,y) \le \frac{1}{4} d(x, \mathcal{E})\}, \quad B_x^+ = \{y \in \mathbb{R}^2 : d(x,y) \le \frac{1}{4} d(x, \mathcal{E}) + 2\}$$

By Lemma 2.9, the SPFK model has uniform exponential decay of finite-volume dual connectivities for the class of all simply lattice-connected subsets of $\mathcal{B}(\mathbb{Z}^2)$ with arbitrary bond boundary conditions. Letting k be the number of stable species, it follows readily that

$$\begin{aligned}
(2.29)\quad & \mathbb{P}\left(\mathcal{E}_+(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}) \cap \Gamma\left(\mathcal{E}, \frac{1}{3}\right)^c = \phi, \mathcal{E}_{++}(\omega_{\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{E}}) \cap \mathcal{F} \neq \phi \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}}, U(\sigma) = U_0\right) \\
& \leq \sup_{\rho} P^{p,q,k,\{h_i\}}_{SPFK,\overline{\mathcal{B}}(\Lambda)\backslash\overline{\mathcal{B}}(\Gamma(\mathcal{E},\frac{1}{3})),\rho}(\text{there exists an open dual path which starts within} \\
& \qquad\qquad \text{distance 1 of } \Gamma(\mathcal{E}, \tfrac{1}{3}) \text{ and ends within distance 1 of } \mathcal{F}) \\
& \leq \sup_{\rho} \sum_{x \in \partial\Gamma^*(\mathcal{E},\frac{2}{3})} P^{p,q,k,\{h_i\}}_{SPFK,\overline{\mathcal{B}}(\Lambda)\backslash\overline{\mathcal{B}}(\Gamma(\mathcal{E},\frac{1}{3})),\rho}(x \overset{*}{\leftrightarrow} B_x^c) \\
& \leq \sup_{\rho} \sum_{x \in \partial\Gamma^*(\mathcal{E},\frac{2}{3})} P^{p,q,k,\{h_i\}}_{SPFK,\overline{\mathcal{B}}(\Lambda)\cap\mathcal{B}(B_x^+),\rho}(x \overset{*}{\leftrightarrow} B_x^c). \\
& \leq \sum_{x \in \partial\Gamma^*(\mathcal{E},\frac{2}{3})} c_{24} e^{-\epsilon_{10} d(x,\mathcal{E})/4}
\end{aligned}$$

where the sup is over all bond boundary conditions, and where for the last inequality we use Theorem 1.1 and the fact that $\overline{\mathcal{B}}(\Lambda) \cap \mathcal{B}(B_x^+)$ is simply lattice-connected. Similarly to (2.28), we have

$$(2.30)\qquad \sum_{x \in \partial\Gamma^*(\mathcal{E},\frac{2}{3})} c_{24} e^{-\epsilon_{10} d(x,\mathcal{E})/4} \leq \sum_{y \in \mathcal{E}, z \in \mathcal{F}} c_{25} e^{-\epsilon_{11} d(y,z)}.$$

Now (2.25)—(2.30) prove that

$$\begin{aligned}
(2.31)\qquad & |\mathbb{P}(F \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}}, U(\sigma) = U_0) - \mathbb{P}(F \mid \omega_{\mathcal{E}} = \rho^1_{\mathcal{E}}, U(\sigma) = \phi)| \\
& \leq c_{26} \sum_{x \in \mathcal{E}, y \in \mathcal{F}} e^{-\epsilon_{12} d(x,y)}.
\end{aligned}$$

Since U_0 is arbitrary, this shows that

$$(2.32)\qquad |P_{\overline{\mathcal{B}}(\Lambda),\eta}(F \mid \omega_{\mathcal{E}} = \rho_{\mathcal{E}}) - P_{\overline{\mathcal{B}}(\Lambda),\eta}(F \mid \omega_{\mathcal{E}} = \rho'_{\mathcal{E}})| \leq c_{26} \sum_{x \in \mathcal{E}, y \in \mathcal{F}} e^{-\epsilon_{12} d(x,y)},$$

which completes the proof. □

To move from strong mixing to ratio strong mixing in our varied contexts, we need to extend some definitions and results in [5]. A *component partition* of a set $\mathcal{E} \subset \mathcal{B}(\mathbb{Z}^n)$ is a partition $\mathcal{E} = \mathcal{E}_1 \cup .. \cup \mathcal{E}_m$ such that each component of $\mathcal{E}$ is contained in some $\mathcal{E}_i$. For a bond percolation model P on $\mathcal{B}(\mathbb{Z}^n)$, for $\Lambda \subset \mathbb{Z}^n$ finite, η a site boundary condition on $\partial\Lambda$, $\mathcal{B} \subset \overline{\mathcal{B}}(\Lambda)$, $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{B} = \mathcal{E} \cup \mathcal{F}$ a component partition, $\epsilon > 0$, and $\rho \in \{0,1\}^{\overline{\mathcal{B}}(\Lambda)}$ we say $\mathcal{B}$ is an

ϵ-near blocking region for $\mathcal{E}, \mathcal{F}$ in ρ if for every configuration ρ' such that $\rho = \rho'$ on $\mathcal{B}$ and every event $A \in \mathcal{G}_{\mathcal{E}}$ we have

$$(1-\epsilon)P_{\Lambda,\eta}(A \mid \omega_{\mathcal{B}\cup\mathcal{F}} = \rho'_{\mathcal{B}\cup\mathcal{F}}) \leq P_{\Lambda,\eta}(A \mid \omega_{\mathcal{B}\cup\mathcal{F}} = \rho_{\mathcal{B}\cup\mathcal{F}}) \leq (1+\epsilon)P_{\Lambda,\eta}(A \mid \omega_{\mathcal{B}\cup\mathcal{F}} = \rho'_{\mathcal{B}\cup\mathcal{F}}).$$

In other words, the configuration on $\mathcal{B}$ blocks the configuration on $\mathcal{F}$ from influencing probabilities for events on $\mathcal{E}$ by more than a factor of $1 \pm \epsilon$. A 0-blocking region is also called a *fully blocking region.* Blocking regions are the finite-volume analogs of contolling regions. Let $\mathfrak{C}$ be a class of subsets of $\mathcal{B}(\mathbb{Z}^n)$. We say that P has *exponentially bounded blocking regions* for the class $\mathfrak{C}$ and metric d if there exist C, λ such that for every Λ, η and $\mathcal{B} \subset \overline{\mathcal{B}}(\Lambda)$ with $(\overline{\mathcal{B}}(\Lambda), \eta) \in \mathfrak{C}$ and every component partition $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{B} = \mathcal{E} \cup \mathcal{F}$, for $\epsilon = C\sum_{x\in V(\mathcal{E}), y\in V(\mathcal{F})} e^{-\lambda d(x,y)}$, we have

$$P_{\overline{\mathcal{B}}(\Lambda),\eta}(\mathcal{B} \text{ is not an } \epsilon\text{-near blocking region for } \mathcal{E}, \mathcal{F}) < \epsilon. \tag{2.33}$$

We say that P has *exponentially bounded fully blocking regions* for the metric d if in place of (2.33) we have

$$P_{\overline{\mathcal{B}}(\Lambda),\eta}(\mathcal{B} \text{ is not a fully blocking region for } \mathcal{E}, \mathcal{F}) < \epsilon.$$

Essentially the same definition applies under bond boundary conditions.

Lemma 2.11. *Let $P = P^{p,q,\{h_i\}}$ be an FK model on $\mathcal{B}(\mathbb{Z}^n)$, let d be the Euclidean or restricted-path metric and let $\mathfrak{C}$ be a class of closure subsets of $\mathbb{Z}^n$ which is inheriting with respect to d.*

(i) If P has uniform exponential decay of finite-volume connectivities for the class $\mathfrak{C}$ with arbitrary site boundary conditions, then $P^{p,q,\{h_i\}}$ has exponentially bounded fully blocking regions for the class $\mathfrak{C}$ with arbitrary site boundary conditions, in the metric d.

(ii) If $n = 2$, $\mathfrak{C}$ is the class of all simply lattice-connected closure subsets and P has exponential decay of connectivities, then $P^{p,q,\{h_i\}}$ has exponentially bounded fully blocking regions for the class $\mathfrak{C}$ with arbitrary site boundary conditions, in the metric d.

(iii) If $n = 2$ and P has a unique stable species, then $P^{p,q,\{h_i\}}$ has exponentially bounded blocking regions for the class $\mathfrak{C}$ with arbitrary single-stable-species site boundary conditions, in the metric d. If d is the Euclidean metric, then arbitrary bond boundary conditions can be included as well.

(iv) If $n = 2$, $\mathfrak{C}$ is the class of all simply lattice-connected subsets and $P^{p,q,\{h_i\}}$ has exponential decay of dual connectivities, then $P^{p,q,\{h_i\}}$ has exponentially bounded blocking regions for the class $\mathfrak{C}$ with free and wired boundary conditions, in the metric d.

Proof. Fix $\Lambda \subset \mathbb{Z}^n$ finite, η a site boundary condition on $\partial\Lambda$, $\mathcal{B} \subset \overline{\mathcal{B}}(\Lambda)$, and $\overline{\mathcal{B}}(\Lambda)\backslash\mathcal{B} = \mathcal{E} \cup \mathcal{F}$ a component partition.

If in a configuration $\omega \in \{0,1\}^{\overline{\mathcal{B}}(\Lambda)}$ there is no open path from $V(\mathcal{E})$ to $V(\mathcal{F})$, then there exists a collection of dual surfaces which together separate $\mathcal{E}$ from $\mathcal{F}$, such that none of these dual surfaces is crossed by an open bond in $\overline{\mathcal{B}}(\Lambda)$. (These dual surfaces may include plaquettes dual to bonds outside $\overline{\mathcal{B}}(\Lambda)$.) It therefore follows from a straightforward minor extension of the Markov property for open dual circuits (cf. (2.12)) that

$$\begin{aligned} (2.34) \qquad & P_{\overline{\mathcal{B}}(\Lambda),\eta}(\mathcal{B} \text{ is not a fully blocking region for } \mathcal{E},\mathcal{F}) \\ & \qquad \leq P_{\overline{\mathcal{B}}(\Lambda),\eta}(x \leftrightarrow y \text{ for some } x \in V(\mathcal{E}), y \in V(\mathcal{F})), \end{aligned}$$

and (i) follows.

Under the assumptions of (ii), it follows from Theorem 1.2 that the assumptions of (i) are satisfied.

Under the assumptions of (iii), with single-stable-species site boundary condition η, let ρ, ρ' be configurations such that $\rho = \rho'$ on $\mathcal{B}$, and let $\Theta = \{x \in \partial\Lambda : \eta_x \neq 0\}$ be the set of boundary sites where the stable species resides. The difference between the measures $P_{\overline{\mathcal{B}}(\Lambda),\eta}(\omega_{\mathcal{E}} \in \cdot \mid \omega_{\mathcal{B}\cup\mathcal{F}} = \rho_{\mathcal{B}\cup\mathcal{F}})$ and $P_{\overline{\mathcal{B}}(\Lambda),\eta}(\omega_{\mathcal{E}} \in \cdot \mid \omega_{\mathcal{B}\cup\mathcal{F}} = \rho'_{\mathcal{B}\cup\mathcal{F}})$ appears only in the weight assigned to clusters of $(\omega_{\mathcal{E}}, \rho_{\mathcal{B}\cup\mathcal{F}})$ which intersect $V(\mathcal{E})$ and $V(\mathcal{F})$ but not Θ. More precisely, considering the first measure, let $\mathcal{C}^0_{\mathcal{E},\mathcal{F}}(\omega_{\mathcal{E}}, \rho_{\mathcal{B}\cup\mathcal{F}})$ denote the set of all such clusters. For $C \in \mathcal{C}^0_{\mathcal{E},\mathcal{F}}(\omega_{\mathcal{E}}, \rho_{\mathcal{B}\cup\mathcal{F}})$ let $\hat{s}(C) = s(C)$ if $C \cap \Theta = \phi$ and $\hat{s}(C) = \infty$ if $C \cap \Theta \neq \phi$. Then the combined weight assigned to all clusters in $\mathcal{C}^0_{\mathcal{E},\mathcal{F}}(\omega_{\mathcal{E}}, \rho_{\mathcal{B}\cup\mathcal{F}})$ under the measure $P_{\overline{\mathcal{B}}(\Lambda),\eta}(\omega_{\mathcal{E}} \in \cdot \mid \omega_{\mathcal{B}\cup\mathcal{F}} = \rho_{\mathcal{B}\cup\mathcal{F}})$ is (cf. (1.12)—(1.14))

$$\begin{aligned} (2.35) \qquad & \prod_{C \in \mathcal{C}^0_{\mathcal{E},\mathcal{F}}(\omega_{\mathcal{E}}, \rho_{\mathcal{B}\cup\mathcal{F}})} \left(1 + (1-p)^{-h_2 \hat{s}(C)} + \ldots + (1-p)^{-h_{q+1}\hat{s}(C)}\right) \\ & \qquad \leq \prod_{C \in \mathcal{C}^0_{\mathcal{E},\mathcal{F}}(\omega_{\mathcal{E}}, \rho_{\mathcal{B}\cup\mathcal{F}})} \left(1 + q(1-p)^{-h_2 s(C)}\right) \\ & \qquad \leq \prod_{x \in V(\mathcal{E})} \left(1 + c_{28} e^{-\epsilon_{13} d(x, V(\mathcal{F}))}\right) \end{aligned}$$

since $h_2 < 0$; here we interpret $(1-p)^\infty$ as 0. Let

$$\delta = \sum_{x \in V(\mathcal{E})} 2c_{28} e^{-\epsilon_{13} d(x, V(\mathcal{F}))}$$

and suppose $\delta \leq 1$. Then by (2.35), the left side of (2.35) is between 1 and $1 + \delta$, and the same holds with $\rho_{\mathcal{B}\cup\mathcal{F}}$ replaced by $\rho'_{\mathcal{B}\cup\mathcal{F}}$. It follows that, in every configuration $\omega_{\overline{\mathcal{B}}(\Lambda)}$, $\mathcal{B}$ is a δ-near blocking region for $\mathcal{E},\mathcal{F}$. On the other hand, if $\delta > 1$, then certainly $P_{\overline{\mathcal{B}}(\Lambda),\eta}(\mathcal{B} \text{ is not a } \delta\text{-near blocking region for } \mathcal{E},\mathcal{F}) < \delta$. This proves (iii) for single-stable-species site boundary conditions.

Next, consider (iii) with d Euclidean and a bond boundary condition $\tilde{\rho}$. We proceed similarly to the site boundary condition case but in place of $\mathcal{C}^0_{\mathcal{E},\mathcal{F}}(\omega_{\mathcal{E}}, \rho_{\mathcal{B}\cup\mathcal{F}})$ we use the class $\mathcal{C}^1_{\mathcal{E},\mathcal{F}}(\omega_{\mathcal{E}}, \rho_{\mathcal{B}\cup\mathcal{F}})$ consisting of all clusters of $(\omega_{\mathcal{E}}, \rho_{\mathcal{B}\cup\mathcal{F}}, \tilde{\rho})$ which intersect both $V(\mathcal{E})$ and $V(\mathcal{F})$.

(Such clusters now need not pass through $\mathcal{B}$ and may connect $V(\mathcal{E})$ to $V(\mathcal{F})$ via the boundary configuration $\tilde{\rho}$.) As before, the difference between the measures given $\rho_{\mathcal{B}\cup\mathcal{F}}$ and given $\rho'_{\mathcal{B}\cup\mathcal{F}}$ lies only in the combined weight assigned to such clusters. Since d is the Euclidean distance, for $C \in \mathcal{C}^1_{\mathcal{E},\mathcal{F}}(\omega_\mathcal{E}, \rho_{\mathcal{B}\cup\mathcal{F}})$ and $x \in C \cap V(\mathcal{E})$ we have $s(C) \geq d(x, V(\mathcal{F}))$. Therefore proceeding as in (2.35) we obtain (iii).

Finally, under the assumptions of (iv), we again consider the class $\mathcal{C}^0_{\mathcal{E},\mathcal{F}}(\omega_\mathcal{E}, \rho_{\mathcal{B}\cup\mathcal{F}})$. Let k be the number of stable species. Suppose there is no open dual path from $V^*(\mathcal{E}^*)$ to $V^*(\mathcal{F}^*)$ in $\mathcal{B}^+(\Lambda)^* \cap \mathcal{B}^*$. If η is wired, this means every cluster intersecting both $V(\mathcal{E})$ and $V(\mathcal{F})$ must also intersect $\partial\Lambda$ in $\mathcal{B}$. This in turn means $\mathcal{C}^0_{\mathcal{E},\mathcal{F}}(\omega_\mathcal{E}, \rho_{\mathcal{B}\cup\mathcal{F}}) = \phi$ and hence $P_{\overline{\mathcal{B}}(\Lambda),\eta}(\omega_\mathcal{E} \in \cdot \mid \omega_{\mathcal{B}\cup\mathcal{F}} = \rho_{\mathcal{B}\cup\mathcal{F}}) = P_{\overline{\mathcal{B}}(\Lambda),\eta}(\omega_\mathcal{E} \in \cdot \mid \omega_{\mathcal{B}\cup\mathcal{F}} = \rho'_{\mathcal{B}\cup\mathcal{F}})$, so $\mathcal{B}$ is a fully blocking region. Thus

$$\begin{aligned} &P_{\overline{\mathcal{B}}(\Lambda),\eta}(\mathcal{B} \text{ is not a fully blocking region for } \mathcal{E},\mathcal{F}) \\ &\qquad \leq P_{\overline{\mathcal{B}}(\Lambda),\eta}(V^*(\mathcal{E}^*) \overset{*}{\leftrightarrow} V^*(\mathcal{F}^*)) \\ &\qquad \leq \sum_{x\in V(\mathcal{E}),y\in V(\mathcal{F})} c_{29} e^{-\epsilon_{14} d(x,y)}, \end{aligned}$$

completing the proof for η wired. If instead η is free, the absense of an open dual path from $V^*(\mathcal{E}^*)$ to $V^*(\mathcal{F}^*)$ in $\mathcal{B}^+(\Lambda)^* \cap \mathcal{B}^*$ means that for each connected component $\mathcal{E}_i$ of $\mathcal{E}$ and each connected component $\mathcal{F}_j$ of $\mathcal{F}$, there is at most one cluster in $\mathcal{C}^0_{\mathcal{E},\mathcal{F}}(\omega_\mathcal{E}, \rho_{\mathcal{B}\cup\mathcal{F}})$ which intersects both $V(\mathcal{E}_i)$ and $V(\mathcal{F}_j)$. If this cluster, call it C_{ij}, exists, the weight assigned to it is

$$k + (1-p)^{-h_{k+1} s(C_{ij})} + \ldots + (1-p)^{-h_{q+1} s(C_{ij})}$$

which is between k and $k + q e^{-\epsilon_{103} d(\mathcal{E}_i,\mathcal{F}_j)}$, uniformly in $\omega_{\overline{\mathcal{B}}(\Lambda)}$. This means that $\mathcal{B}$ is an ϵ-near blocking region for $\mathcal{E},\mathcal{F}$, for

$$\epsilon = \sum_{i,j} \epsilon_{12} e^{-\epsilon_{104} d(\mathcal{E}_i,\mathcal{F}_j)} \leq \sum_{x\in V(\mathcal{E}),y\in V(\mathcal{F})} \epsilon_{12} e^{-\epsilon_{104} d(x,y)}.$$

Thus we have

$$\begin{aligned} &P_{\overline{\mathcal{B}}(\Lambda),\eta}(\mathcal{B} \text{ is not an } \epsilon\text{-near blocking region for } \mathcal{E},\mathcal{F}) \\ &\qquad \leq P_{\overline{\mathcal{B}}(\Lambda),\eta}(V^*(\mathcal{E}^*) \overset{*}{\leftrightarrow} V^*(\mathcal{F}^*)) \\ &\qquad \leq \sum_{x\in V(\mathcal{E}),y\in V(\mathcal{F})} c_{29} e^{-\epsilon_{14} d(x,y)}, \end{aligned}$$

completing the proof for η free. □

The weak-mixing analog of the following theorem is ([5], Theorem 3.3). The proof for strong mixing is essentially the same so we do not include it here.

Theorem 2.12. *Let P be a bond percolation model on $\mathcal{B}(\mathbb{Z}^n)$, let $\mathfrak{C}$ be a class of subsets of $\mathcal{B}(\mathbb{Z}^n)$ together with (site or bond) boundary conditions and let d be the Euclidean or restricted-path metric. Suppose P has the strong mixing property for the class $\mathfrak{C}$ in the metric d, and suppose P has exponentially bounded blocking regions. Then P has the ratio strong mixing property for the class $\mathfrak{C}$ in the metric d.*

Remark 2.13. The obvious spin-system analog of Theorem 2.12 is also valid. In particular, nearest-neighbor systems such as the Potts model always have exponentially bounded blocking regions, so that for such systems, strong mixing implies ratio strong mixing, for any class $\mathfrak{C}$.

Write $x \overset{f}{\leftrightarrow} y$ for the event $[x \leftrightarrow y, x \not\leftrightarrow \infty]$ (if the context is infinite volume) or the event $[x \leftrightarrow y, x \not\leftrightarrow \partial\Lambda]$ (if the context is a finite volume Λ.) Let e_i denote the ith unit coordinate vector. The following lemma was proved in [8] in the absense of external fields. The proof there also works under external fields, but it is complex and interwoven with other proofs, so we present a short direct proof here.

Lemma 2.14. *Consider an FK model $P = P^{p,q,\{h_i\}}$ on $\mathcal{B}(\mathbb{Z}^2)$ with $p > p_c(q, 2, \{h_i\})$. If $P(0 \overset{f}{\leftrightarrow} x)$ decays exponentially in $|x|$, then P has exponential decay of dual connectivities.*

Proof. Suppose the dual connectivity does not decay exponentially, that is, $\tau(x) = 0$ for all x. We claim first that the dual connectivity also does not decay exponentially in halfspaces. Let $\epsilon > 0$ and let $p_\infty = P(0 \to \infty)$, $H_t = \{(x_1, x_2) : x_2 \leq t\}$ and $\tilde{H}_t = \{(x_1, x_2) : x_1 \geq t\}$. Let k be large enough so $P(0^* \overset{*}{\leftrightarrow} (ke_1)^*) \geq e^{-\epsilon k}$. When there is an open dual path from 0^* to $(ke_1)^*$ we choose an uppermost point Y of the path, making an arbitrary choice of path and/or of Y if more than one choice is possible. We may assume $y_1 \geq k/2$; the other case is symmetric. There is also a leftmost point of the path, which we call X. Let

$$c_{30} = \left(\sum_{x \in \mathbb{Z}^2} \frac{1}{(1 + |x|)^3} \right)^{-1}.$$

There must exist $x, y \in (\mathbb{Z}^2)^*$ such that

$$\begin{aligned} &P(0^* \overset{*}{\leftrightarrow} (ke_1)^*, X = x, Y = y) \\ &\quad\geq \frac{c_{30}}{(1+|x|)^3} \frac{c_{30}}{(1+|y-x|)^3} P(0^* \overset{*}{\leftrightarrow} (ke_1)^*) \\ &\quad\geq \frac{c_{30}}{(1+|x|)^3} \frac{c_{30}}{(1+|y-x|)^3} e^{-\epsilon k}. \end{aligned} \tag{2.36}$$

Consider first the case of $|x| \leq |y - x|$. We have two subcases, according to which of $|y_1 - x_1|, |y_2 - x_2|$ is larger; we may assume $|y_1 - x_1| \geq |y_2 - x_2|$, as the other subcase is similar after a 90° rotation. We have

$$P(x \overset{*}{\leftrightarrow} y \text{ via an open dual path in } H_{y_2}) \geq P(0^* \overset{*}{\leftrightarrow} (ke_1)^*, X = x, Y = y) \geq \frac{c_{31}}{|y-x|^6} e^{-\epsilon k}.$$

Let z be the reflection of y across the vertical line through x, and $m = y_1 - x_1$. Then since $m \geq k/2$,

$$\begin{aligned} &P(0^* \overset{*}{\leftrightarrow} (2me_1)^* \text{ via an open dual path in } H_{1/2}) \\ &\quad = P(z \overset{*}{\leftrightarrow} y \text{ via an open dual path in } H_{y_2}) \\ &\quad \geq P(x \overset{*}{\leftrightarrow} y \text{ via an open dual path in } H_{y_2})^2 \\ &\quad \geq \frac{c_{32}^2}{m^{12}} e^{-4\epsilon m}. \end{aligned} \tag{2.37}$$

Since ϵ is arbitrary, it follows that the left side of (2.37) does not decay exponentially in m, proving the claim. The other case, $|y-x| \leq |x|$, is similar, using $0^* \overset{*}{\leftrightarrow} x$ in place of $x \overset{*}{\leftrightarrow} y$.

Since $p \neq p_c(q, 2, \{h_i\})$, random cluster uniqueness holds [7]. Fix $l \geq 1$; we have by the FKG property

$$P(0 \leftrightarrow le_1) \geq P(|C_0| = \infty, |C_{le_1}| = \infty) \geq p_\infty^2.$$

Let $\Theta_{j,r} = [-r, j+r] \times [-r, r]$. By random cluster uniqueness, if r is sufficiently large (depending on l),

$$\inf_\rho P_{\Theta_{l,r},\rho}(0 \leftrightarrow le_1 \text{ via an open path in } \Theta_{l,r}) \geq \frac{1}{2} p_\infty^2,$$

where the inf is over all boundary conditions ρ on $\Theta_{l,r}$. From this we obtain, using the FKG property again and choosing r of form $s + 1/2$ for some integer s, that for $\delta > 0$ and r, n sufficiently large,

$$\begin{aligned} &P(0 \overset{f}{\leftrightarrow} nle_1) \\ &\quad \geq P\Big(0 \leftrightarrow nle_1 \text{via an open path in } \Theta_{nl,r}, \\ &\qquad\qquad (-r,-r) \overset{*}{\leftrightarrow} (-r,r) \overset{*}{\leftrightarrow} (nl+r,r) \overset{*}{\leftrightarrow} (nl+r,-r) \overset{*}{\leftrightarrow} (-r,-r) \\ &\qquad\qquad \text{via open dual paths outside } \Theta_{nl,r}\Big) \\ &\quad \geq \left(\frac{1}{2} p_\infty^2\right)^n e^{-\delta(8r+2nl)}. \end{aligned}$$

Since δ and l are arbitrary, it follows that $P(0 \overset{f}{\leftrightarrow} ne_1)$ does not decay exponentially in n. □

Let $\mu_{\mathbb{Z}^2,i}^{\beta,\{h_i\}}$ denote the infinite-volume Potts model on $\mathbb{Z}^2$ at $(\beta, \{h_i\})$ with species-i boundary condition, for stable i. Part (i) of the next lemma is well-known in the absense of external fields.

Lemma 2.15. *Consider the q-state Potts model $\mu^{\beta,q,\{h_i\}}$ on $\mathbb{Z}^2$. Suppose there are multiple stable species.*

(i) Suppose Gibbs uniqueness holds and $\mu^{\beta,q,\{h_i\}}$ has exponential decay of correlations. Then the corresponding FK model has exponential decay of connectivities.

(ii) Suppose $\beta > \beta_c(q, 2, \{h_i\})$ and $\mu_{\mathbb{Z}^2,1}^{\beta,q,\{h_i\}}$ has exponential decay of correlations. Then the corresponding FK model has exponential decay of dual connectivities.

Proof. Suppose there are k stable species, and let $\mathbb{P}$ denote the distribution of the corresponding Edwards-Sokal joint Potts-FK configuration in a finite volume Λ with all-1 boundary condition. Under $\mathbb{P}$ we have conditional covariance, given the bond configuration, given by

$$\begin{aligned}\operatorname{cov}(\delta_{[\sigma_x=1]}, \delta_{[\sigma_y=1]} \mid \omega) &= \begin{cases} \frac{1}{k+r_n}\left(1 - \frac{1}{k+r_n}\right), & \text{if } x \overset{f}{\leftrightarrow} y \text{ and } s(C_x) = s(C_y) = n \\ 0, & \text{otherwise} \end{cases} \\ &\geq \frac{1}{k+r_1}\left(1 - \frac{1}{k+r_1}\right)\delta_{[x \overset{f}{\leftrightarrow} y]}\end{aligned}$$

and conditional expectation

$$M_x = \mathbb{E}(\delta_{[\sigma_x=1]} \mid \omega) = 1 - \left(1 - \frac{1}{k + r_{s(C_x)}}\right)\delta_{[x \not\leftrightarrow \partial\Lambda]}.$$

Since this conditional expectation is an increasing function of ω, by the FKG property of the FK model we have $\operatorname{cov}(M_x, M_y) \geq 0$ for all x, y. Therefore

$$\operatorname{cov}(\delta_{[\sigma_x=1]}, \delta_{[\sigma_y=1]}) \geq \frac{1}{k+r_1}\left(1 - \frac{1}{k+r_1}\right) P(x \overset{f}{\leftrightarrow} y),$$

where P is the finite-volume FK measure. This same inequality then holds in infinite volume. (Note the infinite-volume FK measure is necessarily unique under both (i) and (ii)—see the remarks preceding Proposition 2.4.) It follows that $P(x \overset{f}{\leftrightarrow} y)$ decays exponentially, This proves (i). Under (ii) there is percolation in the FK model, so (ii) follows from Lemma 2.14. ∎

Proof of Theorem 1.6. The theorem follows from Proposition 2.3, Lemma 2.11(i), and Theorem 2.12. ∎

Proof of Theorem 1.7. (i) is an immediate consequence of Theorems 1.2 and 1.6. (ii) follows from Proposition 2.5, Lemma 2.11(iv), and Theorem 2.12. (iii) follows from Proposition 2.4(iii), Lemma 2.11(iii) and Theorem 2.12. □

Note that we don't need Proposition 2.5 in the proof of Theorem 1.7 if there are no external fields, as (ii) then follows from (i) and duality.

Proof of Theorem 1.8. Let P be the corresponding FK model. Under the hypotheses of (i), by Lemma 2.15(i) P has exponential decay of connectivities. Hence under (i) or (ii), by Proposition 2.4(i), $\mu^{\beta,q,\{h_i\}}$ has the strong mixing property for the class $\mathfrak{C}$ with arbitrary boundary conditions. By Remark 2.13, the desired ratio strong mixing property holds.

Under (iii), Gibbs uniqueness holds, so by Proposition 2.4(iii), $\mu^{\beta,q,\{h_i\}}$ has the strong mixing property for the class $\mathfrak{C}$ with arbitrary boundary conditions. Again by Remark 2.13, the desired ratio strong mixing property holds. □

Proof of Theorem 1.9. By the method of proof of Proposition 2.4(i) (noting that, due to our stronger assumption, we do not need Theorem 1.2), using the FK model, $\mu^{\beta,q,\{h_i\}}$ has the strong mixing property for the class $\mathfrak{C}$ with arbitrary boundary conditions. By Remark 2.13, the desired ratio strong mixing property holds. □

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