On the spectrum of Farey and Gauss maps

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Abstract

In this paper we introduce spaces of holomorphic functions given by generalized Borel and Laplace transforms which are left invariant by the transfer operators of the Farey map and its induced transformation, the Gauss map, respectively. By means of a suitable operator-valued power series we are able to study simultaneously the spectrum of both these operators along with the analytic properties of associated dynamical zeta functions. This construction establishes an explicit connection between previously unrelated results of Mayer and Rugh (see [Ma1] and [Rug]).

1 Preliminaires

We shall first consider the *Farey map* of the interval $[0,1]$ into itself defined as

$$F(x) = \begin{cases} F_0(x), & \text{if } \ 0 \le x \le 1/2 \ , \\ F_1(x), & \text{if } \ 1/2 < x \le 1 \ , \end{cases} \tag{1.1}$$

where

$$F_0(x) := \frac{x}{1-x} \quad \text{and} \quad F_1(x) := F_0(1-x) = \frac{1}{F_0(x)} = \frac{1-x}{x} \cdot \tag{1.2}$$

The inverse branches are

$$\begin{aligned} \Psi_0(x) &\equiv F_0^{-1}(x) = \frac{x}{1+x} = \frac{1}{2} - \frac{1}{2}\left(\frac{1-x}{1+x}\right), \\ \Psi_1(x) &\equiv F_1^{-1}(x) = \frac{1}{1+x} = \frac{1}{2} + \frac{1}{2}\left(\frac{1-x}{1+x}\right). \end{aligned} \tag{1.3}$$

For $x \neq 0$ the map $\Psi_0(x)$ is conjugated to the right translation $x \to S(x) = x+1$, i.e.

$$\Psi_0 = J \circ S \circ J \quad \text{with} \quad J(x) = J^{-1}(x) = 1/x. \tag{1.4}$$

This yields for the n-iterate

$$\Psi_0^n(x) = J \circ S^n \circ J(x) = \frac{x}{1+nx} \cdot \tag{1.5}$$

Moreover $\Psi_1(x)$ satisfies

$$\Psi_1(x) = J \circ S(x). \tag{1.6}$$

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1.1 The induced map

Let $\mathcal{A} = \{A_n\}_{n\geq 1}$ be the countable partition of $[0,1]$ given by $A_n = [1/(n+1), 1/n]$. Setting $A_0 = [0,1]$ it is easy to check that $F(A_n) = A_{n-1}$ for all $n \geq 1$. Let X be the residual set of points in $[0,1]$ which are not preimages of 1 with respect to the map F, namely $X = (0,1] \setminus \{1/n\}_{n\geq 1}$. The *first passage time* $\tau : X \to I\!N$ in the interval A_1 is defined as

$$\tau(x) = 1 + \min\{n \geq 0 \ : \ F^n(x) \in A_1 \ \} = \left[\frac{1}{x}\right], \tag{1.7}$$

where $[a]$ is the integer part of a. We see that A_n is the closure of the set $\{x \in X : \tau(x) = n\}$. On the other hand, the *return time* function $r : A_1 \to I\!N \cup \{\infty\}$ in the interval A_1 is given by

$$r(x) = \min\{n \geq 1 \ : \ F^n(x) \in A_1 \ \} = \tau \circ F(x). \tag{1.8}$$

We now consider the map $G : X \to X$ obtained from F by inducing w.r.t. the first passage time τ, i.e.

$$G(x) = F^{\tau(x)}(x), \tag{1.9}$$

which can be extended to all of $[0,1]$ as $G(0) = 1$, $G(1) = 0$,

$$\lim_{x\uparrow 1/n} G(x) = 0, \quad \lim_{x\downarrow 1/n} G(x) = 1, \quad n > 1,$$

and whenever $x \in (1/(n+1), 1/n)$ we have, using (1.5),

$$G(x) \equiv G_n(x) = F^n(x) = F_1 \circ F_0^{n-1}(x) = \frac{1}{x} - n = \frac{1}{x} - \tau(x). \tag{1.10}$$

In other words the induced map is the celebrated *Gauss map*

$$G(x) = \begin{cases} \left\{\frac{1}{x}\right\}, & \text{if } \ x \neq 0 \ , \\ 0, & \text{if } \ x = 0 \ , \end{cases} \tag{1.11}$$

where $\{a\}$ denotes the fractional part of a. It has countably many inverse branches Φ_n given by

$$\Phi_n(x) = G_n^{-1}(x) = \frac{1}{x+n}, \qquad n \geq 1. \tag{1.12}$$

1.2 Invariant measures

It is an easy task to verify that the σ-finite absolutely continuous measure

$$\nu(dx) \equiv e(x)\, dx = \frac{1}{\log 2} \cdot \frac{dx}{x} \tag{1.13}$$

is invariant for the dynamical system $([0,1], F)$. Note that $\nu(A_n) = (\log 2)^{-1} \log(1 + \frac{1}{n})$ and $\nu([0,1]) = \infty$. Let $B_n = \{x \in A_1 \ : \ r(x) = n\}$. Using (1.8) we have $\overline{F_1(B_n)} = A_n$. We now show that $\nu(A_n) = \sum_{k\geq n} \nu(B_k)$. Indeed, for $n = 1$ we have $\sum_{k\geq 1} \nu(B_k) =$

$\nu(A_1) = 1$. Moreover, since ν is F-invariant, $\nu(A_n) = \nu(F^{-1}(A_n)) = \nu(A_{n+1}) + \nu(B_{n+1})$, and the assertion follows by induction. Therefore the expected return time is infinite:

$$\nu_{A_1}(r) = \int_{A_1} r(x)\,\nu(dx) = \sum_{n\geq 1} n\,\nu(B_n) = \sum_{n\geq 1} \nu(A_n) = \nu([0,1]) = \infty, \tag{1.14}$$

where ν_{A_1} is the conditional probability measure defined as $\nu_{A_1}(E) = \nu(E\cap A_1)/\nu(A_1)$. It is known that in this situation there is the coexistence of two different statistics for the dynamical system $(F,[0,1])$: besides ν, the ergodic means $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{F^i(x)}$ converge weakly to the Dirac delta at 0 (see [Me], [HY]).
Let ρ be the probability measure obtained by pushing forward ν with F_1, i.e.

$$\rho(E) = ((F_1)_*\,\nu)(E) = (\nu \circ \Psi_1)(E). \tag{1.15}$$

Reasoning as above one readily verifies that the converse relation is

$$\nu(E) = \sum_{n\geq 0} (\rho \circ \Psi_0^n)(E). \tag{1.16}$$

In particular we have $\nu(A_n) = \sum_{l\geq n} \rho(A_l)$ and $\rho(A_n) = \rho(F_1(B_n)) = \nu(B_n)$, where B_n is as above. We then have

$$\rho(E) = (\nu \circ \Psi_1)(E) = \sum_{n\geq 0} (\rho \circ \Psi_0^n \circ \Psi_1)(E) = \rho(G^{-1}E), \tag{1.17}$$

which says that ρ is G-invariant. Moreover ρ is ergodic with respect to G (see e.g. [Bi]). Setting $h(x) = \rho(dx)/dx$ we get

$$h = |\Psi_1'| \cdot e \circ \Psi_1, \qquad e = \sum_{k=0}^{\infty} (\Psi_0^k)' \cdot h \circ \psi_0^k, \tag{1.18}$$

which gives the well known result

$$h(x) = \frac{1}{\log 2} \cdot \frac{dx}{(1+x)} \cdot \tag{1.19}$$

The primitive $H(x)$ of $h(x)$, with $H(0) = 0$, is $H(x) = \log(1+x)/\log 2$. Setting $q_n := H(\frac{1}{n+1}) = (\log 2)^{-1} \log(1 + \frac{1}{n+1})$, we have $\nu(A_n) = q_n$ and $\rho(A_n) = q_{n-1} - q_n$. We see that q_n is a (strict) Kaluza sequence, i.e. for all $n \geq 1$

$$1 = q_0 > q_1 > \cdots > q_n > 0 \qquad \text{and} \qquad q_n^2 < q_{n-1}\,q_{n+1}. \tag{1.20}$$

Finally, by (1.7), (1.8), (1.14) and (1.15) we have

$$\rho(\tau) = ((F_1)_*\,\nu)(\tau) = \nu(\tau \circ F_1) = \nu(r) = \infty. \tag{1.21}$$

On the other hand we have the following,

Lemma 1.1 *The function* $\log \tau$ *is in* $L_1(\rho)$ *and satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \tau(G^j(x)) = \rho(\log \tau) = K, \qquad \rho - \text{a.e.}, \tag{1.22}$$

where the positive constant K *is defined by*

$$e^K = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)}\right)^{\frac{\log k}{\log 2}}. \tag{1.23}$$

Proof. We have

$$\begin{aligned} \rho(\log \tau) &= \sum_{k=1}^{\infty} \rho(A_k) \cdot \log k = \sum_{k=1}^{\infty} (q_{k-1} - q_k) \cdot \log k \\ &= \sum_{k=1}^{\infty} \frac{\log k}{\log 2} \cdot \log \left(\left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{k+1}\right)^{-1} \right) \\ &= \sum_{k=1}^{\infty} \frac{\log k}{\log 2} \cdot \log \left(1 + \frac{1}{k(k+2)}\right) = K < \infty. \end{aligned}$$

This computation shows both that $\log \tau \in L_1(\rho)$ and the last equality in (1.22). The first equality in (1.22) now follows from the ergodic theorem [Bi]. $\diamondsuit$

The constant K which appears above is known in number theory as *Khinchin's constant*. This is not a coincidence, as we now briefly explain.

1.3 Connection with number theory

The Farey sum over two rationals $\frac{a}{b}$ and $\frac{a'}{b'}$ is the mediant operation given by [HR]

$$\frac{a''}{b''} = \frac{a+a'}{b+b'}. \tag{1.24}$$

It is easy to see that $\frac{a''}{b''}$ falls in the interval $(\frac{a}{b}, \frac{a'}{b'})$. Now, having fixed $n \geq 0$, let $\mathcal{F}_n$ be the ascending sequence of irreducible fractions between 0 and 1 obtained inductively in the following way. Set first $\mathcal{F}_0 = (\frac{0}{1}, \frac{1}{1})$. Then $\mathcal{F}_n$ is obtained from $\mathcal{F}_{n-1}$ by inserting among each pair of consecutive rationals $\frac{a}{b}$ and $\frac{a'}{b'}$ in $\mathcal{F}_{n-1}$ their mediant $\frac{a''}{b''}$ as above. Thus $\mathcal{F}_1 = (\frac{0}{1}, \frac{1}{2}, \frac{1}{1})$, $\mathcal{F}_2 = (\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1})$, $\mathcal{F}_3 = (\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1})$ and so on. The elements of $\mathcal{F}_n$ are called Farey fractions. The name of the map F can be related to the easily verified observation that the set of pre-images $\cup_{k=0}^{n+1} F^{-k}\{0\}$ coincides with $\mathcal{F}_n$ for all $n \geq 0$.

In particular, this implies that $\cup_{k=0}^{\infty} F^{-k}\{0\} = \mathbb{Q} \cap [0,1]$ (notice that the same is true for the induced map: $\cup_{k=0}^{\infty} G^{-k}\{0\} = \mathbb{Q} \cap [0,1]$).
On the other hand, we recall that every real number $0 < x < 1$ has a continued fraction expansion of the form [Ki]

$$x = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \cdots}}} = [k_1, k_2, k_3, \ldots], \tag{1.25}$$

with $k_i \in I\!N$. By applying Euclid's algorithm one sees that the above expansion terminates if and only if x is a rational number. There is an intimate connection between the partial quotients $k_1, k_2, \cdots$ and the Gauss map G. Indeed, given x as above we can write

$$\begin{aligned} x = \frac{1}{\frac{1}{x}} &= \frac{1}{\left[\frac{1}{x}\right] + \left\{\frac{1}{x}\right\}} = \frac{1}{k_1 + G(x)} = \frac{1}{k_1 + \frac{1}{\frac{1}{G(x)}}} \\ &= \frac{1}{k_1 + \frac{1}{\left[\frac{1}{G(x)}\right] + \left\{\frac{1}{G(x)}\right\}}} = \frac{1}{k_1 + \frac{1}{k_2 + G^2(x)}} = \cdots \end{aligned} \tag{1.26}$$

Therefore, $k_1 = [1/x]$, $k_2 = [1/G(x)]$, $k_3 = [1/G^2(x)]$ and so on. Alternatively,

$$\text{if} \quad x = [k_1, k_2, k_3, \ldots] \quad \text{then} \quad G(x) = [k_2, k_3, \ldots]\,. \tag{1.27}$$

Farey fractions have close relationships with continued fractions. Let us say that a Farey fraction has order n if it belongs to $\mathcal{F}_n \setminus \mathcal{F}_{n-1}$. Given $n \geq 1$ there are exactly 2^{n-1} Farey fractions of order n (they form the set $F^{-(n+1)}\{0\}$) and it is possible to show (see below eq. (1.28)) that the integers k_i in their (finite) continued fraction expansion sum up to $n+1$. Furthermore, it is easy to realize that all Farey fractions which fall in the interval $(1/(n+1), 1/n)$ have order greater than or equal to $n+1$, whereas their continued fraction expansion starts with $k_1 = n$. Thus, the map F acts on Farey fractions by reducing their order of one unit. We can write an explicit expression for the action of F on continued fraction expansions. Indeed, if $1/2 < x \leq 1$ then $k_1 = 1$ and $F(x) = \frac{1}{x} - k_1 = G(x)$. If instead $0 < x \leq 1/2$ then $k_1 > 1$ and $F(x) = 1/(\frac{1}{x} - 1)$. Therefore,

$$\text{if} \quad x = [k_1, k_2, k_3, \ldots] \quad \text{then} \quad F(x) = [k_1 - 1, k_2, k_3, \ldots]\,, \tag{1.28}$$

with $[0, k_2, k_3, \ldots] \equiv [k_2, k_3, \ldots]$ (compare to (1.27)). Now, it is well known that for almost all $x \in (0,1)$ the arithmetic mean of the partial quotients is infinite (see, e.g., [Ki]), i.e.

$$\lim_{n\to\infty} \frac{k_1 + \cdots + k_n}{n} = \infty, \qquad \text{(a.e.)} \tag{1.29}$$

¿From the above discussion and (1.7) we get $k_l = [1/G^{l-1}(x)] = \tau(G^{l-1}(x))$, which for $l > 1$ is the time between the $(l-1)$-st and the l-th passage in A_1 of the orbit of x with F. Therefore, the total number S_n of iterates of F needed to observe n passages in A_1, that is the function

$$S_n(x) = \tau(x) + \tau(G(x)) \cdots + \tau(G^{n-1}(x)), \tag{1.30}$$

satisfies

$$\lim_{n\to\infty} \frac{S_n(x)}{n} = \infty \qquad \text{(a.e.)} \tag{1.31}$$

Since ρ is absolutely continuous w.r.t. the Lebesgue measure on $[0,1]$, the properties expressed by (1.21) and (1.31) can be regarded as an instance of validity of the ergodic theorem for the non-integrable function τ. On can actually say more. As a consequence of ([Ki], Theorem 30) we have that for almost all $x \in (0,1)$ the inequality

$$S_n(x) \geq n \log n \tag{1.32}$$

is satisfied for an infinite number of values of n. On the other hand, Lemma 1.1 can now be rephrased by saying that the geometric mean of the partial quotients has a certain finite value (a.e.). This, in turn, is a corollary of a theorem of Khinchin ([Ki], Theorem 35), which says that for any function $f(k)$ defined on the positive integers and satisfying $f(k) = \mathcal{O}(k^p)$ with $0 \leq p < 1/2$ we have, for almost all $x \in (0,1)$,

$$\left| \frac{1}{n} \sum_{j=i}^{n} f(k_j) - \sum_{k=1}^{\infty} \frac{f(k)}{\log 2} \cdot \log \left(1 + \frac{1}{k(k+2)} \right) \right| \leq \epsilon(n) \tag{1.33}$$

where the error function $\epsilon(n)$ is any positive function decreasing to zero as $n \to \infty$ so that $\sum n^{-2} \cdot \epsilon^{-2}(n) < \infty$. Lemma 1.1 then corresponds to the choice $f(k) = \log k$.

2 Transfer operators

We start by establishing some formal algebraic relations between the the transfer operators $\mathcal{P}$ and $\mathcal{M}$ associated to the map F and G, respectively (see [Ba1]). They are given by

$$\begin{aligned} \mathcal{P}f(x) &= (\mathcal{P}_0 + \mathcal{P}_1)f(x) =: |\Psi_0'(x)| \cdot f(\Psi_0(x)) + |\Psi_1'(x)| \cdot f(\Psi_1(x)) \\ &= \left(\frac{1}{x+1}\right)^2 \left[f\left(\frac{x}{x+1}\right) + f\left(\frac{1}{x+1}\right) \right], \end{aligned} \tag{2.34}$$

and

$$\mathcal{M}f(x) = \sum_{n=1}^{\infty} \mathcal{M}_n\, u(x) =: \sum_{n=1}^{\infty} |\Phi_n'(x)| \cdot f(\Phi_n(x)) = \sum_{n=1}^{\infty} \left(\frac{1}{x+n}\right)^2 f\left(\frac{1}{x+n}\right). \tag{2.35}$$

We first notice that

$$\mathcal{M}_n f(x) = \mathcal{P}^n(f \cdot \chi_n)(x) = \mathcal{P}_1 \mathcal{P}_0^{n-1} f(x), \tag{2.36}$$

where χ_n is the indicator function of A_n. Let $\mathcal{S}f(x) := f \circ S(x) = f(x+1)$ be the shift operator. Let moreover $\mathcal{P}_1^{-1}$ be the operator inverse to $\mathcal{P}_1$, i.e.

$$\mathcal{P}_1^{-1} f(x) = |F_1'(x)|\, f(F_1(x)) = \frac{1}{x^2} f\left(\frac{1-x}{x}\right), \quad x \neq 0. \tag{2.37}$$

¿From (1.4) and (1.6) one obtains

$$\mathcal{P}_0 f(x) = \mathcal{P}_1^{-1}\, \mathcal{S}\, \mathcal{P}_1\, f(x), \tag{2.38}$$

and therefore (2.36) yields

$$\mathcal{M}_n f(x) = \mathcal{P}_1 \mathcal{P}_0^{n-1} f(x) = \mathcal{S}^{n-1} \mathcal{P}_1 f(x). \tag{2.39}$$

More generally, for $z \in \mathbb{C}$, we can shall consider a formal operator-valued power series $\mathcal{M}_z$ defined by

$$\mathcal{M}_z f(x) = \sum_{n=1}^{\infty} z^{\tau(\Phi_n(x))} \cdot |\Phi_n'(x)| \cdot f(\Phi_n(x)) = \sum_{n=1}^{\infty} z^n \mathcal{M}_n f(x) \tag{2.40}$$

so that $\mathcal{M}_1 \equiv \mathcal{M}$. The following operator relations are in force and are independent of the function space the operators are acting on (see also [PS], [HI] and [Is2] for related constructions).

Proposition 2.1 *Let $z \in \mathbb{C}$ be such that (2.40) is absolutely convergent. Then we have*

$$(1 - \mathcal{M}_z)(1 - z\mathcal{P}_0) = 1 - z\mathcal{P} \tag{2.41}$$

and

$$(1 - z\,\mathcal{S})(1 - \mathcal{M}_z) = 1 - z\,\tilde{\mathcal{P}}\,. \tag{2.42}$$

where $\tilde{\mathcal{P}} = \mathcal{S} + \mathcal{P}_1$.

Remark 1 *¿From (2.38) we also have the relation*

$$(1 - z\mathcal{P}_0)^{-1} = \mathcal{P}_1^{-1}\,(1 - z\,\mathcal{S})^{-1}\,\mathcal{P}_1\,, \tag{2.43}$$

valid in the same domain of z-values of Proposition 2.1.

Proof of Proposition 2.1. Using the first identity in (2.39) we get

$$(1 - \mathcal{M}_z)(1 - z\mathcal{P}_0) = (1 - \sum_{n=1}^{\infty} z^n \mathcal{P}_1 \mathcal{P}_0^{n-1})(1 - z\mathcal{P}_0) =$$

$$1 - z\mathcal{P}_0 - \sum_{n=1}^{\infty} z^n \mathcal{P}_1 \mathcal{P}_0^{n-1} + \sum_{n=1}^{\infty} z^{n+1} \mathcal{P}_1 \mathcal{P}_0^{n}$$

=1-zP$_0$ $- z\mathcal{P}_1 = 1 - z\mathcal{P}$.In a similar way, using the second identity in (2.39) one shows (2.42). $\diamondsuit$

Corollary 2.1 *Let $z \neq 0$ be such that (2.40) is absolutely convergent. Then* 1 *is an eigenvalue of $\mathcal{M}_z$ if and only if z^{-1} is an eigenvalue both of $\mathcal{P}$ and $\tilde{\mathcal{P}}$, and they have the same geometric multiplicity. Furthermore, the corresponding eigenfunctions e_z of $\mathcal{P}$ and h_z of $\tilde{\mathcal{P}}$ and $\mathcal{M}_z$ are related by $h_z = (1 - z\mathcal{P}_0)e_z$ or else $e_z = \sum_{k=0}^{\infty} z^k \mathcal{P}_0^k h_z$.*

Proof. Assume that $\mathcal{M}_z h_z = h_z$. From (2.41) it then follows that $(1 - z\mathcal{P}) \sum_{k=0}^{\infty} z^k \mathcal{P}_0^k h_z = 0$. Conversely, assume that $z\mathcal{P}e_z = e_z$, then we have $(1 - \mathcal{M}_z)(1 - z\mathcal{P}_0)e_z = 0$. In the same way, from (2.42) it follows that $\mathcal{M}_z h_z = h_z$ if and only if $\tilde{\mathcal{P}} h_z = z^{-1} h_z$. $\diamondsuit$

Remark 2 *Since the spectral radius of $\mathcal{P}$ in a space as large as $L_1([0,1],dx)$ is equal to one, there are no z-values with $|z|<1$ such that 1 is an eigenvalue of $\mathcal{M}_z$.*

Remark 3 *Setting $z=1$ in Proposition 2.40 we recover (7) with $e\equiv e_1$ and $h\equiv h_1$. In particular we see that the Gauss probability density (1.19) is a fixed point both of $\mathcal{M}$ and $\tilde{\mathcal{P}}$.*

Having fixed an open connected domain $\Omega \subset \mathbb{C}$ let $\mathcal{H}(\Omega)$ be the Fréchet space of functions which are holomorphic in Ω with the topology generated by the family of sup norms on compact subsets of Ω. Moreover, we let $A_\infty(\Omega)\subset\mathcal{H}(\Omega)$ denote the Banach space given by the subset of functions in $\mathcal{H}(\Omega)$ having continuous extension to $\overline{\Omega}$, endowed with the norm

$$\|f\| = \sup_{w\in\overline{\Omega}} |f(w)|, \tag{2.44}$$

(where $w=x+iy$). In the same way as [Ma1] we let first $\mathcal{M}_z$ act on the Banach space $A_\infty(D)$ with $D=\{w\in\mathbb{C} \,:\, |w-1|<3/2\}$. It is easy to verify that $\Phi_n(\overline{D})\subset D$ for all $n\in\mathbb{N}$. Standard arguments (see [Ma2]) then imply that whenever the power series in (2.40) is uniformly convergent $\mathcal{M}_z$ defines a nuclear operator of order zero on $A_\infty(D)$.

Lemma 2.2 *The power series of $\mathcal{M}_z : A_\infty(D)\to A_\infty(D)$ has radius of convergence bounded from below by 1 and, moreover, it converges absolutely at every point of the unit circle.*

Proof. The radius of convergence of $\mathcal{M}_z$ is $\lim_{n\to\infty}\|\mathcal{M}_n\|^{-1/n}$ (here $\|\;\|$ denotes the operator norm as well). We have $\sup_{w\in\overline{D}}|\mathcal{M}_n f(w)|\le C\,n^{-2}\,\|f\|$,

and therefore, $\|\mathcal{M}_n\|\le C\,n^{-2}$. $\diamondsuit$

We now set

$$H_\delta := \{w\in\mathbb{C} : \operatorname{Re} w>\delta\} \tag{2.45}$$

and notice that from the explicit form of the operator $\mathcal{M}_z$ in (2.40) and (2.39) it follows that, for each $z\neq 0$ s.t. $|z|\le 1$, any eigenfunction of $\mathcal{M}_z$ when acting on the space $A_\infty(D)$ must be holomorphic and bounded in every half plane $H_{-1+\epsilon}$ for any $\epsilon>0$. This can be achieved via a generalized Laplace transform as in the following Proposition which generalizes corresponding results obtained by Mayer and Roepstorff (see [MaR1], [MaR2]) for the operator $\mathcal{M}$.

Proposition 2.2 *Let $\mathcal{H}_1\subset A_\infty(D)$ be the space of all complex-valued functions f which are bounded and holomorphic in each of the half planes $H_{-1+\epsilon}$ with $\epsilon>0$, which have a representation as generalized Laplace transform,*

$$f(w) = (\mathcal{L}\,[\varphi])(w) := \int_0^\infty dm(t)\, e^{-tw}\,\varphi(t) \tag{2.46}$$

where dm is the measure on $\mathbb{R}_+$ given by

$$dm(t) = \frac{t}{e^t-1}\,dt\,, \tag{2.47}$$

and $\varphi \in L_2(m)$. Then, for each $z \neq 0$ with $|z| \leq 1$, the space $\mathcal{H}_1$ is invariant under $\mathcal{M}_z$. More precisely we have

$$\mathcal{M}_z \, \mathcal{L}\,[\varphi\,] = \mathcal{L}\,[\,\mathcal{K}_z \varphi\,], \tag{2.48}$$

where $\mathcal{K}_z : L_2(m) \to L_2(m)$ is defined by

$$(\mathcal{K}_z\varphi)(s) := \; \gamma(s,z) \cdot \int_0^\infty dm(t) \, \frac{J_1(2\sqrt{st})}{\sqrt{st}} \varphi(t) \tag{2.49}$$

with

$$\gamma(s,z) := \left(1 + \frac{z-1}{1 - z\,e^{-s}}\right), \tag{2.50}$$

and J_p denotes the Bessel function of order p.

Remark 4 *Setting $z = 1$ we see that the G-invariant density h can be represented as $h = (\log 2)^{-1}\, \mathcal{L}[\,(1 - e^{-t})/t\,]$.*

Proof of Proposition 2.2. Letting $f = \mathcal{L}\,[\varphi\,]$ we have from (2.40) and (2.39)

$$\mathcal{M}_z f(w) = \sum_{n=1}^{\infty} \frac{z^n}{(w+n)^2} \int_0^\infty dm(t)\, e^{-t/(w+n)}\, \varphi(t)\,. \tag{2.51}$$

Clearly, for $|z| \leq 1$, the sum $\sum_{n=1}^{\infty} \frac{z^n}{(w+n)^2} e^{-t/(w+n)}$ is uniformly convergent in $t \in I\!\!R_+$. Therefore, interchanging summation and integration we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{z^n}{(w+n)^2} e^{-t/(w+n)} &= \sum_{k \geq 0} \frac{(-t)^k}{k!} \sum_{n=1}^{\infty} \frac{z^n}{(w+n)^{2+k}} \\ &= \sum_{k \geq 0} \frac{(-t)^k}{k!}\, z\, \Phi(z, k+2, w+1) \end{aligned} \tag{2.52}$$

where $\Phi(z,a,b) = \sum_{n=0}^{\infty} \frac{z^n}{(b+n)^a}$ is the Lerch trascendental function which, for $\mathrm{Re}\, a > 1$, possesses the integral representation

$$\Phi(z,a,b) = \sum_{n=0}^{\infty} \frac{z^n}{(b+n)^a} = \frac{1}{\Gamma(a)} \int_0^\infty \frac{s^{a-1} e^{-(b-1)s}}{e^s - z}\, ds\,. \tag{2.53}$$

This yields

$$\Phi(z, k+2, w+1) = \frac{1}{(k+1)!} \int_0^\infty \frac{s^{k+1} e^{-ws}}{e^s - z}\, ds \, \cdot \tag{2.54}$$

Noting that

$$\sum_{k \geq 0} \frac{(-st)^k}{(k+1)!\, k!} = \frac{J_1(2\sqrt{st})}{\sqrt{st}} \tag{2.55}$$

where $J_1(x)$ is the Bessel function of the first kind, we have thus found that

$$\begin{aligned}\mathcal{M}_z f(w) &= \int_0^\infty ds\, \frac{zs}{(e^s - z)}\, e^{-ws} \int_0^\infty dm(t)\, \frac{J_1(2\sqrt{st})}{\sqrt{st}} \varphi(t) \\ &= \int_0^\infty dm(s)\, e^{-ws}\, (\mathcal{K}_z\varphi)(s) = (\mathcal{L}\,[\mathcal{K}_z\varphi])(w). \end{aligned} \tag{2.56}$$

It is now an easy task to verify that for $|z| \le 1$ and $\varphi \in L_2(m)$ the function $\mathcal{K}_z\varphi$ is in $L_2(m)$ as well. $\diamondsuit$

Remark 5 *As already remarked in* [MaR1], *for $z = 1$ the integral operator $\mathcal{K} \equiv \mathcal{K}_1$ is symmetric, that is $(\mathcal{K}\varphi, \phi) = (\varphi, \mathcal{K}\phi)$ where $(\cdot\,, \cdot)$ denotes the scalar product in $L_2(m)$. Therefore, by the above Proposition,* $\mathrm{sp}(\mathcal{M}) \subset \mathbb{R}$.

But we can say more. First notice that for each $t \in \mathbb{R}_+$ the function $J_1(2\sqrt{st})/\sqrt{st}$ is uniformly bounded and continuous for $s \in \mathbb{R}_+$. Moreover from (2.50) we see that for each fixed $z \in \mathbb{C} \setminus (1, +\infty)$ the function $\gamma(s, z)$ is (continuous and) bounded uniformly for $s \in \mathbb{R}_+$. Therefore, for any $\varphi \in L_2(m)$ the expression (2.49)-(2.50) converges uniformly in any closed region of the complex z-plane not containing points of the ray $(1, +\infty)$. This yields an analytic continuation of $z \to \mathcal{K}_z$ to the entire z-plane with a cross cut along the ray $(1, +\infty)$. Putting together these observations we have thus obtained the following result,

Theorem 2.1 *The operator-valued function $z \to \mathcal{M}_z$, when acting on $\mathcal{H}_1$, can be analytically continued to the entire z-plane with a cross cut along the ray $(1, +\infty)$, and for each z in this domain it is isomorphic to the operator $\mathcal{K}_z : L_2(m) \to L_2(m)$ defined in (2.49). They are both compact operators.*

¿From this we deduce the

Corollary 2.2 *The operator-valued function $z \to (1 - \mathcal{M}_z)^{-1}$, when acting on $\mathcal{H}_1$, is analytic in the open unit disk $\{z : |z| < 1\}$ and can be meromorphically continued to the entire z-plane with a cross cut along the ray $[1, +\infty)$. It has a pole at $z \in \mathbb{C} \setminus [1, +\infty)$ whenever $\mathcal{K}_z$ has 1 as an eigenvalue. If this is the case then $|z| > 1$.*

Proof. The first statement follows immediately from Proposition 2.1 and Remark 2. For the last statement we argue as follows. Let $\mathcal{V}_z : \mathcal{H}_1 \to \mathcal{H}_1$ be the operator defined by the relation

$$\mathcal{M}_z(f \cdot h) = h \cdot \mathcal{V}_z f\,. \tag{2.57}$$

Note that $\mathcal{V}_1 1 = 1$. Clearly $\mathrm{sp}\,(\mathcal{M})_z = \mathrm{sp}\,(\mathcal{V}_z)$ and all eigenvalues have the same geometric multiplicities. Suppose that for $z = e^{i\phi}$ with $\phi(\mathrm{mod} 2\pi) \neq 0$ there exists $f \in \mathcal{H}_1$ so that $\mathcal{M}_z f = f$. Then $\mathcal{V}_z g = g$ with $g = f/h$. ¿From $\mathcal{V}_1 1 = 1$ and $\mathcal{V}_1(e^{i\phi\tau} g) = g$ we obtain $\mathcal{V}_1(e^{i\phi\tau} g - g \circ G) = 0$. Iterating this argument we would get a function $g \in \mathcal{H}_1$ such that $g(G^n(x)) = g(x) \exp\left(i\,\phi\, \sum_{k=0}^{n-1} \tau(G^k(x))\right) + r(x)$ with $r(x) \in \ker(\mathcal{M})$. This would imply

$$\int_0^1 g(G^n(x))\, dx = \int_0^1 g(x)\, e^{i\,\phi\, \sum_{k=0}^{n-1} \tau(G^k(x))}\, dx \not\to \int_0^1 g(x)\, \rho(dx). \tag{2.58}$$

But this is impossible because the dynamical system $([0,1], G, \rho)$ is mixing. Therefore, any $z \neq 1$ such that $\mathcal{M}_z$ has 1 as an eigenvalue must satisfy $|z| > 1$. $\diamondsuit$

Now, from Proposition 2.1 we obtain the following formal relation for the resolvent $\mathcal{R}_\lambda$ of $\mathcal{P}$:

$$\mathcal{R}_\lambda \equiv (\lambda - \mathcal{P})^{-1} = (\lambda - \mathcal{P}_0)^{-1}(1 - \mathcal{M}_{1/\lambda})^{-1}. \tag{2.59}$$

By virtue of (2.43) the analytic properties of the first factor in the r.h.s. can be understood in terms of the spectrum of the translation operator $\mathcal{S}$ when acting on $\mathcal{H}_1$. To this end we notice that for $f = \mathcal{L}[\varphi]$ we have

$$(\mathcal{S}f)(w) = \int_0^\infty dm(t)\, e^{-t(w+1)}\, \varphi(t) = \int_0^\infty dm(t)\, e^{-tw}\, (M\varphi)(t) = (\mathcal{L}\,[M\varphi])(w), \tag{2.60}$$

where $M : L_2(m) \to L_2(m)$ is the multiplication operator acting as $M\varphi(t) = e^{-t}\varphi(t)$. Thus, under generalized Laplace transform $\mathcal{L}$ the restriction of $\mathcal{S}$ to the space $\mathcal{H}_1$ is conjugated to M and hence they have identical spectral properties. On the other hand it is an easy exercise to check that M is self-adjoint and $\mathrm{sp}\,(M)$ is continuous and coincides with the line segment $[0,1]$ (see, e.g., [DeV]).
Furthermore, from (2.41) and the fact that $\mathcal{H}_1$ is invariant under the action of $\mathcal{M}_z$ for all $z \in \mathbb{C} \setminus (1, +\infty)$ (in particular for $z = 1$), the operator $\mathcal{P}$ leaves invariant the space $(1 - \mathcal{P}_0)^{-1}\mathcal{H}_1$. It is easy to check that for $f \in \mathcal{H}_1$ the function $\mathcal{P}_0^n f$ is holomorphic in each half-plane $H_{-\frac{1}{n}+\epsilon}$ and therefore $(1 - \mathcal{P}_0)^{-1} f$ is holomorphic in each half-plane H_ϵ. More specifically, a calculation along the same lines as in the proof of Proposition 2.2 shows that, for $f \in \mathcal{H}_1$ with $f = \mathcal{L}[\varphi]$,

$$(1 - z\,\mathcal{P}_0)^{-1} f(w) = \frac{1}{w^2} \int_0^\infty dm(t) e^{-t/w}\, e^t\, z^{-1}\, (\mathcal{K}_z \varphi)(t)\,. \tag{2.61}$$

Thus, we shall characterize the space $\mathcal{H}_0$ to be acted on by $\mathcal{P}$ as follows:

Definition 2.1 *We denote by $\mathcal{H}_0$ the set of all complex-valued functions f which are bounded and holomorphic in each of the half planes H_ϵ with $\epsilon > 0$ and can be represented as a generalized Borel transform:*

$$f(w) = (\mathcal{B}\,[\varphi])(w) := \frac{1}{w^2} \int_0^\infty dm(t)\, e^{-t/w}\, e^t\, \varphi(t), \quad \varphi \in L_2(m). \tag{2.62}$$

Remark 6 *Another representation can be obtained by a simple change of variable by which (2.62) can be rewritten as*

$$f(w) = \frac{1}{w} \int_0^\infty ds\, e^{-s}\, \psi(sw) \quad \textit{with} \quad \psi(t) = \left(\frac{t}{1 - e^{-t}}\right) \varphi(t)\,. \tag{2.63}$$

Remark 7 *The F-invariant density e (see (1.13)) can be represented as*

$$e = \left(\frac{1}{\log 2}\right) \mathcal{B} \left[\frac{1 - e^{-t}}{t}\right], \tag{2.64}$$

whereas for the G-invariant density h we have (see also Remark 4)

$$h = \left(\frac{1}{\log 2}\right) \mathcal{L}\left[\frac{1-e^{-t}}{t}\right] = \left(\frac{1}{\log 2}\right) \mathcal{B}\left[\frac{(1-e^{-t})^2}{t}\right]. \tag{2.65}$$

In the representation of Remark 6 we have that if $f = e \cdot \log 2$ then $\psi(t) \equiv 1$ whereas for $f = h \cdot \log 2$ we find $\psi(t) = 1 - e^{-t}$. Both these functions can be viewed as ordinary Borel transforms of a sequence $\{a_n\}_{n=0}^{\infty}$, i.e. $\psi(t) = \sum_{n=0}^{\infty} t^n a_n/n!$ so that by (2.63) we have $w \cdot f(w) = \sum_{n=0}^{\infty} w^n a_n$. In the former case we find $a_0 = 1$ and $a_n = 0$ for $n > 0$, in the latter $a_0 = 0$ and $a_n = (-1)^{n-1}$ for $n > 0$.

We now have the following,

Lemma 2.3 *For all $\varphi \in L_2(m)$*

$$\mathcal{L}[\varphi] = \mathcal{B}[(1-M)\mathcal{K}\varphi] \tag{2.66}$$

where $M\varphi(t) = e^{-t}\varphi(t)$ and $\mathcal{K} \equiv \mathcal{K}_1$ is the symmetric integral operator defined in (2.49).

Proof. The proof is an easy calculation based on the identity [GR]

$$\frac{1}{u^{p+1}} \int_0^\infty dt\, e^{-t/u} \varphi(t) = \int_0^\infty dt\, e^{-tu} \int_0^\infty ds \left(\frac{t}{s}\right)^{\frac{p}{2}} J_p(2\sqrt{st})\, \varphi(s), \tag{2.67}$$

with $p = 1$, and therefore we omit it. $\diamondsuit$

Now, for $f = \mathcal{B}[\varphi]$ we get

$$(\mathcal{P}_0 f)(w) = \frac{1}{w^2} \int_0^\infty dm(t)\, e^{-t/w}\, e^t\, (M\varphi)(t) = (\mathcal{B}[M\varphi])(w) \tag{2.68}$$

and

$$(\mathcal{P}_1 f)(w) = \int_0^\infty dm(t)\, e^{-tw}\, \varphi(t) = (\mathcal{L}[\varphi])(w). \tag{2.69}$$

Therefore,

$$(\mathcal{P}_1 \mathcal{P}_0^{n-1} f)(w) = \int_0^\infty dm(t)\, e^{-tw}\, (M^{n-1}\varphi)(t) = (\mathcal{L}[M^{n-1}\varphi])(w), \tag{2.70}$$

so that

$$\mathcal{M}_z f = z \cdot \mathcal{L}[(1-zM)^{-1}\varphi], \qquad f = \mathcal{B}[\varphi]. \tag{2.71}$$

We thus see that $\mathcal{P}_0$ leaves $\mathcal{H}_0$ invariant and by (2.60) its spectral properties in $\mathcal{H}_0$ are identical to those of $\mathcal{S}$ in $\mathcal{H}_1$. Moreover $\mathcal{P}_1$ maps $\mathcal{H}_0$ into $\mathcal{H}_1 \subset \mathcal{H}_0$, and the same does $\mathcal{M}_z$ for all $z \in \mathbb{C} \setminus (1, +\infty)$. Notice that using Lemma 2.3 and (2.71) we immediately recover Proposition 2.2, in that

$$\mathcal{M}_z \mathcal{L}[\varphi] = \mathcal{M}_z \mathcal{B}[(1-M)\mathcal{K}\varphi] = \mathcal{L}[z \cdot (1-zM)^{-1}(1-M)\mathcal{K}\varphi] \equiv \mathcal{L}[\mathcal{K}_z \varphi]. \tag{2.72}$$

We are now in the position to write explicit representations for $\mathcal{P}$ and its resolvent $\mathcal{R}_\lambda$ in the space $\mathcal{H}_0$.

Theorem 2.2 *Let $f \in \mathcal{H}_0$, that is $f = \mathcal{B}[\varphi]$ for some $\varphi \in L_2(m)$, then*

$$\mathcal{P}f = \mathcal{B}[\,(M + (1-M)\mathcal{K}\,)\varphi\,], \tag{2.73}$$

and

$$\mathcal{R}_\lambda f \equiv (\lambda - \mathcal{P})^{-1} f \;= \mathcal{B}[\,(1-\mathcal{K}_{1/\lambda})^{-1}(\lambda - M)^{-1}\varphi\,]. \tag{2.74}$$

Proof. From (2.68) and (2.69) one obtains $\mathcal{P}f = \mathcal{B}[\,M\varphi\,] + \mathcal{L}[\varphi]$, so that (2.73) follows using Lemma 2.3. The expression for $\mathcal{R}_\lambda$ can now be obtained directly from (2.73). But we can also make use of (2.71) and (2.56) to obtain, for a given $f = \mathcal{B}[\varphi]$,

$$\mathcal{M}^n_{1/\lambda} f = \mathcal{L}[\,\mathcal{K}^{n-1}_{1/\lambda}(\lambda - M)^{-1}\varphi\,] \tag{2.75}$$

and therefore

$$(1 - \mathcal{M}_{1/\lambda})^{-1} f = \mathcal{B}[\varphi] + \mathcal{L}[\,(1-\mathcal{K}_{1/\lambda})^{-1}\,(\lambda - M)^{-1}\varphi\,]. \tag{2.76}$$

This expression along with (2.59), (2.61) and (2.68) yield

$$\begin{aligned}(\lambda - \mathcal{P})^{-1} f &= \mathcal{B}[\,(\lambda - M)^{-1}\,\varphi\,] + \mathcal{B}[\,\mathcal{K}_{1/\lambda}(1-\mathcal{K}_{1/\lambda})^{-1}\,(\lambda - M)^{-1}\,\varphi\,] \\ &= \mathcal{B}[\,(1-\mathcal{K}_{1/\lambda})^{-1}(\lambda - M)^{-1}\varphi\,]. \qquad \diamondsuit\end{aligned}$$

The next theorem describes the spectrum of $\mathcal{P}$ in $\mathcal{H}_0$ (see [Rug] for related results).

Theorem 2.3 *The spectrum of the operator $\mathcal{P} : \mathcal{H}_0 \to \mathcal{H}_0$ is given by* $\mathrm{sp}\,(\mathcal{P}) = [0,1]$. *The only (non-zero) eigenvalue of $\mathcal{P}$ is $\lambda = 1$.*

Remark 8 *Note that by Theorem 2.2 the action of transfer operator $\mathcal{P}$ on $\mathcal{H}_0$ can be explicitly expressed in the form*

$$\mathcal{P}\mathcal{B}[\varphi] = \mathcal{B}[\,T\varphi\,], \tag{2.77}$$

with

$$(T\varphi)(t) := e^{-t}\varphi(t) + \int_0^\infty K(s,t)\varphi(s)ds \tag{2.78}$$

and

$$K(s,t) = e^{-t}\left(\frac{e^t - 1}{e^s - 1}\right)\sqrt{\frac{s}{t}}\; J_1(2\sqrt{st}). \tag{2.79}$$

This already shows that the spectrum of $\mathcal{P}$ in $\mathcal{H}_0$ is given by a compact perturbation of the continuous spectrum $\sigma_c = [0,1]$. The above theorem can then be rephrased by saying that the effect of this perturbation is that of adding one simple eigenvalue $\lambda = 1$, so that the spectrum of $\mathcal{P}$ decomposes as $\mathrm{sp}\,(\mathcal{P}) = \sigma_p \cup \sigma_c$ *where σ_p (the pure point spectrum) is such that $\sigma_p = \sigma_p \cap \sigma_c = \{1\}$.*

Remark 9 *The above theorem provides a more complete description of the analytic structure of the operator-valued function $z \to (1 - \mathcal{M}_z)^{-1}$ for the Farey-Gauss pair: in Corollary 2.2 we obtained a meromorphic continuation to the cut plane $\mathbb{C} \setminus [1,\infty)$, but we now see that $(1-\mathcal{M}_z)^{-1}$ is actually analytic in this region.*

Proof of Theorem 2.3. The first assertion is a direct consequence of Corollary 2.2, Theorem 2.2 and the preceeding discussion. For the last assertion we argue as follows. We know from Corollary 2.1 that $\lambda \in \sigma_p$ if and only if $\mathcal{K}_{1/\lambda} : L_2(m) \to L_2(m)$ has 1 as an eigenvalue. Suppose there exists $\varphi \in L_2(m)$ and $\lambda \neq 1$ such that $\mathcal{P} f = \lambda f$ with $f = \mathcal{B}[\varphi]$. Then we would have $\mathcal{K}_{1/\lambda}\varphi(t) = \varphi(t)$, which can also be written as:

$$\mathcal{K}\varphi = (\lambda - M)(1 - M)^{-1}\varphi\,. \tag{2.80}$$

We have seen above that $\lambda_1 = 1$ is a simple eigenvalue of $\mathcal{K}$ with eigenfunction $\varphi_1(t) = (1 - e^{-t})/t$. Furthermore, from [MaR1], we know that the operator $\mathcal{K}$ on $L_2(m)$ admits a spectral representation of the form

$$\mathcal{K}\varphi = \sum_{n=1}^{\infty} \lambda_n (\varphi, \varphi_n)\,\varphi_n \tag{2.81}$$

where the φ_n form an orthonormal basis in $L_2(m)$ and the eigenvalues λ_n are real and satisfy $1 = \lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots$ with $\sum |\lambda_n| < \infty$. Expanding φ on this basis we would obtain that whenever $(\varphi, \varphi_n) \neq 0$ the multiplication operator $(V_\lambda \varphi)(t) = (\lambda - e^{-t})(1 - e^{-t})^{-1}\varphi(t)$ (with $V_1 = \mathrm{Id}$) has eigenvalue λ_n with eigenfunction φ_n. But this is impossible unless $\lambda = 1$ and $\varphi = \varphi_1$. $\diamondsuit$
Second proof. An alternative proof of the last assertion of Theorem 2.3 be achieved using the representation (2.63) of $f \in \mathcal{H}_0$. Indeed, suppose that $\mathcal{P} f = \lambda f$ or explicitly,

$$\lambda f(w) = \left(\frac{1}{w+1}\right)^2 \left[f\left(\frac{w}{w+1}\right) + f\left(\frac{1}{w+1}\right)\right]. \tag{2.82}$$

If we transform this equation by substituting $1/w$ for w and then dividing through w^2 we get

$$\lambda\, w^{-2} f(1/w) = \left(\frac{1}{w+1}\right)^2 \left[f\left(\frac{1}{w+1}\right) + f\left(\frac{w}{w+1}\right)\right]. \tag{2.83}$$

Whence f must satisfy

$$f(w) = w^{-2} f(1/w). \tag{2.84}$$

Putting $f(w) = \frac{1}{w}\int_0^\infty ds\, e^{-s}\,\psi(sw)$ one sees that (2.84) renders either $\psi = 0$ or $\psi = 1$ (a.e.) In the latter case we have, by (2.63) and (2.64), $\lambda = 1$. $\diamondsuit$

Remark 10 *Applying (2.84) to each term of the r.h.s. in (2.82) one obtains, for $\lambda = 1$,*

$$f(w) - f(w+1) = \frac{1}{w^2} f\left(1 + \frac{1}{w}\right). \tag{2.85}$$

This is a particular case of the Lewis functional equation

$$f(w) - f(w+1) = \frac{1}{w^{2(q+1)}} f\left(1 + \frac{1}{w}\right), \tag{2.86}$$

which is related to the so called Maass cusp forms, i.e. PSL(2,$\mathbb{Z}$)-invariant eigenfunctions of the Laplacian on the Poincaré upper half-plane which vanish at the cusp (see

[Le]). Another type of functions equivalent to (even) Maass forms and considered in [Le] are those satisfying an integral equation which in our notation writes

$$g(t) = \int_0^\infty dm(s)\, \frac{J_{2q+1}(2\sqrt{st})}{\sqrt{st}} \left(\frac{s}{t}\right)^q g(s)\,. \tag{2.87}$$

By the foregoing (see Remark 7) we see that for $q = 0$ *we have the relation*

$$f = \mathcal{B}[g]. \tag{2.88}$$

3 Zeta functions

We now consider the dynamical zeta functions ζ_F and ζ_G associated to the maps F and G, respectively, and defined by the following formal series [Ba2]:

$$\zeta_F(z) = \exp \sum_{n=1}^\infty \frac{z^n}{n} Z_n(F) \quad \text{and} \quad \zeta_G(s) = \exp \sum_{n=1}^\infty \frac{s^n}{n} Z_n(G), \tag{3.89}$$

where the 'partition functions' $Z_n(F)$ and $Z_n(G)$ are given by

$$Z_n(F) = \sum_{x=F^n(x)} \prod_{k=0}^{n-1} \frac{1}{|F'(F^k(x))|} \quad \text{and} \quad Z_n(G) = \sum_{x=G^n(x)} \prod_{k=0}^{n-1} \frac{1}{|G'(G^k(x))|}\cdot \tag{3.90}$$

Let us first examine how $\zeta_F(z)$ and $\zeta_G(z)$ are related to one another. Let $\mathrm{Per}\, F$ and $\mathrm{Per}\, G$ denote the sets of all periodic points of the maps F and G, respectively. It is not difficult to realize that, as subsets of $[0,1]$, $\mathrm{Per}\, F \setminus \{0\} = \mathrm{Per}\, G$. Accordingly, given x in either of these sets, we let $p_F(x)$ and $p_G(x)$ denote the periods of x w.r.t. to F and G, respectively. They are related by

$$p_F(x) = \tau(x) + \tau(G(x)) + \cdots + \tau(G^{p_G(x)-1}(x))\,\cdot \tag{3.91}$$

Moreover from the definitions of F and G we have

$$\prod_{k=0}^{p_F(x)-1} \frac{1}{|F'(F^k(x))|} = \prod_{k=0}^{p_G(x)-1} \frac{1}{|G'(G^k(x))|} = \prod_{k=0}^{p_G(x)-1} (G^k(x))^2\,\cdot \tag{3.92}$$

Using this facts we write $Z_n(F)$ as follows:

$$Z_n(F) = 1 + \sum_{m=1}^n \frac{n}{m} \sum_{x=F^n(x)=G^m(x)} \prod_{k=0}^{m-1} (G^k(x))^2\,\cdot \tag{3.93}$$

The second sum ranges over the $\binom{n-1}{m-1}$ ways to write the integer n as a sum of m positive integers. Therefore,

$$\begin{aligned}
\sum_{n=1}^\infty \frac{z^n}{n} Z_n(F) &= \log\left(\frac{1}{1-z}\right) + \sum_{n=1}^\infty \sum_{m=1}^n \frac{1}{m} \sum_{x=F^n(x)=G^m(x)} z^n \prod_{k=0}^{m-1} (G^k(x))^2 \\
&= \log\left(\frac{1}{1-z}\right) + \sum_{\ell=1}^\infty \frac{1}{\ell} \sum_{x=G^\ell(x)} z^{p_F(x)} \prod_{k=0}^{\ell-1} (G^k(x))^2\,\cdot
\end{aligned}$$

We are thus led to study the 'grand partition function' $\Xi_\ell(z)$ given by

$$\Xi_\ell(z) := \sum_{x=G^\ell(x)} z^{p_F(x)} \prod_{k=0}^{\ell-1} (G^k(x))^2 = \sum_{n=0}^{\infty} z^{\ell+n} \sum_{x=G^\ell(x)=F^{\ell+n}(x)} \prod_{k=0}^{\ell-1} (G^k(x))^2. \tag{3.94}$$

The sum over periodic points yields $\binom{n+\ell-1}{\ell-1} = \binom{n+\ell-1}{n}$ terms, corresponding to the number of ways of distributing n identical objects into ℓ distinct boxes. According to (1.27), (1.28) and (3.94) we can also write $\Xi_\ell(z)$ in the following way:

$$\Xi_\ell(z) = \sum_{n=0}^{\infty} z^{\ell+n} \sum_{k_1+\ldots+k_\ell=n+\ell}$$

$\prod_{i=1}^{\ell} x^2_{k_i \ldots k_\ell k_1 \ldots k_{i-1}}$, (3.95)where $x_{k_1\ldots k_\ell} = [\overline{k_1,\ldots,k_\ell}]$ denotes the irrational number whose continued fraction expansion is periodic of period ℓ and starts with the entries $k_1,\ldots,k_\ell$. Putting together the above observations we obtain the next result, to be compared with Proposition 2.1:

Proposition 3.3 *Consider the two-variable zeta function given by*

$$\zeta_2(s,z) = \exp \sum_{\ell=1}^{\infty} \frac{s^\ell}{\ell}\, \Xi_\ell(z). \tag{3.96}$$

Then we have:

$$\zeta_2(1,z) = (1-z)\,\zeta_F(z) \quad \textit{and} \quad \zeta_2(s,1) = \zeta_G(s) \tag{3.97}$$

wherever the series expansions converge absolutely.

In order to study the analytic properties of the function $\zeta_2(s,z)$ we further generalize (2.40) by introducing a family of operator-valued functions $\mathcal{M}_{z,q}$, $q = 0,1,\ldots$, acting as (see also [Ma1])

$$\mathcal{M}_{z,q} f(x) = (-1)^q \sum_{n=1}^{\infty} z^{\tau(\Phi_n(x))} \cdot |\Phi_n'(x)|^{1+q} \cdot f(\Phi_n(x)), \tag{3.98}$$

together with a family of function spaces $\mathcal{H}_{1,q} \subseteq \mathcal{H}_1$ such that a function $f \in \mathcal{H}_{1,q}$ can be represented as

$$f(w) = (\mathcal{L}_q[\varphi])(w) := \int_0^\infty dm(t)\, e^{-tw}\, t^q\, \varphi(t), \quad \varphi \in L_2(m). \tag{3.99}$$

In particular $\mathcal{M}_{z,0} \equiv \mathcal{M}_z$, $\mathcal{L}_0 \equiv \mathcal{L}$ and $\mathcal{H}_{1,0} \equiv \mathcal{H}_1$. We have the following result.

Proposition 3.4 *For any given $q = 0,1\ldots$ the operator valued function $z \to \mathcal{M}_{z,q}$ when acting on $\mathcal{H}_{1,q}$ can be analytically continued to the entire z-plane with a cross cut along the ray $(1,+\infty)$. For each z in this domain we have*

$$\mathcal{M}_{z,q}\, \mathcal{L}_q[\varphi] = \mathcal{L}_q[\mathcal{K}_{z,q}\varphi], \tag{3.100}$$

where $\mathcal{K}_{z,q} : L_2(m) \to L_2(m)$ is given by

$$(\mathcal{K}_{z,q}\varphi)(t) := \gamma(t,z)\,(-1)^q \int_0^\infty dm(s)\, \frac{J_{2q+1}(2\sqrt{st})}{\sqrt{st}}\, \varphi(s)\,. \tag{3.101}$$

The operators $\mathcal{M}_{z,q} : \mathcal{H}_{1,q} \to \mathcal{H}_{1,q}$ and $\mathcal{K}_{z,q} : L_2(m) \to L_2(m)$ are both of trace class.

Proof. The first part follows from a straightforward extension to non zero q values of the arguments of the previous Section. The proof of the last assertion can be extracted from ([Ma1], Theorem 3). $\diamondsuit$

Now, the trace of the operator $\mathcal{K}_{z,q}$ is easily obtained (see also [Ma1]):

$$\begin{aligned}
\mathrm{tr}\ \mathcal{K}_{z,q} &= (-1)^q\, z \int_0^\infty \frac{J_{2q+1}(2t)}{e^t - z}\, dt \\
&= (-1)^q \sum_{k=1}^\infty z^k \int_0^\infty e^{-kt}\, J_{2q+1}(2t)\, dt \\
&= (-1)^q \sum_{k=1}^\infty z^k \frac{x_k^{2(q+1)}}{1+x_k^2},
\end{aligned} \tag{3.102}$$

where the numbers $x_k = \frac{\sqrt{k^2+4}-k}{2} = [k,k,k,\ldots] \equiv [\overline{k}]$ are the fixed points of $G(x)$ and the identity [GR]

$$\int_0^\infty e^{-kt} J_p(2t) dt = \frac{(\sqrt{k^2+4}-k)^p}{2^p\sqrt{k^2+4}}, \quad p = 0, 1, \ldots \tag{3.103}$$

has been used. From (3.102) we immediately obtain the trace formula

$$\Xi_1(z) = \mathrm{tr}\ \mathcal{K}_{z,0} - \mathrm{tr}\ \mathcal{K}_{z,1}. \tag{3.104}$$

But we can say more. Indeed, a straightforward adaptation of ([Ma1], Corollaries 4 and 5) to our z-dependent situation leads to the following general expressions:

$$\Xi_\ell(z) = \mathrm{tr}\ \mathcal{K}^\ell_{z,0} - \mathrm{tr}\ \mathcal{K}^\ell_{z,1} = \mathrm{tr}\ \mathcal{M}^\ell_{z,0} - \mathrm{tr}\ \mathcal{M}^\ell_{z,1}, \tag{3.105}$$

with

$$\mathrm{tr}\ \mathcal{K}^\ell_{z,q} = (-1)^{q\ell} \sum_{k_1,\ldots,k_\ell=1}^\infty z^{k_1+\cdots+k_\ell} \frac{\prod_{i=1}^\ell x^{2(q+1)}_{k_i\ldots k_\ell k_1\ldots k_{\ell-1}}}{1-(-1)^\ell \prod_{i=1}^\ell x^2_{k_i\ldots k_\ell k_1\ldots k_{\ell-1}}}. \tag{3.106}$$

Formula (3.105) along with standard arguments (see [Ma1]) allow us to write the two-variables zeta function (3.96) as a ratio of Fredholm determinants,

$$\zeta_2(s,z) = \exp \sum_{\ell=1}^\infty \frac{s^\ell}{\ell}\, \Xi_\ell(z) = \frac{\det(1 - s\,\mathcal{K}_{z,1})}{\det(1 - s\,\mathcal{K}_{z,0})} = \frac{\det(1 - s\,\mathcal{M}_{z,1})}{\det(1 - s\,\mathcal{M}_{z,0})}, \tag{3.107}$$

where by definition

$$\det(1 - s\,\mathcal{K}_{z,q}) = \exp\left(-\sum_{\ell=1}^\infty \frac{s^\ell}{\ell}\ \mathrm{tr}\ \mathcal{K}^\ell_{z,q}\right) \tag{3.108}$$

is in the sense of Grothendieck [G]. We have thus proved the following result.

Theorem 3.4 *Set $\mathcal{K}_z \equiv \mathcal{K}_{z,0}$, then we have:*

1. for each $s \in \mathbb{C}$, the function $\zeta_2(s,z)$, considered as a function of the variable z, extends to a meromorphic function in the cut plane $\mathbb{C} \setminus [1,\infty)$. Its poles are located among those z-values such that $\mathcal{K}_z : L_2(m) \to L_2(m)$ has $1/s$ as an eigenvalue;

2. for each $z \in \mathbb{C} \setminus (1,\infty)$, the function $\zeta_2(s,z)$, considered as a function of the variable s, extends to a meromorphic function in $\mathbb{C}$. Its poles are located among the inverses of the eigenvalues of $\mathcal{K}_z : L_2(m) \to L_2(m)$.

Putting together the above Theorem and Proposition 3.3 we obtain the following

Corollary 3.3 *The dynamical zeta functions ζ_F and ζ_G of the Farey and Gauss maps have the following properties:*

1. $\zeta_F(z)$ has an analytic extension to the cut plane $\mathbb{C} \setminus [1,\infty)$;

2. $\zeta_G(s)$ has a meromorphic extension to $\mathbb{C}$. All poles are real and are located among the inverses of the eigenvalues λ_n of $\mathcal{K} : L_2(m) \to L_2(m)$.

Remark 11 *Statement 1 of Corollary 3.3 improves for this example a result of Rugh ([Rug], Corollary 1.3) where it is proved that $\zeta_F(z)$ is meromorphic in $\mathbb{C} \setminus [1,\infty)$. Statement 2 was proved by Mayer in [Ma3], where he also showed (using results from [Rue]) that the poles of $\zeta_G(s)$, if arranged in increasing absolute values and according to their order tend to infinity exponentially fast.*

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