Unitary Representations of the Canonical Group, the Semi-Direct Product of the Unitary and Weyl-Heisenberg Group: $\mathcal{C}(1, 3) = \mathcal{U}(1, 3) \otimes_s \mathcal{H}(1, 3)$

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The unitary representations of the Canonical group that is defined to be the semi-direct product of the unitary group with the Weyl-Heisenberg group, $C(1, 3) = \mathcal{U}(1, 3) \otimes_s \mathcal{H}(1, 3)$ are studied. The Canonical group is equivalently written as the semi-direct product of the special unitary group with the Oscillator group $C(1, 3) = \mathcal{SU}(1, 3) \otimes_s \mathcal{O}s(1, 3)$. The non-abelian Weyl-Heisenberg group represents the eight physical degrees of freedom, time, position, momentum and energy. The Weyl-Heisenberg group has 4 canonical abelian 4 dimensional subgroups (i.e. translation groups $\mathcal{T}(4)$) corresponding to the time-position, momentum-energy, time-momentum and position-energy degrees of freedom respectively. In the embedding into $\mathcal{H}(1, 3)$, only one of these translation groups is *diagonal* in a given realization. The $C(1, 3)$ contains four copies of the Poincaré group as subgroups that each have invariant translation subgroups corresponding to these four translation groups. The first two are the familiar groups parameterized by rotations and hyperbolic transformations representing velocity addition bounded by c. The remaining two are parameterized by the same rotations and hyperbolic transformations representing a new physical concept of force addition bounded by a new constant b. For b large such as if it is the Planck scale $b = c^4 / G$, this approximates the usual Euclidean addition law. These new effects would only be apparent in very strongly interacting systems. The representation theory requires generalized *time-like*, *null*, and *space-like* cases, determined by the value of the Hermitian metric, to be considered. The finite dimensional unitary *Little groups* are $\mathcal{SU}(3)$ for the *time-like* case and $\mathcal{SU}(2) \otimes \mathcal{U}(1)$ for the *null* case. The remaining unitary representations are infinite dimensional.

I Introduction

Dirac's transformation theory of quantum mechanics [1] is formulated on the non-abelian position-momentum space of the Weyl-Heisenberg group and it's associated algebra. In this theory, one may choose a realization (which Dirac referred to as a representation) for which the position degrees of freedom are diagonal or a realization for which the momentum degrees of freedom are diagonal, but not both.

Special relativity is formulated on the abelian position-time manifold of the translation group in four dimensions $\mathcal{T}(4)$. The representations of the Poincaré group $\mathcal{P}$ [2], for which the translation group is an invariant abelian subgroup, provide the basic framework of particle physics. The time-like representations of these representations are labeled by the Casimir invariants that may be interpreted in the rest frame as the fundamental concepts of spin and mass. The key Klein-Gordon, Dirac and Rariter-Schwinger equations arise out of the study of these representations.

Particle physics has proven to be much richer than can be encompassed by simply the Poincaré group. $\mathcal{SU}(3)$, that is known to play a fundamental role in hadron physics, simply has no origin in the Poincaré symmetries. Instead, these are regarded as internal, or through the compactification of higher dimensional physical symmetries.

Furthermore, there is an increasing realization that the fundamental characterization of a particle as a point in an abelian manifold is limited. String theory and twister theory, for example, refutes this premise and have given rise to rich, and rather complex theories.

This paper explores the idea of using the Weyl-Heisenberg group $\mathcal{H}(1, 3)$ with an algebra spanned by the $\{T, Q_i, P_i, E\}$ (and the unit element I) corresponding to the time, energy, position and momentum degrees of freedom as the underlying space on which a homogenous group of symmetries act through a semi-direct product. In this way, the fundamental quantum condition arises naturally out of the non-abelian structure $\mathcal{H}(1, 3)$. The natural homogeneous group to consider is $\mathcal{U}(1, 3)$ [3,4]. The resulting group is the Canonical relativistic group $C(1, 3) = \mathcal{U}(1, 3) \otimes_s \mathcal{H}(1, 3)$.

The unitary group $\mathcal{U}(1, 3)$ leaves invariant an orthogonal line element $-T^2 + Q^2/c^2 + P^2/b^2 - E^2/c^2\,b^2$ [5,6] and a symplectic line element $-T \bigwedge E + Q^i \bigwedge P^i$. c is the speed of light and b is a physical constant that is defined by $b = \alpha_G\, c^4/G$ where G is the gravitational constant and α_G is a dimensionless numerical factor. Preserving these metrics is equivalent to preserving a Hermitian line element with a Minkowski signature.

$\mathcal{H}(1, 3)$ is non-abelian and only 4 of the 8 degrees of freedom $\{T, Q_i, P_i, E\}$ can be diagonal in a given realization. The ninth element, I is always diagonal as it is the center. This is equivalent to the observation that there are four canonical copies of the 4 dimensional translation group $\mathcal{T}(4)$ that are embedded as subgroups in $\mathcal{H}(1, 3)$. These correspond to the sets of degrees of freedom $\{T, Q_i\}, \{E, P_i\}, \{T, P_i\}$, and $\{E, Q_i\}$.

The group $C(1, 3)$ contains 4 different copies of the Poincaré group $\mathcal{P} = \mathcal{SO}(1, 3) \otimes_s \mathcal{T}(4)$ with the invariant translation subgroups corresponding to each of the above 4 sets [3]. When embedded as subgroups in $C(1, 3)$, only one of these sets can be diagonal in a given realization. These groups appear in a manner that is completely consistent with Born's notion of reciprocity symmetry [5,6]. It is this *equal standing* of all of the 8 degrees of freedom $\{T, Q_i, P_i, E\}$ together with the invariance of the symplectic and orthogonal metrics that leads to the name, Canonical group, for $C(1, 3)$.

There are 4 different $\mathcal{SO}(1, 3)$ homogeneous groups that correspond to the above Poincaré groups. The pure Lorentz subgroups are defined in terms of the usual hyperbolic rotations that correspond to velocity transformations in the $\{T, Q_i\}$, $\{E, P_i\}$ cases and to force transformations in the $\{T, P_i\}$, and $\{E, Q_i\}$ cases. Clearly the limit of small velocity relative to c and force relative to b result in the usual Euclidean addition laws.

This choice has a number of deep physical consequences. The full group of transformations $\mathcal{U}(1, 3)$ makes it possible to transform energy into time, or more generally momentum-energy into position-time. Clearly, this is unacceptable unless this is does not occur except under conditions that we do not currently generally observe in the universe. However, these effects would only be apparent if the rate of change of momentum, or force of interaction, approaches the value given by the constant b. If α_G is unity (or within a few magnitudes of unity), this constant is very large and the effects would only occur at forces of the Planck scale of natural units. This might occur in the very early universe where a very large energy-momentum degrees of freedom may be transforming, or we could say, condensing, into expanding position-time degrees of freedom.

Another result of the transformation equations is that the rate of change of momentum with time is bounded by a constant b in a manner that is dual to the usual bounding of the rate of change of position by time by c in the time-like case of the representations. This is physically similar to investigations on theories with maximal acceleration such as proposed by Cainiello [7,8]. In previous papers [3,4] it is verified that the transformation equations reduce to the expected transformations if the forces of interaction are small relative to b which is modelled by taking $b \to \infty$ in the group contraction. (The natural 3 dimensional constants that define the dimensions of physics are then c, b and h instead of the usual set of c, G and h.)

Another observation that proves important in the representation theory is that the Canonical group may be written in another form. The unitary group $\mathcal{U}(1, 3)$ is the direct product of the one dimensional unitary group and the special unitary group, $\mathcal{U}(1, 3) = \mathcal{U}(1) \otimes \mathcal{SU}(1, 3)$. Noting that the Oscillator group $\mathcal{O}s(1, 3) = \mathcal{U}(1) \otimes_s \mathcal{H}(1, 3)$, the Canonical relativistic group $\mathcal{C}(1, 3)$ is equivalently $\mathcal{C}(1, 3) = \mathcal{SU}(1, 3) \otimes_s \mathcal{O}s(1, 3)$.

The group $\mathcal{C}(1, 3)$ has a rich structure containing a remarkable number of groups of interest in physics, the Poincaré and Lorentz groups, the Weyl-Heisenberg Group, The Oscillator Group, and the special unitary groups $\mathcal{SU}(3)$ and $\mathcal{SU}(2)$.

Whether this theory does have relevance in the real physical world hinges fundamentally on a study of the representations of the Canonical relativity group. Just as the representations of the Poincaré group provide the framework for physics in the non-strongly interacting case, one would expect the classes of representations to reflect symmetries that encompass these more general cases and be labeled by Casimir invariants with natural physical interpretations .

Wigner provided the framework for the study of semi-direct products through induced representations, for which the Poincaré is a specific example. This theory was generalized by Mackey with the formalism applied to the Poincaré group in a notation accessible to physicist by Moussa and Stora [9]. These enable the full representation theory to be induced by representations of the invariant subgroup, which may be taken to be the Oscillator group, and the corresponding *Little Groups* leaving a representative element invariant.

The representations of the Oscillator group have been studied by Miller [10,11] and Major [12]. It has two Casimir invariants that label the representations. The orbits of the elements of the Oscillator group under the action of the special unitary group $\mathcal{SU}(1, 3)$ are complex hyperboloids. Like the Poincaré group, these hyperboloids fall into classes corresponding to a *time-like, space-like* and *null* cases (in a generalized sense to be defined) and degenerate cases. However, in the Canonical group case, *points* on the time-like case orbit are elements of the Oscillator group $\mathcal{O}s(1, 3)$, not simply points in $\mathcal{T}(4)$. They are characterized by the values of two Casimir invariants as apposed to the single one in the Poincaré case. The representation theory identifies the little group leaving the orbits invariant. For the *time-like* case, this is $\mathcal{SU}(3)$ and the representative point of the orbit may be chosen to be the *time-like* element that is in the $\mathcal{O}s(1)$ subgroup of $\mathcal{O}s(1, 3)$. For the null case, the little group is $\mathcal{O}s(2)$. The unitary representations of $\mathcal{O}s(2)$ are infinite dimensional except for the degenerate case $\mathcal{SU}(2) \otimes \mathcal{U}(1)$ which are finite in a manner that parallels the null representation case of the Poincaré group.

The induced representation theory is reviewed and applied to the Canonical group. The approach is intended to be the more concrete approach for application to physical theories rather than one of mathematical rigor, with a special effort to draw on parallels with the very

familiar Poincaré representation theory. Preliminary physical consequences of this theory are surveyed.

II Unitary Representations of the Canonical Group

A Induced Representations of Semi-Direct Products

Consider a group that is the product $\mathcal{G} = \mathcal{U} \otimes_s \mathcal{A}$. $\mathcal{U}$ and $\mathcal{A}$ are general Lie groups with that are separable and locally compact. $\mathcal{A}$ is the invariant subgroup satisfying the properties that $U\ a \in \mathcal{A}$ for all $U \in \mathcal{U}$. The *Little Group* $\mathcal{U}^\circ$ of a subgroup $\mathcal{A}^\circ$ of $\mathcal{A}$ is defined by:

$$U^\circ\ a^\circ = a^\circ, \quad \text{for all } U^\circ \in \mathcal{U}^\circ \text{ and all } a^\circ \in \mathcal{A}^\circ. \tag{1}$$

Assume $\mathcal{U}$ has the property that every element U of $\mathcal{U}$ can be decomposed into the form $U = Q_a\, U^\circ$ and that the action of $\mathcal{U}$ is onto $\mathcal{A}$. Then for every $a \in \mathcal{A}$, there is a $Q_a \in \mathcal{U} \backslash \mathcal{U}^\circ$ such that $a = Q_a\, a^\circ$.

Let $\rho_\kappa(\mathcal{K}^\circ)$ be a representation $\mathcal{K}^\circ$ in a Hilbert space $\mathcal{H}$. Likewise, let $\rho_\alpha(\mathcal{A})$ be a representation of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$. The representation of $\mathcal{G}^\circ = \mathcal{U}^\circ \otimes \mathcal{A}^\circ$ is then

$$\rho_{\kappa,\alpha}(\mathcal{G}^\circ) = \rho_\kappa(\mathcal{U}^\circ)\, \rho_\alpha(\mathcal{A}). \tag{2}$$

Representations of $\mathcal{G}$ may be induced from representations of $\mathcal{G}^\circ$. Let f be left invariant functions satisfying the property $f(g(U, a)\, U^\circ) = f(g(U, a))\, \rho_\kappa(U^\circ)$. The unitary representation of $\mathcal{G}$ induced by $\mathcal{G}^\circ$ is [9,12]:

$$(\rho_{\kappa,\alpha}(g(U, a))\, f)(z) = \rho_\kappa(Q_a^{-1}\, U\, Q_{U^{-1}a})\, \rho_\alpha(-Q^{-1}_{U^{-1}a}\, U^{-1}\, a)\, f(U^{-1}\, z). \tag{3}$$

To put these results in a more familiar context, consider the example of the time-like case of the unitary representations of the Poincaré group $\mathcal{P} = \mathcal{L} \otimes_s \mathcal{T}(4)$ where clearly $\mathcal{L}$ is the Lorentz group $\mathcal{A} = \mathcal{T}(4)$, the point $p^\circ = (p^\circ, 0, 0, 0)$ and $\mathcal{A}^\circ = \mathcal{T}(1)$. $\mathcal{U} = \mathcal{L}$ and $\mathcal{U}^\circ = \mathcal{R}(3)$, the rotation group. (In the covering case it is $\mathcal{SU}(2)$.) The labels of the representation of the translation group α are the Casimir invariant $m^2 = p^2 = p^{\circ 2}$ and $\rho_{p^\circ}(p) = e^{-i\,(p^0, a)}$ with (,) the Lorentz inner product. Note $(p^\circ, -Q^{-1}_{L^{-1}a}\, L^{-1}\, a) = -(p, a)$. Also, $R = Q_a^{-1}\, L\, Q_{L^{-1}a}$ and the rotation group representations are the *D matrices* labelled by spin s, $D^m_{m'}(s; R)$ and then equation (3) gives the familiar Poincaré group time-like representation with a rotation little group. This, and the other Poincaré representation cases, are fully presented in [9].

B Properties of the Group and Algebra

The group may be parameterized as $g(U, \vartheta, \omega, \iota)$ where $U \in \mathcal{SU}(1, 3)$ has 15 parameters, $\theta, \iota \in \mathbb{R}$ and $\omega \in \mathbb{C}^4$. The invariant Oscillator subgroup is the subgroup $g(I, \vartheta, \omega, \iota)$ and the Weyl-Heisenberg group is $g(I, 0, \omega, \iota)$. The group multiplication law is given by

$$\begin{aligned} &g(U', \vartheta', \omega', \iota')\, g(U, \vartheta, \omega, \iota) = \\ &g\Big(U'\, U, \vartheta' + \vartheta, \omega' + e^{i\vartheta}\, U\, \omega, \iota' + \iota - \tfrac{i}{2}\left((\omega',\, U'\, \omega)\, e^{i\vartheta} - \overline{(\omega',\, U'\, \omega)}\; e^{-i\vartheta}\right)\Big), \end{aligned} \tag{4}$$

where $(\omega, z) = \eta_{a,b}\, \overline{\omega}^a\, z^b$, $(U'\, U)^a_b = U'^a_{\ c}\, U^c_b$ and $(U\, \omega)^a = U^a_b\, \omega^b$.

The inverse group element is

$$g(U, \vartheta, \omega, \iota)^{-1} = g(U^{-1}, -\vartheta, -U^{-1}\, e^{-i\vartheta}\, \omega, -\iota). \tag{5}$$

The algebra of this group is

$$[Z_{ab}, Z_{cd}] = \eta_{bc}\, Z_{ad} - \eta_{ad}\, Z_{cb}, \;\; [A^+_a, A^-_b] = \eta_{ab}\, I, \;\; [Z_{ab}, A^\pm_c] = \mp \eta_{ac}\, A^\pm_b \; . \tag{6}$$

Note that $\overline{A^+_b} = A^-_b$. The algebra of $\mathcal{U}(1, 3)$ factors into the direct sum of the algebra of $\mathcal{U}(1)$ and the algebra of $\mathcal{SU}(1, 3)$. The generator of the $\mathcal{U}(1)$ may be defined as $Z = \eta^{ab}\, Z_{ab}$ and the generators of the~ algebra of $\mathcal{SU}(1, 3)$ are then defined as $\hat{Z}_{ab} = Z_{ab} - \eta_{ab}\, Z/4$. The generators $\hat{Z}_{ab}$ satisfy the same commutation relations as given in equation (6).

The Casimir invariants of the algebra are given by [13].

$$\begin{aligned} c_1(C(1, 3)) &= I, \\ c_2(C(1, 3)) &= \eta^{ab}\, W_{ab}, \\ c_4(C(1, 3)) &= \eta^{\mathrm{ad}}\, \eta^{\mathrm{bc}}\, W_{\mathrm{ab}}\, W_{\mathrm{cd}}, \\ c_6(C(1, 3)) &= \eta^{\mathrm{af}}\, \eta^{\mathrm{bc}}\, \eta^{\mathrm{de}}\, W_{a\,b}\, W_{c\,d}\, W_{e\,f}, \\ c_8(C(1, 3)) &= \eta^{a\,h}\, \eta^{b\,c}\, \eta^{d\,e}\, \eta^{f\,g}\, W_{a\,b}\, W_{c\,d}\, W_{e\,f}\, W_{g\,h}, \end{aligned} \tag{7}$$

where $W_{\mathrm{ab}} = A^+_a\, A^-_b - I\, Z_{a\,b}$.

Note that the first Casimir invariant is simply the Casimir invariant of the Weyl-Heisenberg group and one of the two Casimir invariants of the Oscillator group, $c_1(C(1, 3)) = c_1(\mathcal{H}(1, 3)) = c_1(\mathcal{O}s(1, 3))$ and the second Casimir invariant is the second Casimir invariant of the Oscillator group, $c_2(C(1, 3)) = c_2(\mathcal{O}s(1, 3))$.

C The Bargmann Hilbert Space

The group $C(1, 3)$ may be represented on a 4 dimensional Bargmann space $\mathfrak{B}^4$ [14] that is the Hilbert space of entire analytic functions on $\mathbb{C}^4$ with finite inner product defined by

$$\langle f, h\rangle = \int \overline{f(z)}\, h(z)\, \mathrm{d}\mu(z)\,, \tag{8}$$

with $z \in \mathbb{C}^4$ and $\mathrm{d}\mu(z) = \pi^{-4}\, e^{-(z,z)}\, d\,x^0 \ldots d\,x^3\, d\,y^0 \ldots d\,y^3$ with $z^a = x^a + i\, y^a$.

An orthonormal basis of this space is given by $\xi_m(z) = \frac{(z^0)^{m_0}}{\sqrt{m_0!}} \ldots \frac{(z^3)^{m_3}}{\sqrt{m_3!}}$, $m_a = 0, 1, 2, \ldots$ and it follows that $\langle \xi_m, \xi_n\rangle = \delta_{m,n}$ where $\delta_{m,n} = \delta_{m_0,n_0} \ldots \delta_{m_3,n_3}$.

The Bargmann space is related to the usual Hilbert space $\mathfrak{H}^4 = L^2(\mathbb{R}^4)$ of square integral functions with inner product

$$\langle \varphi, \psi\rangle = \int \overline{\varphi(x)}\, \psi(x)\, d^4\, x\,, \tag{9}$$

where $x \in \mathbb{R}^4$ through the Bargmann transform

$$\begin{aligned} f(z) &= (\boldsymbol{B}\,\psi)\,(z) = \int B(z, x)\, \psi(x)\, d^4\, x \\ \psi(x) &= (\overline{\boldsymbol{B}}\, f)\,(x) = \int \overline{B(z, x)}\, f(x)\, \mathrm{d}\mu(z). \end{aligned} \tag{10}$$

The kernel of the integral transform is given by

$$B(z, x) = \pi^{-1}\, e^{-\frac{1}{2}(z^2+x^2)+\sqrt{2}\, z\, q}\,. \tag{11}$$

In particular, the orthonormal basis $\eta_m(x)$ of $\mathfrak{H}^4$ obtained by transforming the orthonormal basis vectors $\xi_m(z)$ of $\mathfrak{B}^4$, $\eta_m(x) = (\overline{\boldsymbol{B}}\, \xi_m)\,(x)$ are given in terms of Hermite polynomials

$$\begin{aligned} \eta_m(x) &= \eta_{m_0}(x) \ldots \eta_{m_3}(x), \\ \eta_{m_a}(x) &= \left(2^{m_a}\, m_a!\, \sqrt{m_a}\right)^{-1/2} e^{-x^2/2}\, H_{m_a}(x). \end{aligned} \tag{12}$$

D Unitary Irreducible Representations of $Os(1, 3)$

The Lie algebra can be realized as operators on this Bargmann space [13,14]

$$A^+_a = \kappa_0\, z^a,\ A^-_a = \tfrac{\partial}{\partial z^a},\ Z = -\kappa_1 + z^a\, \tfrac{\partial}{\partial z^a},\ I = \kappa_0\,. \tag{13}$$

Using this realization of the algebra, the action of the algebra of $Os(1, 3)$ on the basis is given by:

$$A^+{}_a \left| \xi_{m_0, \ldots m_a \ldots m_3} \right\rangle = \sqrt{\kappa_0 (m_a + 1)}\, \left| \xi_{m_0, \ldots m_a + 1 \ldots m_3} \right\rangle$$

$$A^-{}_a \left| \xi_{m_0, \ldots m_a \ldots m_3} \right\rangle = \sqrt{\kappa_0\, m_a}\, \left| \xi_{m_0, \ldots m_a - 1 \ldots m_3} \right\rangle$$

$$Z\,|\,\xi_{m_0,\,..\,m_a...\,m_3}\rangle \;=\; (-\kappa_1 - m_0 + m_1 + m_2 + m_3)\,|\,\xi_{m_0,\,..\,m_a...\,m_3}\rangle$$

$$I\,|\,\xi_{m_0,\,..\,m_a...\,m_3}\rangle \;=\; \kappa_0\,|\,\xi_{m_0,\,..\,m_a...\,m_3}\rangle .$$

The Casimir invariants of $\mathcal{O}s(1,\,3)$ are therefore

$$\begin{aligned} I\,|\,\xi_{m_0,\,..\,m_a...\,m_3}\rangle &= \kappa_0\,|\,\xi_{m_0,\,..\,m_a...\,m_3}\rangle \\ (\eta^{a,b}\,A^-{}_a\,A^+{}_b - I\,Z)\,|\,\xi_{m_0,\,..\,m_a...\,m_3}\rangle &= \kappa_0\;\kappa_1\,|\,\xi_{m_0,\,..\,m_a...\,m_3}\rangle . \end{aligned} \tag{15}$$

The representation of the Oscillator group on the Bargmann space are labelled by the Casimir invariants:

$$(\rho_{\kappa_0,\kappa_1}(g(I,\,\vartheta,\,\omega,\,\iota))\,f)\,(z) \;=\; e^{\kappa_0(i\,\iota-(\omega,z-\omega/2))-i\,\kappa_1\,\vartheta}\;\; f(e^{-i\,\vartheta}(z+\omega)), \tag{16}$$

where $\kappa_0,\;\kappa_1 \in \mathbb{R},\;\kappa_0 > 0$ and $f \in \mathcal{B}^4$. Note that from the realization (13) that this can be written as the action of the algebra

$$(\rho_{\kappa_0,\kappa_1}(g(I,\,\vartheta,\,\omega,\,\iota))\,f)\,(z) \;=\; e^{I(i\,\iota+(\omega,\omega)/2)-(\omega,A^+)}\;\; e^{(\overline{\omega},A^-)}\,e^{i\,\vartheta\,Z}\,f(z). \tag{17}$$

In the case $\kappa_0 = 0$, the representation is

$$(\rho_\kappa(g(I,\,\vartheta,\,\omega,\,0))\,f)\,(z) = e^{-i\,\kappa(\omega\,z\,+\,\overline{\omega}\,z^{-1})/2}\,f(e^{-i\,\vartheta}\,z). \tag{18}$$

E Unitary Irreducible Representations of $C(1,\,3)$

Noting that $g(U^{-1},\,0,\,0,\,0)\,g(I,\,\vartheta,\,\omega,\,\iota)\,g(U,\,0,\,0,\,0) \;=\; g(I,\,\vartheta,\,U\,\omega,\,\iota)$, the orbits of the action of $\mathcal{SU}(1,\,3)$ on the representations of $\mathcal{O}s(1,\,3)$ are given by

$$((g(U,\,0,\,0,\,0)\,\rho_{\kappa_0,\kappa_1}(g(I,\,\vartheta,\,\omega,\,\iota)))\,f)\,(z) \;=\; (g(I,\,\vartheta,\,U\,\omega,\,\iota)\,f)\,(z)\;. \tag{19}$$

Consequently, orbits are the hermitian hyperboloids that leave invariant the Hermitian inner product $(\omega,\omega) = \mu^2,\;\mu^2 \in \mathbb{R}$. Note that while μ is an invariant under the action of $\mathcal{SU}(1,\,3)$, it is not a Casimir invariant of the full group. In fact, $\mu^2 = m_0 - m_1 - m_2 - m_3$ in the Oscillator group realization given above in equation (14).

The representations therefore break into categories depending on whether μ^2 is positive, null or negative. In addition, there are degenerate cases where the first Casimir invariantκ_0 is zero and $\omega = 0$.

- **Degenerate Case:** $\omega = 0$

In the degenerate case, the representation simply reduces to the representation of $\mathcal{SU}(1,\,3)$. The unitary representations of this noncompact group are infinite dimensional.

- $\mu^2 > 0$ **Case**

The representative point in the orbit may be taken to be the *rest frame* for which only the 0^{th} component has non-zero values. That is, these are points in the Oscillator group for which $\{\omega^a\} = \{\omega\circ, 0, 0, 0\}$ that defines the one dimensional Oscillator subgroup $\mathcal{O}s(1)$ of $\mathcal{O}s(1, 3)$. The 4 generators $\{A^+{}_0, A^-{}_0, Z, I\}$ define the algebra of the one dimensional Oscillator group $\mathcal{O}s(1)$.

The little group U° is the $\mathcal{SU}(3)$ subgroup of $\mathcal{SU}(1, 3)$. The algebra of this group is spanned by the 8 generators $\{\hat{Z}_{ij}\}$ that satisfy the additional traceless condition $\delta^{ij} Z_{ij} = 0$. $\mathcal{SU}(3)$ has, of course been extensively studied. The representation is labelled by the two Casimir invariants:

$$\begin{aligned} c_2(\mathcal{SU}(3)) &= \delta^{ij}\,\delta^{kl}\,\hat{Z}_{ik}\,\hat{Z}_{lj}, \\ c_3(\mathcal{SU}(3)) &= \delta^{ij}\,\delta^{kl}\,\delta^{mn}\,\hat{Z}_{ik}\,\hat{Z}_{lm}\,\hat{Z}_{nj}. \end{aligned} \tag{20}$$

The respective eigenvalues κ_2, κ_3 may be expressed in terms of the highest weights of the representation [15]

$$\kappa_2 = \tfrac{1}{3}\,(n_{\max}{}^2 + n_{\max}\,m_{\max} + m_{\max}{}^2 + n_{\max} + m_{\max}), \tag{21}$$

with $n_{\max}, m_{\max}$ integer and a similar expression for κ_3. The representations of $\mathcal{SU}(3)$ are

$$\rho_{\kappa_2,\kappa_3}(U^\circ)\,|\,m, n\rangle = D^{m,n}_{m'.n'}(\kappa_2, \kappa_3;\ U^\circ)\,|\,m, n\rangle, \tag{22}$$

where $D^{m,n}_{m'.n'}(\kappa_2, \kappa_3;\ U^\circ)$ are the $\mathcal{SU}(3)$ *D Matrices*.

The Casimir invariants of the full algebra may be evaluated at a representative point in the isometry subgroup $\mathcal{SU}(3) \otimes \mathcal{O}s(1)$. As these are invariants, $c_n(C(1, 3)) = c_n(\mathcal{SU}(3) \otimes \mathcal{O}s(1))$

$$\begin{aligned} c_1(C(1, 3)) &= c_1(\mathcal{O}s(1)) = \kappa_0, \\ c_2(C(1, 3)) &= c_2(\mathcal{O}s(1)) = \kappa_0(\kappa_1 + 4), \\ c_4(C(1, 3)) &= c_2(\mathcal{O}s(1))^2 + c_1(\mathcal{O}s(1))^2\, c_2(\mathcal{SU}(3)) = \kappa_0{}^2(\kappa_1{}^2 + \kappa_2), \\ c_6(C(1, 3)) &= c_2(\mathcal{O}s(1))^3 + c_1(\mathcal{O}s(1))^3\, c_3(\mathcal{SU}(3)) = \kappa_0{}^3(\kappa_1{}^3 + \kappa_3). \end{aligned} \tag{23}$$

The corresponding representations of $C(1, 3)$ may be labelled by the 4 Casimir invariants $\kappa = \{\kappa_0, \kappa_1, \kappa_2, \kappa_3\}$. These representations are induced by the sections mapping the isotropy group into the full group

$$\begin{aligned} &(\rho_{\kappa_0,\kappa_1,\kappa_2,\kappa_3}(g(U, \vartheta, \omega, \iota))\, f)(z) = \\ &D(\kappa_1; \vartheta)\, D^{m,n}_{m'.n'}(\kappa_2, \kappa_3;\ U^\circ)\, e^{\kappa_0(i\,\iota + (\omega,\,\omega)/2 - (\omega, z))}\ f_{m.,n}(U^{-1}\, e^{-i\,\vartheta}(z + \omega)), \end{aligned} \tag{24}$$

where $U^\circ = Q_{U^{-1}z}{}^{-1}\, U^{-1}\, Q_z$ and $D(\kappa_1; \vartheta) = e^{i\kappa_1\vartheta}$ is the $\mathcal{U}(1)$ *D matrix*.

- $\mu^2 < 0$ **Case**

The representation theory of this case parallels the case $\mu^2 > 0$ except that the little group is the group $\mathcal{SU}(1, 2)$ rather than $\mathcal{SU}(3)$ and so the unitary representations are not finite dimensional.

- $\mu^2 = 0$ **Case**

In the null case, the representative point of the orbit may be taken to be $\tilde{\omega}^\circ = \{\tilde{\omega}\circ, 0, 0, \tilde{\omega}\circ\}$. The 3 generators $\{A^{\circ\pm}, A^{\circ\pm}, Z\}$ with $A^{\circ\pm} = \frac{1}{2}(A_3^\pm - A_1^\pm)$ satisfy the algebra

$$[A^{\circ\pm}, Z] = \pm A^{\circ\pm}, \quad [A^{\circ+}, A^{\circ-}] = 0, \tag{25}$$

which is the algebra of the $C\mathcal{E}(2)$ complex Euclidean group in 2 dimensions. The Casimir invariant is

$$c_2(C\mathcal{E}(2)) = A^{\circ+}\, A^{\circ-}. \tag{26}$$

The representations of this group are just the representations of the Oscillator group with $\kappa_0 = 0$ given in equation (18).

$$(\rho_{\tilde{\kappa}_0}(g(I, \tilde{\vartheta}, \tilde{\omega}^\circ, 0))\, f)(z) = e^{-i\tilde{\kappa_0}(\tilde{\omega}^\circ z + \overline{\tilde{\omega}^\circ} z^{-1})/2}\, f\left(e^{-i\tilde{\vartheta}}\, z\right). \tag{27}$$

In addition there is the degenerate case where $\omega^\circ = 0$ in which case the representation is simply the representation of $\mathcal{U}(1)$ with an algebra spanned by Z.

The algebra of the little group must commute with the generators $\{A^{\circ\pm}, A^{\circ\pm}, Z\}$. The maximal set of such generators are $\{C_i^\pm,\ Z_{i,j},\ I^\circ\}$ where $i = 1, 2$ in this section and

$$C_i^+ = Z_{i,0} - Z_{i,3}, \quad C_i^- = Z_{0,i} - Z_{3,i}, \quad I^\circ = Z - Z_{3,3}\ , \tag{28}$$

where in this section $i,\ j,\ \ldots = 1, 2$. These generators satisfy the algebra

$$[Z_{i,j}, Z_{k\,l}] = \delta_{j\,k}\, Z_{l\,i} - \delta_{i\,l}\, Z_{j\,k}, \quad [C_i^+, C_j^-] = \delta_{i\,j}\, I^\circ, \quad [Z_{i\,j}, C_k^\pm] = \pm\delta_{i\,k}\, C_i^\pm. \tag{29}$$

This is the algebra of the Canonical group in two dimension, $C(2)$. Thus, $C(2)$ appears as the Little group. (This is analogous to the two dimensional Euclidean group appearing in the null representation case of the Poincaré group.). $C(2)$ can be factored into $C(2) = \mathcal{SU}(2) \otimes_s \mathcal{O}s(2)$. The corresponding parameters of the group space are $\{\tilde{U}\}$ for $\mathcal{SU}(2)$ and $\{\tilde{\vartheta}, \tilde{\omega}_i, \tilde{\imath}\}$ for $\mathcal{O}s(2)$.

The general little group $\mathcal{O}s(2)$ has a representation of the form given in equation (24), but in two dimensions using the $\mathcal{SU}(2)$ *D matrices*:

$$(\rho_{\tilde{\kappa}_1,\tilde{\kappa}_2,\tilde{\kappa}_3}(g(\tilde{U},\,\tilde{\vartheta},\,\tilde{\omega},\,\tilde{\imath}))\,f)\,(z)\;=$$
$$D^{m}_{m'}\,(\tilde{\kappa}_2;\;\tilde{U})\,e^{\tilde{\kappa}_0(i\,\tilde{\imath}+(\tilde{\omega},\,\tilde{\omega})/2-(\tilde{\omega},z))+i\,\tilde{\kappa}_1\,\tilde{\vartheta}}\;f_{m.,n}\Big(\tilde{U}^{-1}\,e^{-i\,\tilde{\vartheta}}(z+\tilde{\omega})\Big). \tag{30}$$

This representation of the little group, together with the representation of the invariant subgroup $\mathcal{CE}(2)$, can be used to induce the full representation using equation (*3*). However, as the little groups are infinite dimensional, these probably are not physically interesting. However, analogous to the *Poincaré* representation theory, this group has the degenerate case $\mathcal{U}(1)\otimes\mathcal{SU}(2)$ that clearly has finite dimensional unitary representations in terms of the *D matrices* of $\mathcal{U}(1)$ and $\mathcal{SU}(2)$

$$(\rho_{\tilde{\kappa}_1,\tilde{\kappa}_2}(g(\tilde{U},\,\tilde{\theta}))\,f)\,(z)\;=D(\tilde{\kappa}_1;\;\tilde{\theta})\,D^{m}_{m'}\,(\tilde{\kappa}_2;\;\tilde{U}). \tag{31}$$

These representations are lower dimensional cases of representations that have already been computed. First, the degenerate case is given simply be the *D matrices* of the $\mathcal{U}(2)=\mathcal{U}(1)\otimes\mathcal{SU}(2)$ group:

$$(\rho_{\tilde{\kappa}_1,\tilde{\kappa}_2}(g(\tilde{U},\,\tilde{\theta}))\,f)\,(z)\;=D(\tilde{\kappa}_1;\;\tilde{\theta})\,D^{m}_{m'}\,(\tilde{\kappa}_2;\;\tilde{U})\,f_m(z). \tag{32}$$

Again, this can be used together with the representation of $\mathcal{E}(2)$ in equation (27) to induce the full representation using equation (3). The general representation is

$$(\rho_{\tilde{\kappa}_0,\tilde{\kappa}_1,\tilde{\kappa}_2}(g(\tilde{U},\,\tilde{\theta},\,\tilde{\vartheta},\,\tilde{\omega},\,\tilde{\imath}))\,f)\,(z)\;=$$
$$D(\tilde{\kappa}_2;\;\tilde{\vartheta})\,D^{m}_{m'}\,(\tilde{\kappa}_2;\;\tilde{U})\,e^{-i\,\tilde{\kappa}_0(\tilde{\omega}\,\circ\,z+\overline{\tilde{\omega}\,\circ\,z^{-1}})/2}\,f_m\Big(\tilde{U}^{-1}\,e^{-i\,\tilde{\theta}}\,e^{-i\,\tilde{\vartheta}}(z+\tilde{\omega})\Big), \tag{33}$$

III Discussion and Conclusions

A Parallels with the Poincaré Representation theory

The induced representation theory was developed in the context of the Poincaré group $\mathcal{P} = \mathcal{L} \otimes_s \mathcal{T}(4)$. As is well known, its representation theory divides into the time-like, space-like, null and degenerate cases based on the values of the Casimir invariant $m^2 = Y \cdot Y$ where Y_a are the generators of the algebra of $\mathcal{T}(4)$. These define the mass and null hyperboloids that are generated by applying the pure Lorentz transformations Q_Y to the rest frame point on the hyperboloid $Y^\circ = (m, 0, 0, 0)$ which is simply an orbit in $\mathbb{R}^4$, $Y = Q_Y Y^\circ$. The physically interesting cases are the time-like and null cases. The time-like little group is the rotation group $\mathcal{R}(3)$, or in the case of the covering group, $\mathcal{SU}(2)$. This has finite dimensional unitary representation in terms of the usual *D Matrices* labelled by the spin that is the eigenvalue of the Casimir invariant of the group. The null little group is $\mathcal{E}(2) = \mathcal{U}(1) \otimes_s \mathcal{T}(2)$. No finite dimensional representations exist for the full group except for the degenerate case $\mathcal{U}(1)$.

The $C(1, 3) = \mathcal{SU}(1, 3) \otimes_s \mathcal{O}s(1, 3)$ representation theory follows the same overall structure with key differences. In this case, the invariant defined by the Hermitian inner product (A^+, A^+) is not a Casimir invariant, but it is invariant under the action of $\mathcal{SU}(1, 3)$. The hyperboloids are orbits in $\mathcal{O}s(1, 3)$ that are generated by the $\mathcal{SU}(1, 3)$ applied to the points in $\mathcal{O}s(1, 3)$. Thus points on the hyperboloids are quantum oscillators; they are the primitive elements or *points* in the theory. The hyperboloids are generated by the action of the pure boosts applied to the generalized rest frame element $\{I, Z, A^{+\circ}, I\}$ with $A^{+\circ} = \{A_0^+, 0, 0, 0\}$. That is, $\{I, Z, A^+, I\} = Q(\xi)\{I, Z, A^{+\circ}, I\} = \{I, Z, Q(\xi) A^{+\circ}, I\}$ as I, Z, I are invariant under the action of $Q(\xi)$. $\xi^i = \beta^i + i\gamma^j$ is the complex hyperbolic angle defining the pure boost hyperbolic rotation, $i, j = 1, 2, 3$.

The boosts may be computed straightforwardly. First, define $A_a^+ = X_a + i\, Y_a$ where $\{X_a, Y_a\}$ are real. These can be further defined in terms of $\{T, Q_i, P_i, E\}$ that have the dimensions of time, position, momentum and energy respectively. Introduce the Plank scales for the dimensions of the 4 physical dimensional degrees of freedom:

$$\lambda_t = \sqrt{h/b\,c},\ \lambda_q = \sqrt{h\,c/b},\ \lambda_p = \sqrt{h\,b/c},\ \lambda_e = \sqrt{h\,b\,c}\ . \tag{34}$$

where c is the speed of light, h is Planck's constant and b is a constant defined by $b = \alpha_G\, c^4 / G$ where G is the gravitational constant and α_G is a dimensionless constant. Actually, these are strictly only the Planck scales if $\alpha_G = 1$ but we allow the more general case. In fact, the transformation equations that follow lead naturally to the conclusion that b with the dimensions of rate of change of momentum with respect to time is the third

fundamental dimensional constant and that G is defined by $G = \alpha_G c^4 / b$. Straightforward computation given in [3] yields the expression for the transformation of A^+ under the action of the pure boost $Q(\xi)$.

$$
\begin{aligned}
T' &= \cosh\zeta\, T + \tfrac{\sinh\zeta}{\zeta}\,(\beta^i Q_i / c^2 + \gamma^i P_i / b^2),\\
E' &= \cosh\zeta\, E + \tfrac{\sinh\zeta}{\zeta}\,(-\gamma^i Q_i / c^2 + \beta^i P_i / b^2),\\
Q_i' &= \tfrac{\cosh\zeta - 1}{\zeta^2}\,(\beta^i \beta^j / c^2 + \gamma^i \gamma^j / b^2)\, Q_j + \tfrac{\sinh\zeta}{\zeta}\,(\beta^i T - \gamma^i E / b^2),\\
P_i' &= \tfrac{\cosh\zeta - 1}{\zeta^2}\,(\beta^i \boldsymbol{\beta}^j / c^2 + \gamma^i \gamma^j / b^2)\, P_j + \tfrac{\sinh\zeta}{\zeta}\,(\beta^i E / c^2 - \gamma^i T).
\end{aligned}
\tag{35}
$$

with $\zeta = \sqrt{\beta^i \beta^i / c^2 + \gamma^i \gamma^i / b^2}$.

Analysis of these equations shows that in the limits $b \to \infty$ this reduces to the usual pure Lorentz boosts on the $\{T, Q_i\}$ and $\{E, P_i\}$ degrees of freedom. Correspondingly, the limit $c \to \infty$ yields *Lorentz boosts* on the $\{T, P_i\}$ and $\{E, Q_i\}$ degrees of freedom. These correspond to a generalized notation of the addition of rates of change of momentum that are bounded by b.

B Conclusions

The Canonical Group warrants further study if only from the fact that almost all of the major groups that appear in current physical theories are woven into it. As we have noted, there are 4 canonical copies of the Poincaré group embedded as subgroups in the Canonical group. Each of these Poincaré groups has a translation subgroup that is one of the 4 canonical four dimensional translation groups embedded as subgroups in the Weyl-Heisenberg group. In a given realization of the Canonical group, only one of these four translation subgroups can have an algebra with diagonal generators.

It contains a copy of the Weyl-Heisenberg group and the Oscillator group. The Little groups includes the special unitary groups $\mathcal{SU}(3)$, $\mathcal{SU}(2)$ and $\mathcal{U}(1)$.

From a physical perspective, the theory introduces a physical constant b that together with c, h define the 3 fundamental dimensional constants for physics. The effects of the full $C(1, 3)$ transformations become most significant at very high rates of change or momentum, or forces. As has been noted, in a very strongly interacting system, one has the notion of momentum and energy degrees of freedom transforming, or you could say, condensing, into position and time degrees of freedom. This has certain parallels with the mixing of position and time degrees of freedom in the Lorentz transformations. The pure boosts in full $C(1, 3)$ case reduce in the limit to the Lorentz boosts.

The representation theory of $C(1, 3)$ has time-like, space-like, null and degenerate cases based on the values of the quasi-invariant μ^2. The little groups of the time-like case are $\mathcal{SU}(3)$. The null case has the little group $C(2) = \mathcal{U}(2) \otimes_s \mathcal{H}(2)$. Like the Poincaré case, this group has no finite dimensional unitary representations except for the degenerate case $\mathcal{U}(1) \otimes \mathcal{SU}(2)$.

The next step in this investigation is to study further the limiting behavior of the full representation theory in the limit $b \to \infty$. The problem is then to construct a physical theory that arise from these representations and that, at least in the limit, corresponds to existing theories.

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