Long-range scattering of three-body quantum systems, II

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1 Introduction and results

This paper is the second in a series of two papers on asymptotic completeness for (generalized) three-body quantum systems with long-range interaction. Asymptotic completeness is henceforth abbreviated AC; an "effective" version of AC is the existence of the limits (1.4) and (1.5) given below. For a full account of the statement we refer the reader to our first paper [S1]. We shall in this paper prove results in the regime "$\mu \in (0, \frac{1}{2}]$" with μ measuring the decay of the "pair potentials" at infinity as in (1.2) (given below). All examples in the literature on AC for many-body systems seem so far to be restricted to μ greater than one half. Moreover Yafaev ([Y]) constructed counter-examples to AC for any $\mu \in \left(0, \frac{1}{2}\right)$ in systems of one-dimensional particles. On the other hand there are many papers in the literature on AC for two-body systems with arbitrary $\mu > 0$. We have results for two different classes of potentials: 1) One-dimensional potentials with a negative upper bound near infinity (like those considered in [S1]). 2) Potentials (in any dimension) with a positive lower bound near infinity.

We recall the basic model and a reduction scheme for AC; see [S1] for more details. We consider a finite family of subspaces $\{X_a | a \in \mathcal{F}\}$ of a finite dimensional Euclidean space X. By definition $a_{\min}, a_{\max} \in \mathcal{F}$ are given by $X_{a_{\min}} = X$ and $X_{a_{\max}} = \{0\}$, respectively, and for a and b different from $a_{\min}$ the "three-body" condition $X_a \cap X_b = \{0\}$ is imposed. The position and momentum operators on the basic Hilbert space $\mathcal{H} = L^2(X)$ are denoted by x and p, respectively. The orthogonal complement of X_a in X is denoted by X^a. The corresponding components of x and p are denoted by x_a, p_a and x^a, p^a, respectively.

The basic Hamiltonian on $\mathcal{H}$ is

$$H = \tfrac{1}{2}p^2 + V;\ V(x) = \sum_{a \in \mathcal{F}} V^a(x^a), \tag{1.1}$$

where each "pair potential" V^a is assumed to be a real-valued smooth function on X^a obeying for some $\mu > 0$ (independent of a) and all multiindices β

$$\partial^\beta_{x^a} V^a(x^a) = O\Big(|x^a|^{-\mu-|\beta|}\Big). \tag{1.2}$$

We consider for $a \neq a_{\max}, a_{\min}$ the propagator $U_a(t)$ generated by $H_a(t) = H^a + \frac{1}{2}p_a^2 + I_a(t,x)$ (i.e. $i\partial_t U_a(t) = H_a(t)U_a(t)$ and $U_a(1) = I$), where by definition $H^a = \frac{1}{2}(p^a)^2 + V^a$ and $I_a(t,x) = J\big(\frac{x}{t}\big)I_a(x)$ with J an arbitrary C_0^∞ cutoff function supported in $Y_a = X\backslash\big(\cup_{b\not\subset a} X_b\big)$ and with $I_a(x) = V(x) - V^a(x^a)$. (Here $b \subset a \Leftrightarrow X^b \subset X^a$.) Clearly we have the bound

$$\partial^\beta_x I_a(t,x) = O\Big(t^{-\mu-|\beta|}\Big) \text{ uniformly in } x. \tag{1.3}$$

There exists the asymptotic energy $H^{a+} = \lim\limits_{t\to+\infty} U_a(t)^\star H^a U_a(t)$ (understood in the strong resolvent sense). Let $\tilde{U}_a(t)$ denote the propagator generated by $H^a + \frac{1}{2}p_a^2 + I_a(t,x_a)$.

The notation $E_\Omega(D)$ denotes the spectral projection for a self-adjoint operator D corresponding to a Borel set $\Omega \subseteq \mathbf{R}$.

Now, the "effective" version of AC that we are going to address in this paper is the following statement: For all $\phi_a^+ \in E_{\{0\}}\big(H^{a+}\big)$ there exists the limit

$$\tilde{\phi}_a^+ = \lim_{t\to+\infty} \tilde{U}_a(t)^* U_a(t)\phi_a^+; \tag{1.4}$$

and for all $\tilde{\phi}_a^+ \in E_{\{0\}}(H^a)$ there exists the limit

$$\phi_a^+ = \lim_{t\to+\infty} U_a(t)^* \tilde{U}_a(t)\tilde{\phi}_a^+. \tag{1.5}$$

1.1 Negative potentials

In addition to (1.2) we shall need the following negativity condition of [S1] for a (fixed) $a \neq a_{\max}, a_{\min}$: For some $c, R > 0$

$$V^a(x^a) \leq -c|x^a|^{-\mu},\ |x^a| \geq R. \tag{1.6}$$

Moreover we shall assume that the subspace X^a is one-dimensional. In our last set of conditions (1.7)-(1.11) stated below we identify the part of x denoted by x^a by a coordinate for this vector given by fixing a basis vector for the subspace X^a. Suppose

$$V^a(x^a) = V_1^a(x^a) + V_2^a(x^a) + V_3^a(x^a), \tag{1.7}$$

where $V_1^a(x^a)$, $V_2^a(x^a)$ and $V_3^a(x^a)$ obey (1.2),

$$\begin{aligned} &-C_r|x^a|^{-\mu-2} \leq V_1^{a\prime\prime}(x^a) \leq -c_r|x^a|^{-\mu-2};\ x^a \geq R, \\ &c_r, C_r > 0 \text{ and } C_r < 2^{-1}(2+\mu)^2 c_r, \end{aligned} \tag{1.8}$$

$$
\begin{aligned}
&-C_l|x^a|^{-\mu-2} \le V_1^{a\prime\prime}(x^a) \le -c_l|x^a|^{-\mu-2};\ x^a \le -R,\\
&c_l, C_l > 0 \text{ and } C_l < 2^{-1}(2+\mu)^2 c_l,
\end{aligned} \tag{1.9}
$$

and for some $\epsilon > 0$

$$
V_2^{a\prime}(x^a) = O\big(|x^a|^{-1-\mu-\epsilon}\big), \tag{1.10}
$$

and

$$
V_3^a(x^a) = O\Big(|x^a|^{-\frac{1+\epsilon}{\alpha}}\Big). \tag{1.11}
$$

In (1.11) and henceforth $\alpha = 2(2+\mu)^{-1}$.

Now, suppose the conditions (1.2), (1.3) and (1.6) for some $a \neq a_{\max}, a_{\min}$ with $\dim X^a = 1$; all conditions with the same (fixed) $\mu \in (0, \frac{1}{2}]$. Then one can introduce, cf. [S1],

$$
\begin{aligned}
P^{a+} &= P_r^{a+} + P_l^{a+},\\
P_r^{a+} &= s - \lim_{t\to+\infty} U_a(t)^* E_{[t^{\alpha-\epsilon}, t^{\alpha+\epsilon}]}(x^a) U_a(t) E_{\{0\}}\big(H^{a+}\big),\\
P_l^{a+} &= s - \lim_{t\to+\infty} U_a(t)^* E_{[-t^{\alpha+\epsilon}, -t^{\alpha-\epsilon}]}(x^a) U_a(t) E_{\{0\}}\big(H^{a+}\big).
\end{aligned} \tag{1.12}
$$

It was proven that these limits are independent of (small) $\epsilon > 0$ and that $P^{a+} = E_{\{0\}}\big(H^{a+}\big)$. We shall show the existence of (1.4) by proving that $P^{a+} = 0$, cf. [S1].

Our main result is the following.

Theorem 1.1 *Under the conditions (1.2), (1.3), (1.6)-(1.11) for a fixed* $a \neq a_{\max}, a_{\min}$ *with* $\dim X^a = 1$ *and for* $\mu \in (0, \frac{1}{2}]$

$$
E_{\{0\}}\big(H^{a+}\big) = 0; \tag{1.13}
$$

in particular the existence of (1.4) holds.

The existence of (1.5) with these assumptions follows from the fact that in this case $E_{\{0\}}(H^a) = 0$, cf. [O, Theorem 2.2 p. 196]. Combined with (1.13), AC follows.

Remark 1.2 For simplicity of presentation we shall prove Theorem 1.1 with the additional assumption that $V_2^a = V_3^a = 0$. The general case may be treated in the following fashion: First we may assume $V_3^a = 0$ since V_3^a is "short-range". Next we keep V_2^a in the analysis (of Sections 2–7). We define the classical orbit in (2.3) in terms of the "dominating" term V_1^a only. Keeping track of contributions from error terms coming from V_2^a yields a weaker localization than (2.4), but strong enough for the arguments of Section 7 ((7.2) needs to be replaced by a weaker estimate).

As indicated in the above remark we devote Sections 2–7 to a proof of a slightly simplified version of Theorem 1.1. Our basic strategy is similar to one applied to a different problem, although with common spirit, in [HS]. We compare the evolution of a state $\phi_a^+ \in P_r^{a+}\mathcal{H}$ (or $\phi_a^+ \in P_l^{a+}\mathcal{H}$) with a simplified evolution in terms of a relative wave operator. Setting up this wave operator is the content of Sections 2–6. Our techniques at this point resemble at many points those applied in [HS]. In Section 7 we verify conditions of [S2] for the simplified evolution. Using the relevant result of [S2] we infer that indeed states propagated with this evolution *cannot* be localized to regions of the configuration space that the projection P_r^{a+} a priori prescribes. Consequently $\phi_a^+ = 0$. Notice that this is a purely quantum statement; it *doesn't* have an analogue in classical mechanics. The technique of Section 7 differs completely from the one applied at the similar step in [HS]. We remark that one may modify the latter technique of [HS] as to provide another approach to our second step in Section 7. However it is more complicated. Morever we remark that neither of these approaches seem to be optimal. For example we consider the conditions (1.8) and (1.9) to be "technical"; the condition (2.1) stated below should suffice. As another open problem motivated by the analysis of [S1] we mention AC for negative potentials in higher dimensions with spherical symmetry.

In Appendix A we prove bounds for classical orbits of some one-dimensional quadratic Hamilton functions, that are needed at various points in Sections 5–7.

1.2 Positive potentials

In addition to (1.2) for a $\mu \in (0, \frac{1}{2}]$ we shall need the following positivity condition for a (fixed) $a \neq a_{\max}, a_{\min}$: For some $\mu^+ \in [\mu, \frac{2\mu}{1-\mu})$ and $R > 0$

$$V^a(x^a) \geq |x^a|^{-\mu^+}, \ |x^a| \geq R. \tag{1.14}$$

Our main result is the following.

Theorem 1.3 *Under the conditions (1.2), (1.3), (1.14) for a fixed $a \neq a_{\max}, a_{\min}$ and with $\mu \in (0, \frac{1}{2}]$, the limits (1.4) and (1.5) exist; in particular AC holds.*

The proof of Theorem 1.3 is given in Section 8. It is based on some energy bounds which may be viewed as modifications of results of [S1]. Conceptually and technically it is much simpler than the proof of AC for negative potentials due to the fact that there are no classical orbits (for the internal dynamics) at infinity with zero energy. To put our result and method into perspective we also give a proof of Wang's result for $\mu \in (\frac{1}{2}, \sqrt{3} - 1]$, [W], referred to in [S1].

Obviously Theorems 1.1 and 1.3 can be combined to obtain asymptotic completeness as defined in [S1] for families of pair potentials of mixed type, each either negative or positive (at infinity) with further properties as specified in the theorems.

2 Preliminary estimates (negative potentials)

In addition to (1.2), (1.3) and (1.6) with $\dim X^a = 1$ and $\mu \in (0, \frac{1}{2}]$, we shall in Sections 2–6 need the concavity assumption

$$V^{a\prime\prime}(x^a) \leq 0 \text{ for } |x^a| \geq R. \tag{2.1}$$

Only in Section 7 the stronger conditions (1.8) and (1.9) are needed (to verify (7.5)). Recall that we put $V_2^a = V_3^a = 0$, cf. Remark 1.2.

We aim at showing that for any given $\phi_a^+ \in P_r^{a+}\mathcal{H}$ indeed $\phi_a^+ = 0$. Since our proof can be adapted for P_l^{a+} we then conclude (1.13).

In this section we are going to use the freedom to change $I_a(t,x)$ to the effect

$$I_a(t,x) = I_a(t,x)F_+\left(\tfrac{x^a}{t^{\alpha-\epsilon}}\right) + I_a(t,x^a=0,x_a)F_-\left(\tfrac{x^a}{t^{\alpha-\epsilon}}\right) \tag{2.2}$$

for $\epsilon > 0$ chosen arbitrarily small, cf. [S1, Section 5]. (Here and henceforth we adapt the notation F_+ and F_- of [S1, Definitions 2.1].) The proof of [S1, Lemma 4.5] (with $j = 1$) yields the following improved (and classically "optimal") localization (cf. the proof of [S1, (5.6)]). We remark that we are not going to use the full strength of the result; it is stated here only for completeness of presentation.

Lemma 2.1 *Let L be the solution of the initial value problem*

$$\tfrac{d}{dt}L(t) = \sqrt{-2V^a(L(t))},\ L(0) = R;\ t \geq 0. \tag{2.3}$$

Then for all $\varepsilon > 0$ and $\phi_a^+ \in P_r^{a+}\mathcal{H}$

$$||F_+\big(t^{\mu-\varepsilon-1}|x^a - L|\big)\phi_a^+(t)|| = o\big(t^0\big), \tag{2.4}$$

where $\phi_a^+(t) = U_a(t)\phi_a^+$.

Proof Given $\varepsilon > 0$ we may assume (2.2) for a small $\epsilon > 0$ obeying $\frac{3}{2}\mu\epsilon < \varepsilon$. We modify the proof of [S1, (5.6)] by introducing for small $\sigma > 0$ obeying $\frac{3}{2}\mu\epsilon < 3\sigma < \varepsilon$

$$\begin{aligned}
&\alpha_0 = \alpha + \frac{\sigma}{\mu},\\
&\gamma_1 = \frac{\mu}{2}\alpha_0 + \sigma = \frac{\mu}{2}\alpha + \frac{3}{2}\sigma,\\
&\alpha_1 = 1 - \gamma_1 - \sigma = \alpha - \frac{5}{2}\sigma,\\
&\delta = \mu + \frac{\mu}{2}\alpha - \sigma,\\
&\beta_1 = 1 - \mu + \frac{\mu}{2}\alpha + 3\sigma.
\end{aligned} \tag{2.5}$$

All requirements of the proof of [S1, (5.6)] including [S1, (4.36)-(4.38)] are fulfilled for the choice (2.5) with $\sigma > 0$ small enough yielding the following statement:

$$\lim_{t\to+\infty} ||\phi_a^+(t) - \bar{B}\big(t, t^{\beta_1}\big)\phi_a^+(t)|| = 0, \tag{2.6}$$

where for parameters given by (2.5) and $K(x^a)$ being the inverse of $L(t)$

$$\begin{aligned} &\bar{B}\Big(t, t^{\beta_1}\Big) = \bar{F}_0 F_1 F_2 F_3 F_4; \\ &\bar{F}_0 = F_-\Big(t^{-\beta_1}|t - K(x^a)|\Big), \\ &F_1 = F_+\left(\frac{x^a}{t^{\alpha_1}}\right),\ F_2 = F_-\left(\frac{x^a}{t^{\alpha_0}}\right),\ F_3 = F_+(t^{\gamma_1}p^a),\ F_4 = F_-\Big(|t^\delta H^a|\Big). \end{aligned} \tag{2.7}$$

(In the present context we dont need [S1, (4.37)] though. We can use the fact that $\phi_a^+ \in P_r^{a+}\mathcal{H} \subseteq E_{\{0\}}\big(H^{a+}\big)\mathcal{H}$ and [S1, Lemma 2.3] to avoid a certain symmetrizing under use of [S1, (4.37)].)

□

We shall need some operators r_t, b_t and $\tilde{b}_t$ which are modelled after constructions in [D] and [HS]:

$$r_t(x^a) = f\, r\big(f^{-1}(x^a - L)\big);\ f = t^{\frac{3}{4}},\ r(y) = \langle y\rangle = \big(1 + |y|^2\big)^{\frac{1}{2}}. \tag{2.8}$$

We compute its Heisenberg derivative (with $\mathbf{D} = \frac{d}{dt} + i[H_a(t), \cdot])$

$$\begin{aligned} &b_t = \mathbf{D}r_t = \frac{1}{2}\Big(r'\big(f^{-1}(x^a - L)\big)\Big(p^a - \dot{L}\Big) + h.c.\Big) + \dot{f}d_t; \\ &d_t = f r_t^{-1},\ \dot{L} = \frac{d}{dt}L(t). \end{aligned} \tag{2.9}$$

Furthermore (with dots used again for time-derivatives and $V(t) = V^a(x^a) + I_a(t,x)$)

$$\begin{aligned} &\mathbf{D}b_t = \mathbf{D}^2 r_t = f^{-1}c_t + \ddot{f}d_t - f^{-3}e_t - r'\big(f^{-1}(x^a - L)\big)\left(\frac{\partial}{\partial x^a}V(t) + \ddot{L}\right); \\ &c_t = P^*\left(\frac{d^2}{dx^{a2}}r\right)\big(f^{-1}(x^a - L)\big)P,\ P = p^a - \dot{L} - \frac{\dot{f}}{f}(x^a - L), \\ &e_t = 4^{-1}\left(\frac{d^4}{dx^{a4}}r\right)\big(f^{-1}(x^a - L)\big). \end{aligned} \tag{2.10}$$

Clearly both b_t and $\mathbf{D}b_t$ are bounded relatively to $p^{a2} = (p^a)^2$. Also we notice the non-negativity of c_t and the uniform boundedness (with respect to t) of the terms d_t and e_t.

We may estimate using also the fact that

$$\frac{\partial}{\partial x^a}V(t) + \ddot{L} = \frac{\partial}{\partial x^a}I_a(t,x) + (x^a - L)\int_0^1 ds V^{a\prime\prime}(L + s(x^a - L)) \tag{2.11}$$

and (2.1),

$$\mathbf{D}b_t \geq -r'\big(f^{-1}(x^a - L)\big)\frac{\partial}{\partial x^a}I_a(t,x) + C_1\ddot{f} \geq -C_2 t^{-\min\left(\mu,\frac{1}{4}\right)-1}. \tag{2.12}$$

We introduce the regularizations

$$\begin{array}{l} \tilde{b}_t = N_t^{-1}b_t N_t^{-1}\ (= O(t^\nu)),\ \tilde{c}_t = N_t^{-1}c_t N_t^{-1}\ \big(= O\big(t^{2\nu}\big)\big); \\ N_t = I + t^{-2\nu}p^{a2},\ \nu > 0. \end{array} \tag{2.13}$$

We also introduce for small $\sigma > 0$

$$\begin{array}{l} g = t^{\rho_1};\ \rho_1 = \dfrac{\mu}{2}\alpha + 3\sigma, \\ h = t^{1-\rho_2} = t^{\alpha - 2\sigma};\ \rho_2 = \dfrac{\mu}{2}\alpha + 2\sigma. \end{array} \tag{2.14}$$

Obviously (for future reference) $f = o(h)$ and $\rho_2 < \rho_1$. Moreover $C_1 t^\alpha \leq L \leq C_2 t^\alpha$, cf. [S1, (5.8)].

To start out the analysis we shall use the following weaker localization than the one presented in Lemma 2.1:

$$||\phi_a^+(t) - F_-\big(h^{-1}r_t\big)\phi_a^+(t)|| = o\big(t^0\big). \tag{2.15}$$

Notice that this localization is very weak as opposed to the "optimal" one of Lemma 2.1. Similarly the following localization result is very weak; the "optimal" bound follows readily from the proof:

Lemma 2.2 *Let ϕ_a^+ be given as in Lemma 2.1. Then for all small $\sigma > 0$ (and all $\nu > 0$)*

$$||F_+\Big(g\tilde{b}_t\Big)\phi_a^+(t)|| = o\big(t^0\big). \tag{2.16}$$

Proof As in the proof of Lemma 2.1 we may assume (2.2) with $\epsilon > 0$ small. We use the proof of this lemma. By (2.6) it suffices to estimate

$$||g\tilde{b}_t\bar{B}\big(t,t^{\beta_1}\big)\phi_a^+(t)|| = o\big(t^0\big). \tag{2.17}$$

By commutation (2.17) will follow from

$$||g\Big(p^a - \dot{L}\Big)\bar{B}\big(t,t^{\beta_1}\big)\phi_a^+(t)|| = o\big(t^0\big). \tag{2.18}$$

To show (2.18) we may insert $F_+ = F_+(4t^{\gamma_1}p^a)$ to the left and then write

$$F_+\Big(p^a - \dot{L}\Big) = F_+\Big(p^a + \dot{L}\Big)^{-1} 2(H^a - (V^a(x^a) - V^a(L))).$$

The contribution from H^a may by commutation be shown to be $O\big(t^{3\sigma-\delta+\mu\alpha}\big)$, while the one from the second term to the right is $O\Big(t^{6\sigma-\mu+\frac{\mu}{2}\alpha}\Big)$ due to the presence of the factor $\bar{F}_0$ and the formula $V^a(x^a) - V(L) = O\Big(t^{-\alpha(1+\mu)}\Big)(x^a - L)$. □

For any given $\phi_a^+ \in P_r^{a+}\mathcal{H}$ we aim at proving the existence of the limit

$$\lim_{t\to+\infty} \breve{U}_a^*(t)\phi_a^+(t), \tag{2.19}$$

for some comparison dynamics $\breve{U}_a(t)$ to be defined in Section 4 and for which a result of [S2] can be applied (to conclude $\phi_a^+ = 0$).

3 Integral estimates

In order to prove the existence of (2.19) we need certain integral estimates for the full dynamics $U_a(t)$, cf. [D] and [HS]. Henceforth we shall not use or assume (2.2). Nevertheless we are going to apply some of the parameters of (2.5). The δ is changed to

$$\delta = \mu - \sigma. \tag{3.1}$$

Lemma 3.1 *For all small enough $\sigma, \nu > 0$ (depending only on μ) and with $F(t) = F_1F_2F_3F_4$, where the factors to the right are given by (2.7) with $\alpha_0, \gamma_1,$ and α_1 given by (2.5) and δ by (3.1), and with the expectation value $\langle\cdot\rangle_\phi$ given in the state $\phi = \phi_a^+(t) = U_a(t)\phi_a^+$ for any $\phi_a^+ \in P_r^{a+}\mathcal{H}$*

$$\int_1^\infty \Big\langle \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)A(t)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\Big\rangle_{G(t)\phi_a^+(t)} dt < \infty;$$

$$A(t) = gf^{-1}\tilde{c}_t + t^{-1}I - gN_t^{-1}r'\big(f^{-1}(x^a - L)\big)F_+\left(4\frac{x^a}{t^{\alpha_1}}\right)\Big(V^{a\prime} + \ddot{L}\Big)N_t^{-1}, \tag{3.2}$$

$$G(t) = G_+(t) = F_+\big(h^{-1}r_t\big)N_t^{-1}F(t) \text{ or}$$

$$G(t) = G_-(t) = F_-\big(h^{-1}r_t\big)N_t^{-1}F(t),$$

and

$$\int_1^\infty \Big\langle F_-^2\Big(g\tilde{b}_t\Big)\Big(t^{-1-\sigma}\Big(2I-g\tilde{b}_t\Big)+t^{-1}I\Big)\Big\rangle_{H(t)\phi_a^+(t)}dt<\infty;$$

$$H(t)=H_+(t)=\big(F_+^{2\prime}\big)^{\frac{1}{2}}\big(h^{-1}r_t\big)F(t)\ \textit{or}$$

(3.3)

$$H(t)=H_-(t)=\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\big(h^{-1}r_t\big)F(t).$$

Proof We notice that indeed the expression $A(t)$ of (3.2) is a sum of non-negative terms, cf. (2.1) and (2.11). A similar remark is due for the integrand of (3.3).

For (3.2) we claim that the estimate with $G(t)=N_t^{-1}F(t)$ follows by considering the "propagation observable" (more precisely the uniformly bounded family of observables)

$$\Phi(t)=\big(N_t^{-1}F(t)\big)^*F_-^2\Big(g\tilde{b}_t\Big)N_t^{-1}F(t).$$

Let us compute the "leading" term coming from differentiating the middle term. It suffices to differentiate w.r.t. the Heisenberg derivative $\mathbf{D}_t=\frac{d}{dt}+i\Big[\frac{1}{2}p^2+V^a(x^a)F_+\Big(4\frac{x^a}{t^{\alpha_1}}\Big)+I_a(t,x),\cdot\Big]$; the remainder is integrable. We compute

$$\mathbf{D}_tF_-^2\Big(g\tilde{b}_t\Big)=-\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\Big(\mathbf{D}_t\Big(g\tilde{b}_t\Big)\Big)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)+R_1(t),\tag{3.4}$$

with

$$||R_1(t)||\leq C||\Big[g\tilde{b}_t,\Big(\mathbf{D}_t\Big(g\tilde{b}_t\Big)\Big)\Big]||,\tag{3.5}$$

cf. [DG, Lemma C.4.1]. Using (3.6) given below (and concrete expressions for the derivatives involved) we readily estimate the right hand side by a constant times $g^2t^{2\nu}f^{-2}$ which is in $L^1(dt)$ for $\nu>0$ small enough.

Obviously

$$\mathbf{D}_t\Big(g\tilde{b}_t\Big)=\dot{g}\tilde{b}_t+gN_t^{-1}(\mathbf{D}_tb_t)N_t^{-1}+2g\mathrm{Re}\big(\big(\mathbf{D}_tN_t^{-1}\big)b_tN_t^{-1}\big).\tag{3.6}$$

The contribution from the first two terms on the right hand side of (3.6) to the Heisenberg derivative of $\Phi(t)$ is

$$T_1(t)+R_2(t);$$

$$T_1(t)=-B(t)^*\Big\{\dot{g}\tilde{b}_t+gN_t^{-1}(\mathbf{D}b_t)N_t^{-1}\Big\}B(t),\ B(t)=\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)N_t^{-1}F(t),$$

$$||R_2(t)||\in L^1(dt).$$

Here we used again that the functions $F_1=F_+\Big(\frac{x^a}{t^{\alpha_1}}\Big)$ and $F_-\Big(4\frac{x^a}{t^{\alpha_1}}\Big)$ have disjoint support. Commutation picks up an integrable term. (In fact $R_2(t)=O\big(t^{-\infty}\big)$.) As for

the contribution from the last term on the right hand side of (3.6) we compute for a suitable real-valued $F_c \in C_0^\infty(\mathbf{R})$

$$
\begin{aligned}
&-2gB(t)^*\mathrm{Re}\big(\big(\mathbf{D}_t N_t^{-1}\big) b_t N_t^{-1}\big) B(t) = T_2(t) + R_3(t) + R_4(t);\\
&T_2(t) = -4\nu B(t)^* F_c\Big(g\tilde{b}_t\Big) t^{-2\nu-1} p^{a2} N_t^{-1} F_c\Big(g\tilde{b}_t\Big) B(t),\\
&||R_3(t)|| \le C||\Big[F_c\Big(g\tilde{b}_t\Big), t^{-2\nu-1}p^{a2}N_t^{-1}\Big]|| = O\big(gf^{-1}t^{-1}\big) = O\Big(t^{\frac{\mu}{2}\alpha-\frac{7}{4}+3\sigma}\Big),\\
&||R_4(t)|| \le C||\left[V^a(x^a)F_+^2\left(4\frac{x^a}{t^{\alpha_1}}\right) + I_a(t,x), N_t^{-1}\right]N_t|| = O\Big(t^{-\nu-(1+\mu)\alpha_1}\Big).
\end{aligned}
$$

Clearly it follows that $||R_3(t)||$, $||R_4(t)|| \in L^1(dt)$.

By (2.10), (2.11) and (2.12)

$$
T_1(t) + T_2(t) \le -CB(t)^*A(t)B(t) + R(t), \tag{3.7}
$$

where $C > 0$ and $||R(t)|| \in L^1(dt)$.

Next we look at the contributions

$$
T_3(t) = \big(N_t^{-1}F(t)\big)^* F_-^2\Big(g\tilde{b}_t\Big)\big(\mathbf{D}_t N_t^{-1}\big)F(t) + h.c., \tag{3.8}
$$

and

$$
T_4(t) = \big(N_t^{-1}F(t)\big)^* F_-^2\Big(g\tilde{b}_t\Big) N_t^{-1}\mathbf{D}F(t) + h.c.,
$$

to the Heisenberg derivative of $\Phi(t)$:

The first term on the right hand side of the formula

$$
\mathbf{D}_t N_t^{-1} = \frac{d}{dt}N_t^{-1} + i\left[V^a(x^a)F_+^2\left(4\frac{x^a}{t^{\alpha_1}}\right) + I_a(t,x), N_t^{-1}\right]
$$

contributes to (3.8) by a term that is $O\big(t^{-1-2\nu}\big)$ (since p^{a2} may be bounded by the factor F_4). Obviously by the above bound for $R_4(t)$ the second term is integrable.

As for the term $T_4(t)$ the derivatives of the factors of $F(t)$ are readily handled (i.e. proven integrable) by using various estimates of [S1], cf. the proof of Lemma 2.1. Straightforward computations of commutators with the middle term $N_t^{-1}F_-^2\Big(g\tilde{b}_t\Big)N_t^{-1}$ needed when symmetrizing expressions of the derivatives of the factors F_1 and F_2 show that those contribute by integrable terms. We refer to (5.5) and (5.6) for a similar issue.

In combination with (3.7) we finally conclude the estimate

$$
\int_1^\infty \Big\langle \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big) A(t) \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\Big\rangle_{N_t^{-1}F(t)\phi_a^+(t)} dt < \infty. \tag{3.9}
$$

To obtain (3.2) for $G(t) = G_-(t)$ it suffices by (3.9) to show the statement for $G(t) = G_+(t)$. We show the latter and (3.3) for $H(t) = H_+(t)$ in one stroke by considering the propagation observable

$$\Phi(t) = G_+(t)^* F_-^2\Big(g\tilde{b}_t\Big) G_+(t).$$

We notice that

$$\begin{aligned}
&\mathbf{D}F_+\big(h^{-1}r_t\big) = \frac{1}{2}F_+'\big(h^{-1}r_t\big)\Big(h^{-1}\mathbf{D}r_t - \dot{h}h^{-2}r_t\Big) + h.c.\\
&= h^{-1}b_t F_+'\big(h^{-1}r_t\big) - \frac{\dot{h}}{h}\big(h^{-1}r_t\big)F_+'\big(h^{-1}r_t\big) + O\big(h^{-2}\big),
\end{aligned}$$

tends to be negative when sandwitched by factors of $F_-\Big(g\tilde{b}_t\Big)$, cf. the proof of [D, Proposition 5.6]. Explicitly, by commutation cf. (5.6) given below,

$$\begin{aligned}
&N_t^{-1}\big(\mathbf{D}F_+\big(h^{-1}r_t\big)\big)F_-^2\Big(g\tilde{b}_t\Big)F_+\big(h^{-1}r_t\big)N_t^{-1} + h.c.\\
&= -(gh)^{-1}A_1^*A_1 - \frac{\dot{h}}{h}A_2^*A_2 + O\Big((gh)^{-1}\Big);\\
&A_1 = f_1\Big(g\tilde{b}_t\Big)\big(F_+^{2\prime}\big)^{\frac{1}{2}}\big(h^{-1}r_t\big),\ A_2 = f_2\big(h^{-1}r_t\big)F_-\Big(g\tilde{b}_t\Big)N_t^{-1},\\
&f_1(x) = \sqrt{2-x}F_-(x),\ f_2(x) = \sqrt{x}\big(F_+^{2\prime}\big)^{\frac{1}{2}}(x).
\end{aligned} \tag{3.10}$$

The previous arguments for the contribution from $\mathbf{D}_t F_-^2\Big(g\tilde{b}_t\Big)$ applies again (this term contributes by another non-positive term).

To obtain (3.3) for $H(t) = H_-(t)$ we may differentiate $\Phi(t) = G_-(t)^* F_-^2\Big(t^{\rho_1}\tilde{b}_t\Big)G_-(t)$ under use of similar computations as for the first estimate of (3.3), and we use (3.2) for $G(t) = G_-(t)$. □

Remark 3.2 We are only going to use the bounds of Lemma 3.1 for the cases $G(t) = G_-(t)$ and $H(t) = H_-(t)$.

In the first case the contribution to the integral from the third term of the expression $A(t)$ of (3.2) simplifies as follows: First we replace the factor $F_+\Big(4\frac{x^a}{t^{\alpha_1}}\Big)$ by $F_-\big(4^{-1}h^{-1}r_t\big)$. This can be done up to an integrable error due to the presence of the factor $F_-\big(h^{-1}r_t\big)$ in the product $G_-(t)$. Next we use a Taylor expansion (cf. Section 4) to write

$$\begin{aligned}
&-gN_t^{-1}r'\big(f^{-1}(x^a - L)\big)F_-\big(4^{-1}h^{-1}r_t\big)\Big(V^{a\prime} + \ddot{L}\Big)N_t^{-1}\\
&= \tilde{\gamma}_t^* F_-\big(4^{-1}h^{-1}r_t\big)\tilde{\gamma}_t + O\big(gft^{-2}\big) + O\Big(gt^{-(3+\mu)\alpha}h^2\Big);\\
&\tilde{\gamma}_t = \big(-V^{a\prime\prime}(L)gr_t\big)^{\frac{1}{2}}N_t^{-1}.
\end{aligned}$$

Clearly the second term $O\big(gft^{-2}\big)$ is in $L^1(dt)$, and since $gt^{-(3+\mu)\alpha}h^2 = t^{-1-\sigma}$ the same holds for the third term. Finally we may remove the factor $F_-\big(h^{-1}r_t\big)$ and obtain consequently, that we can replace the third term of the expression $A(t)$ by $\tilde{\gamma}_t^*\tilde{\gamma}_t$ for the estimate of (3.2) with $G(t) = G_-(t)$.

We also notice that the functions F_1 and F_2 of Lemma 3.1 are one on a neighborhood of the support of the functions $F_-\big(h^{-1}r_t\big)$ and $\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\big(h^{-1}r_t\big)$ of the products $G_-(t)$ and $H_-(t)$, respectively. Consequently we may remove the factors F_1 and F_2 in the applications given in the coming sections.

4 A simplified comparison dynamics

We introduce a comparison dynamics $\breve{U}_a(t)$ by

$$i\tfrac{d}{dt}\breve{U}_a(t) = \breve{H}_a(t)\breve{U}_a(t),\ \breve{U}_a(1) = I, \tag{4.1}$$

and

$$\begin{aligned}
&\breve{H}_a(t) = \frac{1}{2}p_a^2 + \breve{H}^a(t) + \breve{R}(t,x);\\
&\breve{H}^a(t) = \frac{1}{2}p^{a2} + V^a(L) + (x^a - L)V^{a\prime}(L) + 2^{-1}V^{a\prime\prime}(L)(x^a - L)^2,\\
&\breve{R}(t,x) = I_a(t,x) + \breve{R}^a(t,x^a),\\
&\breve{R}^a = \breve{F}^*(x^a - L)^3\int_0^1 \frac{(1-s)^2}{2}\left(\frac{d^3}{dx^{a3}}V^a\right)(L + s(x^a - L))ds\breve{F},\\
&\breve{F} = F_-\big(4^{-1}h^{-1}r_t\big).
\end{aligned} \tag{4.2}$$

Clearly for all $k \in \mathbf{N}\cup\{0\}$

$$\partial_{x^a}^k\breve{R}^a = O\Big(t^{-(3+\mu)\alpha}h^{3-k}\Big), \tag{4.3}$$

uniformly in x.

It is known that $\breve{U}_a(t)$ preserves the domain of $p^2 + x^2$, see [S2, Section 4].

5 Further integral estimates

We need further integral estimates for the full dynamics. To motivate those we consider the "model Hamiltonian" $\breve{H}^a(t)$ of (4.2). Let $\breve{U}^a(t)$ denote the corresponding propagator, i.e. $i\partial_t\breve{U}^a(t) = \breve{H}^a(t)\breve{U}^a(t)$ and $\breve{U}^a(1) = I$.

We shall introduce "radiation operators" for the generator. This is done in terms of two solutions $\alpha^+(t)$ and $\alpha^-(t)$ to the Riccati equation

$$\dot{\alpha} = -V^{a\prime\prime}(L) - \alpha^2$$

with the properties that for some $C > 1$ and all large t

$$t^{-1} \leq \alpha^+(t) \leq Ct^{-1},\ 0 \leq -\alpha^-(t) \leq Ct^{-1}. \tag{5.1}$$

(See Appendix A for an elaboration.)

We define

$$\eta^+ = \Big(p^a - \dot{L}\Big) - \alpha^+(x^a - L),\ \eta^- = \Big(p^a - \dot{L}\Big) - \alpha^-(x^a - L),$$

and notice that

$$\breve{\mathbf{D}}^a \eta^+ = -\alpha^+ \eta^+,\ \breve{\mathbf{D}}^a \eta^- = -\alpha^- \eta^-, \tag{5.2}$$

where $\breve{\mathbf{D}}^a$ refers to the Heisenberg derivative with respect to $\breve{H}^a(t)$.

Let

$$N_t^+ = \big(\eta^+\big)^2,\ N_t^- = \big(\eta^-\big)^2,$$

and

$$G_1(t) = F_-\Big(g\tilde{b}_t\Big) F_-^2\big(h^{-1}r_t\big) N_t^{-1} F_3 F_4,$$

$$G_2(t) = F_-\Big(g\tilde{b}_t\Big) F_-^2\big(N_t^-\big) F_-^2\big(N_t^+\big) N_t^{-1},$$

$$G_3(t) = F_-\big(h^{-1}r_t\big) N_t^{-1} F_3 F_4,$$

where F_3 and F_4 (and the other factors) are given as in Lemma 3.1.

Lemma 5.1 *With $\phi_a^+(t)$ given as in Lemma 3.1*

$$\int_1^\infty t^{-1} \big\langle -F_-^{2\prime}\big(N_t^+\big)\big\rangle_{F_-\left(g\tilde{b}_t\right) G_3(t)\phi_a^+(t)} dt < \infty, \tag{5.3}$$

and

$$\int_1^\infty \alpha^-(t) \big\langle F_-^{2\prime}\big(N_t^-\big)\big\rangle_{F_-\left(g\tilde{b}_t\right) G_3(t)\phi_a^+(t)} dt < \infty. \tag{5.4}$$

Proof We consider for (5.3) the propagation observable

$$\Phi_1(t) = G_3(t)^* F_+\big(N_t^+\big) F_-^2\Big(g\tilde{b}_t\Big) F_+\big(N_t^+\big) G_3(t).$$

To treat the contribution from the Heisenberg derivatives of the two factors $F_+\big(N_t^+\big)$ in $\Phi_1(t)$ we introduce the modified derivative $\breve{\mathbf{D}}_t = \frac{d}{dt} + i\Big[\breve{H}_a(t), \cdot\Big]$. Due to the fact that the function $\breve{F}$ of (4.2) is one on a neighborhood of the support of the function

$F_-\big(h^{-1}r_t\big)$ of the product $G_3(t)$ it suffices to consider $\breve{\mathbf{D}}_t F_+\big(N_t^+\big)$: We compute using (5.2)

$$\breve{\mathbf{D}}_t F_+\big(N_t^+\big) = -2\alpha^+ N_t^+ F_+'\big(N_t^+\big) + i\Big[\breve{R}(t,x), F_+\big(N_t^+\big)\Big],$$

cf. (5.6) stated below. By (1.3) and (4.3) the second term on the right hand side is $O\big(t^{-1-\mu}\big) + O\Big(t^{-(3+\mu)\alpha}h^2\Big)$ and hence in particular integrable. Next we compute

$$\begin{aligned} &- 2\alpha^+ N_t^+ F_+'\big(N_t^+\big) F_-^2\Big(g\tilde{b}_t\Big) F_+\big(N_t^+\big) + h.c. \\ &= F_-\Big(g\tilde{b}_t\Big)\big(-2\alpha^+ N_t^+ F_+^{2\prime}\big(N_t^+\big)\big) F_-\Big(g\tilde{b}_t\Big) + O\big(t^{-1}gf^{-1}t^\nu\big), \end{aligned} \tag{5.5}$$

by first representing with $B = F_-\Big(g\tilde{b}_t\Big)$

$$\begin{aligned} &\big[B, F\big(N_t^+\big)\big] \\ &= -\frac{1}{\pi}\int\limits_{\mathbf{C}} \Big(\bar{\partial}\tilde{F}\Big)(w)\big(N_t^+ - w\big)^{-1}\big[B, N_t^+\big]\big(N_t^+ - w\big)^{-1} G_-(t)dudv;\ w = u + iv, \end{aligned} \tag{5.6}$$

cf. [DG, Appendix C.3], and then using the representation formula once again for the commutator to the right.

The first term on the right hand side of (5.5) is non-positive; the second term is integrable.

Next to treat the contribution from the Heisenberg derivative of the factor $F_-^2\Big(g\tilde{b}_t\Big)$ we notice that the proof of Lemma 3.1 yields the upper bound

$$\begin{aligned} &G_3(t)^* F_+\big(N_t^+\big)\Big(\mathbf{D}F_-^2\Big(g\tilde{b}_t\Big)\Big) F_+\big(N_t^+\big) G_3(t) \\ &\leq -C G_3(t)^* F_+\big(N_t^+\big)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big) A(t)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big) F_+\big(N_t^+\big) G_3(t) + R(t), \end{aligned} \tag{5.7}$$

for a positive C and with $R(t) \in L^1(dt)$.

Notice that the first term on the right hand side of (5.7) is non-positive.

Assuming for the moment that the contribution from the Heisenberg derivatives of the two factors $G_3(t)$ is integrable we obtain from (5.5) and (5.7) that

$$\int\limits_1^\infty \Big\langle \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big) A(t)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\Big\rangle_{F_+\left(N_t^+\right)G_3(t)\phi_a^+(t)} dt < \infty. \tag{5.8}$$

Combining (5.8) and Lemma 3.1 leads to the estimate

$$\int\limits_1^\infty \Big\langle \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big) A(t)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\Big\rangle_{F_-\left(N_t^+\right)G_3(t)\phi_a^+(t)} dt < \infty, \tag{5.9}$$

cf. Remark 3.2.

It remains to check the integrability of the contribution from the derivatives of the two factors $G_3(t)$.

The contributions from the derivatives of the factors of N_t^{-1}, F_3 and F_4 are integrable due to [S1, Lemmas 2.3 and 4.1], cf. the proof of Lemmas 2.1 and 3.1.

As for the contribution from the derivatives of the two factors $F_-\big(h^{-1}r_t\big)$ we compute for a positive C, cf. (3.10),

$$
\begin{aligned}
&N_t^{-1}\big(\mathbf{D}F_-\big(h^{-1}r_t\big)\big)F_+\big(N_t^+\big)F_-^2\Big(g\tilde{b}_t\Big)F_+\big(N_t^+\big)F_-\big(h^{-1}r_t\big)N_t^{-1}+h.c.\\
&=(gh)^{-1}A_1^*F_+^2\big(N_t^+\big)A_1+\frac{\dot{h}}{h}A_2^*F_+^2\big(N_t^+\big)A_2+O\big(t^{-1-\sigma}\big)\\
&\leq (gh)^{-1}A_1^*A_1+\frac{\dot{h}}{h}A_2^*A_2+O\big(t^{-1-\sigma}\big);\\
&A_1=f_1\Big(g\tilde{b}_t\Big)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\big(h^{-1}r_t\big),\ A_2=f_2\big(h^{-1}r_t\big)F_-\Big(g\tilde{b}_t\Big)N_t^{-1},\\
&f_1(x)=\sqrt{2-x}F_-(x),\ f_2(x)=\sqrt{x}\big(-F_-^{2\prime}\big)^{\frac{1}{2}}(x).
\end{aligned}
\tag{5.10}
$$

Combining (5.10) and Lemma 3.1 (and Remark 3.2) leads to the wanted integrability and therefore (5.9).

Next we consider the Heisenberg derivative of

$$\Phi_2(t)=G_3(t)^*F_-\big(N_t^+\big)F_-^2\Big(g\tilde{b}_t\Big)F_-\big(N_t^+\big)G_3(t).$$

We compute the main contribution from the derivatives of the factors $F_-\big(N_t^+\big)$:

$$
\begin{aligned}
&G_3(t)^*\big(-2\alpha^+N_t^+F_-'\big(N_t^+\big)\big)F_-^2\Big(g\tilde{b}_t\Big)F_-\big(N_t^+\big)G_3(t)+h.c.\\
&=G_3(t)^*F_-\Big(g\tilde{b}_t\Big)\big(-2\alpha^+N_t^+F_-^{2\prime}\big(N_t^+\big)\big)F_-\Big(g\tilde{b}_t\Big)G_3(t)+O\big(t^{-1}gf^{-1}t^{2\nu}\big),
\end{aligned}
\tag{5.11}
$$

cf. (5.5).

The contribution from the derivative of the factor $F_-^2\Big(g\tilde{b}_t\Big)$ is integrable by (5.9). The contribution from the derivatives of the two factors $G_3(t)$ is integrable by the same arguments used for the same factors in $\Phi_1(t)$.

Finally we conclude (5.3) by integrating (5.11).

The estimate (5.4) follows by considering the observable

$$\Phi(t)=G_3(t)^*F_-\big(N_t^-\big)F_-^2\Big(g\tilde{b}_t\Big)F_-\big(N_t^-\big)G_3(t);$$

the computations and arguments are similar as for those given above for $\Phi_1(t)$.

□

By (2.6), (2.15) and (2.16)

$$||\phi_a^+(t)-G_2(t)^*G_1(t)\phi_a^+(t)||=o\big(t^0\big).$$

Therefore to show the existence of (2.19) it suffices to show the existence of

$$\lim_{t\to+\infty} \breve{U}_a(t)^* G_2(t)^* G_1(t)\phi_a^+(t). \tag{5.12}$$

For future reference we notice that

$$||F_+\big(4^{-1}h^{-1}r_t\big)G_2(t)^*G_1(t)|| \in L^1(dt). \tag{5.13}$$

6 Integral estimates for the comparison dynamics

We shall need the following estimates for $\breve{U}_a(t)$, cf. Lemmas 3.1 and 5.1. (The operator $\tilde{\gamma}_t$ in (6.3) is defined in Remark 3.2.)

Lemma 6.1 *For all small enough $\sigma, \nu > 0$ and all $\phi \in \mathcal{H}$ the following estimates hold with $\breve{\phi}_a(t) = \breve{U}_a(t)\phi$:*

$$\int_1^\infty t^{-1}\big\langle -F_-^{2\prime}\big(N_t^+\big)\big\rangle_{\breve{\phi}_a(t)}dt \le C||\phi||^2, \tag{6.1}$$

$$\int_1^\infty \alpha^-(t)\big\langle F_-^{2\prime}\big(N_t^-\big)\big\rangle_{\breve{\phi}_a(t)}dt \le C||\phi||^2, \tag{6.2}$$

$$\begin{aligned}
&\int_1^\infty \left\langle \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)B(t)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\right\rangle_{H_1(t)\breve{\phi}_a(t)} dt \le C||\phi||^2;\\
&B(t) = gf^{-1}\tilde{c}_t + t^{-1}I + \tilde{\gamma}_t^*\tilde{\gamma}_t,\\
&H_1(t) = N_t^{-1}F_-\big(N_t^-\big)F_-\big(N_t^+\big),
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
&\int_1^\infty \left\langle F_-^2\Big(g\tilde{b}_t\Big)\Big(t^{-1-\sigma}\Big(2I - g\tilde{b}_t\Big) + t^{-1}I\Big)\right\rangle_{H_2(t)\breve{\phi}_a(t)} dt \le C||\phi||^2;\\
&H_2(t) = \big(-F_-^{2\prime}\big)^{\frac{1}{2}}\big(h^{-1}r_t\big)F_-\big(N_t^-\big)F_-\big(N_t^+\big),
\end{aligned} \tag{6.4}$$

Proof As for (6.1) and (6.2) we notice that the estimates follow from the proof of Lemma 5.1.

For (6.3) we consider the observable

$$\Phi(t) = H_1(t)^* F_-^2\Big(g\tilde{b}_t\Big)H_1(t).$$

We use the representation formula applied for (5.6), the proof of Lemma 3.1, (6.1) and (6.2), the bounds

$$\sup_{t\ge 1} ||p^{a2}F_-\big(N_t^-\big)F_-\big(N_t^+\big)|| < \infty, \tag{6.5}$$

(6.6) $$\sup_{t\geq 1}||\tfrac{|x^a|}{t}F_-\big(N_t^-\big)F_-\big(N_t^+\big)|| < \infty,$$

and that $g||\partial_{x^a}\breve{R}(t,x)|| = O\big(t^{-1-\sigma}\big)\big(\in L^1(dt)\big)$. We notice that the bound (6.5) compensates for energy-localization. The bound (6.6) is used to bound (by commutation) certain expressions containing the unbounded factor $\frac{d}{dx^a}\Big(2^{-1}V^{a\prime\prime}(L)(x^a-L)^2\Big)$.

For (6.4) we first consider the observable

$$\Phi_1(t) = H_1(t)^*F_+\big(h^{-1}r_t\big)F_-^2\Big(g\tilde{b}_t\Big)F_+\big(h^{-1}r_t\big)H_1(t),$$

cf. the scheme of the proof of Lemmas 3.1 and 5.1. The arguments above combined with (3.10) and (6.3) yields

(6.7) $$\int_1^\infty\Big\langle\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)B(t)\big(-F_-^{2\prime}\big)^{\frac{1}{2}}\Big(g\tilde{b}_t\Big)\Big\rangle_{F_-(h^{-1}r_t)H_1(t)\breve{\phi}_a(t)}dt \leq C||\phi||^2.$$

Next we first consider the observable

$$\Phi_2(t) = H_1(t)^*F_-\big(h^{-1}r_t\big)F_-^2\Big(g\tilde{b}_t\Big)F_-\big(h^{-1}r_t\big)H_1(t).$$

Finally by combining previous arguments, (6.7) and a computation similar to (5.10) we conclude (6.4).

□

7 Proof of $\phi_a^+ = 0$ (negative potentials)

The first step of the proof of the statement, that any given $\phi_a^+ \in P_r^{a+}\mathcal{H}$ indeed must vanish, is the following result.

Lemma 7.1 *The limit (5.12) exists.*

Proof We prove the existence of the limit using the integral estimates for $U_a(t)$ and $\breve{U}_a(t)$ proven in Sections 3, 5 and 6. Due to the presence of the factor $F_-\big(h^{-1}r_t\big)$ in the product $G_1(t)$ it suffices to consider

(7.1)
$$\begin{aligned}&\mathbf{D}_t(H(t))F_3F_4 + H(t)(\mathbf{D}_t(F_3)F_4 + F_3\mathbf{D}F_4);\\ &H(t) = N_t^{-1}F_-^2\big(N_t^+\big)F_-^2\big(N_t^-\big)F_-^2\Big(g\tilde{b}_t\Big)F_-^2\big(h^{-1}r_t\big)N_t^{-1},\end{aligned}$$

where the derivative $\mathbf{D}_t$ is given as in the proof of Lemma 3.1, cf. (5.13): For the first term on the right hand side of (7.1) we compute the derivative of all factors (cf. the proof of Lemma 3.1 and Remark 3.2). After symmetrizing all "leading" terms (i.e. those that are not in $L^1(dt)$) we may invoke combinations of Lemmas 3.1 and 5.1 for $U_a(t)$ and (6.1)-(6.4) for $\breve{U}_a(t)$. We notice that in some cases we use the estimates for some functions F_- replaced by F_-^2 (which are just special cases). By construction the contribution from the derivative of $F_-^2\Big(g\tilde{b}_t\Big)$ doesn't need to be commuted. In other cases that is needed; the errors are easily shown to be in $L^1(dt)$.

For the second term on the right hand side of (7.1) we use [S1, Lemmas 2.3 and 4.1]. □

The second step is to invoke [S2, Theorem 1.2]. We consider a state $\breve{\phi}_a^+$ for which for all $\varepsilon > 0$

$$F_+\big(t^{\mu-\varepsilon-1}|x^a - L|\big)\breve{\phi}_a^+(t) = o\big(t^0\big), \tag{7.2}$$

where $\breve{\phi}_a^+(t) = \breve{U}_a(t)\breve{\phi}_a^+$ (cf. Lemma 2.1). We need to show that

$$\breve{\phi}_a^+ = 0. \tag{7.3}$$

For that we introduce the asymptotic velocity

$$\breve{x}^+ = s - C_\infty - \lim_{t\to+\infty} \breve{U}_a(t)^*\Big(\tfrac{x^a - L(t)}{x^+(t)}, \tfrac{x_a}{t}\Big)\breve{U}_a(t), \tag{7.4}$$

where $x^+(t) = \exp\int_1^t \alpha^+ dt'$ with α^+ given as in (5.1) (see also Appendix A). Suppose for the moment the conditions of [S2, Theorem 1.2] so that this asymptotic velocity is well-defined and absolutely continuous w.r.t. the Lebesgue measure on X (the latter by the conclusion of the theorem). Then since $x^+(t) \geq t$ we obviously get from (7.2) that $\breve{\phi}_a^+ \in E_{\{0\}\times X_a}\big(\breve{x}^+\big)\mathcal{H}$, whence we conclude from the stated absolute continuity that indeed (7.3) holds.

Now to verify the conditions of [S2, Theorem 1.2] we notice that

$$\sup_x |\partial_x \breve{R}(t,x)| \in L^1(dt),$$

cf. (4.3) and (1.3). (This bound is sufficient for the existence of $\breve{x}^+$.) As for the second order derivatives [S2, (1.6)] we use (4.3) and the bound (A.5) to estimate $\partial_{x^a}^2 \breve{R}^a(t,x^a)$ as follows:

$$|\partial_{x^a}^2 \breve{R}^a(t,x^a)| \leq O\big(t^{-2-2\sigma}\big) \leq x^+(t)^{-1}x^-(t)^{-1}O\big(t^{-1-2\sigma}\big),$$

uniformly in x. The condition [S2, (1.6)] follows for this second order derivative since $t^{-1-2\sigma} \in L^1(dt)$. We treat $\partial_{x^a}^2 I(t,x)$ and $\partial_{x_a}^2 I(t,x)$ similarly.

It remains to bound

$$|\partial_{x^a}\partial_{x_a} I(t,x)| \leq x^+(t)^{-1}h_2(t), \tag{7.5}$$

for some $h_2 \in L^1(dt)$. For that we notice that the lower bound,

$$V^{a\prime\prime}(L) \geq -\tilde{\mu}t^{-2} \text{ for some } \tilde{\mu} \geq 0, \tag{7.6}$$

implies the bound

$$x^+(t) \leq t^{\frac{1+\sqrt{1+4\tilde{\mu}}}{2}}; \tag{7.7}$$

see Appendix A.

The following computations show that under assumption (1.8), (7.6) holds for some $\tilde{\mu} \geq 0$ with

(7.8) $\quad \frac{1+\sqrt{1+4\tilde{\mu}}}{2} < 1+\mu.$

Obviously we may replace (1.8) by:

(7.9) $\quad V^{a\prime\prime}(x^a) \leq -c_r\mu(\mu+1)|x^a|^{-\mu-2};\ x^a \geq R,\ c_r > 0,$

(7.10) $\quad V^{a\prime\prime}(x^a) \geq -C_r\mu(\mu+1)|x^a|^{-\mu-2};\ x^a \geq R,\ C_r > 0,$

and

(7.11) $\quad C_r < 2^{-1}(2+\mu)^2 c_r.$

By integrating (7.9) twice (to infinity) we get

$$V^a(x^a) \leq -c_r|x^a|^{-\mu}.$$

Thus by the formula $t = \int\limits_R^L (-2V^a(x))^{-\frac{1}{2}}dx$ we can estimate

$$t \leq \int\limits_R^L \left(2c_r x^{-\mu}\right)^{-\frac{1}{2}}dx \leq \left(1+\frac{\mu}{2}\right)^{-1}(2c_r)^{-\frac{1}{2}}L^{1+\frac{\mu}{2}},$$

yielding

(7.12) $\quad \left((2c_r)^{\frac{1}{2}}\alpha^{-1}t\right)^{\alpha} \leq L.$

Next we use (7.10) for $x^a = L$ and estimate the right hand side by (7.12) yielding

(7.13) $\quad V^{a\prime\prime}(L) \geq -\frac{C_r}{c_r}\mu(\mu+1)\frac{2}{(2+\mu)^2}t^{-2}.$

Finally by combining (7.11) and (7.13) we get (7.6) for some

(7.14) $\quad \tilde{\mu} < \mu(\mu+1).$

We notice that (7.14) implies (7.8).

By combining (7.7) and (7.8) we get that $x^+(t) \leq t^{1+\mu-\epsilon}$ for some $\epsilon > 0$. Therefore

$$|\partial_{x^a}\partial_{x_a}I(t,x)| \leq O\left(t^{-2-\mu}\right) \leq x^+(t)^{-1}O\left(t^{-1-\epsilon}\right),$$

yielding (7.5).

We have verified the conditions of [S2, Theorem 1.2] and hence proved the absolute continuity of $\breve{x}^+$.

8 Positive potentials

We shall prove Theorem 1.3. We start out somewhat more generally assuming (1.2) for an arbitrary $\mu > 0$. Suppose in addition that $a \neq a_{\max}, a_{\min}$ is given such that for some $\mu^- > \mu$,

$$V^a(x^a) \geq -|x^a|^{-\mu^-}, \ |x^a| \geq R. \tag{8.1}$$

We define $\langle w\rangle = \big(1+|w|^2\big)^{\frac{1}{2}}$; $w \in \mathbf{C}$. Then we have the following modification of [S1, Lemma 2.2], which follows from the same proof.

Lemma 8.1 *Let $\tilde{\alpha} \geq 0$ and $\kappa, t \geq 1$. Then (with the above assumptions) for all $w \in \mathbf{C}$*

$$||F\big(t^{-\tilde{\alpha}}|x^a| > 1\big)p^a(\kappa H^a - w)^{-1}|| \leq C\frac{\langle w\rangle^{\frac{1}{2}}}{|Im w|}\Big(t^{-\tilde{\alpha}\mu^-} + t^{-2\tilde{\alpha}} + \kappa^{-1}\Big)^{\frac{1}{2}}. \tag{8.2}$$

We obtain the following modification of [S1, Lemma 2.3].

Lemma 8.2 *Suppose (8.1) and that for some positive $\breve{\alpha} < \mu$*

$$I(t,x) = I(t,x_a) \text{ for } |x^a| < 2t^{\breve{\alpha}}. \tag{8.3}$$

Suppose $\phi_a^+ \in E_{\{0\}}\big(H^{a+}\big)$. Then with $\phi_a^+(t) = U(t)\phi_a^+$ and any positive δ and σ with

$$\delta \leq \min\Big(2(\mu - 2\sigma), \mu - 2\sigma + \breve{\alpha}\min\Big(\tfrac{\mu^-}{2}, 1\Big)\Big), \tag{8.4}$$

there holds the bound

$$||F_+\big(|t^\delta H^a|\big)\phi_a^+(t)|| \leq Ct^{-\sigma}. \tag{8.5}$$

Proof Following the proof of [S1, Lemma 2.3] using now Lemma 8.1 we get

$$\begin{aligned}
&\langle F_+(|\kappa H^a|)\rangle_{\phi_a^+(t)} \\
&\leq \int_t^\infty \kappa C_1 s^{-(1+\mu)}\Big(s^{-\tilde{\alpha}\mu^-} + s^{-2\tilde{\alpha}} + \kappa^{-1}\Big)^{\frac{1}{2}} ds \\
&\leq C_2\kappa t^{-\Big(\mu+\tilde{\alpha}\min\Big(\frac{\mu^-}{2},1\Big)\Big)} + C_3\kappa^{\frac{1}{2}}t^{-\mu}.
\end{aligned}$$

Let $\kappa = t^\delta$. □

We give a new proof of the main result of [W], cf. [S1, Remark 3.4].

Theorem 8.3 ([W]) *Under the conditions (1.2), (1.3) and (8.1) for a fixed $a \neq a_{\max}, a_{\min}$ and with $\mu \in (\frac{1}{2}, \sqrt{3}-1]$ and $\mu^- > 2\mu^{-1}(1-\mu)$, the limits (1.4) and (1.5) exist; in particular AC holds.*

Proof We shall only prove the existence of the limit (1.4); the existence of (1.5) can be shown completely similarly. We may assume (8.3) for any fixed $\breve{\alpha} < \mu$, cf. [E]. Let $\breve{\alpha} = \mu - \epsilon$ and $\delta = 2(1-\mu) + 5\epsilon$ for a small $\epsilon > 0$. In addition we can assume (8.4) (for σ small enough) and

$$\min\left(\tfrac{\breve{\alpha}\mu^-}{2}, \breve{\alpha}, \tfrac{\delta}{2}\right) \geq 1 + \epsilon - \breve{\alpha}. \tag{8.6}$$

By Lemma 8.2 and by mimicking the proof of [S1, Lemma 3.1] under use of (8.6) and Lemma 8.1 we obtain

$$\phi_a^+(t) \approx F_-\big(|t^\delta H^a|\big) F_-\big(t^{-\breve{\alpha}}|x^a|\big) F_-\big(|t^\delta H^a|\big)\phi_a^+(t) \text{ as } t \to +\infty. \tag{8.7}$$

Moreover,

$$\int_1^\infty t^{-1} |\big\langle F\big(t^\delta |H^a| \approx 1\big)\big\rangle_{\phi_a^+(t)}| dt < \infty, \tag{8.8}$$

cf. [S1, Lemma 2.4], and

$$\int_1^\infty t^{-1} |\big\langle F\big(t^{-\breve{\alpha}} |x^a| \approx 1\big)\big\rangle_{F_-(|t^\delta H^a|)\phi_a^+(t)}| dt < \infty, \tag{8.9}$$

cf. [S1, Lemma 3.3].

The analogous statements for $\tilde{U}_a(t)$ read with $\tilde{\phi}(t) = \tilde{U}_a(t)\tilde{\phi}$:

$$\int_1^\infty t^{-1} |\big\langle F\big(t^\delta |H^a| \approx 1\big)\big\rangle_{\tilde{\phi}(t)}| dt \leq C||\tilde{\phi}||^2, \tag{8.10}$$

and

$$\int_1^\infty t^{-1} |\big\langle F\big(t^{-\breve{\alpha}} |x^a| \approx 1\big)\big\rangle_{F_-(|t^\delta H^a|)\tilde{\phi}(t)}| dt \leq C||\tilde{\phi}||^2. \tag{8.11}$$

Substituting (8.7) in the expression

$$\tilde{U}_a(t)^* \phi_a^+(t),$$

and then differentiating yields integrable terms: The one with the potentials is bounded by

$$||(I_a(t,x_a) - I_a(t,x))F_-\Big(t^{-\breve{\alpha}}|x^a|\Big)|| + ||\Big[I_a(t,x), F_-\Big(|t^\delta H^a|\Big)\Big]|| = O\big(t^{-1-2\sigma}\big);$$

notice that the first term to the left vanishes by (8.3) and that the bound of the second term comes about by using the proof of Lemma 8.2. Clearly the bound is integrable.

The contribution from the derivative of the factor $F_-\big(t^{\epsilon-\mu}|x^a|\big)$,

$$\mathbf{D}F_-\Big(t^{-\breve{\alpha}}|x^a|\Big) = \frac{1}{2}\bigg(t^{-\breve{\alpha}}F'_-(\cdot)\frac{x^a}{|x^a|}\cdot p^a + h.c.\bigg) - \frac{\breve{\alpha}}{t}F'_-(\cdot)(\cdot),$$

is handled as follows: Using (8.6) and Lemma 8.1 again we infer that the first term on the right hand side is $O\big(t^{-1-\epsilon}\big)$. The second term is treated by (8.9) and (8.11).

The contributions from the two factors $\frac{d}{dt}F_-\big(|t^\delta H^a|\big)$ are treated by (8.8) and (8.10) after commutations under use of (8.6) and Lemma 8.1.

□

Now we consider the regime $\mu \in (0, \frac{1}{2}]$:

Proof of Theorem 1.3: We shall only prove the existence of the limit (1.4); the existence of (1.5) can be shown completely similarly. We pick $\epsilon > 0$ so small that

$$(1-\mu+2\epsilon)\mu^+ \leq 2\mu - 3\epsilon. \tag{8.12}$$

Let $\breve{\alpha} = \mu - \epsilon$ and $\delta = 2\breve{\alpha}$. We can assume (8.3). Clearly (8.4) holds for σ small enough.

Proceeding exactly as in the beginning of the proof of Theorem 8.3 we conclude that

$$\phi_a^+(t) \approx F_-\big(t^{\mu-1-2\epsilon}|x^a|\big)F_-\big(|t^\delta H^a|\big)\phi_a^+(t) \text{ as } t \to +\infty. \tag{8.13}$$

The next result doesn't have an analogue in the previous proof. Proceeding as the proof of [S1, Lemma 4.1] we introduce for $\psi \in \mathcal{H}$

$$\tilde{\psi} = F_1F_2F_3\psi;$$

$$F_1 = F\Big(t^{-\breve{\alpha}}|x^a| > 1\Big),\ F_2 = F\big(t^{\mu-1-2\epsilon}|x^a| < 1\big),\ F_3 = F\Big(|t^\delta H^a| < 1\Big).$$

Suppose we know that $||F_1F_2F_3|| = O\big(t^{-s}\big)$, then we shall show the bound with the right hand side replaced by $O\big(t^{-s-\frac{\epsilon}{2}}\big)$, leading inductively to the conclusion that

$$||F_1F_2F_3|| = O\big(t^{-\infty}\big). \tag{8.14}$$

We estimate, cf. [S1, (4.11)],

$$\begin{aligned}
&\langle 2H^a\rangle_{\tilde{\psi}} = \bigg\langle H^aF_1^2F_2^2 + F_1^2F_2^2H^a + \Big(t^{-\breve{\alpha}}F_1'F_2 + t^{\mu-1-2\epsilon}F_1F_2'\Big)^2\bigg\rangle_{F_3\psi}\\
&\leq 2t^{-\delta}||F_1F_2\Big(t^\delta H^aF_3\Big)\psi||\,||\tilde{\psi}||\\
&+ 2t^{-2\breve{\alpha}}||F_1'F_2F_3\psi||^2 + 2t^{2(\mu-1-2\epsilon)}||F_1F_2'F_3\psi||^2,
\end{aligned} \tag{8.15}$$

and, using here (8.12),

$$\langle 2H^a\rangle_\psi \geq Ct^{-(1-\mu+2\epsilon)\mu^+}||\tilde{\psi}||^2 \geq Ct^{-2\mu+3\epsilon}||\tilde{\psi}||^2. \tag{8.16}$$

Combining (8.15) and (8.16) leads to $||\tilde{\psi}|| \leq Ct^{-s-\frac{\epsilon}{2}}||\psi||$, whence we conclude (8.14).

Combining (8.13) and (8.14) yields

$$\phi_a^+(t) \approx F_-\Big(|t^\delta H^a|\Big)F_-\Big(t^{-\breve{\alpha}}|x^a|\Big)F_-\Big(|t^\delta H^a|\Big)\phi_a^+(t).$$

Following now the last part of the proof of Theorem 8.3 we differentiate the expression

$$\tilde{U}_a(t)^* F_-\Big(|t^\delta H^a|\Big)F_-\Big(t^{-\breve{\alpha}}|x^a|\Big)F_-\Big(|t^\delta H^a|\Big)\phi_a^+(t),$$

and pick up integrable terms: The contribution with the potentials is estimated by

$$||\Big[I_a(t,x), F_-\Big(|t^\delta H^a|\Big)\Big]|| = O\big(t^{-1-2\sigma}\big) \in L^1(dt),$$

cf. the proof of Lemma 8.2. The one from the Heisenberg derivative of the factor $F_-\big(t^{-\breve{\alpha}}|x^a|\big)$ is clearly integrable by (8.14). The same conclusion holds for the contributions from the two terms with a factor $\frac{d}{dt}F_-\big(|t^\delta H^a|\big)$; this follows by using the bounds (8.8) and (8.10) (with the present δ) after commutations under use of (8.14) again.

□

A Classical orbits

In this Appendix we shall construct solutions $\alpha^+(t)$ and $\alpha^-(t)$ to the Riccati equation

$$\dot{\alpha} = q(t)^2 - \alpha^2 \tag{A.1}$$

where $q(t)^2$ is a continuous (non-negative) function. For the example $q(t)^2 = -V^{a\prime\prime}(L)$ of Section 5 (cf. (2.1)) we establish (5.1) using in this case the upper bound

$$q(t)^2 \leq Ct^{-2}, \tag{A.2}$$

cf. [S1, (5.8)] (or (7.12)).

First we construct two solutions $\alpha^+(t)$ and $\alpha^-(t)$ satisfying the bounds

$$t^{-1} \leq \alpha^+(t) \text{ and } \alpha^-(t) \leq 0.$$

This will be done without using (A.2).

To find $\alpha^+(t)$ we notice that $\beta(t) = t^{-1}$ is a solution to

$$\dot{\beta} = -\beta^2.$$

Clearly we can solve (A.1) in a neighborhood of $t = 1$ with the initial condition $\alpha(1) = \beta(1) = 1$. By the standard comparison theorem (see for example [BR, Theorem 1.8]) we conclude that $\alpha(t) \geq t^{-1}$ for $t \geq 1$. Using the equation (A.1) we readily continue $\alpha(t)$ to the whole half-axis. The obtained solution is denoted by $\alpha^+(t)$.

It remains to construct $\alpha^-(t) \leq 0$. For that we consider the Schrödinger equation

$$-x''(t) - V^{a\prime\prime}(L(t))x(t) = 0. \tag{A.3}$$

From the solution $x^+(t) = \exp\left(\int_1^t \alpha^+ dt'\right)$ we obtain another one, cf. [BR, Section 2.5], by the formula

$$x^-(t) = x^+(t) \int_t^\infty x^+\left(t'\right)^{-2} dt'.$$

Let us note the following bounds

$$\begin{aligned} 0 < x^-(t) &= x^+(t)^{-1} \int_t^\infty e^{-2\int_t^{t'} \alpha^+ dt''} dt' \\ &\leq x^+(t)^{-1} \int_t^\infty e^{-2\int_t^{t'} \frac{1}{t''} dt''} dt' = x^+(t)^{-1} t \leq 1. \end{aligned} \tag{A.4}$$

From (A.3) we get $x^-(t)'' \geq 0$. Consequently if $x^-(t)'$ if positive for some t the solution $x^-(t)$ will grow at least linearly contradicting (A.4). Therefore $x^-(t)' \leq 0$ and we conclude that

$$\alpha^-(t) := \frac{x^-(t)'}{x^-(t)} \leq 0.$$

We shall now show the bounds

$$\alpha^+(t) \leq C't^{-1} \text{ and } -C't^{-1} \leq \alpha^-(t)$$

under the assumption (A.2).

To get the upper bound of $\alpha^+(t)$ we introduce the function

$$\beta(t) = \frac{1+\sqrt{1+4C}}{2}t^{-1}$$

which satisfies $\beta(1) \geq \alpha^+(1) = 1$ and solves

$$\dot{\beta} = Ct^{-2} - \beta^2.$$

By another comparison we conclude that $\alpha^+(t) \leq \beta(t)$.

To get the lower bound $\alpha^-(t) \geq -C''t^{-1}$ we compute using the upper bound $\alpha^+(t) \leq C't^{-1}$

$$\alpha^-(t) = \alpha^+(t) - \left(\int_t^\infty e^{-2\int_t^{t'} \alpha^+ dt''} dt' \right)^{-1}$$

$$\geq t^{-1} - \left(\int_t^\infty e^{-2\int_t^{t'} \frac{C'}{t''} dt''} dt' \right)^{-1} = -2\big(C'-1\big)t^{-1}.$$

The proof of the bounds is completed. In particular we have verified (5.1).

For an application in Section 7 we notice that the formula

$$x^+x^- = \frac{x^+x^-}{x^{+\prime}x^- - x^{-\prime}x^+} = \frac{1}{\alpha^+ - \alpha^-},$$

and (5.1) yield the bounds

$$(2C)^{-1}t \leq x^+(t)x^-(t) \leq t \tag{A.5}$$

for C given as in (5.1).

References

[BR] G. Birkhoff, G.C. Rota: *Ordinary differential equations,* 4. edition, New York, John Wiley & Sons, 1989.

[D] J. Dereziński: *Asymptotic completeness for N-particle long-range quantum systems,* Ann. Math., 138 (1993), 427–476.

[DG] J. Dereziński, C. Gérard: *Scattering theory of classical and quantum N-particle systems*, Texts and Monographs in Physics, Springer, Berlin Heidelberg New York, 1997.

[E] V. Enss: *Long-range scattering of two- and three-body quantum systems*, Journées Equations à Dérivées Partielles, Saint Jean de Monts, Juin, 1989, Publ. Ecole Polytechnique, Palaiseau, 1989, 1–31.

[G] C. Gérard: *Asymptotic completeness for 3–particles systems*, Invent. Math. 114 (1993), 333–397.

[HS] I. Herbst, E. Skibsted: *Quantum scattering for homogeneous of degree zero potentials: Absence of channels at local maxima and saddle points,* MaPhySto preprint 24, 1999.

[H] L. Hörmander: *The analysis of linear partial differential operators III*, Berlin, Springer-Verlag, 1985.

[O] F.W.J. Olver: *Asymptotic and special functions,* New York, Academic Press 1974.

[S1] E. Skibsted: *Long-range scattering of three-body quantum systems, I,* MaPhySto preprint.

[S2] E. Skibsted: *Asymptotic absolute continuity for perturbed time-dependent quadratic Hamiltonians,* MaPhySto preprint 12, 2001, to appear in Mathematical section of the Proceedings of the Indian Academy of Sciences.

[W] X.P. Wang: *On the three body long-range scattering problems,* Reports Math. Phys., 25 (1992), 267–276.

[Y] D. Yafaev: *New channels of scattering for three-body quantum systems with long-range petentials,* Duke Math. J., 82 no.3 (1996), 553–584.