# HOMOGENIZATION OF RANDOM WALK IN ASYMMETRIC RANDOM ENVIRONMENT

JOSEPH G. CONLON

Abstract. In this paper, the author investigates the scaling limit of a partial difference equation on the d dimensional integer lattice $\mathbf{Z}^d$, corresponding to a translation invariant random walk perturbed by a random vector field. In the case when the translation invariant walk scales to a Cauchy process he proves convergence to an effective equation on $\mathbf{R}^d$. The effective equation corresponds to a Cauchy process perturbed by a constant vector field. In the case when the translation invariant walk scales to Brownian motion he proves that the scaling limit, if it exists, depends on dimension. For $d = 1, 2$ the scaling limit cannot be diffusion.

## 1. Introduction.

In this paper we shall be concerned with homogenization of a nondivergence form elliptic equation. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $\mathbf{b} : \Omega \to \mathbf{R}^d$ be a bounded measurable function $\mathbf{b}(\omega) = (b_1(\omega), ..., b_d(\omega))$ with

$$|\mathbf{b}(\omega)|^2 = \sum_{i=1}^{d} |b_i(\omega)|^2 \le 1 \quad , \omega \in \Omega \ . \tag{1.1}$$

We assume that $\mathbf{Z}^d$ acts on $\Omega$ by translation operators $\tau_x : \Omega \to \Omega$, $x \in \mathbf{Z}^d$, which are measure preserving ergodic and satisfy the properties $\tau_x \tau_y = \tau_{x+y}$, $\tau_0 =$ identity, $x, y \in \mathbf{Z}^d$. Using these operators we can define a measurable function $\mathbf{b} : \mathbf{Z}^d \times \Omega \to \mathbf{R}^d$ by $\mathbf{b}(x, \omega) = \mathbf{b}(\tau_x \omega),\ x \in \mathbf{Z}^d,\ \omega \in \Omega$.

For $\varepsilon > 0$ let $\mathbf{Z}^d_\varepsilon = \varepsilon \mathbf{Z}^d$ be the $\varepsilon$ scaled integer lattice. We define the integral of a function $g : \mathbf{Z}^d_\varepsilon \to \mathbf{R}$ by

$$\int_{\mathbf{Z}^d_\varepsilon} g(x) dx \quad \stackrel{\text{def.}}{=} \quad \varepsilon^d \sum_{x \in \mathbf{Z}^d_\varepsilon} g(x) \ .$$

The space $L^2(\mathbf{Z}^d_\varepsilon)$ is then the space of square integrable functions $u : \mathbf{Z}^d_\varepsilon \to \mathbf{C}$ with norm, $\|u\|_\varepsilon$ satisfying

$$\|u\|^2_\varepsilon = \int_{\mathbf{Z}^d_\varepsilon} |u(x)|^2 dx.$$

For $i = 1, ..., d$, let $\mathbf{e}_i \in \mathbf{Z}^d$ be the element with entry 1 in the ith position and 0 in the other positions. Suppose $\gamma \in \mathbf{C}$ and $f : \mathbf{R}^d \to \mathbf{R}$ is a $C^\infty$ function with compact support. Let $p : \mathbf{Z}^d \to \mathbf{R}$ be a probability density, whence

$$p(x) \ge 0,\ x \in \mathbf{Z}^d, \quad \sum_{x \in \mathbf{Z}^d} p(x) = 1 \ .$$

With any such $p$ we can associate a translation invariant operator $A_p$ on functions $u_\varepsilon : \mathbf{Z}_\varepsilon^d \to \mathbf{R}$ by

$$A_p u_\varepsilon(x) = u_\varepsilon(x) - \sum_{y\in\mathbf{Z}^d} p(y) u_\varepsilon(x+\varepsilon y).$$

For some probability densities $p$ we can define the degree of the symbol for $A_p$. We denote this degree by $|A_p|$. If $p$ is given by

$$p(\pm\mathbf{e}_i) = 1/2d, \quad i = 1, ..., d \ , \tag{1.2}$$

then $|A_p| = 2$. We can also define a $p$ with $|A_p| = 1$. To do this consider the standard nearest neighbor random walk on $\mathbf{Z}^{d+1}$ started at the point $(0, -1)$, $0 \in \mathbf{Z}^d$. Then

{ (y,0) : y $\in$ $\mathbf{Z}^d$} in the point $(x, 0)$, $x$ $\in$ $\mathbf{Z}^d$.

We shall be interested in solutions to the equation,

$$\begin{aligned} A_p u_\varepsilon(x,\omega) \quad &- \quad \gamma \sum_{i=1}^d b_i\left(\frac{x}{\varepsilon},\omega\right)\left[u_\varepsilon(x+\varepsilon\mathbf{e}_i,\omega) - u_\varepsilon(x-\varepsilon\mathbf{e}_i,\omega)\right] \\ + \varepsilon^{|A_p|} u_\varepsilon(x,\omega) \quad &= \quad \varepsilon^{|A_p|} f(x), \ x \in \mathbf{Z}_\varepsilon^d \ , \quad \omega\in\Omega, \end{aligned} \tag{1.3}$$

when $p$ is given by either (1.2) or (**??**). For a function $g : \mathbf{X} \to \mathbf{C}$ where $\mathbf{X} = \mathbf{R}^d$ or $\mathbf{X} = \mathbf{Z}_\varepsilon^d$, let $\hat{g}$ be its Fourier transform,

$$\hat{g}(\xi) = \int_{\mathbf{X}} g(x) e^{ix\cdot\xi} dx, \quad \xi \in \mathbf{R}^d.$$

Note that when $\mathbf{X} = \mathbf{Z}_\varepsilon^d$, then $\hat{g}(\xi)$ is periodic in $\xi$ whence we can restrict $\xi$ to the cube $[-\pi/\varepsilon, \ \pi/\varepsilon]^d$. Suppose now $\mathbf{b} \equiv 0$ in (1.3) and $p$ is given by (1.2). Then it is easy to see that (1.3) has a unique solution $u_\varepsilon(x)$ in $L^2(\mathbf{Z}_\varepsilon^d)$. Further, if $u(x), \ x \in \mathbf{R}^d$, is the function which satisfies

$$[1 + |\xi|^2/2d]\hat{u}(\xi) = \hat{f}(\xi), \quad \xi \in \mathbf{R}^d,$$

then $\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_\varepsilon = 0$. Similarly if $\mathbf{b} \equiv 0$ and $p$ is given by (**??**) then (1.3) has a unique solution $u_\varepsilon(x)$ in $L^2(\mathbf{Z}_\varepsilon^d)$. We show in §2 that if $u(x), x \in \mathbf{R}^d$, is the function which satisfies

$$[1 + |\xi|]\hat{u}(\xi) = \hat{f}(\xi), \quad \xi \in \mathbf{R}^d \ , \tag{1.4}$$

then

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_\varepsilon = 0 \ . \tag{1.5}$$

We can also prove a corresponding theorem when $\mathbf{b} \not\equiv 0$ in the case $p$ is given by (**??**). In the following the expectation value of a random variable $X$ on $\Omega$ is denoted by $< X >$.

**Theorem 1.1.** *For $0 < \varepsilon \le 1$, there exists a constant $\gamma_d > 0$, depending only on $d$, such that if $\gamma \in \mathbf{C}$ satisfies $|\gamma| < \gamma_d$ then (1.3) has a unique solution $u_\varepsilon(\cdot,\omega) \in L^2(\mathbf{Z}_\varepsilon^d)$, $\omega \in \Omega$. There exists a vector $q_\gamma \in \mathbf{C}^d$, $|q_\gamma| < 1$, analytic in $\gamma$, such that if $u(x), \ x \in \mathbf{R}^d$, is the function which satisfies*

$$[1 + |\xi| + iq_\gamma\cdot\xi]\hat{u}(\xi) = \hat{f}(\xi), \quad \xi \in \mathbf{R}^d, \tag{1.6}$$

*then there is the limit*

$$\lim_{\varepsilon\to 0} \left\langle \|u_\varepsilon(\ ,\cdot) - u\|_\varepsilon^2 \right\rangle = 0. \tag{1.7}$$

Theorem 1.1 is a homogenization result for $p$ given by (**??**). It says that the scaling limit of the random walk, with transition probability (**??**), in a random environment described by $\mathbf{b}(x,\omega)$, is a Cauchy process with a constant drift $q_\gamma$. Corresponding results for uniformly elliptic partial differential equations in divergence form were first proved by Kozlov [10] and later by Papanicolaou and Varadhan [14]. In this case the scaling limit is Brownian motion. One can also prove a Brownian motion scaling limit for parabolic equations in divergence form when the coefficients are random in time as well as space [13]. It is possible to relax somewhat the uniform ellipticity assumption and still prove the scaling limit [5]. For a comprehensive survey of homogenization results in partial differential equations see the book of Oleinik at al [22]. Here we obtain the scaling limit of an asymmetric partial difference equation. The scaling limit of a divergence form partial difference equation has been obtained by Künnemann [11] and of a symmetric partial difference equation in non-divergence form by Lawler [12].

Evidently $q_\gamma$ is in general nonzero even in the case $<\mathbf{b}(\cdot)>= 0$. Nevertheless it is possible to impose some general conditions on $\mathbf{b}(\cdot)$ to ensure $q_\gamma = 0$. We say that the vector field $\mathbf{b}(\cdot)$ is reflection invariant if it has the property that

$$\left\langle \prod_{i=1}^n b_{k_i}(\tau_{x_i}\cdot) \right\rangle = (-1)^n \left\langle \prod_{i=1}^n b_{k_i}(\tau_{-x_i}\cdot) \right\rangle, \tag{1.8}$$
$$x_i \in \mathbf{Z}^d,\ 1 \le k_i \le d,\quad i = 1,...n,\quad n \ge 1.$$

Clearly (1.8) implies that $<\mathbf{b}(\cdot)>= \mathbf{0}$. We show at the end of §2 that if $\mathbf{b}(\cdot)$ satisfies (1.8) then $q_\gamma = 0$ in (1.6).

Suppose now (1.8) holds. Let $u_\varepsilon(x) =< u_\varepsilon(x,\cdot) >,\ x \in \mathbf{Z}^d_\varepsilon$, where $u_\varepsilon(x,\omega)$ is the unique solution to (1.3) given by Theorem 1.1. Suppose $\hat{p} : [-\pi,\pi]^d \to \mathbf{R}$ is the Fourier transform of the function $p$ of (**??**) and $\hat{f}_\varepsilon$ the Fourier transform of the function $f$ restricted to the lattice $\mathbf{Z}^d_\varepsilon$. If $\hat{u}_\varepsilon(\xi)$ is the Fourier transform of $u_\varepsilon(x)$, then one can see that there is a $d \times d$ matrix $q_{\gamma,\varepsilon}(\zeta),\ \zeta \in [-\pi,\ \pi]^d$, continuous in $\zeta$ and analytic in $\gamma$, such that

$$\hat{u}_\varepsilon(\xi)[1 + \varepsilon^{-1}\{1 - \hat{p}(\varepsilon\xi)\} - \varepsilon^{-1}e(\varepsilon\xi)q_{\gamma,\varepsilon}(\varepsilon\xi)e(-\varepsilon\xi)] = \hat{f}_\varepsilon(\xi)\ ,\ \xi \in [-\pi/\varepsilon,\ \pi/\varepsilon]\ , \tag{1.9}$$

where $e(\zeta),\ \zeta \in \mathbf{R}^d$, is the vector $e(\zeta) = (e_1(\zeta), \cdots, e_d(\zeta))$, with $e_k(\zeta) = 1 - e^{i\mathbf{e}_k\cdot\zeta}$.

**Theorem 1.2.** *Suppose the variables* $\mathbf{b}(\tau_x\cdot),\ x \in \mathbf{Z}^d$, *are independent with* $<\mathbf{b}(\cdot)>= 0$ *and* $q_{\gamma,\varepsilon}(\zeta)$ *is defined by (1.9) for* $|\gamma| < \gamma_d$, *where* $\gamma_d$ *is the constant of Theorem 1.1. Then there exists a positive constant* $\gamma'_d < \gamma_d$, *depending only on* $d$, *such that if* $|\gamma| < \gamma'_d$ *the matrix* $q_{\gamma,\varepsilon}(\zeta)$ *converges uniformly in* $\zeta$ *as* $\varepsilon \to 0$ *to a matrix* $q_\gamma(\zeta),\ \zeta \in [-\pi,\ \pi]^d$.

Theorem 1.2 is a considerably deeper theorem than Theorem 1.1. One has to use the Calderon-Zygmund theorem [8, 19] that the Hilbert transform is bounded on $L^p$ spaces for $p \ne 2$. In contrast, Theorem 1.1 can be proved by just using $L^2$ theory.

Next we consider the situation when $p$ is given by (1.2) and $\mathbf{b} \not\equiv 0$. We then have the following theorem:

**Theorem 1.3.** *For $0 < \varepsilon \le 1$ and $\gamma \in \mathbf{C}$ satisfying $|\gamma| < \varepsilon/\sqrt{2d}$, equation (1.3) has a unique solution $u_\varepsilon(\cdot,\omega) \in L^2(\mathbf{Z}^d_\varepsilon)$, $\omega \in \Omega$. If (1.8) holds then there is a $d \times d$ matrix $q_{\gamma,\varepsilon}(\zeta)$, $\zeta \in [-\pi,\ \pi]^d$, continuous in $\zeta$ and analytic in $\gamma$, such that the Fourier transform $\hat{u}_\varepsilon(\xi)$ of $< u_\varepsilon(x,\cdot) >$ satisfies,*

(1.10)
$$\hat{u}_\varepsilon(\xi)[1+\varepsilon^{-2}\{1-\hat{p}(\varepsilon\xi)\} - \varepsilon^{-2} e(\varepsilon\xi) q_{\gamma,\varepsilon}(\varepsilon\xi) e(-\varepsilon\xi)] = \hat{f}_\varepsilon(\xi) \ , \ \xi \in [-\pi/\varepsilon,\ \pi/\varepsilon] \ .$$

Note that (1.10) is simply the analogue for $p$ given by (1.2) of (1.9) for $p$ given by (**??**). Since $q_{\gamma,\varepsilon}(\zeta)$ is analytic in $\gamma$, we can write it in a series expansion,

$$q_{\gamma,\varepsilon}(\zeta) = \sum_{m=2}^{\infty} \gamma^m q_{m,\varepsilon}(\zeta) \ , \tag{1.11}$$

which converges for $|\gamma| < \varepsilon/\sqrt{2d}$. In §5 we investigate the limiting behavior of $q_{m,\varepsilon}$ as $\varepsilon \to 0$. We restrict ourselves to the situation when the $\mathbf{b}(\tau_x\cdot)$, $x \in \mathbf{Z}^d$, are given by independent Bernoulli variables. Thus we assume that $b_j(\cdot) \equiv 0$, $j > 1$, and $b_1(\tau_x\cdot) = Y_x$, $x \in \mathbf{Z}^d$, where the variables $Y_x$, $x \in \mathbf{Z}^d$, are assumed to be i.i.d. Bernoulli, $Y_x = \pm 1$ with equal probability. In that case we have the following:

**Theorem 1.4.** *For $m = 2, 3, \cdots$ the $d \times d$ matrix $q_{m,\varepsilon}(\zeta)$ has the properties:*
*(a) $q_{m,\varepsilon}(\zeta) \equiv 0$ if $m$ is odd or $m = 2$.*
*(b) The entries of the matrix $q_{m,\varepsilon}(\zeta) = [q_{m,\varepsilon,k,k'}(\zeta)]$, $1 \le k, k' \le d$, are zero if $k, k' > 1$.*
*(c) The entries of the matrix $q_{m,\varepsilon}(0) = [q_{m,\varepsilon,k,k'}(0)]$ are zero if $k + k' > 2$.*
*(d) Assume $d = 1$ and $\mathcal{K} \subset \mathbf{R}$ is a compact subset. Then $\varepsilon^{m/2} q_{m,\varepsilon}(\varepsilon\xi)$ converges uniformly for $\xi \in \mathcal{K}$ to a function $q_m(\xi)$, as $\varepsilon \to 0$. The function $q_4(\xi)$ is given by the formula,*

$$q_4(\xi) = \frac{32}{18+\xi^2} \ . \tag{1.12}$$

*(e) If $d = 2$ then $q_{4,\varepsilon,1,1}(0)$ converges to $+\infty$ as $\varepsilon \to 0$.*
*(f) If $d \ge 3$ then $q_{4,\varepsilon}(\zeta)$ converges uniformly for $\zeta \in [-\pi,\pi]^d$ to a matrix $q_4(\zeta) = [q_{4,k,k'}(\zeta)]$ satisfying $q_{4,1,1}(0) > 0$, as $\varepsilon \to 0$.*

Unlike in the previous theorems, the results of Theorem 1.4 depend on the dimension $d$. For $d = 1$ the theorem suggests that if we set $\gamma = \gamma'\sqrt{\varepsilon}$ in (1.3), fix $\gamma'$and let $\varepsilon \to 0$, then we should obtain a scaling limit. Note however that one expects the resulting series obtained from (1.11) by letting $\varepsilon \to 0$ to be at most asymptotic in $\gamma'$, not analytic. The limit here is related to the limits obtained by Sinai[18] and Kesten[9] for one dimensional random walk in random environment. For $d = 2$ the theorem suggests that if we fix $\gamma$ in (1.3) and let $\varepsilon \to 0$ no scaling limit exists. This appears to contradict a conjecture of Fisher[6] that two dimensional random walk in random environment has a diffusive scaling limit. In contrast Derrida and Luck[4] conjecture a nondiffusive scaling limit in two dimensions. A numerical study of the two dimensional equation (1.3) was made in [3] using a multigrid algorithm. It was observed that the algorithm gave considerable acceleration over pure iterative methods such as Gauss-Seidel. This suggests that there is some stability in the Fourier space behavior of the solution to (1.3) as $\varepsilon \to 0$. For $d \ge 3$ the theorem suggests a diffusive scaling limit for $d$ dimensional random walk in random environment and that the effective diffusion constant is smaller than for the zero noise case.

There has been some recent work [15, 20, 21] on (1.3) with $p$ given by (1.2) and under the assumption $<\mathbf{b}(\cdot)>\neq 0$. This situation is very different to the situation studied in Theorem 1.4 since one expects now the drift to dominate diffusion. The methods used in [20, 21] are related to methods used to prove Anderson localisation for the random Schrodinger equation.

## 2. Proof of Theorem 1.1

We first prove (1.5). Thus we consider (1.3) with $\mathbf{b}\equiv 0$ and $p$ given by (**??**). To obtain an expression for $p$, let us denote a point in $\mathbf{Z}^{d+1}$ by $(x,y)$ with $x\in\mathbf{Z}^d,\ y\in\mathbf{Z}$. For $n=0,1,2,...$ let $G(x,y,n)$ be the probability density of the standard random walk in $\mathbf{Z}^{d+1}$, started at the origin, after $n$ steps. Thus

$$\sum_{x\in\mathbf{Z}^d}\sum_{y\in\mathbf{Z}} G(x,y,n)e^{i[x\cdot\xi+y\zeta]}=\left\{\frac{1}{(d+1)}\left[\sum_{i=1}^d\cos(\mathbf{e}_i\cdot\xi)+\cos\zeta\right]\right\}^n. \tag{2.1}$$

By the reflection principle, the probability that the $d+1$ dimensional walk, started at $(0,-1)$ first hits the hyperplane $\{(y,0):y\in\mathbf{Z}^d\}$ after $n+1$ steps at the point $(x,0)$ is given by

$$\frac{1}{2(d+1)}\left[G(x,0,n)-G(x,2,n)\right].$$

Hence we have

$$p(x)=\frac{1}{2(d+1)}\sum_{n=0}^{\infty}\left[G(x,0,n)-G(x,2,n)\right] \tag{2.2}$$

We can compute the Fourier transform of $p$ by using (2.1). Thus

(2.3)
$$\hat{p}(\xi)=\sum_{x\in\mathbf{Z}^d}p(x)e^{ix\cdot\xi}=\frac{1}{4\pi(d+1)}\int_{-\pi}^{\pi}d\zeta\frac{1-\cos 2\zeta}{1-\left[\sum_{i=1}^d\cos(\mathbf{e}_i\cdot\xi)+\cos\zeta\right]/(d+1)}.$$

**Lemma 2.1.** *The limit (1.5) holds.*

*Proof.* We rewrite the formula of (2.3) for $\hat{p}(\xi)$ as

$$\hat{p}(\xi)=1-\frac{1}{2\pi}\int_{-\pi}^{\pi}d\zeta\frac{(1+\cos\zeta)\sum_{i=1}^d[1-\cos(\mathbf{e}_i\cdot\xi)]}{\left[(1-\cos\zeta)+\sum_{i=1}^d[1-\cos(\mathbf{e}_i\cdot\xi)]\right]}, \tag{2.4}$$

whence, on performing the integration with respect to $\zeta$, we obtain the formula,

$$1-\hat{p}(\xi)=\left\{\left[1+\frac{2}{\sum_{i=1}^d[1-\cos(\mathbf{e}_i\cdot\xi)]}\right]^{1/2}-1\right\}\sum_{i=1}^d[1-\cos(\mathbf{e}_i\cdot\xi)]. \tag{2.5}$$

It follows from (2.3), (2.4) that $0<\hat{p}(\xi)\leq 1$. We can also easily see from (2.5) that

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}[1-\hat{p}(\varepsilon\xi)]=|\xi|,\qquad \xi\in\mathbf{R}^d. \tag{2.6}$$

Consider now (1.3) with $\mathbf{b} \equiv 0$. Taking the Fourier transform of (1.3), we see that (1.3) is equivalent to the equation,

$$[1 + \varepsilon - \hat{p}(\varepsilon\xi)]\hat{u}_\varepsilon(\xi) = \varepsilon \hat{f}_\varepsilon(\xi), \quad \xi \in \left[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^d, \tag{2.7}$$

where $\hat{f}_\varepsilon$ is the Fourier transform of $f$ regarded as a function on $\mathbf{Z}^d_\varepsilon$. Since $f$ is $C^\infty$ of compact support it is easy to see that (2.7) is uniquely solvable in $L^2(\left[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^d)$ and

$$\hat{u}_\varepsilon(\xi) = \hat{f}_\varepsilon(\xi)/[1 + \varepsilon^{-1}\{1 - \hat{p}(\varepsilon\xi)\}], \quad \xi \in \left[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^d. \tag{2.8}$$

Let $\hat{u}(\xi)$ be the function given in (1.4). We decompose its Fourier inverse $u(x)$, $x \in \mathbf{R}^d$, as a sum, $u(x) = v_\varepsilon(x) + w_\varepsilon(x)$, where

$$v_\varepsilon(x) = \frac{1}{(2\pi)^d} \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} \hat{u}(\xi) e^{-ix\cdot\xi} d\xi.$$

Since $\hat{f}(\xi)$ is rapidly decreasing in $\xi$ it follows that $\lim_{\varepsilon\to 0} \|w_\varepsilon\|_\varepsilon = 0$. We can see this from the identity

$$\|w_\varepsilon\|_\varepsilon^2 = \frac{1}{(2\pi)^d} \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} \left| \sum_{n\in\mathbf{Z}^d\backslash\{0\}} \hat{u}(\xi + \frac{2\pi n}{\varepsilon}) \right|^2 d\xi.$$

Thus we have

$$\begin{aligned} \lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_\varepsilon &= \lim_{\varepsilon\to 0} \|u_\varepsilon - v_\varepsilon\|_\varepsilon, \\ \|u_\varepsilon - v_\varepsilon\|_\varepsilon^2 &= \frac{1}{(2\pi)^d} \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} |\hat{u}_\varepsilon(\xi) - \hat{u}(\xi)|^2 d\xi, \end{aligned}$$

where $\hat{u}_\varepsilon(\xi)$ is given by (2.8) and $\hat{u}(\xi)$ by (1.4). The result follows from (2.6) and the dominated convergence theorem. ☐

Next we wish to prove existence of a solution to (1.3) provided $\gamma$ is sufficiently small, depending only on $d$. To do this we need to define the Green's function $G_{p,\eta}(x)$, $x \in \mathbf{Z}^d$, associated with $p$. Thus for $\eta > 0$, $G_{p,\eta}(x)$ is the solution to the equation

$$A_p\, G_{p,\eta}(x) + \eta\, G_{p,\eta}(x) = \delta(x), \quad x \in \mathbf{Z}^d, \tag{2.9}$$

where $\delta$ is the Kronecker $\delta$ function, $\delta(0) = 1$, $\delta(x) = 0$, $x \neq 0$. The Fourier transform $\hat{G}_{p,\eta}(\xi)$ of $G_{p,\eta}(x)$ is given by the formula

$$\hat{G}_{p,\eta}(\xi) = [1 + \eta - \hat{p}(\xi)]^{-1}, \quad \xi \in [-\pi, \pi]^d. \tag{2.10}$$

It is easy to see from (2.5) that

$$0 \le \hat{G}_{p,\eta}(\xi) \le C/[\eta + |\xi|],$$

for some universal constant $C$. We can also obtain a corresponding decay rate on the function $G_{p,\eta}(x)$.

**Lemma 2.2.** *Let $G_{p,\eta}(x)$, $x \in \mathbf{Z}^d$, be the solution to (2.9), and $0 < \eta \le 1$. Then there is a constant $C_d$, depending only on d such that if $d \ge 2$,*

$$0 \le G_{p,\eta}(x) \le C_d / \left[1 + \eta^2|x|^2\right]\left[1 + |x|^{d-1}\right], \quad x \in \mathbf{Z}^d. \tag{2.11}$$

*Proof.* Consider the random walk on $\mathbf{Z}^d$ with transition probability $p$ given by (2.2). Let $p_n(x), n = 0, 1, 2, ...$ be the probability that the walk, started at the origin hits $x$ after $n$ steps. Thus $p_0(x) = \delta(x)$ and $p_1(x) = p(x), \quad x \in \mathbf{Z}^d$. It is easy to see that the solution of (2.9) is given by the formula,

$$G_{p,\eta}(x) = \sum_{m=0}^{\infty} (1+\eta)^{-m-1} p_m(x), \qquad x \in \mathbf{Z}^d. \tag{2.12}$$

We can derive a formula for $p_m(x), \quad m \geq 2$, similar to the formula (2.2) for $p(x)$. In fact we have

$$p_m(x) = \frac{1}{2(d+1)} \sum_{n=0}^{\infty} [G(x, m-1, n) - G(x, m+1, n)]. \tag{2.13}$$

It is easy to see from this and (2.12) that

$$G_{p,0}(x) = \delta(x) + \frac{1}{2(d+1)} \sum_{n=0}^{\infty} [G(x, 0, n) + G(x, 1, n)]. \tag{2.14}$$

If we use now the bound

$$0 \leq G(x, y, n) \leq \frac{C_d}{1 + n^{(d+1)/2}} \exp\left[-\min\{|x| + |y|, (|x|^2 + |y|^2)/(n+1)\}/C_d\right],$$

for the standard random walk on $\mathbf{Z}^{d+1}$, then the inequality (2.11) for $\eta = 0$ follows from (2.14). To obtain (2.11) for $\eta > 0$ we need to use the inequality,

$$\mathrm{C}_d y \overline{1+n^{(d+3)/2} \exp[-\min\{|x|+|y|,\ (|x|^2+|y|^2)/(n+1)\}/C_d],}$$

x $\in$ $\mathbf{Z}^d$, $y$ $\in$ $\mathbf{Z}$, $y$ $\geq$ 0.

In follows from this and (2.13) that

$$0 \leq p_m(x) \leq \frac{C_d m}{1 + [|x|^2 + m^2]^{(d+1)/2}}, \qquad m \geq 1.$$

The inequality (2.11) follows now from this last inequality by estimating the sum (2.12). $\square$

**Lemma 2.3.** *Suppose $d \geq 2$ and $f : \mathbf{R}^d \to \mathbf{R}$ is a $C^\infty$ function with compact support. Then there exists a constant $\gamma_d$, depending only on d such that if $|\gamma| < \gamma_d$ and $0 < \varepsilon \leq 1$, then (1.3) has a unique solution $u_\varepsilon(x, \omega)$ in $L^2(\mathbf{Z}^d_\varepsilon)$. Further, there is a constant $C_d$, depending only on d, such that $\|u_\varepsilon(\cdot, \omega)\|_\varepsilon \leq C_d \|f\|_\varepsilon$ .*

*Proof.* Suppose $u_\varepsilon(x, \omega)$ is in $L^2(\mathbf{Z}^d_\varepsilon)$ and satisfies (1.3). Let $v_\varepsilon(x, \omega)$ be defined by

$$v_\varepsilon(x, \omega) = \varepsilon^{-1}[A_p u_\varepsilon(x, \omega) + \varepsilon u_\varepsilon(x, \omega)].$$

It is easy to see that $v_\varepsilon(x, \omega) \in L^2(\mathbf{Z}^d_\varepsilon)$. In view of (2.9) it follows that

$$u_\varepsilon(x, \omega) = \varepsilon^{1-d} \int_{\mathbf{Z}^d_\varepsilon} dy\ G_{p,\varepsilon}\left(\frac{x-y}{\varepsilon}\right) v_\varepsilon(y, \omega). \tag{2.15}$$

Now for $i = 1, ..., d$ let us define operators $T_i$ on $L^2(\mathbf{Z}^d_\varepsilon)$ by

$$T_i w(x) = \varepsilon^{-d} \int_{\mathbf{Z}^d_\varepsilon} dy \left[G_{p,\varepsilon}\left(\frac{x-y}{\varepsilon} + \mathbf{e}_i\right) - G_{p,\varepsilon}\left(\frac{x-y}{\varepsilon} - \mathbf{e}_i\right)\right] w(y) \ .$$

It is easy to see that $T_i$ is a bounded operator on $L^2(\mathbf{Z}^d_\varepsilon)$. It follows from (2.10) that the norm of $T_i$ is bounded by

$$\|T_i\| \le \sup_{\xi\in[-\pi,\pi]^d}\left[\frac{2|\sin(\mathbf{e}_i\cdot\xi)|}{1+\varepsilon-\hat{p}(\xi)}\right]. \tag{2.16}$$

We can see from (2.5) that there is a constant $c_d$, $0<c_d<1$, depending only on $d$ such that

$$1-\hat{p}(\xi)\ge c_d|\xi|, \quad \xi\in[-\pi,\pi]^d. \tag{2.17}$$

It follows from this and (2.16) that $\|T_i\|\le 2/c_d$. Observe now that (1.3) implies that $v_\varepsilon(x,\omega)\in L^2(\mathbf{Z}^d_\varepsilon)$ satisfies the equation

$$v_\varepsilon(x,\omega)-\gamma\sum_{i=1}^d b_i(x/\varepsilon,\omega)T_i v_\varepsilon(x,\omega)=f(x). \tag{2.18}$$

In view of (1.1), if we take $\gamma_d=c_d/2$ it follows that for $|\gamma|<\gamma_d$ the function $v_\varepsilon(x,\omega)$ is uniquely determined by (2.18). In view of (2.15) it follows that $u_\varepsilon(x,\omega)\in L^2(\mathbf{Z}^d_\varepsilon)$ is the unique solution to (1.3). To prove existence note that (2.18) has a solution $v_\varepsilon(x,\omega)\in L^2(\mathbf{Z}^d_\varepsilon)$ provided $|\gamma|<\gamma_d$. If we then define $u_\varepsilon(x,\omega)$ by (2.15) one can easily see that $u_\varepsilon(x,\omega)$ satisfies (1.3). □

We have proven existence of a unique solution $u_\varepsilon(x,\omega)$ to (1.3) provided $|\gamma|<\gamma_d$. Next we follow a development similar to the one in [2]. Thus we put $v_\varepsilon(x,\omega)=u_\varepsilon(x,\tau_{-x/\varepsilon}\omega)$, $x\in\mathbf{Z}^d_\varepsilon,\omega\in\Omega$, whence $u_\varepsilon(x,\omega)=v_\varepsilon(x,\tau_{x/\varepsilon}\omega)$. Rewriting (1.3) in terms of $v_\varepsilon$, we obtain the equation,

$$\begin{aligned} v_\varepsilon(x,\tau_{x/\varepsilon}\omega) \quad &- \quad \sum_{y\in\mathbf{Z}^d} p(y)v_\varepsilon(x+\varepsilon y,\tau_y\tau_{x/\varepsilon}\omega) \\ &- \quad \gamma\sum_{i=1}^d b_i(\tau_{x/\varepsilon}\omega)[v_\varepsilon(x+\varepsilon\mathbf{e}_i,\tau_{\mathbf{e}_i}\tau_{x/\varepsilon}\omega) \\ &- \quad v_\varepsilon(x-\varepsilon\mathbf{e}_i,\tau_{-\mathbf{e}_i}\tau_{x/\varepsilon}\omega)]+\varepsilon v_\varepsilon(x,\tau_{x/\varepsilon}\omega)=\varepsilon f(x), \quad x\in\mathbf{Z}^d_\varepsilon, \quad \omega\in\Omega. \end{aligned}$$

This last equation is evidently the same as

$$\begin{aligned} v_\varepsilon(x,\omega) \quad &- \quad \sum_{y\in\mathbf{Z}^d} p(y)v_\varepsilon(x+\varepsilon y,\tau_y\omega) \\ &- \quad \gamma\sum_{i=1}^d b_i(\omega)[v_\varepsilon(x+\varepsilon\mathbf{e}_i,\tau_{\mathbf{e}_i}\omega)-v_\varepsilon(x-\varepsilon\mathbf{e}_i,\tau_{-\mathbf{e}_i}\omega)] \\ &+ \quad \varepsilon v_\varepsilon(x,\omega)=\varepsilon f(x), \quad x\in\mathbf{Z}^d_\varepsilon, \quad \omega\in\Omega. \end{aligned} \tag{2.19}$$

We write now $v_\varepsilon(x,\omega)=u_\varepsilon(x)+\psi_\varepsilon(x,\omega)$, $x\in\mathbf{Z}^d_\varepsilon$, $\omega\in\Omega$, where $<\psi_\varepsilon(x,\cdot)>=0$, $x\in\mathbf{Z}^d_\varepsilon$. It follows that

$$u_\varepsilon(x)=\Big\langle v_\varepsilon(x,\cdot)\Big\rangle=\Big\langle u_\varepsilon(x,\cdot)\Big\rangle, \quad x\in\mathbf{Z}^d_\varepsilon\ .$$

Substituting into (2.19) we obtain the equation

$$u_\varepsilon(x) \quad - \quad \sum_{y\in\mathbf{Z}^d} p(y)u_\varepsilon(x+\varepsilon y) \tag{2.20}$$

$$
\begin{aligned}
&- \quad \gamma \sum_{i=1}^{d} b_i(\omega)[u_\varepsilon(x+\varepsilon\mathbf{e}_i) - u_\varepsilon(x-\varepsilon\mathbf{e}_i)] + \varepsilon u_\varepsilon(x) \\
&+ \quad \psi_\varepsilon(x,\omega) - \sum_{y\in\mathbf{Z}^d} p(y)\psi_\varepsilon(x+\varepsilon y, \tau_y\omega) \\
&- \quad \gamma \sum_{i=1}^{d} b_i(\omega)\left[\psi_\varepsilon(x+\varepsilon\mathbf{e}_i, \tau_{\mathbf{e}_i}\omega) - \psi_\varepsilon(x-\varepsilon\mathbf{e}_i, \tau_{-\mathbf{e}_i}\omega)\right] \\
&+ \quad \varepsilon\psi_\varepsilon(x,\omega) = \varepsilon f(x), \quad x \in \mathbf{Z}^d_\varepsilon, \quad \omega \in \Omega.
\end{aligned}
$$

Taking the expected value of this last equation, we obtain the equation,

$$
\begin{aligned}
\text{(2.21)} \quad A_p u_\varepsilon(x) \quad &- \quad \gamma \sum_{i=1}^{d} < b_i(\cdot) > [u_\varepsilon(x+\varepsilon\mathbf{e}_i) - u_\varepsilon(x-\varepsilon\mathbf{e}_i)] + \varepsilon u_\varepsilon(x) \\
&- \quad \gamma \left\langle \sum_{i=1}^{d} b_i(\cdot)\left[\psi_\varepsilon(x+\varepsilon\mathbf{e}_i, \tau_{\mathbf{e}_i}\cdot) - \psi_\varepsilon(x-\varepsilon\mathbf{e}_i, \tau_{-\mathbf{e}_i}\cdot)\right] \right\rangle \\
&= \quad \varepsilon f(x), \quad x \in \mathbf{Z}^d_\varepsilon \ .
\end{aligned}
$$

If we subtract now (2.21) from (2.20) we obtain

$$
\begin{aligned}
\text{(2.22)} \quad &\psi_\varepsilon(x,\omega) - \sum_{y\in\mathbf{Z}^d} p(y)\psi_\varepsilon(x+\varepsilon y, \tau_y\omega) \\
&\qquad - \gamma P \sum_{i+1}^{d} b_i(\omega)[\psi_\varepsilon(x+\varepsilon\mathbf{e}_i, \tau_{\mathbf{e}_i}\omega) - \psi_\varepsilon(x-\varepsilon\mathbf{e}_i, \tau_{-\mathbf{e}_i}\omega)] \\
&+ \varepsilon\psi_\varepsilon(x,\omega) = \gamma \sum_{i=1}^{d} [b_i(\omega) - < b_i(\cdot) >][u_\varepsilon(x+\varepsilon\mathbf{e}_i) - u_\varepsilon(x-\varepsilon\mathbf{e}_i)], \ x \in \mathbf{Z}^d_\varepsilon, \ \omega \in \Omega,
\end{aligned}
$$

where $P : L^2(\Omega) \to L^2(\Omega)$ is the projection orthogonal to the constant function. Thus a solution $u_\varepsilon(x,\omega)$ of (1.3) which is in $L^2(\mathbf{Z}^d_\varepsilon \times \Omega)$ yields functions $u_\varepsilon(x)$ in $L^2(\mathbf{Z}^d_\varepsilon)$ and $\psi_\varepsilon(x,\omega)$ in $L^2(\mathbf{Z}^d_\varepsilon \times \Omega)$ which satisfy $< \psi_\varepsilon(x,\cdot) >= 0, \ \ x \in \mathbf{Z}^d_\varepsilon$, and the equations (2.21), (2.22). Conversely, if we can find functions $u_\varepsilon(x)$ in $L^2(\mathbf{Z}^d_\varepsilon)$ and $\psi_\varepsilon(x,\omega)$ in $L^2(\mathbf{Z}^d_\varepsilon \times \Omega)$ which have the property that $< \psi_\varepsilon(x,\cdot) >= 0, \ \ x \in \mathbf{Z}^d_\varepsilon$, and satisfy (2.21), (2.22) then we can construct from them the solution $u_\varepsilon(x,\omega)$ of (1.3) and $u_\varepsilon(x,\omega)$ is in $L^2(\mathbf{Z}^d_\varepsilon \times \Omega)$.

We concentrate now on finding solutions to the system (2.21), (2.22) of equations. To do this we Fourier transform the equations. The Fourier transform of (2.21) is given by

$$
\begin{aligned}
\text{(2.23)} \quad &\hat{u}_\varepsilon(\xi)\left\{ 1 + \varepsilon^{-1}\left[1 - \hat{p}(\varepsilon\xi)\right] + 2\gamma i \sum_{j=1}^{d} < b_j(\cdot) > \frac{\sin(\mathbf{e}_j \cdot \varepsilon\xi)}{\varepsilon} \right\} \\
&+ \gamma\varepsilon^{-1}\left\langle \sum_{j=1}^{d} b_j(\cdot)\left[ e^{i\varepsilon\mathbf{e}_j\cdot\xi}\hat{\psi}_\varepsilon(\xi, \tau_{-\mathbf{e}_j}\cdot) - e^{-i\varepsilon\mathbf{e}_j\cdot\xi}\hat{\psi}_\varepsilon(\xi, \tau_{\mathbf{e}_j}\cdot)\right] \right\rangle = \hat{f}_\varepsilon(\xi), \ \ \xi \in \left[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^d.
\end{aligned}
$$

The Fourier transform of (2.22) is given by

$$+\gamma P\sum_{j=1}^d b_j(\omega)\left[e^{i\varepsilon\mathbf{e}_j\cdot\xi}\hat{\psi}_\varepsilon(\xi,\tau_{-\mathbf{e}_j}\omega)-e^{-i\varepsilon\mathbf{e}_j\cdot\xi}\hat{\psi}_\varepsilon(\xi,\tau_{\mathbf{e}_j}\omega)\right]$$
$$+\varepsilon\hat{\psi}_\varepsilon(\xi,\omega)$$
$$=\gamma\sum_{j=1}^d\left[b_j(\omega)-\langle b_j(\cdot)\rangle\right]\left[e^{-i\varepsilon\mathbf{e}_j\cdot\xi}-e^{i\varepsilon\mathbf{e}_j\cdot\xi}\right]\hat{u}_\varepsilon(\xi),\quad \xi\in\left[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]^d,\quad \omega\in\Omega.$$

Suppose now that $\Psi_\varepsilon^k(\zeta,\omega)$, $1\le k\le d$, $\zeta\in[-\pi,\pi]^d$, $\omega\in\Omega$, are functions in $L^2([-\pi,\pi]^d\times\Omega)$ which satisfy the equation,

$$\Psi_\varepsilon^k(\zeta,\omega)-\sum_{y\in\mathbf{Z}^d}p(y)e^{-iy\cdot\zeta}\Psi_\varepsilon^k(\zeta,\tau_y\omega)$$
$$+\gamma P\sum_{j=1}^d b_j(\omega)\left[e^{i\mathbf{e}_j\cdot\zeta}\Psi_\varepsilon^k(\zeta,\tau_{-\mathbf{e}_j}\omega)-e^{-i\mathbf{e}_j\cdot\zeta}\Psi_\varepsilon^k(\zeta,\tau_{\mathbf{e}_j}\omega)\right]$$
$$+\varepsilon\Psi_\varepsilon^k(\zeta,\omega)=b_k(\omega)-<b_k(\cdot)>,\ \zeta\in[-\pi,\pi]^d,\quad \omega\in\Omega,\ 1\le k\le d. \tag{2.24}$$

Then it is clear that the function

$$\hat{\psi}_\varepsilon(\xi,\omega)=-2i\gamma\hat{u}_\varepsilon(\xi)\sum_{k=1}^d\sin(\varepsilon\mathbf{e}_k\cdot\xi)\Psi_\varepsilon^k(\varepsilon\xi,\omega) \tag{2.25}$$

is in $L^2(\left[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]^d\times\Omega)$ provided $\hat{u}_\varepsilon(\xi)$ is bounded. Furthermore $\hat{\psi}_\varepsilon(\xi,\omega)$ given by (2.24), (2.25) satisfies (**??**). If we substitute for $\hat{\psi}_\varepsilon$ from (2.25) into (2.23) then we obtain the equation,

$$\hat{u}_\varepsilon(\xi)\Bigg\{1+\varepsilon^{-1}\left[1-\hat{p}(\varepsilon\xi)\right]+2\gamma i\sum_{j=1}^d<b_j(\cdot)>\frac{\sin(\varepsilon\mathbf{e}_j\cdot\xi)}{\varepsilon}$$
$$-2i\gamma^2\Bigg\langle\sum_{j,k=1}^d b_j(\cdot)\frac{\sin(\varepsilon\mathbf{e}_k\cdot\xi)}{\varepsilon}\Bigg[e^{i\varepsilon\mathbf{e}_j\cdot\xi}\Psi_\varepsilon^k(\varepsilon\xi,\tau_{-\mathbf{e}_j}\cdot)$$
$$-e^{-i\varepsilon\mathbf{e}_j\cdot\xi}\Psi_\varepsilon^k(\varepsilon\xi,\tau_{\mathbf{e}_j}\cdot)\Bigg]\Bigg\rangle\Bigg\}=\hat{f}_\varepsilon(\xi),\quad \xi\in\left[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]^d. \tag{2.26}$$

**Lemma 2.4.** *Suppose $d\ge 2$ and $\gamma_d=c_d/2$, where $c_d$ satisfies (2.17). Then if $|\gamma|<\gamma_d$ and $0<\varepsilon\le 1$, the equation (2.24) has a unique solution $\Psi_\varepsilon^k(\zeta,\omega)\in L^2(\Omega)$. Furthermore, $\Psi_\varepsilon^k(\zeta,\omega)$ viewed as a function from $[-\pi,\pi]^d$ to $L^2(\Omega)$ is continuous and satisfies an inequality,*

$$\varepsilon^2\sup_{\zeta\in[-\pi,\pi]^d}\left\langle|\Psi_\varepsilon^k(\zeta,\cdot)|^2\right\rangle\le C_d, \tag{2.27}$$

*where $C_d$ depends only on $d$.*

*Proof.* We follow an argument similar to that in Lemma 2.3. Suppose $\Psi_\varepsilon^k(\zeta,\omega)\in L^2(\Omega)$ and satisfies (2.24). Let $\Phi_\varepsilon^k(\zeta,\omega)$ be defined by

$$\Phi_\varepsilon^k(\zeta,\omega)=\Psi_\varepsilon^k(\zeta,\omega)-\sum_{y\in\mathbf{Z}^d}p(y)e^{-iy\cdot\zeta}\Psi_\varepsilon^k(\zeta,\tau_y\omega)+\varepsilon\Psi_\varepsilon^k(\zeta,\omega). \tag{2.28}$$

It is easy to see that $\Phi_\varepsilon^k(\zeta,\omega)\in L^2(\Omega)$. Further, it follows from (2.9) that

$$\Psi_\varepsilon^k(\zeta,\omega)=\sum_{x\in\mathbf{Z}^d}G_{p,\varepsilon}(x)e^{-ix\cdot\zeta}\Phi_\varepsilon^k(\zeta,\tau_x\omega). \tag{2.29}$$

Now for $\eta > 0,\ \zeta \in [-\pi,\pi]^d,\ j = 1,...,d$ we define operators $T_{j,\eta,\zeta}$ on $L^2(\Omega)$ by

$$T_{j,\eta,\zeta}\varphi(\omega) = \sum_{x\in\mathbf{Z}^d} [G_{p,\eta}(x-\mathbf{e}_j) - G_{p,\eta}(x+\mathbf{e}_j)]\, e^{-ix\cdot\zeta}\varphi(\tau_x\omega). \tag{2.30}$$

We can see from Bochner's theorem [17] that $T_{j,\eta,\zeta}$ is bounded on $L^2(\Omega)$ and

$$\|T_{j,\eta,\zeta}\| \le \sup_{\zeta'\in[-\pi,\pi]^d} \frac{2|\sin(\mathbf{e}_j\cdot\zeta')|}{1+\eta-\hat{p}(\zeta')} \le \frac{2}{c_d}\ , \tag{2.31}$$

where the constant $c_d$ is from (2.17). Further, (2.24), (2.28), (2.29) imply that $\Phi^k_\varepsilon(\zeta,\omega)$ satisfies the equation,

$$\Phi^k_\varepsilon(\zeta,\omega) - \gamma P\sum_{j=1}^d b_j(\omega)T_{j,\varepsilon,\zeta}\Phi^k_\varepsilon(\zeta,\omega) = b_k(\omega) - < b_k(\cdot) > . \tag{2.32}$$

Hence if we take $\gamma_d = c_d/2$, it follows that for $|\gamma| < \gamma_d$ the function $\Phi^k_\varepsilon(\zeta,\omega)$ is uniquely determined by the equation (2.32). In view of (2.29) it follows that $\Psi^k_\varepsilon(\zeta,\omega)$ is the unique solution of (2.24) in $L^2(\Omega)$. We can similarly prove existence for $|\gamma| < \gamma_d$. The inequality of (2.27) follows from (2.29) on using Bochner's theorem and the fact that $\Phi^k_\varepsilon(\zeta,\cdot) \in L^2(\Omega)$ is bounded independently of $\zeta,\varepsilon$. □

It is clear now that we can construct the solutions to (2.21), (2.22), by using Lemma 2.4 and the equation (2.25), (2.26). Next we wish to show that the limit of the coefficient of $\hat{u}_\varepsilon(\xi)$ in (2.26) as $\varepsilon \to 0$ exists. Evidently

$$\lim_{\varepsilon\to 0}\left\{1+\varepsilon^{-1}[1-\hat{p}(\varepsilon\xi)] + 2\gamma i\sum_{j=1}^d < b_j(\cdot) > \frac{\sin(\varepsilon\mathbf{e}_j\cdot\xi)}{\varepsilon}\right\}$$
$$= 1+|\xi| + 2\gamma i\sum_{j=1}^d < b_j(\cdot) > (\mathbf{e}_j\cdot\xi)\ ,$$

so we are left to show existence of the term involving the function $\Psi^k_\varepsilon$. To prove this first observe from (2.29), (2.30) that

$$e^{-i\mathbf{e}_j\cdot\zeta}\Psi^k_\varepsilon(\zeta,\tau_{\mathbf{e}_j}\omega) - e^{i\mathbf{e}_j\cdot\zeta}\Psi^k_\varepsilon(\zeta,\tau_{-\mathbf{e}_j}\omega) = T_{j,\varepsilon,\zeta}\Phi^k_\varepsilon(\zeta,\omega).$$

Next we prove a version of the von Neumann ergodic theorem [17].

**Lemma 2.5.** *Let $\varphi \in L^2(\Omega)$ and $\xi \in \mathbf{R}^d$. Then if $<\varphi>= 0$ the function $T_{j,\varepsilon,\varepsilon\xi}\varphi$ converges in $L^2(\Omega)$ to the function $T_{j,0,0}\varphi$.*

*Proof.* First note that we can define the operator $T_{j,0,\zeta},\ \zeta \in [-\pi,\pi]^d$, by (2.30) and from (2.31) it is a bounded operator on $L^2(\Omega)$. To be more precise, observe that for $\eta > 0,\ T_{j,\eta,\zeta}\varphi = 0$ if $\varphi \in L^2(\Omega)$ satisfies $\partial_{j,\zeta}\varphi = 0$, where $\partial_{j,\zeta}$ is the operator on $L^2(\Omega)$ given by

$$\partial_{j,\zeta}\varphi(\omega) = e^{-i\mathbf{e}_j\cdot\zeta}\varphi(\tau_{\mathbf{e}_j}\omega) - e^{i\mathbf{e}_j\cdot\zeta}\varphi(\tau_{-\mathbf{e}_j}\omega).$$

Observe that the adjoint $\partial^*_{j,\zeta}$ of $\partial_{j,\zeta}$ is given by $\partial^*_{j,\zeta} = -\partial_{j,\zeta}$. Suppose now $\varphi = \partial_{j,\zeta}\psi$ where $\psi \in L^2(\Omega)$. Then from (2.30) we have that

$$T_{j,\eta,\zeta}\varphi(\omega) = \sum_{x\in\mathbf{Z}^d} [G_{p,\eta}(x+2\mathbf{e}_j) + G_{p,\eta}(x-2\mathbf{e}_j) - 2G_{p,\eta}(x)]\, e^{-ix\cdot\zeta}\psi(\tau_x\omega). \tag{2.33}$$

We can readily see, using the argument of Lemma 2.2 that there is a constant $C_d$, depending only on $d$ such that

$$\begin{aligned}|G_{p,\eta}(x+2\mathbf{e}_j) \quad &+ \quad G_{p,\eta}(x-2\mathbf{e}_j)-2G_{p,\eta}(x)| \\ &\le \quad C_d/[1+\eta^2|x|^2][1+|x|^{d+1}], \quad x\in\mathbf{Z}^d.\end{aligned}$$

Hence if $\varphi = \partial_{j,\zeta}\psi,\ \psi \in L^2(\Omega)$, we can define $T_{j,0,\zeta}\varphi$ by setting $\eta = 0$ in the formula (2.33). If $\partial_{j,\zeta}\varphi = 0$ we set $T_{j,0,\zeta}\varphi = 0$. In view of (2.31) this defines $T_{j,0,\zeta}$ as a bounded operator on $L^2(\Omega)$ which has the property that, for any $\varphi \in L^2(\Omega)$, $T_{j,\eta,\zeta}\ \varphi$ converges in $L^2(\Omega)$ to $T_{j,0,\zeta}\varphi$ as $\eta \to 0$.

So far we have not used the ergodicity of the translation operators $\tau_x,\ x \in \mathbf{Z}^d$. Suppose now $\varphi \in L^2(\Omega)$ and $<\phi>=0$. Then ergodicity of the translation operators implies that for any $\delta > 0$ there exists $\psi_\delta \in L^2(\Omega)$ such that $\|\varphi - \partial_{j,0}\psi_\delta\| < \delta$. Now using the fact that

$$\partial_{j,0}\psi_\delta = \partial_{j,\varepsilon\xi}\psi_\delta + \psi'_{\delta,\varepsilon}$$

where $\psi'_{\delta,\varepsilon} \in L^2(\Omega)$ satisfies $\|\psi'_{\delta,\varepsilon}\| \le 2\varepsilon|\mathbf{e}_j\cdot\xi|\|\psi_\delta\|$, it follows from (2.31), (2.33) that $T_{j,\varepsilon,\varepsilon\xi}[\partial_{j,0}\psi_\delta]$ converges in $L^2(\Omega)$ as $\varepsilon \to 0$ to the function $T_{j,0,0}[\partial_{j,0}\psi_\delta]$. The result then follows from (2.31) on letting $\delta \to 0$. □

Lemma 2.5 enables us to define the vector $q_\gamma \in \mathbf{R}^d$ of (1.6).

**Lemma 2.6.** *There exists a constant $\gamma_d > 0$, depending only on $d$, such that if $|\gamma| < \gamma_d$ and $0 < \varepsilon \le 1$, the equation (2.26) has a unique solution $\hat{u}_\varepsilon \in L^2([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d)$. There is a vector $q_\gamma \in \mathbf{C}^d$, analytic in $\gamma$ and satisfying $|q_\gamma| < 1$, such that if $u(x),\ x \in \mathbf{R}^d$, has Fourier transform satisfying (1.6) and $u_\varepsilon(x),\ x \in \mathbf{Z}^d_\varepsilon$, is the Fourier inverse of $\hat{u}_\varepsilon$, then*

$$\lim_{\varepsilon\to 0}\|u_\varepsilon - u\|_\varepsilon = 0. \tag{2.34}$$

*Proof.* First observe that we can rewrite (2.26) as

$$\begin{aligned}\hat{u}_\varepsilon(\xi)\Big\{1+\varepsilon^{-1}\left[1-\hat{p}(\varepsilon\xi)\right]+2\gamma i\sum_{j=1}^d <b_j(\cdot)>\frac{\sin(\varepsilon\mathbf{e}_j\cdot\xi)}{\varepsilon} \\ +2i\gamma^2\Big\langle\sum_{j,k=1}^d b_j(\cdot)\frac{\sin(\varepsilon\mathbf{e}_k\cdot\xi)}{\varepsilon}T_{j,\varepsilon,\varepsilon\xi}\Phi^k_\varepsilon(\varepsilon\xi,\cdot)\Big\rangle\Big\} = \hat{f}_\varepsilon(\xi), \quad \xi\in\left[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]^d,\end{aligned} \tag{2.35}$$

where $\Phi^k_\varepsilon(\zeta,\omega)$ is the solution of (2.32). Note that from (2.35),(2.17) and (2.31) one has that $\hat{u}_\varepsilon \in L^2([\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d)$ provided $0<\varepsilon\le 1$, $|\gamma|<\gamma_d$ and $\gamma_d$ is sufficiently small, depending only on $d$. It is also clear from Lemma 2.5 and (2.31) that for any $\varphi \in L^2(\Omega)$ with $<\varphi>=0$ and nonnegative integer $m$,

$$\lim_{\varepsilon\to 0} T^m_{j,\varepsilon,\varepsilon\xi}\varphi = T^m_{j,0,0}\varphi\ ,$$

where the convergence is in the $L^2$ norm. It follows then from (2.32) that

$$\lim_{\varepsilon\to 0} T_{j,\varepsilon,\varepsilon\xi}\Phi^k_\varepsilon(\varepsilon\xi,\cdot) = T_{j,0,0}\Phi^k_0(0,\cdot).$$

Hence from (2.35) we define the vector $q_\gamma = (q_1,...,q_d)$ by

$$q_k = 2\gamma <b_k(\cdot)> +2\gamma^2\sum_{j=1}^d\left\langle b_j(\cdot)T_{j,0,0}\Phi^k_0(0,\cdot)\right\rangle. \tag{2.36}$$

Evidently we may choose $\gamma_d$ sufficiently small, depending only on $d$, such that $|q_\gamma| < 1$. The limit (2.34) now follows from the argument of Lemma 2.1. $\square$

**Lemma 2.7.** *Let $\Psi_\varepsilon^k(\zeta,\omega)$ be the function defined in Lemma 2.4 for $|\gamma| < \gamma_d$, $0 < \varepsilon \le 1$. Then for any $\xi \in \mathbf{R}^d$, there is the limit,*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \left\langle |\Psi_\varepsilon^k(\varepsilon\xi,\cdot)|^2 \right\rangle = 0. \tag{2.37}$$

*Proof.* We first consider the zero*th* order contribution in $\gamma$ to $\Psi_\varepsilon^k$. Thus in (2.32) we make the approximation $\Phi_\varepsilon^k(\zeta,\omega) \simeq b_k(\omega) - < b_k(\cdot) >$. In this approximation we have from (2.29) and Bochner's theorem the identity,

$$\varepsilon^2 \left\langle |\Psi_\varepsilon^k(\varepsilon\xi,\cdot)|^2 \right\rangle = \int_{[-\pi,\pi]^d} d\mu_\varphi(\zeta) \left[ \frac{\varepsilon}{1+\varepsilon-\hat{p}(\zeta-\varepsilon\xi)} \right]^2, \tag{2.38}$$

where $d\mu_\varphi$ is the finite positive Borel measure on $[-\pi,\pi]^d$, satisfying

$$\left\langle \varphi(\tau_x\cdot)\overline{\varphi(\cdot)} \right\rangle = \int_{[-\pi,\pi]^d} e^{ix\cdot\zeta} d\mu_\varphi(\zeta), \quad x \in \mathbf{Z}^d\ ,$$

and $\varphi(\omega) = b_k(\omega) - < b_k(\cdot) >$. Observe now that

$$\left[ \frac{\varepsilon}{1+\varepsilon-\hat{p}(\zeta-\varepsilon\xi)} \right]^2 \ \le\ 1, \quad \zeta \in [-\pi,\pi]^d, \tag{2.39}$$

$$\lim_{\varepsilon\to 0} \left[ \frac{\varepsilon}{1+\varepsilon-\hat{p}(\zeta-\varepsilon\xi)} \right]^2 \ =\ 0, \quad \zeta \in [-\pi,\pi]^d \backslash \{0\}. \tag{2.40}$$

Since $< \varphi >= 0$, one has by ergodicity of the translation operators $\tau_x$, $x \in \mathbf{Z}^d$, that $\mu_\varphi(\{0\}) = 0$. It follows now by dominated convergence that (2.37) holds in this approximation.

To deal with the general case note that (2.38) holds with $\varphi(\omega) = \varphi_\varepsilon(\omega) = \Phi_\varepsilon^k(\varepsilon\xi,\omega)$. The result follows then from the above argument if we use the fact that $d\mu_{\phi_\varepsilon} \le 2d\mu_{\phi_0} + 2d\mu_{\phi_\varepsilon-\phi_0}$ and the fact that $\lim_{\varepsilon\to 0}\mu_{\phi_\varepsilon-\phi_0}([-\pi,\pi]^d) = \|\phi_\varepsilon - \phi_0\|^2 = 0$. $\square$

*Proof of Theorem 1.1*: We just need to prove (1.7). In view of Lemma 2.6 and (2.25) it will be sufficient to show that

$$\lim_{\varepsilon\to 0} \int_{[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}]^d} |\hat{f}_\varepsilon(\xi)|^2 |\xi|^2 \varepsilon^2 \left\langle |\Psi_\varepsilon^k(\varepsilon\xi,\cdot)|^2 \right\rangle d\xi = 0.$$

This follows by dominated convergence from (2.27) and Lemma 2.7. $\square$

Finally we show that if (1.8) holds then the vector $q_\gamma$ defined by (2.36) is zero.

**Lemma 2.8.** *Suppose (1.8) holds and the conditions of Theorem 1.1. Then $q_\gamma = 0$ in (1.6).*

*Proof.* In view of (2.36) we need to show that

$$\left\langle b_j(\cdot) T_{j,0,0} \Phi_0^k(0,\cdot) \right\rangle = 0, \quad 1 \le j,k \le d, \tag{2.41}$$

where $\Phi_0^k$ satisfies (2.32). To do this let $L^2_R(\Omega)$ be the subset of functions $\Phi \in L^2(\Omega)$ which have the property that

$$\left\langle \Phi(\tau_x\cdot) \prod_{i=1}^n b_{k_i}(\tau_{x_i}\cdot) \right\rangle = (-1)^{n+1} \left\langle \Phi(\tau_{-x}\cdot) \prod_{i=1}^n b_{k_i}(\tau_{-x_i}\cdot) \right\rangle,$$

$$x, x_i \in \mathbf{Z}^d, \quad 1 \le k_i \le d, \quad i = 1, ..., n, \quad n \ge 0 .$$

Evidently $\Phi \in L^2_R(\Omega)$ implies $< \Phi >= 0$. In view of (1.8) $b_i(\cdot) \in L^2_R(\Omega),\ 1 \le i \le d$. Observe now that for any $\eta > 0$ the operator $b_j(\omega)T_{j,\eta,0}$ takes $L^2_R(\Omega)$ into $L^2_R(\Omega)$ . This follows from the fact that $G_{p,\eta}(x) = G_{p,\eta}(-x),\ x \in \mathbf{Z}^d$. Now let $\Phi^k_\eta(\omega)$ be the solution in $L^2_R(\Omega)$ to the equation,

$$\Phi^k_\eta(\omega) - \gamma \sum_{j=1}^d b_j(\omega) T_{j,\eta,0} \Phi^k_\eta(\omega) = b_k(\omega). \tag{2.42}$$

Clearly a unique solution $\Phi^k_\eta$ exists if $|\gamma| < \gamma_d$. Comparing (2.42) and (2.32) one sees that $\Phi^k_\varepsilon(0,\omega) = \Phi^k_\varepsilon(\omega) \in L^2_R(\Omega)$. Taking the limit as $\varepsilon \to 0$ we conclude that $\Phi^k_0(0,\omega) \in L^2_R(\Omega)$, whence (2.41) follows. $\square$

## 3. Proof of Theorem 1.2

Since $< \mathbf{b}(\cdot) >= 0$ we have from (2.32), (2.35) that

$$e(\varepsilon\xi) q_{\gamma,\varepsilon}(\varepsilon\xi) e(-\varepsilon\xi) = -2i \sum_{k=1}^d \sin(\varepsilon \mathbf{e}_k \cdot \xi) \sum_{n=1}^\infty \gamma^{n+1} \left\langle \left[ \sum_{j=1}^d b_j(\cdot) T_{j,\varepsilon,\varepsilon\xi} P \right]^n b_k(\cdot) \right\rangle .$$

It is easy to see from the argument of Lemma 2.8 that if (1.8) holds, then

$$\left\langle \left[ \sum_{j=1}^d b_j(\cdot) T_{j,\varepsilon,0} P \right]^n b_k(\cdot) \right\rangle = 0, \quad 1 \le k \le d, \quad n \ge 1. \tag{3.1}$$

Hence we may write

$$\begin{aligned} e(\varepsilon\xi) q_{\gamma,\varepsilon}(\varepsilon\xi) e(-\varepsilon\xi) = -2i \sum_{k=1}^d \sin(\varepsilon \mathbf{e}_k \cdot \xi) \sum_{n=0}^\infty \gamma^{n+2} \sum_{m=0}^n \\ \left\langle \left[ \sum_{j=1}^d b_j(\cdot) T_{j,\varepsilon,0} P \right]^m \left[ \sum_{j=1}^d b_j(\cdot) \left\{ T_{j,\varepsilon,\varepsilon\xi} - T_{j,\varepsilon,0} \right\} P \right] \right. \\ \left. \left[ \sum_{j=1}^d b_j(\cdot) T_{j,\varepsilon,\varepsilon\xi} P \right]^{n-m} b_k(\cdot) \right\rangle . \end{aligned} \tag{3.2}$$

If $x \in \mathbf{Z}^d$ with $x \cdot \mathbf{e}_1 > 0$, we denote by $L_1(x)$ all the points of $\mathbf{Z}^d$ which lie on the line segment joining the point $x - (x \cdot \mathbf{e}_1)\mathbf{e}_1$ and $x$, excluding the point $x$. If $x \cdot \mathbf{e}_1 = 0$, then $L_1(x)$ is the empty set. If $x \cdot \mathbf{e}_1 < 0$, then $L_1(x)$ is all the points of $\mathbf{Z}^d$ which lie on the line segment joining the point $x - (x \cdot \mathbf{e}_1)\mathbf{e}_1$ and $x$, excluding the point $x - (x \cdot \mathbf{e}_1)\mathbf{e}_1$. Similarly if $x \cdot \mathbf{e}_2 > 0$ we denote by $L_2(x)$ all the points of $\mathbf{Z}^d$ which lie on the line segment joining the point $x - (x \cdot \mathbf{e}_1)\mathbf{e}_1 - (x \cdot \mathbf{e}_2)\mathbf{e}_2$ and $x - (x \cdot \mathbf{e}_1)\mathbf{e}_1$, excluding the point $x - (x \cdot \mathbf{e}_1)\mathbf{e}_1$ etc. For $\eta > 0$, $\zeta \in [-\pi, \pi]^d$, $k = 1, \cdots, d$, we define operators $S_{k,\eta,\zeta}$ on $L^2(\Omega)$ by

(3.3)

$$S_{k,\eta,\zeta}\varphi(\omega) = \sum_{j=1}^d b_j(\omega) \sum_{x \in \mathbf{Z}^d} \left[ G_{p,\eta}(x - \mathbf{e}_j) - G_{p,\eta}(x + \mathbf{e}_j) \right] \operatorname{sgn}(x \cdot \mathbf{e}_k) \sum_{y \in L_k(x)} e^{-iy\cdot\zeta} \varphi(\tau_x \omega) .$$

We can see now from (3.2) and (2.30) that the matrix $q_{\gamma,\varepsilon}(\zeta)=[q_{\gamma,\varepsilon,k,k'}(\zeta)],\ 1\le k,k'\le d$, is given by the formula

$$-\sum_{n=0}^{\infty}\gamma^{n+2}\sum_{m=0}^{n}\Big\langle\Big[\sum_{j=1}^d b_j(\cdot)T_{j,\varepsilon,0}P\Big]^m S_{k',\varepsilon,\zeta}P\Big[\sum_{j=1}^d b_j(\cdot)T_{j,\varepsilon,\zeta}P\Big]^{n-m} b_k(\cdot)\Big\rangle$$
$$\Big\langle\Big[\sum_{j=1}^d b_j(\cdot)T_{j,\varepsilon,0}P\Big]^m e^{-i\mathbf{e}_k\cdot\zeta}S_{k,\varepsilon,\zeta}P\Big[\sum_{j=1}^d b_j(\cdot)T_{j,\varepsilon,\zeta}P\Big]^{n-m} b_{k'}(\cdot)\Big\rangle \quad .$$

We proceed as in [2] to consider the case when the $\mathbf{b}(\tau_x\cdot),\ x\in\mathbf{Z}^d$, are given by independent Bernoulli variables. Thus we assume that $b_j(\cdot)\equiv 0,\ j>1$, and $b_1(\tau_x\cdot)=Y_x\ ,x\in\mathbf{Z}^d$, where the variables $Y_x,\ x\in\mathbf{Z}^d$, are assumed to be i.i.d. Bernoulli, $Y_x=\pm1$ with equal probability. Using the notation of §4 of [2] we introduce the Fock spaces $\mathcal{F}^r(\mathbf{Z}^d),\ 1<r<\infty$, with $\mathcal{F}^2(\mathbf{Z}^d)$ isomorphic to $L^2(\Omega)$. Now we have seen from (2.31) that $T_{1,\varepsilon,\zeta}$ is a bounded operator on $L^2(\Omega)$ with norm $\|T_{1,\varepsilon,\zeta}\|_2\le 2/c_d$. We have also seen in Lemma 2.5 that

$$\lim_{\varepsilon\to0}\ \|[T_{1,\varepsilon,\zeta}-T_{1,0,\zeta}]\varphi\|_2=0,\quad \varphi\in L^2(\Omega). \tag{3.4}$$

We note now as in [2] that the Calderon-Zygmund theorem implies that a similar result holds for all Fock spaces $\mathcal{F}^r(\mathbf{Z}^d),\ 1<r<\infty$. We shall show that the limit in (3.4) is uniform for $\zeta\in[-\pi,\pi]^d$, provided one projects orthogonal to the constant function.

**Lemma 3.1.** *Suppose $1<r<\infty$ and regard $T_{1,\varepsilon,\zeta}$ as an operator on $\mathcal{F}^r(\mathbf{Z}^d)$. Then there is a constant $C_{r,d}$ depending only on $r,d$ such that $\|T_{1,\varepsilon,\zeta}\|_r\le C_{r,d}$. There is also the limit,*

$$\lim_{\varepsilon\to0}\left\{\sup_{\zeta\in[-\pi,\pi]^d}\|[T_{1,\varepsilon,\zeta}P-T_{1,0,\zeta}P]\varphi\|_r\right\}=0,\quad \varphi\in\mathcal{F}^r(\mathbf{Z}^d). \tag{3.5}$$

*Proof.* Let $T_{\varepsilon,\zeta}$ be the integral operator on $L^r(\mathbf{Z}^d)$ defined by

$$T_{\varepsilon,\zeta}f(x)=\sum_{y\in\mathbf{Z}^d}K_\varepsilon(x-y)e^{-i(x-y)\cdot\zeta}f(y),\ x\in\mathbf{Z}^d\ ,$$

where $K_\varepsilon(x)=G_{p,\varepsilon}(x-\mathbf{e}_1)-G_{p,\varepsilon}(x+\mathbf{e}_1)$. It follows from (2.17) that

$$|\hat{K}_\varepsilon(\xi)|\le 2/c_d,\quad \xi\in[-\pi,\pi]^d\ . \tag{3.6}$$

We may also apply the argument of Lemma 2.2 to see that there is a constant $C_d>0$, depending only on $d$, such that

$$\begin{aligned}|K_\varepsilon(x)| &\le C_d/[1+\varepsilon^2|x|^2][1+|x|^d],\quad x\in\mathbf{Z}^d\ ,\\ |K_\varepsilon(x+\mathbf{e}_i)-K_\varepsilon(x)| &\le C_d/[1+\varepsilon^2|x|^2][1+|x|^{d+1}],\quad x\in\mathbf{Z}^d\ .\end{aligned} \tag{3.7}$$

It follows now from (3.6), (3.7) that $T_{\varepsilon,\zeta}$ is a bounded operator on $L^r(\mathbf{Z}^d)$ and there is a constant $C_{r,d}$ depending only on $r,d$ such that $\|T_{\varepsilon,\zeta}\|_r\le C_{r,d}$. This is a discrete version of the Calderon-Zygmund Theorem [19]. Now, arguing as in [2], we see that this also implies $\|T_{1,\varepsilon,\zeta}\|_r \le C_{r,d}$.

Next we show that

$$\lim_{\varepsilon\to 0}\left\{\sup_{\zeta\in[-\pi,\pi]^d}\|[T_{\varepsilon,\zeta}-T_{0,\zeta}]f\|_r\right\}=0,\quad f\in L^r(\mathbf{Z}^d). \tag{3.8}$$

We first show that (3.8) holds for $r=2$. We can do this by going to the Fourier representation of $T_{\varepsilon,\zeta}$, which we denote $\hat{T}_{\varepsilon,\zeta}$. Thus,

$$\hat{T}_{\varepsilon,\zeta}\hat{f}(\xi)=\hat{K}_\varepsilon(\xi-\zeta)\hat{f}(\xi),\ \xi\in[-\pi,\pi]^d\ . \tag{3.9}$$

Observe now that since $\hat{f}\in L^2([-\pi,\pi]^d)$, for any $\delta>0$ there exists $\nu>0$ such that

$$\int_{|\xi-\zeta|<\nu}|\hat{f}(\xi)|^2d\xi\ <\ \delta,\quad \zeta\in[-\pi,\pi]^d\ . \tag{3.10}$$

It is easy to see that for any $\nu>0$, $\hat{K}_\varepsilon(\xi)$ converges uniformly to $\hat{K}_0(\xi)$ in the region $\xi\in[-\pi,\pi]^d$, $|\xi|>\nu$. It follows from this and (3.6), (3.9), (3.10) that (3.8) holds for $r=2$.

To prove (3.8) for general $r$, $1<r<\infty$, we define for $n=1,2\cdots$, $\zeta\in[-\pi,\pi]^d$ an operator $A_{n,\zeta}$ on functions $f:\mathbf{Z}^d\to\mathbf{C}$ by

$$A_{n,\zeta}f(x)=\sum_{z\in\mathbf{Z}^d}\varphi_n(z)e^{iz\cdot\zeta}f(x+z),\quad x\in\mathbf{Z}^d\ .$$

Here $\varphi_n$ is the probability density for the standard random walk in $\mathbf{Z}^d$ after $n$ steps of the walk. Thus $\varphi_n(z)\ge 0$, $z\in\mathbf{Z}^d$, and

$$\sum_{z\in\mathbf{Z}^d}\varphi_n(z)=1\ .$$

It follows that $A_{n,\zeta}$ is a bounded operator on $L^r(\mathbf{Z}^d)$ with $\|A_{n,\zeta}\|_r\le 1$. Observe now that

$$T_{\varepsilon,\zeta}[f-A_{n,\zeta}f](x)=\sum_{y\in\mathbf{Z}^d}K_{\varepsilon,n}(x-y)e^{-i(x-y)\cdot\zeta}f(y),\quad x\in\mathbf{Z}^d,$$

where the function $K_{\varepsilon,n}$ is given by

$$K_{\varepsilon,n}(x)=\sum_{z\in\mathbf{Z}^d}[K_\varepsilon(x)-K_\varepsilon(x+z)]\varphi_n(z)\ .$$

It is easy to see from (3.7) that $K_{\varepsilon,n}\in L^1(\mathbf{Z}^d)$ with norm $\|K_{\varepsilon,n}\|_1$ satisfying $\|K_{\varepsilon,n}\|_1\le C_{d,n}$, where the constant $C_{d,n}$ depends only on $d$ and $n$. Further, for any $x\in\mathbf{Z}^d$, $K_{\varepsilon,n}(x)$ converges to $K_{0,n}(x)$. We conclude that for any function $f\in L^r(\mathbf{Z}^d)$ one has

$$\lim_{\varepsilon\to 0}\left\{\sup_{\zeta\in[-\pi,\pi]^d}\|[T_{\varepsilon,\zeta}-T_{0,\zeta}][f-A_{n,\zeta}f]\|_r\right\}=0,$$

for any $n=1,2,\cdots$. The result (3.8) follows then if we can show that for any $f\in L^r(\mathbf{Z}^d)$, $1<r<\infty$, and $\delta>0$, there exists a positive integer $N$ such that

$$\|A_{n,\zeta}f\|_r<\delta,\quad n\ge N,\quad \zeta\in[-\pi,\pi]^d\ .$$

This last inequality follows by approximating $f\in L^r(\mathbf{Z}^d)$ with a function of finite support. A similar argument implies (3.5). □

**Lemma 3.2.** *Suppose $d \ge 2$, $1 < r < d$ and $q < \infty$ satisfies $1/r > 1/d + 1/q$. Then the operator $S_{k,\eta,\zeta}$ of (3.3) is a bounded operator from the space $\mathcal{F}^r(\mathbf{Z}^d)$ to $\mathcal{F}^q(\mathbf{Z}^d)$. There is a constant $C_{r,q,d}$, depending only on $r, q, d$, such that the norm $\|S_{k,\eta,\zeta}\|_{r,q}$ of the operator $S_{k,\eta,\zeta}$ satisfies the inequality $\|S_{k,\eta,\zeta}\|_{r,q} \le C_{r,q,d}$. There is also the limit,*

$$\lim_{\varepsilon\to 0}\left\{\sup_{\zeta\in[-\pi,\pi]^d} \|[S_{k,\varepsilon,\zeta}P - S_{k,0,\zeta}P]\varphi\|_{r,q}\right\} = 0, \quad \varphi \in \mathcal{F}^r(\mathbf{Z}^d). \tag{3.11}$$

*Proof.* We proceed as in Lemma 3.1. Let $S_{\varepsilon,\zeta}$ be the operator on functions $f \in L^r(\mathbf{Z}^d)$ defined by

$$S_{\varepsilon,\zeta}f(x) = \sum_{y\in\mathbf{Z}^d} K_{\varepsilon,\zeta}(x-y)f(y), \ x \in \mathbf{Z}^d \ ,$$

where

$$K_{\varepsilon,\zeta}(x) = [G_{p,\varepsilon}(x-\mathbf{e}_1) - G_{p,\varepsilon}(x+\mathbf{e}_1)]\mathrm{sgn}(x\cdot\mathbf{e}_k) \sum_{z\in L_1(x)} e^{-iz\cdot\zeta} \ .$$

It follows from (3.7) that there is a constant $C_d > 0$, depending only on $d$, such that

$$|K_{\varepsilon,\zeta}(x)| \le C_d/[1+|x|^{d-1}], \quad x \in \mathbf{Z}^d \ .$$

Hence $K_{\varepsilon,\zeta} \in L^s(\mathbf{Z}^d)$ for any $s > d/(d-1)$. It follows now by Young's inequality that $S_{\varepsilon,\zeta}f \in L^q(\mathbf{Z}^d)$ for any $q > 0$ satisfying $1/r > 1/d + 1/q$. Further, there is a constant $C_{r,q,d}$, depending only on $r, q, d$, such that the norm $\|S_{\varepsilon,\zeta}\|_{r,q}$ of $S_{\varepsilon,\zeta}$ as an operator from $L^r(\mathbf{Z}^d)$ to $L^q(\mathbf{Z}^d)$ satisfies $\|S_{\varepsilon,\zeta}\|_{r,q} \le C_{r,q,d}$. It is easy to see that for any $x \in \mathbf{Z}^d$,

$$\lim_{\varepsilon\to 0}\left\{\sup_{\zeta\in[-\pi,\pi]^d} |K_{\varepsilon,\zeta}(x) - K_{0,\zeta}(x)|\right\} = 0 \ .$$

Hence, if $S_{0,\zeta}$ is the operator with kernel $K_{0,\zeta}$, one has

$$\lim_{\varepsilon\to 0}\left\{\sup_{\zeta\in[-\pi,\pi]^d} \|[S_{\varepsilon,\zeta} - S_{0,\zeta}]f\|_q\right\} = 0, \quad f \in L^r(\mathbf{Z}^d) \ ,$$

provided $q < \infty$ satisfies $1/r > 1/d + 1/q$. Now, arguing as in [2] we obtain the result. □

**Corollary 3.1.** *Suppose $b_j(\cdot) \equiv 0$, $j > 1$ and $b_1(\tau_x\cdot) = Y_x$, $x \in \mathbf{Z}^d$, where the $Y_x$, $x \in \mathbf{Z}^d$, are independent Bernoulli variables with zero mean. Then Theorem 1.2 holds.*

*Proof.* Observe that $b_k(\cdot) \in \mathcal{F}^r(\mathbf{Z}^d)$ for all $r \ge 1$. The result follows now from Lemmas 3.1, 3.2 and the representation (**??**) for $q_{\gamma,\varepsilon}(\zeta)$. □

Next we wish to generalize our method to all $\mathbf{b}(\cdot)$ such that $\mathbf{b}(\tau_x\cdot)$, $x \in \mathbf{Z}^d$, are independent random variables. To do this we pursue the method developed in §5 of [2]. Thus we define a spin space $S$ as all $s = (m_1, ..., m_d)$ where the $m_i$ are nonnegative integers satisfying $|s| = m_1 + ... + m_d \ge 1$. For $s \in S$ we define a random variable $b_s$ by

$$b_s(\cdot) = \prod_{i=1}^d b_i(\cdot)^{m_i} \ , \quad s = (m_1, ..., m_d),$$

and a random variable $Y_{0,s} = b_s - < b_s >$. Then for $s \in S$, $n \in \mathbf{Z}^d$ we define the variable $Y_{n,s}$ to be the translate of $Y_{0,s}$ i.e. $Y_{n,s}(\cdot) = Y_{0,s}(\tau_n \cdot)$. For $1 < r < \infty$ we may define Fock spaces $\mathcal{F}^r_S(\mathbf{Z}^d)$ of many particle wave functions where the particles move in $\mathbf{Z}^d$ and have spin in $S$. We can also, for any $\delta > 0$, define a mapping $U_\delta$ from $\mathcal{F}^2_S(\mathbf{Z}^d)$ to functions on $\Omega$ by

$$U_\delta \psi = \psi_0 + \sum_{N=1}^{\infty} \sum_{\substack{\{n_1,\dots,n_N\} \in \mathbf{Z}^{d,N} \\ s_i \in S,\ 1 \le i \le N}} \psi_N(n_1, s_1, n_2, s_2, \dots, n_N, s_N) \times \delta^{|s_1| + \dots + |s_N|}\ Y_{n_1,s_1} Y_{n_2,s_2} \dots Y_{n_N,s_N} \ . \tag{3.12}$$

Following the argument of Lemma 5.1 of [2] we have

**Lemma 3.3.** *There exists $\delta > 0$, depending only on $d$, such that the mapping $U_\delta$ of (3.12) is a bounded operator from $\mathcal{F}^2_S(\mathbf{Z}^d)$ to $L^2(\Omega)$ satisfying $\|U_\delta\| \le 1$.*

We shall use the operator $U_\delta = U$ of Lemma 3.3 to transfer the equation of (2.32) from $L^2(\Omega)$ to $\mathcal{F}^2_S(\mathbf{Z}^d)$. Thus we shall need to construct operators $P_{\mathcal{F}}$, $B_{j,\mathcal{F}}$, $T_{j,\varepsilon,\zeta,\mathcal{F}}$, $1 \le j \le d$, with the property that they are bounded operators on $\mathcal{F}^2_S(\mathbf{Z}^d)$ and satisfy

$$UP_{\mathcal{F}} = PU, \ \ UB_{j,\mathcal{F}} = b_j(\cdot)U, \ \ UT_{j,\varepsilon,\zeta,\mathcal{F}} = T_{j,\varepsilon,\zeta}U \ . \tag{3.13}$$

The simplest of these operators to construct is $P_{\mathcal{F}}$. Thus if $\psi = \{\psi_N : N = 0,1,2,\dots\} \in \mathcal{F}^2_S(\mathbf{Z}^d)$ then $P_{\mathcal{F}}\psi = \{(P_{\mathcal{F}}\psi)_N : N = 0,1,2,\dots\}$, where $(P_{\mathcal{F}}\psi)_0 = 0$, $(P_{\mathcal{F}}\psi)_N = \psi_N$, $N \ge 1$. Similarly we have $T_{j,\varepsilon,\zeta,\mathcal{F}}\psi = \{(T_{j,\varepsilon,\zeta,\mathcal{F}}\psi)_N : N = 0,1,2,\dots\}$, where

$$(T_{j,\varepsilon,\zeta,\mathcal{F}}\psi)_0 = \left[e^{-i\zeta\cdot\mathbf{e}_j} - e^{i\zeta\cdot\mathbf{e}_j}\right] \hat{G}_{p,\varepsilon}(-\zeta)\psi_0,$$

$$(T_{j,\varepsilon,\zeta,\mathcal{F}}\psi)_N(n_1, s_1, \dots, n_N, s_N) =$$

$$\sum_{x \in \mathbf{Z}^d} \left[G_{p,\varepsilon}(x - \mathbf{e}_j) - G_{p,\varepsilon}(x + \mathbf{e}_j)\right] e^{-ix\cdot\zeta} \psi_N(n_1 - x, s_1, \dots, n_N - x, s_N), \quad N \ge 1.$$

The most complicated of the operators to define is $B_{j,\mathcal{F}}$. We have

$$(B_{j,\mathcal{F}}\psi)_N(n_1, s_1, \dots, n_N, s_N) = < b_j(\cdot) > \psi_N(n_1, s_1, \dots, n_N, s_N)$$

$$+ \sum_{s \in S} \left[< b_j b_s > - < b_j >< b_s >\right] \delta^{|s|} \psi_{N+1}(0, s, n_1, s_1, \dots, n_N, s_N),$$

$$\text{if} \quad n_k \ne 0, \quad 1 \le k \le N, \quad N \ge 0;$$

$$\begin{aligned}
(B_{j,\mathcal{F}}\psi)_N(0, s_1, n_2, s_2, \dots, n_N, s_N) &= 0, \\
\text{if} \ \ n_k \ne 0, \ \ 2 \le k \le N, \ \ s_1 &= (m_1, \dots, m_d), \ m_j = 0; \\
(B_{j,\mathcal{F}}\psi)_N(0, s_1, n_2, s_2, \dots, n_N, s_N) &= \delta^{-1}\psi_N(0, s_1', n_2, s_2, \dots, n_N, s_N), \\
\text{if} \ \ n_k \ne 0, \ \ 2 \le k \le N, \ \ s_1 &= (m_1, \dots, m_d), \ m_j > 0, \ |s_1| > 1, \\
s_1' &= (m_1, \dots, m_{j-1}, m_j - 1, m_{j+1}, \dots, m_d); \\
(B_{j,\mathcal{F}}\psi)_N(0, s_1, n_2, s_2, \dots, n_N, s_N) &= \delta^{-1}\psi_{N-1}(n_2, s_2, \dots, n_N, s_N)
\end{aligned}$$

$$- \sum_{s \in S} < b_s > \delta^{|s|-1}\psi_N(0, s, n_2, s_2, \dots, n_N, s_N),$$

$$\text{if} \quad n_k \neq 0, \quad 2 \le k \le N \quad \text{and} \quad s_1 = (0,0,..,1,0,...,0),$$

with 1 in the ith position.

**Lemma 3.4.** *For $1 \le k \le d$ let $\varphi_k \in \mathcal{F}_S^2(\mathbf{Z}^d)$ be the wave function $\varphi_k = \{\varphi_{k,N} : N = 0,1,2,...\}$ where $\varphi_{k,N} \equiv 0$ if $N \neq 1$ and $\varphi_{k,1}$ is defined by*

$$\varphi_{k,1}(n,s) = 1 \quad \text{if} \quad n = 0, \quad s = (0,..,0,1,0,...,0),$$

*with 1 in the kth position, $\varphi_{k,1}(n,s) = 0$, otherwise. Then there exists a constant $\gamma_d$ depending only on $d$ such that if $|\gamma| < \gamma_d$, the equation*

$$\Phi^k_{\varepsilon,\mathcal{F}}(\zeta) - \gamma P_{\mathcal{F}} \sum_{j=1}^d B_{j,\mathcal{F}} T_{j,\varepsilon,\zeta,\mathcal{F}} \Phi^k_{\varepsilon,\mathcal{F}}(\zeta) = \varphi_k \ , \tag{3.14}$$

*has a unique solution $\Phi^k_{\varepsilon,\mathcal{F}}(\zeta) \in \mathcal{F}_S^2(\mathbf{Z}^d)$. Further the function $\Phi^k_\varepsilon(\zeta,\omega) = U\Phi^k_{\varepsilon,\mathcal{F}}(\zeta) \in L^2(\Omega)$ is the solution to (2.32).*

*Proof.* Evidently $P_{\mathcal{F}}$ is a bounded operator on $\mathcal{F}_S^2(\mathbf{Z}^d)$ with norm 1. We can also easily see that $T_{j,\varepsilon,\zeta,\mathcal{F}}$ is a bounded operator on $\mathcal{F}_S^2(\mathbf{Z}^d)$ with norm bounded by the RHS of (2.31). Also $B_{j,\mathcal{F}}$ is bounded with norm depending only on $d$. Hence we may choose $\gamma_d$ small enough, depending only on $d$, such that (3.14) is uniquely solvable. The fact that $U\Phi^k_{\varepsilon,\mathcal{F}}(\zeta)$ solves (2.32) comes from (3.13) and $U\varphi_k = b_k(\cdot) - \langle b_k \rangle$. $\square$

*Proof of Theorem 1.2*: Observe from (2.35) that $q_{\varepsilon,\gamma}(\zeta)$ is defined in terms of $\Phi^k_\varepsilon(\zeta,\omega)$. In view of Lemma 3.4 we get a representation of $q_{\varepsilon,\gamma}(\zeta)$ similar to (**??**) but with the operators $T_{j,\varepsilon,\zeta},\ S_{k,\varepsilon,\zeta},\ b_j(\cdot)$ on $L^2(\Omega)$ replaced by the corresponding operators $T_{j,\varepsilon,\zeta,\mathcal{F}},\ S_{k,\varepsilon,\zeta,\mathcal{F}},\ B_{j,\mathcal{F}}$ on Fock space. Here the operator $S_{k,\varepsilon,\zeta,\mathcal{F}}$ satisfies $US_{k,\varepsilon,\zeta,\mathcal{F}} = S_{k,\varepsilon,\zeta}U$. We now argue exactly as in Corollary 3.1. $\square$

## 4. Proof of Theorem 1.3

We assume in this section that the function $p$ is given by (1.2). Let $G_{p,\eta}(x)$ be the function defined by (2.9). Then $\hat{G}_{p,\eta}(\xi)$ satisfies the inequality,

$$0 \le \hat{G}_{p,\eta}(\xi) \le C/[\eta + |\xi|^2] \ ,$$

for some universal constant $C$. One can also see that if $d \ge 2$ and $0 < \eta < 1$, there is a constant $C_d$, depending only on $d$, such that

$$|G_{p,\eta}(x - \mathbf{e}_k) - G_{p,\eta}(x + \mathbf{e}_k)| \le C_d \exp\left[-\sqrt{\eta}|x|/C_d\right]/[1 + |x|^{d-1}] \ . \tag{4.1}$$

Next we prove the analogue of Lemma 2.3.

**Lemma 4.1.** *Suppose $d \ge 2$ and $f : \mathbf{R}^d \to \mathbf{R}$ is a $C^\infty$ function with compact support. Then if $\gamma \in \mathbf{C}$ satisfies $|\gamma| < \varepsilon/\sqrt{2d}$ and $0 < \varepsilon \le 1$, the equation (1.3) has a unique solution $u_\varepsilon(x,\omega)$ in $L^2(\mathbf{Z}^d_\varepsilon)$. Further, there is a constant $C_d$, depending only on $d$, such that $\|u_\varepsilon(\cdot,\omega)\|_\varepsilon \le C_d \|f\|_\varepsilon$ .*

*Proof.* Suppose $u_\varepsilon(x,\omega)$ is in $L^2(\mathbf{Z}^d_\varepsilon)$ and satisfies (1.3). Let $v_\varepsilon(x,\omega)$ be defined by

$$v_\varepsilon(x,\omega) = \varepsilon^{-2}[A_p u_\varepsilon(x,\omega) + \varepsilon^2 u_\varepsilon(x,\omega)].$$

It is easy to see that $v_\varepsilon(x,\omega) \in L^2(\mathbf{Z}^d_\varepsilon)$. In view of (2.9) it follows that

$$u_\varepsilon(x,\omega) = \varepsilon^{2-d} \int_{\mathbf{Z}^d_\varepsilon} dy \ G_{p,\varepsilon^2}\left(\frac{x-y}{\varepsilon}\right) v_\varepsilon(y,\omega). \tag{4.2}$$

Now for $i = 1, ..., d$ let us define operators $T_i$ on $L^2(\mathbf{Z}^d_\varepsilon)$ by

$$T_i w(x) = \varepsilon^{1-d} \int_{\mathbf{Z}^d_\varepsilon} dy \left[ G_{p,\varepsilon^2}\left( \frac{x-y}{\varepsilon} + \mathbf{e}_i \right) - G_{p,\varepsilon^2}\left( \frac{x-y}{\varepsilon} - \mathbf{e}_i \right) \right] w(y) \ .$$

It is easy to see that $T_i$ is a bounded operator on $L^2(\mathbf{Z}^d_\varepsilon)$. It follows from (2.10) that the norm of $T_i$ is bounded by

$$\|T_i\| \le \sup_{\xi \in [-\pi,\pi]^d} \left[ \frac{2\varepsilon |\sin(\mathbf{e}_i \cdot \xi)|}{1+\varepsilon^2 - \hat{p}(\xi)} \right] . \tag{4.3}$$

Now (4.3) and (1.1) imply that the operator $\sum_{i=1}^d b_i(\cdot,\omega)T_i$ is bounded on $L^2(\mathbf{Z}^d_\varepsilon)$ and has norm satisfying the inequality,

$$\begin{aligned} \|\sum_{i=1}^d b_i(\cdot,\omega)T_i\| \quad &\le \quad \sup_{\xi\in[-\pi,\pi]^d} \left[ \frac{2\varepsilon \left\{ \sum_{i=1}^d \sin^2(\mathbf{e}_i\cdot\xi) \right\}^{1/2}}{1+\varepsilon^2-\hat{p}(\xi)} \right] \\ &\le \quad 4\sup_{z>0} \left[ \frac{\varepsilon z}{\varepsilon^2 + 2z^2/d} \right] \ = \ \sqrt{2d} \ . \end{aligned}$$

Observe now that (1.3) implies that $v_\varepsilon(x,\omega) \in L^2(\mathbf{Z}^d_\varepsilon)$ satisfies the equation

$$v_\varepsilon(x,\omega) - \gamma\varepsilon^{-1} \sum_{i=1}^d b_i(x/\varepsilon,\omega) T_i v_\varepsilon(x,\omega) = f(x). \tag{4.4}$$

Hence if $|\gamma| < \varepsilon/\sqrt{2d}$, the function $v_\varepsilon(x,\omega)$ is uniquely determined by (4.4). In view of (4.2) it follows that $u_\varepsilon(x,\omega) \in L^2(\mathbf{Z}^d_\varepsilon)$ is the unique solution to (1.3). To prove existence note that (4.4) has a solution $v_\varepsilon(x,\omega) \in L^2(\mathbf{Z}^d_\varepsilon)$ provided $|\gamma| < \varepsilon/\sqrt{2d}$. If we then define $u_\varepsilon(x,\omega)$ by (4.2) one can easily see that $u_\varepsilon(x,\omega)$ satisfies (1.3). □

In order to obtain the equation (1.10) we follow a development similar to the one following Lemma 2.3. Thus we put $v_\varepsilon(x,\omega) = u_\varepsilon(x, \tau_{-x/\varepsilon}\omega)$, $x \in \mathbf{Z}^d_\varepsilon, \omega \in \Omega$, and write $v_\varepsilon(x,\omega) = u_\varepsilon(x) + \psi_\varepsilon(x,\omega)$, $x \in \mathbf{Z}^d_\varepsilon$, $\omega \in \Omega$, where $<\psi_\varepsilon(x,\cdot)>= 0$, $x \in \mathbf{Z}^d_\varepsilon$. It follows that

$$u_\varepsilon(x) = \Big\langle v_\varepsilon(x,\cdot) \Big\rangle = \Big\langle u_\varepsilon(x,\cdot) \Big\rangle, \ \ x \in \mathbf{Z}^d_\varepsilon \ .$$

Furthermore, the Fourier transform $\hat{u}_\varepsilon(\xi)$ of $u_\varepsilon(x)$ satisfies the equation,

$$\begin{aligned} &\hat{u}_\varepsilon(\xi) \left\{ 1 + \varepsilon^{-2}[1-\hat{p}(\varepsilon\xi)] + 2\gamma i \sum_{j=1}^d <b_j(\cdot)> \frac{\sin(\mathbf{e}_j\cdot\varepsilon\xi)}{\varepsilon^2} \right\} \\ &+\gamma\varepsilon^{-2} \left\langle \sum_{j=1}^d b_j(\cdot) \left[ e^{i\varepsilon\mathbf{e}_j\cdot\xi} \hat{\psi}_\varepsilon(\xi, \tau_{-\mathbf{e}_j}\cdot) - e^{-i\varepsilon\mathbf{e}_j\cdot\xi}\hat{\psi}_\varepsilon(\xi,\tau_{\mathbf{e}_j}\cdot) \right] \right\rangle = \hat{f}_\varepsilon(\xi), \ \ \xi \in \left[ \frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon} \right]^d . \end{aligned} \tag{4.5}$$

The Fourier transform of $\psi_\varepsilon(x,\omega)$ is given in terms of $\hat{u}_\varepsilon(\xi)$ by

$$\hat{\psi}_\varepsilon(\xi,\omega) = -2i\gamma\hat{u}_\varepsilon(\xi) \sum_{k=1}^d \sin(\varepsilon\mathbf{e}_k\cdot\xi)\Psi^k_\varepsilon(\varepsilon\xi,\omega) \tag{4.6}$$

Here $\Psi^k_\varepsilon(\zeta,\omega)$, $1 \le k \le d$, $\zeta \in [-\pi,\pi]^d$, $\omega \in \Omega$, are functions in $L^2([-\pi,\pi]^d \times \Omega)$ which satisfy the equation,

$$\begin{aligned}(4.7)\quad \Psi_\varepsilon^k(\zeta,\omega) &- \sum_{y\in\mathbf{Z}^d} p(y)e^{-iy\cdot\zeta}\Psi_\varepsilon^k(\zeta,\tau_y\omega) \\ &+ \gamma P\sum_{j=1}^d b_j(\omega)\left[e^{i\mathbf{e}_j\cdot\zeta}\Psi_\varepsilon^k(\zeta,\tau_{-\mathbf{e}_j}\omega) - e^{-i\mathbf{e}_j\cdot\zeta}\Psi_\varepsilon^k(\zeta,\tau_{\mathbf{e}_j}\omega)\right] \\ &+ \varepsilon^2\Psi_\varepsilon^k(\zeta,\omega) = b_k(\omega) - < b_k(\cdot) >,\ \zeta\in[-\pi,\pi]^d,\quad \omega\in\Omega,\ 1\le k\le d.\end{aligned}$$

If we substitute for $\hat\psi_\varepsilon$ from (4.6) into (4.5) then we obtain the equation,

$$\begin{aligned}(4.8)\quad \hat u_\varepsilon(\xi)\Bigg\{ 1 &+ \varepsilon^{-2}\left[1-\hat p(\varepsilon\xi)\right] + 2\gamma i\sum_{j=1}^d < b_j(\cdot) > \frac{\sin(\varepsilon\mathbf{e}_j\cdot\xi)}{\varepsilon^2} \\ &- 2i\gamma^2\Bigg\langle \sum_{j,k=1}^d b_j(\cdot)\frac{\sin(\varepsilon\mathbf{e}_k\cdot\xi)}{\varepsilon^2}\Bigg[e^{i\varepsilon\mathbf{e}_j\cdot\xi}\Psi_\varepsilon^k(\varepsilon\xi,\tau_{-\mathbf{e}_j}\cdot) \\ &- e^{-i\varepsilon\mathbf{e}_j\cdot\xi}\Psi_\varepsilon^k(\varepsilon\xi,\tau_{\mathbf{e}_j}\cdot)\Bigg]\Bigg\rangle\Bigg\} = \hat f_\varepsilon(\xi),\quad \xi\in\left[\frac{-\pi}{\varepsilon},\frac{\pi}{\varepsilon}\right]^d.\end{aligned}$$

Let us assume now that (1.8) holds. Then, following the development at the beginning of §3, we have that the matrix $q_{\gamma,\varepsilon}(\zeta) = [q_{\gamma,\varepsilon,k,k'}(\zeta)],\ 1\le k,k'\le d$ of (1.10) is given by the formula

$$-\sum_{n=0}^\infty \gamma^{n+2}\sum_{m=0}^n \Bigg\langle \left[\sum_{j=1}^d b_j(\cdot)T_{j,\varepsilon^2,0}P\right]^m S_{k',\varepsilon^2,\zeta}P\left[\sum_{j=1}^d b_j(\cdot)T_{j,\varepsilon^2,\zeta}P\right]^{n-m} b_k(\cdot)\Bigg\rangle \Bigg\langle \left[\sum_{j=1}^d b_j(\cdot)T_{j,\varepsilon^2,0}P\right]^m e^{-i\mathbf{e}_k\cdot\zeta}S_{k,\varepsilon^2,\zeta}P\left[\sum_{j=1}^d b_j(\cdot)T_{j,\varepsilon^2,\zeta}P\right]^{n-m} b_{k'}(\cdot)\Bigg\rangle,$$

where the operator $T_{j,\eta,\zeta}$ is defined by (2.30) and $S_{k,\eta,\zeta}$ by (3.3), with $p$ given by (1.2). One can see from the proof of Lemma 4.1 that $q_{\gamma,\varepsilon}(\zeta)$ is continuous in $\zeta$ and analytic in $\gamma$ provided $|\gamma| < \varepsilon/\sqrt{2d}$.

## 5. Proof of Theorem 1.4

The matrix $q_{m,\varepsilon}(\zeta)$ is given by the coefficient of $\gamma^m$ in the expansion (**??**). Since the $\mathbf{b}(\tau_x\cdot)$ are given by Bernoulli variables, (a) and (b) of Theorem 1.4 follow easily from (**??**). Next we obtain a formula for $q_{4,\varepsilon}(\zeta)$. Let $K_\varepsilon(x)$ be defined by

$$(5.1)\qquad K_\varepsilon(x) = G_{p,\varepsilon^2}(x-\mathbf{e}_1) - G_{p,\varepsilon^2}(x+\mathbf{e}_1),\ x\in\mathbf{Z}^d\ .$$

For $x\in\mathbf{Z}^d$, let $x^R$ be the reflection of $x$ in the plane through the origin with normal $\mathbf{e}_1$. It is clear that

$$(5.2)\qquad K_\varepsilon(x^R) \ = \ -K_\varepsilon(x),\ x\in\mathbf{Z}^d\ .$$

For $1 \le k \le d,\ \zeta \in [-\pi,\pi]^d$, define $a_k(\zeta),\ b_k(\zeta)$ by

$$
\begin{aligned}
a_{k,\varepsilon}(\zeta) &= \sum_{x\in\mathbf{Z}^d} [K_\varepsilon(x)]^3 \mathrm{sgn}(x\cdot\mathbf{e}_k) \sum_{y\in L_k(x)} e^{-iy\cdot\zeta} \ , \\
b_{k,\varepsilon}(\zeta) &= \sum_{x\in\mathbf{Z}^d} [K_\varepsilon(x)]^3 e^{ix\cdot\zeta} \mathrm{sgn}(x\cdot\mathbf{e}_k) \sum_{y\in L_k(x)} e^{-iy\cdot\zeta} \ .
\end{aligned} \tag{5.3}
$$

Note that

$$
a_{k,\varepsilon}(0) = b_{k,\varepsilon}(0) = \sum_{x\in\mathbf{Z}^d} [K_\varepsilon(x)]^3 (x\cdot\mathbf{e}_k) \ . \tag{5.4}
$$

Then one can see from (**??**) that

$$
q_{4,\varepsilon,1,1}(\zeta) = [1+e^{-i\mathbf{e}_1\cdot\zeta}][2a_{1,\varepsilon}(\zeta) - b_{1,\varepsilon}(\zeta)] \ . \tag{5.5}
$$

One also has for $1 < k \le d$,

$$
\begin{aligned}
q_{4,\varepsilon,1,k}(\zeta) &= 2a_{k,\varepsilon}(\zeta) - b_{k,\varepsilon}(\zeta) \ , \\
q_{4,\varepsilon,1,k}(\zeta) &= e^{-i\mathbf{e}_k\cdot\zeta}[2a_{k,\varepsilon}(\zeta) - b_{k,\varepsilon}(\zeta)] \ .
\end{aligned} \tag{5.6}
$$

Since $a_{k,\varepsilon}(0) = b_{k,\varepsilon}(0)$ it follows that $q_{4,\varepsilon,1,1}(0) = 2a_{1,\varepsilon}(0)$. From (5.2) and (5.4) we see that $a_{k,\varepsilon}(0) = 0$ if $k > 1$. Hence $q_{4,\varepsilon,k,k'}(0) = 0$ for $k + k' > 2$.

**Lemma 5.1.** *(a) Let $d = 1$ and $\mathcal{K} \subset \mathbf{R}$ be a compact set. Then $\varepsilon^2 a_{1,\varepsilon}(\varepsilon\xi)$ converges uniformly for $\xi \in \mathcal{K}$ to a function $a_1(\xi)$, as $\varepsilon \to 0$. Similarly $\varepsilon^2 b_{1,\varepsilon}(\varepsilon\xi)$ converges to a function $b_1(\xi)$.*
*(b) If $d \ge 3$ and $1 \le k \le d$, then $a_{k,\varepsilon}(\zeta)$ converges uniformly for $\zeta \in [-\pi,\pi]$ to a function $a_k(\zeta)$, as $\varepsilon \to 0$. Similarly $b_{k,\varepsilon}(\zeta)$ converges to a function $b_k(\zeta)$.*

*Proof.* (a) It is easy to see that for $d = 1,\ \eta > 0$, one has

$$
G_{p,\eta}(x) = \frac{\left[1+\eta-\sqrt{2\eta+\eta^2}\right]^{|x|}}{\sqrt{2\eta+\eta^2}} \ , \quad x \in \mathbf{Z} \ .
$$

Hence if for $\varepsilon > 0$, we define $\nu(\varepsilon)$ by

$$
\nu(\varepsilon) = 1 + \varepsilon^2 - \varepsilon\sqrt{2+\varepsilon^2} \ , \tag{5.7}
$$

then the function $K_\varepsilon(x)$ of (5.1) is given by

$$
K_\varepsilon(x) = 2\nu(\varepsilon)^{|x|},\ x \in \mathbf{Z},\ x > 0 \ . \tag{5.8}
$$

Substituting from (5.8) into (5.3) we obtain the identity,

$$
a_{1,\varepsilon}(\zeta) = \frac{8\nu(\varepsilon)^3}{1-\nu(\varepsilon)^3}\left[\frac{1}{1-\nu(\varepsilon)^3 e^{-i\zeta}} + \frac{e^{i\zeta}}{1-\nu(\varepsilon)^3 e^{i\zeta}}\right] \ .
$$

One can see that

$$
\lim_{\varepsilon\to 0} \frac{\varepsilon}{1-\nu(\varepsilon)^3 e^{-i\varepsilon\xi}} = \frac{1}{3\sqrt{2}+i\xi} \ .
$$

We conclude that

$$
\lim_{\varepsilon\to 0} \varepsilon^2 a_{1,\varepsilon}(\varepsilon\xi) = \frac{16}{18+\xi^2} \ .
$$

Since for $d = 1$ one has $a_{1,\varepsilon}(\zeta) = b_{1,\varepsilon}(-\zeta)$, the formula (1.12) follows now from the last equation and (5.5).
(b) This follows from the fact that for $d \ge 3$, the limit $\lim_{\varepsilon\to 0} K_\varepsilon(x) = K_0(x)$ exists for all $x \in \mathbf{Z}^d$, and the bound (4.1). ◻

**Lemma 5.2.** *(a) For $d \ge 3$, one has $a_1(0) > 0$.*
*(b) For $d = 2$ one has $\lim_{\varepsilon\to 0} a_{1,\varepsilon}(0) = +\infty$.*

*Proof.* (a) In view of (5.2) and (5.4) it is sufficient to show that $K_0(x) > 0$ if $x \in \mathbf{Z}^d$ satisfies $x \cdot \mathbf{e}_1 > 0$. To see this observe from (2.9) and (5.2) that $K_\varepsilon(x)$ satisfies the equation

$$A_p K_\varepsilon(x) + \varepsilon^2 K_\varepsilon(x) = g(x), \quad x \in \mathbf{Z}^d,\ x \cdot \mathbf{e}_1 > 0\ ,$$

with Dirichlet boundary condition $K_\varepsilon(x) = 0,\ x \cdot \mathbf{e}_1 = 0$. Here $g$ is the function, $g(x) = 1,\ x = \mathbf{e}_1,\ g(x) = 0$ otherwise. The positivity of $K_0(x)$ follows now from the maximum principle.
(b) We shall show that there exists $\alpha > 0$ and a positive constant $c_\alpha$, depending only on $\alpha$ such that for $d = 2$,

$$\lim_{\varepsilon\to 0} K_\varepsilon(n\mathbf{e}_1 + m\mathbf{e}_2) \ge \frac{c_\alpha}{n}, \qquad |m| < \alpha n,\ n = 1, 2, \cdots\ . \tag{5.9}$$

To prove (5.9) we use the fact that for $d = 1$ one has

$$K_\varepsilon(n) = 2\nu(\varepsilon)^n = \frac{2}{\pi}\int_0^\pi \frac{\sin(n\zeta)\sin(\zeta)}{2\sin^2(\zeta/2) + \varepsilon^2} d\zeta \qquad n \ge 1\ .$$

Since one also has for $d = 2$ the representation,

$$K_\varepsilon(n\mathbf{e}_1 + m\mathbf{e}_2) = \frac{2}{\pi^2}\int_0^\pi\int_0^\pi \frac{\cos(m\xi)\sin(n\zeta)\sin(\zeta)}{\sin^2(\zeta/2) + \sin^2(\xi/2) + \varepsilon^2} d\zeta d\xi\ ,$$

we obtain for $d = 2$ the formula,

$$\lim_{\varepsilon\to 0} K_\varepsilon(n\mathbf{e}_1 + m\mathbf{e}_2) = \frac{4}{\pi}\int_0^\pi \nu(\sqrt{2}\sin(\xi/2))^n \cos(m\xi) d\xi\ , \quad m, n \in \mathbf{Z},\ n \ge 1.$$

Now there exist universal positive constants $c, C$ such that,

$$\exp[-Cn\xi] \le \nu(\sqrt{2}\sin(\xi/2))^n \le \exp[-cn\xi], \quad n \ge 1,\ 0 \le \xi \le \pi.$$

We conclude that the LHS of (5.9) is bounded below by

$$\frac{4}{\pi}\int_0^\pi \exp[-Cn\xi]\ d\xi = \frac{4}{C\pi n}\left\{1 - \exp[-C\pi n]\right\}\ ,$$

if $m = 0$, and by

$$\frac{2\sqrt{2}}{\pi}\int_0^{\pi/4|m|} \exp[-Cn\xi]\ d\xi - \frac{4}{\pi}\int_{\pi/4|m|}^\infty \exp[-cn\xi]\ d\xi$$
$$= \frac{2\sqrt{2}}{C\pi n}\left\{1 - \exp\left[-\frac{C\pi n}{4|m|}\right]\right\} - \frac{4}{c\pi n}\exp\left[-\frac{c\pi n}{4|m|}\right]\ ,$$

if $m \neq 0$. The inequality (5.9) clearly follows from the last two identities. The result follows now from (5.4) and (5.9) since the argument of part (a) implies $K_\varepsilon(n\mathbf{e}_1 + m\mathbf{e}_2) > 0,\ m, n \in \mathbf{Z},\ n > 0$. □

We are left now to prove (c) and (d) of Theorem 1.4 for $m > 4$. In order to do this we need to obtain a suitable representation of $q_{m,\varepsilon}(\zeta)$ for general $m$. We first consider the situation $\zeta = 0$. Recall that we are assuming the $b_j(\cdot) \equiv 0,\ j > 1$, and $b_1(\tau_x \cdot) = Y_x,\ x \in \mathbf{Z}^d$, where the variables $Y_x,\ x \in \mathbf{Z}^d$, are assumed to be i.i.d. Bernoulli, $Y_x = \pm 1$ with equal probability.

**Lemma 5.3.** *Suppose $m \ge 4$ is even. Then $q_{m,\varepsilon,k,k'}(0) = 0$ if $k + k' > 2$ and $q_{m,\varepsilon,1,1}(0)$ has the representation*

$$q_{m,\varepsilon,1,1}(0) = -2 \sum_{x_1,\cdots,x_{m-1}\in\mathbf{Z}^d} \left(\sum_{\alpha=1}^{m-1} x_\alpha\cdot\mathbf{e}_1\right) K_\varepsilon(x_1)\cdots K_\varepsilon(x_{m-1}) \left\langle Y_0 Y_{x_1} Y_{x_1+x_2}\cdots Y_{x_1+\cdots+x_{m-1}}\right\rangle \ . \tag{5.10}$$

*Proof.* First note that by the argument of Lemma 2.8 the representation (**??**) for $q_{\gamma,\varepsilon}(0)$ continues to hold when we delete the projection operator $P$ from the RHS of the equation. It is easy to see from this that (5.10) holds. If $k > 1$ then one has a similar representation to (5.10) for $q_{m,\varepsilon,1,k}(0)$ and $q_{m,\varepsilon,k,1}(0)$ but with $\mathbf{e}_1$ in (5.10) replaced by $\mathbf{e}_k$. In that case (5.2) implies $q_{m,\varepsilon,k,1}(0) = q_{m,\varepsilon,1,k}(0) = 0$. □

In order to prove (d) of Theorem 1.4 for general $m$ we use (5.10) to obtain a representation of $q_{m,\varepsilon,1,1}(0)$ as a sum indexed by certain types of graph. We shall use the terminology of [1]. For $q = 2, 3, \cdots$ let $\mathcal{F}_q$ be the set of unlabeled, connected, directed multigraphs on $q$ vertices with the properties:
(A) The graph has no loops,
(B) Each vertex has equal indegree and outdegree,
(C) The degree of each vertex is a multiple of 4,
(D) The number of edges in the graph is $4q$.

For a graph $G \in \mathcal{F}_q$ let $V[G]$ denote the vertex set of $G$ and $E[G]$ be the set of directed edges of $G$. Each directed edge $e$ has two vertices $e_+$ and $e_-$, with the direction of the edge being from $e_-$ to $e_+$. We associate with $G$ and a directed edge $e \in E[G]$, a number $K_\varepsilon(G,e)$ defined by,

$$K_\varepsilon(G,e) = \sum_{\{y_v\in\mathbf{Z}^d: v\in V[G]\}} (y_{e_-}\cdot\mathbf{e}_1)\delta(y_{e_+}) \prod_{e'\in E[G],\ e'\ne e} K_\varepsilon(y_{e'_+} - y_{e'_-}) \ . \tag{5.11}$$

In (5.11) $\delta(x),\ x \in \mathbf{Z}^d$ is the Kronecker delta, $\delta(0) = 1,\ \delta(x) = 0,\ x \ne 0$.

**Lemma 5.4.** *Let $m \ge 4$ be an even integer and $q = m/2$. Then*

$$q_{m,\varepsilon,1,1}(0) = \sum_{G\in\mathcal{F}_q,\ e\in E[G]} c(G,e) K_\varepsilon(G,e) \ , \tag{5.12}$$

*for some integers $c(G,e)$ depending only on the graph $G$ and the directed edge $e \in E[G]$.*

*Proof.* Observe that

$$K_\varepsilon(x_1)\cdots K_\varepsilon(x_{m-1}) \left\langle Y_0 Y_{x_1} Y_{x_1+x_2}\cdots Y_{x_1+\cdots+x_{m-1}}\right\rangle \ne 0,$$

only if the sequence $0, x_1, x_1+x_2, \cdots, x_1+\cdots+x_{m-1}$ of points in $\mathbf{Z}^d$ has the property that it consists of $q$ pairs of points with the property that no two adjacent points of the sequence are the same. The $q$ pairs of points donot have to be distinct. Hence we may write,

$$q_{m,\varepsilon,1,1}(0) = -2\sum_{i=1}^{q} a_i \ , \tag{5.13}$$

where $a_i$ corresponds to the term on the RHS of (5.10) where $i$ of the $q$ pairs are distinct. For a graph $G \in \mathcal{F}_q$ let $K'_\varepsilon(G,e)$ be defined in the same way as $K_\varepsilon(G,e)$ except that the summation on the RHS of (5.11) is over all $y_v \in \mathbf{Z}^d : v \in V[G]$ such that the $q$ points $y_v \in \mathbf{Z}^d$ are distinct. Define $\mathcal{F}'_q$ as the subset of $\mathcal{F}_q$ consisting of regular graphs of degree 4 i.e. the degree of each vertex is exactly 4. Now by (B) of the definition of $\mathcal{F}_q$ and Euler's theorem [1] a graph $G \in \mathcal{F}_q$ has an Eulerian path. Let $c(G)$ be the number of Eulerian paths. For a graph $G \in \mathcal{F}_q$, one can group directed edges into equivalence classes where two directed edges $e, e'$ are equivalent if there is a graph isomorphism of $G$ taking $e$ to $e'$. Let $E'[G]$ be this set of equivalence classes. Then we may define $K'_\varepsilon(G,\hat{e})$ for $\hat{e} \in E'[G]$ by $K'_\varepsilon(G,\hat{e}) = K'_\varepsilon(G,e)$ for any $e \in \hat{e}$. Then one can see that $a_q$ of (5.13) is given by,

$$a_q = \sum_{G\in\mathcal{F}'_q,\ \hat{e}\in E'[G]} c(G) K'_\varepsilon(G,\hat{e}) \ .$$

Since we can obtain a similar representation for the $a_i,\ 1 \le i < q$, we have the formula (5.12) up to replacing $K'_\varepsilon(G,e)$ by $K_\varepsilon(G,e)$. The result therefore holds if we can show that for any graph $G \in \mathcal{F}_q$ and $e \in E[G]$ there exist integers $c_{G,e}(G',e'),\ G' \in \mathcal{F}_q,\ e' \in E[G']$ such that

$$K'_\varepsilon(G,e) = \sum_{G'\in\mathcal{F}_q,\ e'\in E[G']} c_{G,e}(G',e') K_\varepsilon(G',e') \ . \tag{5.14}$$

The previous identity follows from Mayer's trick to obtain an expansion for an non-ideal gas [7]. That is for a function $\varphi : \mathbf{Z}^d \to \mathbf{R}$, one writes

$$\exp\left[-\sum_{1\le i<j\le N} \varphi(y_i - y_j)\right] = \prod_{1\le i<j\le N} [1 + f_{i,j}] \ , \tag{5.15}$$

where $f_{i,j} = \exp[-\varphi(y_i - y_j)] - 1$, and expands the RHS of (5.15) out. Now (5.14) follows from (5.15) on taking $\varphi$ to be the function, $\varphi(y) = 0,\ y \ne 0,\ \varphi(0) = +\infty$. □

**Lemma 5.5.** *Suppose $d = 1,\ q \ge 2$ and $G \in \mathcal{F}_q,\ e \in G$. If $|G|$ is the number of vertices of $G$, then the limit,*

$$\lim_{\varepsilon\to 0} \varepsilon^{|G|} K_\varepsilon(G,e) = K(G,e) \tag{5.16}$$

*exists.*

*Proof.* Evidently it is sufficient to show that (5.16) holds with $K_\varepsilon(G,e)$ replaced by $K'_\varepsilon(G,e)$. Suppose $|G| = k \le q$ and label the vertices of $G$ as $1, 2, \cdots, k$ with $e_+ = 1,\ e_- = k$. Then

$$K'_\varepsilon(G,e) = \sum_{\pi\in\mathcal{S}_k} K_{\varepsilon,\pi}(G,e) \ ,$$

where $\mathcal{S}_k$ is the group of permutations on $1, 2, \cdots, k$ and

$$K_{\varepsilon,\pi}(G,e) = \sum_{y_{\pi 1}<y_{\pi 2}<\cdots<y_{\pi k}} \delta(y_1) y_k \prod_{e'\in E[G],\ e'\ne e} K_\varepsilon(y_{e'_+} - y_{e'_-}) \ .$$

In view of (5.8) there are positive integers $N_j,\ 1 \le j \le k-1$ such that

$$K_{\varepsilon,\pi}(G,e) = \pm \sum_{y_{\pi 1}<y_{\pi 2}<\cdots<y_{\pi k}} \delta(y_1) y_k \prod_{j=1}^{k-1} 2^{N_j} \nu(\varepsilon)^{N_j[y_{\pi(j+1)} - y_{\pi(j)}]} \ .$$

Suppose now that $\pi k_1 = 1,\ \pi k_2 = k$ and $k_1 < k_2$. Then

$$K_{\varepsilon,\pi}(G,e) = \sum_{i=k_1}^{k_2-1} K_{\varepsilon,\pi,i}(G,e) \ , \tag{5.17}$$

where

$$K_{\varepsilon,\pi,i}(G,e) = \pm \sum_{y_{\pi 1}<y_{\pi 2}<\cdots<y_{\pi k}} \delta(y_1)[y_{\pi(i+1)} - y_{\pi(i)}] \prod_{j=1}^{k-1} 2^{N_j}\nu(\varepsilon)^{N_j[y_{\pi(j+1)}-y_{\pi(j)}]} \ .$$

It is easy to see now just as in Lemma 5.1 that the limit

$$\lim_{\varepsilon\to 0} \varepsilon^k K_{\varepsilon,\pi,i}(G,e)$$

exists. Hence from (5.17) the limit,

$$\lim_{\varepsilon\to 0} \varepsilon^k K_{\varepsilon,\pi}(G,e) \tag{5.18}$$

exists if $k_1 < k_2$. A similar argument shows that the limit (5.18) also exists if $k_1 > k_2$. It follows then from the previous identities that the limit (5.16) exists. $\square$

**Remark 1.** *Lemma 5.5 proves that (d) of Theorem 1.4 holds with $\xi = 0$. Observe that the number of graphs in $\mathcal{F}_q$ is of order $q!$. One can see this from an asymptotic formula for the number $A_q$ of labeled 4-regular simple graphs on $q$ vertices* [16],

$$A_q \sim [(4q)!e^{-15/4}]/[(2q)!96^q] \ .$$

Next we wish to obtain the analogue of Lemma 5.4 for $q_{m,\varepsilon,k,k'}(\zeta)$ with $\zeta \in [-\pi,\pi]^d$ and at least one of $k, k'$ being 1. To do this we define for any $G \in \mathcal{F}_q$, $e \in E[G]$, a function $K_\varepsilon(G,e,\zeta)$ by

$$K_\varepsilon(G,e,\zeta) = \sum_{\{y_v\in\mathbf{Z}^d:v\in V[G]\}} \exp[-iy_{e_-}\cdot\zeta]\delta(y_{e_+}) \prod_{e'\in E[G],\ e'\neq e} K_\varepsilon(y_{e'_+} - y_{e'_-}) \ . \tag{5.19}$$

Note that $K_\varepsilon(G,e,\zeta)$ is defined similarly to $K_\varepsilon(G,e)$ of (5.11) and has the property $K_\varepsilon(G,e,0) = 0$. For $1 \le k \le d$ we define $K_{\varepsilon,k}(G,e,\zeta)$ by

$$\begin{aligned} K_{\varepsilon,k}(G,e,\zeta) = \sum_{\{y_v\in\mathbf{Z}^d:v\in V[G]\}} \delta(y_{e_+}) \left[\prod_{e'\in E[G],\ e'\neq e} K_\varepsilon(y_{e'_+} - y_{e'_-})\right] \\ \left[\sum_{e'\in E[G],\ e'\neq e} \exp[i(y_{e'_+} - y_{e_-})\cdot\zeta]\ \mathrm{sgn}[(y_{e'_+} - y_{e'_-})\cdot\mathbf{e}_k] \sum_{y\in L_k(y_{e'_+}-y_{e'_-})} e^{-iy\cdot\zeta}\right] \ . \end{aligned} \tag{5.20}$$

Note that $K_\varepsilon(G,e)$ of (5.11) is given by $K_\varepsilon(G,e) = K_{\varepsilon,1}(G,e,0)$. In the following lemma we need to define $\mathcal{F}_q$ for $q = 0, 1$. The set $\mathcal{F}_1$ is the single point vertex graph $G$ with $K_\varepsilon(G,e,\zeta) = 0$. The set $\mathcal{F}_0$ is also the single point vertex graph $G$ with $K_\varepsilon(G,e,\zeta) = 1$.

**Lemma 5.6.** *Let $m \ge 4$ be an even integer and $q = m/2$. Define $A_{m,\varepsilon,k}(\zeta)$ by*

$$\begin{aligned} A_{m,\varepsilon,k}(\zeta) = \sum_{q'=2}^{q} \sum_{G\in\mathcal{F}_{q'},\ G'\in\mathcal{F}_{q-q'}} \\ \sum_{e\in E[G],\ e'\in E[G']} c(G,e,G',e')K_{\varepsilon,k}(G,e,\zeta)K_\varepsilon(G',e',\zeta) \ , \end{aligned} \tag{5.21}$$

*for suitable integers $c(G,e,G',e')$ depending only on the graphs $G,G'$ and the directed edges $e \in E[G]$, $e' \in E[G']$. Then,*

$$\begin{array}{rcll} q_{m,\varepsilon,1,1}(\zeta) & = & [1+e^{-i\mathbf{e}_1\cdot\zeta}]\ A_{m,\varepsilon,1}(\zeta)\ , & \\ q_{m,\varepsilon,1,k}(\zeta) & = & A_{m,\varepsilon,k}(\zeta)\ , & 1<k\le d, \\ q_{m,\varepsilon,k,1}(\zeta) & = & e^{-i\mathbf{e}_k\cdot\zeta}A_{m,\varepsilon,k}(\zeta)\ , & 1<k\le d. \end{array}$$

*Proof.* We define $A_{m,\varepsilon,k'}(\zeta)$ as the coefficient of $\gamma^m$ in the first term on the RHS of (**??**). Evidently the formulas for $q_{m,\varepsilon,k,k'}(\zeta)$ in the statement of the lemma follow from this. We need then to establish (5.21). This follows by the same argument as in Lemma 5.4. Note now that, unlike in Lemmas 5.3 and 5.4, the projection operators $P$ in (**??**) make a contribution. Thus we obtain the factorisations $K_{\varepsilon,k}(G,e,\zeta)K_\varepsilon(G',e',\zeta)$ in (5.21) with $G \in \mathcal{F}_{q'}$, $G' \in \mathcal{F}_{q''}$, where $q'+q''=q$. □

**Lemma 5.7.** *Suppose $d=1$ and $\mathcal{K} \subset \mathbf{R}$ is a compact set. Let $q \ge 2$ and $G \in \mathcal{F}_q$, $e \in G$. If $|G|$ is the number of vertices of $G$, then the limits,*

$$\lim_{\varepsilon\to 0} \varepsilon^{|G|-1} K_\varepsilon(G,e,\varepsilon\xi) \quad = \quad K(G,e,\xi)\ , \tag{5.22}$$

$$\lim_{\varepsilon\to 0} \varepsilon^{|G|} K_{\varepsilon,1}(G,e,\varepsilon\xi) \quad = \quad K_1(G,e,\xi)\ , \tag{5.23}$$

*exist, uniformly for $\xi \in \mathcal{K}$.*

*Proof.* The proof of (5.22) follows exactly the proof of Lemma 5.5. To prove (5.23) we proceed similarly. Thus we focus on one term in the summation over $e' \in E[G]$, $e' \ne e$ in (5.20). As in Lemma 5.5 we fix a permutation $\pi$ and consider the sum over $y_{\pi 1} < y_{\pi 2} < \cdots < y_{\pi k}$. Suppose that $\pi k_1 = e'_-$, $\pi k_2 = e'_+$ and $k_1 < k_2$. Then

$$\operatorname{sgn}(y_{e'_+} - y_{e'_-}) \sum_{y\in L_1(y_{e'_+}-y_{e'_-})} e^{-iy\zeta} =$$

$$\sum_{i=k_1}^{k_2-1} \left( \exp[-i(y_{\pi(i)} - y_{\pi(k_1)})\zeta] \sum_{z=0}^{y_{\pi(i+1)}-y_{\pi(i)}-1} e^{-iz\zeta} \right) .$$

If we consider a fixed $i$ on the RHS of the previous equation and sum over $y_{\pi 1} < y_{\pi 2} < \cdots < y_{\pi k}$, then we see as in Lemma 5.5 that the contribution of this summation to (5.23) converges uniformly for $\xi \in \mathcal{K}$ as $\varepsilon \to 0$. Since we can similarly argue for all the other contributions we conclude that the uniform limit (5.23) exists. □

It is clear that (d) of Theorem 1.4 follows from the previous two Lemmas.

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University of Michigan, Department of Mathematics, Ann Arbor, MI 48109-1109
*E-mail address*: `conlon@math.lsa.umich.edu`