**The Relativistic Theory of the Asymmetric Field**

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*Abstract*

The fundamental concepts of General Relativity have been reexamined. The heuristic value of the concept of general covariance in seeking new laws of nature is acknowledged, in that any new tensor, which arises naturally within the development, is deemed as a candidate for representing a physical quantity. The inclusion of the anti-symmetric part of the metric connection is seen to lead to field equations which do not require the ad hoc introduction of an energy tensor of matter. The natural appearance of this tensor shows that the field equations of General Relativity may be derived from a geometrical tensor identity. The theory indicates, even more than ever, how the physics which can occur in the Universe coincides with the geometry of space-time.

*Keywords*

Christoffel tensor, geodesic equation, transposition invariance, curvature tensor, cosmological constant, energy tensor, field theory.

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*Introduction*

One of the earliest references on the foundations of General Relativity is Einstein's book [1]. Here, as a continuation of his work, the expansion of the equations of the gravitational field has led to new possibilities for using a 4-dimensional field theory as a means of unification. Previously, systems with more than four dimensions have been given more emphasis since it seemed that electromagnetism could not be included in a natural way into the 4-dimensional system of gravitational equations. Unification methods which contract the metric connection to a trace alone will lose much of the physical structure which may be present. Within this field theory, however, the unification of electromagnetism and gravitation arises from the coupling of the metric connection and the four-potential. The expanded system of equations must eventually be shown to be consistent with the known facts of physical cosmology [2].

*Remarks on the Displacement Field*

The 'displacement field' is also known as the 'metric connection' and the anti-symmetric part of the displacement field will be represented below by a capital 'sigma'. This anti-symmetric part of the connection transforms as a tensor which suggests that it may have some physical existence in its own right, other than being just a part of the connection. This is the basis of the current work, however, it differs from Einstein's non-symmetric theory, in that the symmetry of the metric tensor itself is retained. The heading for this section is taken from a sub-heading in the main reference, where it is remarked that the lower indices of the connection, in the equation for the increment of a vector, 'play quite different roles' [1]. This indicated that the requirement of symmetry for the metric connection may have been too restrictive.

*The Asymmetric Connection*

The symbol 'gamma' has been retained for the symmetric part of the displacement field and the symbol 'sigma' has been introduced for the anti-symmetric part. The word 'asymmetric' is used to distinguish this connection from the 'non-symmetric' connection used in Appendix II of the main reference. The theory of the 'non-symmetric field' introduced a metric tensor [3] which was also non-symmetric. The metric tensor occurs in the separation formula for a pair of events in space-time and it is difficult to see why the quadratic terms should not be symmetric in the coordinate indices, so here the symmetry of the metric tensor itself is regarded as essential. The covariant derivative of a covariant vector in the asymmetric theory is now given by

$$B_{\mu;\nu} = B_{\mu,\nu} - \left(\Gamma^{\alpha}_{\mu\nu} + \Sigma^{\alpha}_{\mu\nu}\right) B_{\alpha} \tag{1}$$

where we have used the semicolon to represent covariant differentiation and comma subscript 'nu' to represent partial differentiation with respect to x subscript 'nu'.

*The Natural Anti-Symmetric Tensor*

In order to generate the simplest anti-symmetric tensor which can be formed from the covariant derivative, the symmetric part of the tensor in equation (1) is removed by forming a difference.

$$\overline{\phi}_{\mu\nu} = B_{\mu;\nu} - B_{\nu;\mu} \tag{2}$$

This, however, retains the anti-symmetric part of the connection, while the symmetric part cancels.

$$\overline{\phi}_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - 2\Sigma^{\alpha}_{\mu\nu} B_{\alpha} \tag{3}$$

The first result for the asymmetric theory is to note the presence of the third term in (3). This term occurs in the same position as gauge theory extensions to the electromagnetic field [4]. This method of tensor formation from the four-potential was used to include the electromagnetic field into General Relativity. However, the third term is absent in the symmetric theory and there is no coupling between the metric connection and electromagnetism unless it is introduced to the field equations as an energy tensor. The expression will be rewritten using the traditional notation for the electromagnetic field.

$$\overline{\phi}_{\mu\nu} = \phi_{\mu\nu} - 2\Sigma^{\alpha}_{\mu\nu} B_{\alpha} \tag{4}$$

$$\phi_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu}$$

*Maxwell's Equations*

The second of Maxwell's equations can be written as

$$\frac{\partial \phi_{\mu\nu}}{\partial x_{\sigma}} + \frac{\partial \phi_{\nu\sigma}}{\partial x_{\mu}} + \frac{\partial \phi_{\sigma\mu}}{\partial x_{\nu}} = 0 \tag{5}$$

With the usual identification of the electric and magnetic vector components with elements of the electromagnetic field tensor [4], this equation will still allow the 'curl' of the electric field to equal the rate of change of the magnetic field. However, the other 'curl' equation is usually derived from the divergence of the field. We may now speculate that, although Maxwell's equations were derived from empirical observations, the observable quantities may not have been of a fundamental nature. In the asymmetric theory, we now take equation (4) as an expression for the natural field tensor, and set its divergence to zero.

$$\frac{\partial \overline{\phi}_{\mu\nu}}{\partial x_\nu} = \frac{\partial \phi_{\mu\nu}}{\partial x_\nu} - \frac{\partial \left( 2\Sigma^\alpha_{\mu\nu} B_\alpha \right)}{\partial x_\nu} = 0 \tag{6}$$

The above equation will allow the 'curl' of the magnetic field to equal the rate of change of the electric field plus the current vector, if we now identify the current with the term in 'sigma'.

$$J_\mu = \frac{\partial \left( 2\Sigma^\alpha_{\mu\nu} B_\alpha \right)}{\partial x_\nu} \tag{7}$$

These equations begin the unification of electromagnetism and gravitation which will be completed if some deeper unification is possible with the 'sigma' tensor. It would be important to show that the current can be represented as above in Minkowski space-time. This is left as an open question for now as we move on to consider gravitation.

*The Geodesic Equation*

In the symmetric theory this equation defines particle paths under the influence of inertia and gravitation alone [5]. The generalization to the asymmetric field is given by

$$\frac{d^2 x^\mu}{ds^2} + \left( \Gamma^\mu_{\alpha\beta} + \Sigma^\mu_{\alpha\beta} \right) \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \tag{8}$$

When expanded, the term containing 'sigma' is found to be equal to zero by interchanging 'alpha' and 'beta'. The geodesic equation retains its original form although, as will be seen below, the symmetric 'gamma' is not the same 'gamma' as in General Relativity.

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \tag{9}$$

*The Covariant Derivative of the Metric*

This is the starting point for attempts to express the 'gamma' in terms of the metric tensor. For the symmetric theory the method has been described in many standard texts where the covariant derivative of the metric tensor is set to zero.

$$\frac{\partial g_{\mu\nu}}{\partial x_\alpha} - g_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} - g_{\nu\beta}\Gamma^{\beta}_{\mu\alpha} = 0 \tag{10}$$

The covariant derivative is now given using the fully asymmetric connection.

$$\frac{\partial g_{\mu\nu}}{\partial x_\alpha} - g_{\mu\beta}\left(\Gamma^{\beta}_{\nu\alpha} + \Sigma^{\beta}_{\nu\alpha}\right) - g_{\nu\beta}\left(\Gamma^{\beta}_{\mu\alpha} + \Sigma^{\beta}_{\mu\alpha}\right) = 0 \tag{11}$$

In this case it is not possible to express the 'gamma' or the 'sigma' in terms of the metric tensor alone. However, if we follow the same method as before, using a cyclic interchange of indices with some addition and subtraction of equations we obtain,

$$\frac{\partial g_{\nu\alpha}}{\partial x_\mu} + \frac{\partial g_{\alpha\mu}}{\partial x_\nu} - \frac{\partial g_{\mu\nu}}{\partial x_\alpha} = +2g_{\alpha\beta}\Gamma^{\beta}_{\mu\nu} + 2g_{\nu\beta}\Sigma^{\beta}_{\alpha\mu} + 2g_{\mu\beta}\Sigma^{\beta}_{\alpha\nu} \tag{12}$$

This indicates that the 'gamma' should now be expressed in terms of the familiar derivatives of the metric tensor and a new symmetric tensor derived from the 'sigma' and the metric.

*The Symmetric Connection and the Christoffel Tensor*

The symmetric part of the connection is the sum of the Christoffel tensor of the second kind and the new tensor.

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\alpha}\left(\frac{\partial g_{\nu\alpha}}{\partial x_\mu} + \frac{\partial g_{\alpha\mu}}{\partial x_\nu} - \frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right) + S^{\sigma}_{\mu\nu} \tag{13}$$

where the additional contribution to the symmetric part of the connection is given by

$$S^{\sigma}_{\mu\nu} = g^{\sigma\alpha}\left(g_{\nu\beta}\Sigma^{\beta}_{\mu\alpha} + g_{\mu\beta}\Sigma^{\beta}_{\nu\alpha}\right) \tag{14}$$

This is a significant result. It is telling us that, if we try to introduce an anti-symmetric part to the connection, while retaining a symmetric metric tensor, then there will be a symmetric contribution to the metric connection. This also allows the geodesic equation to retain its original form so that, although the anti-symmetric part of the connection cancels explicitly in equation (8), there is still a new contribution to the geometry of the geodesic paths [6].

*The Contracted Symmetric Connection*

Contracting equation (13) gives

$$\Gamma^{\nu}_{\mu\nu} = \frac{1}{2} g^{\nu\alpha} \frac{\partial g_{\nu\alpha}}{\partial x_{\mu}} + \delta^{\alpha}_{\beta} \Sigma^{\beta}_{\mu\alpha} + g^{\nu\alpha} g_{\mu\beta} \Sigma^{\beta}_{\nu\alpha} \tag{15}$$

The third term on the right-hand side is found to be equal to zero by interchanging 'nu' and 'alpha'. The second term becomes what can be described as 'the trace of the sigma tensor'. It appears to be related closely to the determinant of the metric.

$$\Gamma^{\nu}_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x_{\mu}} + \Sigma^{\alpha}_{\mu\alpha} \tag{16}$$

It is now confirmed that the dimensions of 'sigma' are those of reciprocal length.

*The Transposition Invariant Derivative*

We now introduce transposition of the lower indices which will be required later for evaluation of curvature. Firstly, we write the partial derivative of equation (16) with respect to x, using the comma notation for partial derivatives.

$$\Gamma^{s}_{is,k} = \frac{\partial}{\partial x_k} \left( \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x_i} + \Sigma^{s}_{is} \right) \tag{17}$$

Transposing indices i and k, gives 'gamma' tilde.

$$\tilde{\Gamma}^{s}_{is,k} = \frac{\partial}{\partial x_i} \left( \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x_k} + \Sigma^{s}_{ks} \right) \tag{18}$$

A transposition invariant derivative of the contracted symmetric connection is given by

$$\Gamma^{s}_{is,k} + \tilde{\Gamma}^{s}_{is,k} = 2G_{ik} + H_{ik} \tag{19}$$

where the two tensors on the right hand side are given by equations (20) and (21)

$$G_{ik} = \frac{\partial}{\partial x_i}\left(\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x_k}\right) \tag{20}$$

$$H_{ik} = \frac{\partial \Sigma^s_{is}}{\partial x_k} + \frac{\partial \Sigma^s_{ks}}{\partial x_i} \tag{21}$$

The latter quantity is a transposition invariant derivative of the sigma tensor trace.

*The Curvature Tensor*

The derivation of the contracted curvature tensor is independent of the symmetry properties of the connection. For the asymmetric connection we write the negative as

$$-K_{ik} = \left(\Gamma^s_{ik} + \Sigma^s_{ik}\right)_{,s} - \left(\Gamma^s_{is} + \Sigma^s_{is}\right)_{,k} - \left(\Gamma^s_{it} + \Sigma^s_{it}\right)\left(\Gamma^t_{sk} + \Sigma^t_{sk}\right) + \left(\Gamma^s_{ik} + \Sigma^s_{ik}\right)\left(\Gamma^t_{st} + \Sigma^t_{st}\right) \tag{22}$$

*Transposition Invariance*

We now appeal to Einstein's comments on transposition invariance [1]. This is the parting of the ways for the non-symmetric theory and the asymmetric theory. By persisting with the calculation in terms of the 'gamma' and 'sigma' we will find a transposition symmetric tensor which may be used in field equations. Since the connection has both symmetric and antisymmetric parts, to form a transposition symmetric tensor from the contracted curvature tensor we take

$$-2M_{ik} = -\left(K_{ik} + \tilde{K}_{ik}\right) \tag{23}$$

Rather than give the explicit version of this in terms of 'gamma' and 'sigma', the method of evaluation of some of the terms which occur in the expansion will be given.

To evaluate some of the terms containing products, we consider various values of a parametric tensor 'pi' and its transpose 'pi' tilde.

If $\Pi_{ik} = \Gamma^s_{it}\Gamma^t_{sk}$ then

$\tilde{\Pi}_{ik} = \Gamma^s_{kt}\Gamma^t_{si} = \Gamma^s_{ti}\Gamma^t_{ks} = \Gamma^s_{it}\Gamma^t_{sk}$ and

$\tilde{\Pi}_{ik} = \Pi_{ik}$

This product is transposition symmetric, so the value is doubled in the expansion of equation (23).

Also,
$\Pi_{ik} = \Gamma^s_{ik}\Gamma^t_{st}$ is transposition symmetric since 'gamma' is symmetric.

If
$\Pi_{ik} = \Sigma^s_{it}\Gamma^t_{sk} + \Sigma^t_{sk}\Gamma^s_{it}$
then
$\tilde{\Pi}_{ik} = \Sigma^t_{ks}\Gamma^s_{ti} + \Sigma^s_{ti}\Gamma^t_{ks}$

$\tilde{\Pi}_{ik} = -\Sigma^t_{sk}\Gamma^s_{it} - \Sigma^s_{it}\Gamma^t_{sk}$
and
$\tilde{\Pi}_{ik} = -\Pi_{ik}$
so the terms cancel in the expansion of equation (23).

If
$\Pi_{ik} = \Sigma^s_{ik}\Gamma^t_{st} + \Sigma^t_{st}\Gamma^s_{ik}$ then
$\Pi_{ik} + \tilde{\Pi}_{ik} = 2\Sigma^t_{st}\Gamma^s_{ik}$

If
$\Pi_{ik} = \Sigma^s_{it}\Sigma^t_{sk}$
then
$\tilde{\Pi}_{ik} = \Sigma^s_{kt}\Sigma^t_{si} = \Sigma^s_{ti}\Sigma^t_{ks}$
and
$\tilde{\Pi}_{ik} = \Pi_{ik}$

If
$\Pi_{ik} = \Sigma^s_{ik}\Sigma^t_{st}$
then
$\tilde{\Pi}_{ik} = \Sigma^s_{ki}\Sigma^t_{st} = -\Sigma^s_{ik}\Sigma^t_{st}$
$\tilde{\Pi}_{ik} = -\Pi_{ik}$

To evaluate some of the differential terms, we consider various values of a parametric tensor 'delta' and its transpose 'delta' tilde.

If
$\Delta_{ik} = \Gamma^s_{ik,s}$
then
$\tilde{\Delta}_{ik} = \Gamma^s_{ki,s} = \Gamma^s_{ik,s}$
and
$\tilde{\Delta}_{ik} = \Delta_{ik}$
so the value is doubled in the expansion of equation (23).

If
$\Delta_{ik} = \Gamma^s_{is,k}$
then
$\tilde{\Delta}_{ik} = \Gamma^s_{ks,i}$
and using equation (19),
$\Delta_{ik} + \tilde{\Delta}_{ik} = 2G_{ik} + H_{ik}$

If
$\Delta_{ik} = \Sigma^s_{ik,s}$
then
$\tilde{\Delta}_{ik} = -\Delta_{ik}$

If $\Delta_{ik} = \Sigma^s_{is,k}$
then, using equation (21),
$\Delta_{ik} + \tilde{\Delta}_{ik} = H_{ik}$
a tensor which will play a part in the field equations.

*The Transposition Symmetric Curvature Tensor*

We now return to the expansion of equation (23) of the previous section, and using the results of that section we simplify,

$$-2M_{ik} = 2\Gamma^s_{ik,s} - (2G_{ik} + H_{ik}) - 2\Gamma^s_{it}\Gamma^t_{sk} + 2\Gamma^s_{ik}\Gamma^t_{st} - H_{ik} - 2\Sigma^s_{it}\Sigma^t_{sk} + 2\Sigma^t_{st}\Gamma^s_{ik}$$

Collecting some of the familiar terms containing the 'gamma',

$$M_{ik} = \overline{R}_{ik} + H_{ik} + P_{ik} - Q_{ik} \tag{24}$$

$$\overline{R}_{ik} = -\Gamma^s_{ik,s} + G_{ik} + \Gamma^s_{it}\Gamma^t_{sk} - \Gamma^s_{ik}\Gamma^t_{st} \tag{25}$$

$$H_{ik} = \Sigma^s_{is,k} + \Sigma^s_{ks,i} \tag{26}$$

$$P_{ik} = \Sigma^s_{it}\Sigma^t_{sk} \tag{27}$$

$$Q_{ik} = \Sigma^t_{st}\Gamma^s_{ik} \tag{28}$$

Equation (25) is recognized as the contracted curvature tensor found in the field equations of General Relativity if we use equation (20), however the 'gamma' are now defined by equation (13).

*Field Equations*

There is some freedom of choice in selecting the form of field equations. We could follow Einstein's heuristic approach [1], and introduce the 'energy tensor of matter'. Alternatively, we can just form a generally covariant equation from the transposition symmetric curvature tensor, and explore the consequences.

$$M_{ik} - \frac{1}{2}g_{ik}M = 0 \tag{29}$$

where the scalar of the symmetric curvature is given by

$$M = \overline{R} + H + P - Q \tag{30}$$

and the four scalars on the right-hand side are found by contracting equations (25) to (28) with the contravariant metric tensor.

Any new field equations must reduce to those of General Relativity, as a zero order test, when the metric connection is completely symmetric so we attempt to recover the cosmological equations by setting

$$\overline{R}_{ik} - \frac{1}{2} g_{ik} \overline{R} + \Lambda g_{ik} = -\kappa T_{ik} \tag{31}$$

where the bar indicates that the functions depend upon a symmetric part of the connection which has been augmented, as in equation (13). The cosmological constant is then given by

$$\Lambda = -\frac{1}{2} (H + P - Q) \tag{32}$$

This equation shows how the cosmological constant will be related to the fundamental constants of nature once the scalars for the energy tensor have been identified. The algebraic sum of scalars with opposite signs implies that the cosmological constant may change in sign as the Universe evolves.

The energy tensor has been included in equation (31) to show where it must be introduced within the equations of General Relativity in order to represent the sources of gravitation. Here, the term represents forms of energy which have not been introduced explicitly within the unification.

$$\kappa T_{ik} = H_{ik} + P_{ik} - Q_{ik} \tag{33}$$

We have identified equation (33) as an expression for the energy tensor of the asymmetric field. It may yet transpire that we have the liberty to introduce particular forms of the energy tensor in order to study specific problems.

*Conclusion*

Here we have a relativistic field theory which does not require the ad hoc introduction of either the current vector or the energy tensor of matter. The natural appearance of the energy tensor confirms that the asymmetric field has non-zero energy. The derivation of transposition symmetric field equations which include the anti-symmetric part of the metric connection is complete. The field equations of General Relativity are contained within the asymmetric theory as a special case of complete symmetry.

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