PERTURBATION THEORY FOR LYAPUNOV EXPONENTS OF A TORAL MAP: EXTENSION OF A RESULT OF SHUB AND WILKINSON.

by David Ruelle*.

Abstract. Starting from a hyperbolic toral automorphism, we obtain, for a small volume preserving perturbation, an exact and rigorous second order perturbation expansion of the Lyapunov exponents.

Keywords: Lyapunov exponent, toral automorphism, hyperbolicity.

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We consider volume preserving perturbations F of a diffeomorphism $F_0 = (\Phi, J)$ of $\mathbf{T}^{m+1} = \mathbf{T}^m \times \mathbf{T}$, where Φ is a hyperbolic automorphism of $\mathbf{T}^m$, and J is a translation of $\mathbf{T}$. Writing $F = F_0 + aF'$, we shall show that the Lyapunov exponents for (F, volume) can be expanded to second order in a (Theorem 1). In particular, the central Lyapunov exponent λ^c of (F, volume), to second order in a, is generally $\neq 0$ (Corolary 11). For a special family of perturbations one obtains particularly simple formulae, first noted by Shub and Wilkinson [10]. We recover their result in Theorem 12. We deviate from [10] mostly in that we don't have differentiability of λ^c, only a second order expansion around $a = 0$. The ideas used here are largely those in Shub and Wilkinson [10], and can be appreciated in the background provided by Hirsh, Pugh and Shub [6], Burns and Wilkinson [3], Ruelle and Wilkinson [9]. For recent work related to Lyapunov exponents, see also Bonatti, Gómez-Mont and Viana [2], Avila and Bochi [1].

After completing the writing of this paper, the author received a preprint by D. Dolgopyat [4], which develops similar ideas in a more general setting, but without the specific formulas we obtain here.

1. Theorem.

Let Φ be a hyperbolic automorphism of $\mathbf{T}^m$, and $J : y \mapsto y + \alpha \pmod 1$ a translation of $\mathbf{T}$. Define $F_0 = (\Phi, J)$, and let $F = F_0 + aF'$ be a C^2 volume preserving perturbation of F_0. (We take $F' : \mathbf{T}^m \mapsto \mathbf{R}^m$ and $F_0\xi + aF'(\xi)$ has to be understood (mod 1) in each component). Let $\lambda_1 < \lambda_2 < \dots$ be the Lyapunov exponents of (F_0, volume) and $m_1, m_2, \dots$ their multiplicities (the exponent $= 0$ occurs with multiplicity 1). Also let $\lambda_a^{(1)} \le \lambda_a^{(2)} \le \dots$ be the Lyapunov exponents of (F, volume) repeated according to multiplicity. Then we have the second order expansion

$$\sum_{\ell=m_1+\dots+m_{r-1}+1}^{m_1+\dots+m_r} \lambda_a^{(\ell)} = m_r\lambda_r + a^2 L_r + o(a^2)$$

If $m_r = 1$, and writing $\lambda_r = \lambda_0^{(\ell)}$, we have

$$\lambda_a^{(\ell)} = \lambda_0^{(\ell)} + a^2 L^{(\ell)} + o(a^2)$$

(this applies in particular to $\lambda^c = \lambda_a^{(\ell)}$ for $\lambda_0^{(\ell)} = 0$).

An explicit expression for L_r can be obtained (see Proposition 9). We do not assume ergodicity of (F, volume), and therefore we use *integrated* Lyapunov exponents (averaged over the volume), see however Remark 15(a).

2. Normal hyperbolicity.

As in [10], we invoque the theory of normal hyperbolicity of [6]. We start from the fact that F_0 is normally hyperbolic to the smooth fibration of $\mathbf{T}^{m+1}$ by circles $\{x\} \times \mathbf{T}$. Taking some $k \ge 2$ we apply [6] Theorems (7.1), (7.2). Thus we obtain a C^1 neighborhood U of F_0 in the C^k diffeomorphisms of $\mathbf{T}^{m+1}$ such that, for $F \in U$, there is an equivariant fibration $\pi : \mathbf{T}^{m+1} \to \mathbf{T}^m$ with

$$\pi F = \Phi \pi$$

The fibers $\pi^{-1}\{x\}$ are C^k circles forming a continuous fibration of $\mathbf{T}^{m+1}$ (this fibration is in general not smooth). Furthermore there is a TF-invariant continuous splitting of $T\mathbf{T}^{m+1}$ into three subbundles:

$$T\mathbf{T}^{m+1} = E^s + E^u + E^c$$

such that E^c is 1-dimensional tangent to the circles $\pi^{-1}\{x\}$, E^s is m^s-dimensional contracting and E^u is m^u-dimensional expanding for TF.

If $\lambda_r < 0$ (and F is in a suitable C^1-small neighborhood U of F_0), we can introduce a continuous vector subbundle E^r of $T\mathbf{T}^{m+1}$ which consists of vectors contracting under TF^n faster than $(\lambda_r + \epsilon)^n$ where $\epsilon > 0$ and $\lambda_r + \epsilon < \lambda_{r+1}$. In fact E^r is a hyperbolic (attracting) fixed point for the action induced by TF^{-1} on the bundle of $m_1 + \ldots + m_r$ dimensional linear subspaces of $T\mathbf{T}^{m+1}$ (over F^{-1} acting on T^{m+1}).

If $\lambda_r > 0$, replacement of F by F^{-1} similarly yields a continuous subbundle $\bar{E}^r$ of $m_r + \ldots$ dimensional subspaces.

3. Proposition.

Assume that F is of class C^k, $k \geq 2$, and that F is C^k close to F_0. The bundles E^r, $\bar{E}^r$, when restricted to a circle $\pi^{-1}\{x\}$ are of class C^{r-1}, continuously in x.

If $\mathcal{G}$ denotes the (Grassmannian) manifold of $m_1+\ldots+m_r$ dimensional linear subspaces of $\mathbf{R}^{m+1}$, we may identify the bundle of $m_1 + \ldots + m_r$ dinensional linear subspaces of $T\mathbf{T}^{m+1}$ with $\mathbf{T}^{m+1} \times \mathcal{G}$. We denote by $\mathcal{E} \in \mathcal{G}$ the spectral subspace of the matrix defining Φ corresponding to the smallest $m_1+\ldots+m_r$ eigenvalues (in absolute value, and repeated according to multiplicity).

If $\mathcal{F}_0$ is the action defined by TF_0 on $T\mathbf{T}^{m+1} \times \mathcal{G}$, the circles $\{x\} \times \mathbf{T} \times \{\mathcal{E}\}$ form an $\mathcal{F}_0$ invariant fibration of $\mathbf{T}^{m+1} \times \{\mathcal{E}\}$, to which $\mathcal{F}_0$ is normally hyperbolic. If F is C^k close to F_0, the corresponding C^{k-1} action $\mathcal{F}$ is normally hyberbolic to a pertubed fibration where $\{x\} \times \mathbf{T} \times \{\mathcal{E}\}$ is replaced by $E^r|\pi^{-1}\{x\}$. According to [6] Theorem 7.4, Corollary (8.3) and the following Remark 2, we find that the C^{k-1} circle $E^r|\pi^{-1}\{x\} \subset \mathbf{T}^{m+1} \times \mathcal{G}$ depends continuously on $x \in \mathbf{T}^{m+1}$. Similarly for $\bar{E}$. □

Note that in [10], the C^r section theorem is used in a similar situation, giving estimates uniform in x. However, continuity in x (not just uniformity) will be essential for us in what follows.

4. Corollary.

The splitting $T\mathbf{T}^{m+1} = E^s + E^u + E^c$ when restricted to a circle $\pi^{-1}\{x\}$ is of class C^{k-1}, continuously in x.

It is clear that $E^c|\pi^{-1}\{x\}$ is of class C^{k-1} because it is the tangent bundle to the C^k circle $\pi^{-1}\{x\}$. As to E^s, E^u, they are special cases of E^r, $\bar{E}^r$. □

Notation.

Remember that $F = F_0 + aF'$, and fix F'. We shall use the notation π_a, E^r_a, ... to indicate the a-dependence of π, E^r, ...

5. Proposition.

For small $\epsilon > 0$ there is a continuous function $x \mapsto \gamma_x$ from $\mathbf{T}^m$ to $C^k(\mathbf{T} \times (-\epsilon, \epsilon) \to \mathbf{T}^m)$ such that $\gamma_x(y, 0) = 0$ and $\pi_a^{-1}\{x\} = \{x + \gamma_x(y, a) : y \in \mathbf{T}\}$.

To see this define $\tilde{F} : \mathbf{T}^{m+1} \times (-\epsilon, \epsilon) \to \mathbf{T}^{m+1} \times (-\epsilon, \epsilon)$ by $\tilde{F}(\xi, a) = ((F_0 + F')(\xi), a)$ and observe that $\tilde{F}$ is normally hyperbolic to the 2-dimensional manifolds

$$\cup_{a \in (-\epsilon, \epsilon)} (\pi_a^{-1}\{x\}, a)$$

and these are thus C^k 2-dimensional submanifolds of $\mathbf{T}^{m+1} \times (-\epsilon, \epsilon)$. □

We may in the same manner replace $\pi_a^{-1}\{x\}$ by $\cup_{a \in (-\epsilon, \epsilon)} (\pi_a^{-1}\{x\}, a)$ in Proposition 3 and Corollary 4. Writing E_a for E_a^r, $\bar{E}_a^r$, E_a^s, E_a^u, E_a^c, we obtain that $(\cdot, a) \mapsto E_a(\cdot)$, when restricted from $\mathbf{T}^{m+1} \times (-\epsilon, \epsilon)$ to $\cup_{a \in (-\epsilon, \epsilon)} (\pi_a^{-1}\{x\}, a)$ is of class C^{k-1}. We rephrase this as follows:

6. Proposition.

The map

$$x \mapsto \{(y, a) \mapsto E_a(x + \gamma(y, a))\}$$

where E_a stands for E_a^r, $\bar{E}_a^r$, E_a^s, E_a^u, E_a^c, is continuous $\mathbf{T}^m \to \mathrm{C}^{k-1}(\mathbf{T} \times (-\epsilon, \epsilon) \to$ Grassmannian of $\mathbf{R}^{m+1})$ where we have used the identification $T\mathbf{T}^{m+1} = \mathbf{T}^{m+1} \times \mathbf{R}^{m+1}$. □

Notation.

From now on we write E_a for E_a^r, $\bar{E}_a^r$, E_a^s, E_a^u, E_a^c. When $a = 0$, E_0 is a trivial subbundle of $T\mathbf{T}^{m+1} = \mathbf{T}^{m+1} \times \mathbf{R}^{m+1}$, and we shall write $E_0 = \mathbf{T}^{m+1} \times \mathcal{E}$, denoting thus by $\mathcal{E}$ a spectral subspace of the matrix on $\mathbf{R}^{m+1}$ defining $(\Phi, 1)$. We denote by $\mathcal{E}^\perp$ the complementary spectral subspace.

Taking $k = 2$ we have then:

7. Corollary.

There are linear maps $G(x, y), R(x, y, a) : \mathcal{E} \to \mathcal{E}^\perp$ such that $G(x, y)$ depends continuously on $(x, y) \in \mathbf{T}^m \times \mathbf{T}$, $R(x, y, a)$ on $(x, y, a) \in \mathbf{T}^m \times \mathbf{T} \times (-\epsilon, \epsilon)$,

$$E_a(x + \gamma_x(y, a), y) = \{X + aG(x, y)X + R(x, y, a)X : X \in E_0\}$$

and $||R(x, y, a)||$ is $o(a)$ uniformly in x, y. □

Notice now that, if $\tilde{x} = \pi_a(x, y)$, then $x = \tilde{x} + \gamma_{\tilde{x}}(y, a)$, where $\gamma_{\tilde{x}}(y, a) = O(a)$. Now

$$E_a(x, y) = E_a(\tilde{x} + \gamma_{\tilde{x}}(y, a), y) = \{X + aG(\tilde{x}, y)X + R(\tilde{x}, y, a)X : X \in \mathcal{E}\}$$

differs from

$$E_a(x + \gamma_x(y, a), y) = \{X + aG(x, y)X + R(x, y, a)X : X \in \mathcal{E}\}$$

by the replacement $\tilde{x} \to x$ in the right-hand side, and since $\mathrm{dist}(\tilde{x}, x) = O(a)$, we find that $\mathrm{dist}(E_a(x,y), E_a(x + \gamma_x(y,a), y)) = o(a)$. Therefore, changing the definition of R, we can again write:

8. Corollary.

There are linear maps $G(x,y), R(x,y,a) : \mathcal{E} \to \mathcal{E}^\perp$, depending continuously on their arguments, such that

$$E_a(x,y) = \{X + aG(x,y)X + R(x,y,a)X : X \in \mathcal{E}\}$$

and $||R(x,y,a)||$ is $o(a)$ uniformly in x, y. □

We may write $T_\xi F = T_\xi(F_0 + aF') = D_0 + aD'(\xi)$ where D_0 does not depend on ξ and preserves the decomposition $T_\xi M = \mathcal{E} + \mathcal{E}^\perp$. If we apply TF to an element $X + aGX + RX$ of E_a (as in Corollary 8) we obtain $X_1 +$ element of $\mathcal{E}^\perp \in E_a$, with $X_1 \in \mathcal{E}$:

$$X_1 = D_0 X + aD'X + a^2 D'GX + aD'RX \qquad \text{projected on } \mathcal{E} \tag{1}$$

Under $(TF)^\wedge$, the volume element θ in $E_a(\xi)$ is multiplied by a factor $M(\xi, a)$, and the projection in $\mathcal{E}$ of $(TF)^\wedge\theta$ is equal to the projection in $\mathcal{E}$ of θ multiplied by a factor $N(\xi, a)$ such that

$$M(\xi, a) = N(\xi, a) + \ell_a(\xi) - \ell_a(F\xi)$$

for suitable ℓ_a. We may compute N from (1):

$$N(\xi, a) = N_{(0)} + aN_{(1)}(\xi) + a^2 N_{(2)}(\xi) + o(a^2)$$

To proceed we take now $E_a = E_a^r$, and assume $\lambda_r < 0$. We have then, writing $d\xi$ for the volume element in T^{m+1},

$$L_a = \sum_{\ell=1}^{m_1+\ldots+m_r} \lambda_a^{(\ell)} = \int d\xi \, \log M(\xi, a) = \int d\xi \, \log N(\xi, a)$$

$$= L_{(0)} + aL_{(1)}(\xi) + a^2 L_{(2)}(\xi) + o(a^2) \tag{2}$$

More precisely, we shall prove

9. Proposition.

If $\lambda_r < 0$, we have

$$\sum_{\ell=1}^{m_1+\ldots+m_r} \lambda_a^{(\ell)} = \sum_{k=1}^{r} m_k \lambda_k + a^2 L + o(a^2)$$

where

$$L = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(F_0^n \xi)) \geq 0$$

and $\mathcal{E}$ is the spectral subspace of the matrix defining Φ corresponding to the smallest $m_1 + \ldots + m_r$ eigenvalues (in absolute value, and repeated according to multiplicity).

The proof that $L \geq 0$ is postponed to Remark 15(b).

The proposition is obtained by comparing formula (2) with the formula (5) below, which we shall obtain by a second order perturbation calculation.

To first order in a we have

$$F^n = (F_0 + aF')^n = F_0^n + a\sum_{j=1}^{n} F_0^{n-j} \circ F' \circ F_0^{j-1}$$

hence

$$T_\xi F^n = D_0^n + a\sum_{j=1}^{n} D_0^{n-j} D'(F^{j-1}\xi) D_0^{j-1}$$

If we apply TF^n to $X + aGX + RX \in E_a$ we obtain $X_n +$ element of $\mathcal{E}^\perp \in E_a$, with $X_n \in \mathcal{E}$. To zero-th order in a, $X_n = D_0^n X$, so we may write to first order $X_n = D_0^n X + aY_n(\xi)$. Therefore, to first order in a,

$$D_0^n X + aY_n(\xi) + aG(F^n\xi)D_0^n X = D_0^n X + a\sum_{j=1}^{n} D_0^{n-j} D'(F^{j-1}\xi) D_0^{j-1} X + aD_0^n G(\xi)X$$

and, taking the components along $\mathcal{E}^\perp$,

$$G(F^n\xi)D_0^n X = \sum_{j=1}^{n} D_0^{n-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X + D_0^n G(\xi)X$$

where $D'_\perp(.)$ is $D'(.)$ followed by taking the component along $\mathcal{E}^\perp$, or

$$\sum_{j=1}^{n} D_0^{-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X + G(\xi)X = D_0^{-n} G(F^n\xi) D_0^n X$$

When $n \to \infty$, the right-hand side tends to zero (exponentially fast, remember that $X \in \mathcal{E}$, $GX \in \mathcal{E}^\perp$). Therefore

$$G(\xi)X = -\sum_{j=1}^{\infty} D_0^{-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X$$

which we shall use in the form

$$G(\xi)X = -\sum_{n=0}^{\infty} D_0^{-n-1} D'_\perp(F_0^n\xi) D_0^n X \tag{3}$$

where we have written F_0^n instead of F^n since G is evaluated to order 0 in a. (The right-hand side is an exponentially convergent series).

Returning to (1) we see that, to second order in a,

$$X_1 = D_0 X + aD'(\xi)X + a^2 D'(\xi)G(\xi)X \qquad \text{projected on } \mathcal{E}$$

$$= D_0(1 + aD_0^{-1}D'(\xi) + a^2 D_0^{-1}D'(\xi)G(\xi))X \qquad \text{projected on } \mathcal{E}$$

Let now $(u^{(i)})$ and $(u^{(i)\perp})$ be conjugate bases of $\mathcal{E}$. Also let $\delta^{(i)}$ for $i = 1, \ldots, m_1 + \ldots + m_r$ be the eigenvalues of D_0 restricted to $\mathcal{E}$. Then, to second order in a,

$$N(\xi, a) \wedge_1^{m_1+\ldots+m_r} u^{(\ell)}$$

is, up to a factor of absolute value 1,

$$(\prod_{\ell=1}^{m_1+\ldots+m_r} \delta^{(\ell)})[1 + a \sum_{i=1}^{m_1+\ldots+m_r} (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)})$$

$$+a^2 \sum_{i<j} ((u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(j)})$$

$$-(u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(j)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(i)}) + a^2 \sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)G(\xi)u^{(i)})] \wedge_\ell u^{(\ell)}$$

so that

$$N(\xi, a) = (\prod_{\ell=1}^{m_1+\ldots+m_r} |\delta^{(\ell)}|)[1 + \{a \sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)})$$

$$+a^2 \sum_{i<j} ((u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(j)})$$

$$-(u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(j)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(i)})) + a^2 \sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)G(\xi)u^{(i)})\}]$$

Since $\log |\delta^{(\ell)}| = \lambda_0^{(\ell)}$ we obtain, to second order in a,

$$L_a = \int d\xi \, \log N(\xi, a) = m_1\lambda_1 + \ldots + m_r\lambda_r + \int d\xi \, [\{\ldots\} - \frac{a^2}{2}(\sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)}))^2]$$

where $\{\ldots\}$ has the same meaning as above. Write

$$\Psi_i(\sum_\ell \xi_\ell u^{(\ell)}) = (u^{(i)\perp}, D_0^{-1}F'(\sum_\ell \xi_\ell u^{(\ell)}))$$

The first term of $\int d\xi \, \{\ldots\}$ is

$$a \sum_i \int d\xi \, (u^{(i)\perp}, D_0^{-1}TF'(\xi)u^{(i)}) = a \sum_i \int d\xi \, \frac{\partial}{\partial \xi_i}\Psi_i$$

which vanishes because $\int d\xi \, \frac{\partial}{\partial \xi_i} \ldots = 0$. The next term in $\int d\xi \, \{\ldots\}$ is

$$a^2 \sum_{i<j} \int d\xi \, ((\frac{\partial \Psi_i}{\partial \xi_i})(\frac{\partial \Psi_j}{\partial \xi_j}) - (\frac{\partial \Psi_i}{\partial \xi_j})(\frac{\partial \Psi_j}{\partial \xi_i})) = a^2 \sum_{i<j} \int d\xi \, (\frac{\partial}{\partial \xi_i}(\Psi_i \frac{\partial \Psi_j}{\partial \xi_j}) - \frac{\partial}{\partial \xi_j}(\Psi_i \frac{\partial \Psi_j}{\partial \xi_i}))$$

which vanishes as above. Thus we are left with

$$L_a - (m_1 \lambda_1 + \ldots + m_r \lambda_r)$$

$$= a^2 \int d\xi \, [\sum_i (u^{(i)\perp}, D_0^{-1} D'(\xi) G(\xi) u^{(i)}) - \frac{1}{2} (\sum_i (u^{(i)\perp}, D_0^{-1} D'(\xi) u^{(i)}))^2] \qquad (4)$$

and we may write, using (3),

$$\sum_i (u^{(i)\perp}, D_0^{-1} D'(\xi) G(\xi) u^{(i)}) = -\sum_{n=0}^{\infty} \sum_i (u^{(i)\perp}, D_0^{-1} D'(\xi) D_0^{-n-1} D'_\perp(F_0^n) D_0^n u^{(i)})$$

$$= -\sum_{n=0}^{\infty} \sum_i \sum_j{}^* (u^{(i)\perp}, D_0^{-1} D'(\xi) u^{(j)}) (u^{(j)\perp}, D_0^{-n-1} D'(F_0^n \xi) D_0^n u^{(i)})$$

where we have introduces conjugate bases $(u^{(j)})$, $(u^{(j)\perp})$ of $\mathcal{E}$, indexed by $j = m_1 + \ldots + m_r + 1, \ldots, m+1$, and $\sum_i$ is over $i \leq m_1 + \ldots + m_r + 1$, $\sum_j^*$ is over $j > m_1 + \ldots + m_r + 1$. The above expression is also

$$= -\sum_{n=0}^{\infty} \sum_i \sum_j{}^* \frac{\partial}{\partial \xi_j} (u^{(i)\perp}, D_0^{-1} F'(\sum_\ell \xi_\ell u^{(\ell)})) \frac{\partial}{\partial \xi_i} (u^{(j)\perp}, D_0^{-n-1} F'(F_0^n \sum_\ell \xi_\ell u^{(\ell)}))$$

and integration by part gives thus

$$\int d\xi \, \sum_i (u^{(i)\perp}, D_0^{-1} D'(\xi) G(\xi) u^{(i)})$$

$$= -\sum_{n=0}^{\infty} \int d\xi \, \sum_i \frac{\partial}{\partial \xi_i} (u^{(i)\perp}, D_0^{-1} F'(\sum_\ell \xi_\ell u^{(\ell)})) \sum_j{}^* \frac{\partial}{\partial \xi_j} (u^{(j)\perp}, D_0^{-n-1} F'(F_0^n \sum_\ell \xi_\ell u^{(\ell)}))$$

$$= -\sum_{n=0}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \mathrm{Tr}_{\mathcal{E}^\perp}(D_0^{-n-1} D'(F_0^n \xi) D_0^n)$$

$$= -\sum_{n=0}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \mathrm{Tr}_{\mathcal{E}^\perp}(D_0^{-1} D'(F_0^n \xi))$$

The fact that $F = F_0 + aF'$ is volume preserving is expressed by $\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1} D'(\xi)) = 0$ hence

$$\int d\xi \, \sum_i (u^{(i)\perp}, D_0^{-1} D'(\xi) G(\xi) u^{(i)})$$

$$= \sum_{n=0}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(F_0^n \xi))$$

and introducing this in (4) yields

$$L_a - (m_1 \lambda_1 + \ldots + m_r \lambda_r)$$

$$= a^2 [\sum_{n=1}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(F_0^n \xi)) + \frac{1}{2} \int d\xi \, (\mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)))^2]$$

$$= \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(F_0^n \xi)) \tag{5}$$

where the last step used the invariance of $d\xi$ under F_0^n. □

10. Proof of Theorem 1.

We use Proposition 9, the corresponding result with F replaced by F^{-1}, and the fact that $\sum_{\ell=1}^{m} \lambda_a^{(\ell)} = 0$ (because F is volume preserving). This gives an estimate of all the sums of $\lambda_a^{(\ell)}$ that occur in Theorem 1. □

11. Corollary.

In the situation of Theorem 1, the central Lyapunov exponent is

$$\lambda^c = \frac{a^2}{2} \sum_{-\infty}^{\infty} \int d\xi \, [\mathrm{Tr}^u(D_0^{-1} D'(\xi)) \mathrm{Tr}^u(D_0^{-1} D'(F_0^n \xi)) - \mathrm{Tr}^s(D_0^{-1} D'(\xi)) \mathrm{Tr}^s(D_0^{-1} D'(F_0^n \xi))]$$

$$= \frac{a^2}{2} \sum_{-\infty}^{\infty} \int d\xi \, [\mathrm{Tr}^s(D_0^{-1} D'(\xi)) - \mathrm{Tr}^u(D_0^{-1} D'(\xi))] \mathrm{Tr}^c(D_0^{-1} D'(F_0^n \xi))$$

where Tr^s, Tr^u, Tr^c *denote the traces over the spectral subspaces* $\mathcal{E}^s$, $\mathcal{E}^u$, $\mathcal{E}^c$ *of* D_0 *corresponding to eigenvalues* < 1, > 1, *or* $= 1$ *in absolute value* ($\mathcal{E}^c$ *is one dimensional).*

Since F preserves the volume, the sum of all Lyapunov exponents vanishes. Therefore λ^c is minus the sum of the negative Lyapunov exponents, given by (5), minus the sum of the positive Lyapunov exponents. Note that replacing F by F^{-1}, $\mathcal{E}^s$ by $\mathcal{E}^u$ (and, to the order considered, $D'(\xi)$ by $-D'(\xi)$) replaces the sum of the negative Lyapunov exponents by minus the sum of the positive exponents. This gives the first formula for λ^c.

To obtain the second formula, express $\mathrm{Tr}^u \mathrm{Tr}^u - \mathrm{Tr}^s \mathrm{Tr}^s$ in terms of $\mathrm{Tr}^u \pm \mathrm{Tr}^s$, and remember that (because F preserves the volume) $\mathrm{Tr}^s + \mathrm{Tr}^u + \mathrm{Tr}^c = 0$ when applied to $D_0^{-1} D'(\xi)$. □

The above formula (5) takes a particularly simple form in a special case described in the next theorem.

12. Theorem.

Let Φ be a hyperbolic automorphism of $\mathbf{T}^m$, with stable and unstable dimensions m^s and $m^u = m - m^s$, and with entropy λ_0^u. Let $J : y \to y + \alpha \pmod 1$ be a translation of $\mathbf{T}$, and $\phi : \mathbf{T}^m \to \mathbf{T}$ a morphism $\neq 0$. Finally let $\psi : \mathbf{T} \to \mathbf{R}^m$ be a nullhomotopic C^2 function.

Define $h, g_a : \mathbf{T}^m \times \mathbf{T} \to \mathbf{T}^m \times \mathbf{T}$ by

$$h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \Phi x \\ Jy + \phi\Phi x - \phi x \end{pmatrix} \qquad , \qquad g_a\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + a\psi(y) \pmod 1 \\ y \end{pmatrix}$$

and let $f_a = g_a \circ h$.

Denote by λ_a^s (resp. λ_a^u) the sum of the smallest m^s (resp. the largest m^u) Lyapunov exponents for (f_a, volume). Also let $\lambda_a^c = -\lambda_a^s - \lambda_a^u$ be the "central exponent". Then λ_a^s, λ_a^u, λ_a^c have expansions of order 2 in a:

$$\lambda_a^s = -\lambda_0^u + \frac{a^2}{2}\int_{\mathbf{T}} dy\,((\nabla\phi)\psi'^s(y))^2 + o(a^2)$$

$$\lambda_a^u = \lambda_0^u - \frac{a^2}{2}\int_{\mathbf{T}} dy\,((\nabla\phi)\psi'^u(y))^2 + o(a^2)$$

$$\lambda_a^c = \frac{a^2}{2}\int_{\mathbf{T}} dy\,[((\nabla\phi)\psi'^u(y))^2 - ((\nabla\phi)\psi'^s(y))^2] + o(a^2)$$

Here $\psi'^s(y)$ and $\psi'^u(y)$ are the components of the derivative $\psi'(y) \in \mathbf{R}^m$ in the stable and unstable subspaces $\mathcal{E}^s$ and $\mathcal{E}^u$ for Φ. Also, we have used $\nabla\phi : \mathbf{R}^m \to \mathbf{R}$ to denote the derivative of the map $\phi : \mathbf{T}^m \to \mathbf{T}$ with the obvious identifications.

This theorem is a simple (but nontrivial) extension of the result proved by Shub and Wilkinson [10]. In the situation that they consider $\Phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, J =identity, $\phi = (1, 0)$, $\psi' = \psi'^u$. [Remark that, in the notation of [10], $u_0 = ((1,1).v_0)/(m-1) = ((1,0).v_0)$ so that the formula given in Proposition II of [10] agrees with our result above].

Notation.

We shall henceforth omit the (mod 1). We shall keep ∇ to denote the derivative in $\mathbf{T}^m$. With obvious abuses of notation, the reader may find it convenient to think of Φ or $\nabla\Phi$ as an $m \times m$ matrix (with integer entries and determinant ± 1), and ϕ or $\nabla\phi$ as a row m-vector (with integer entries not all zero).

13. Reformulation of the problem.

Note that $f_a^{-1} = h^{-1} \circ g_a^{-1}$ where h^{-1}, g_a^{-1} are obtained from h, g_a by the replacements $\Phi, J, \phi, \psi \to \Phi^{-1}, J^{-1}, \phi, -\psi$. These replacements also interchange the stable and unstable subspaces for Φ and replace λ^s, λ^u by $-\lambda^u$, $-\lambda^s$. Therefore the formula for λ^u in the theorem follows from the formula for λ^s. And the formula for $\lambda^c = -\lambda^s - \lambda^u$ also follows. To complete the proof of the theorem we turn now to the formula for λ^s.

Define

$$\hat{\phi}\binom{x}{y} = \binom{x}{y+\phi x}$$

then

$$F_0\binom{x}{y} = \hat{\phi}^{-1} h \hat{\phi}\binom{x}{y} = \binom{\Phi x}{Jy}$$

$$\hat{g}_a\binom{x}{y} = \hat{\phi}^{-1} g_a \hat{\phi}\binom{x}{y} = \binom{x + a\psi(y+\phi x)}{y - a(\nabla\phi)\psi(y+\phi x)}$$

so that

$$F\binom{x}{y} = \hat{\phi}^{-1} f_a \hat{\phi}\binom{x}{y} = \hat{g}_a F_0\binom{x}{y} = \binom{\Phi x + a\psi(Jy+\phi\Phi x)}{Jy - a(\nabla\phi)\psi(Jy+\phi\Phi x)}$$

Finally, $F = F_0 + aF'$ with

$$F_0\binom{x}{y} = \binom{\Phi x}{Jy} \qquad , \qquad F'\binom{x}{y} = \binom{\psi(Jy+\phi\Phi x)}{-(\nabla\phi)\psi(Jy+\phi\Phi x)}$$

Since F is conjugate (linearly) to f_a, we may compute λ^s from F instead of f_a.

14. Proof of Theorem 12.

Write $\mathbf{R}^{m+1} = \mathcal{E}^s + \mathcal{E}^u + \mathbf{R}$. We shall apply Proposition 9 with $\mathcal{E} = \mathcal{E}^s$, $\mathcal{E}^\perp = \mathcal{E}^u + \mathbf{R}$. Using $\xi = (x, y)$ and $X \in \mathcal{E}^s$, $Y \in \mathcal{E}^u$, $Z \in \mathbf{R}$ we may write

$$D_0\binom{X+Y}{Z} = \binom{(\nabla\Phi)(X+Y)}{Z}$$

$$D'(\xi)\binom{X+Y}{Z} = \binom{\psi'(Jy+\phi\Phi x)((\nabla\phi\Phi)(X+Y)+Z)}{-(\nabla\phi)\psi'(Jy+\phi\Phi x)((\nabla\phi\Phi)(X+Y)+Z)}$$

where ψ' denotes the derivative of ψ. Therefore

$$\mathrm{Tr}_{\mathcal{E}}(D'(\xi)D_0^{-1}) = (\nabla\phi)\psi'^s(Jy+\phi\Phi x)$$

and (5)contains the integrals

$$\int d\xi\, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n\xi))$$

$$= \int d\xi\, [(\nabla\phi)\psi'^s(Jy+\phi\Phi x)][(\nabla\phi)\psi'^s(J^{n+1}y+\phi\Phi^{n+1}x)]$$

Performing a change of variables $\bar{x} = \Phi x$, $\bar{y} = Jy + \phi\Phi x$ we find that this is

$$= \int d\bar{x}\, d\bar{y}\, [(\nabla\phi)\psi'^s(\bar{y})][(\nabla\phi)\psi'^s(J^n\bar{y}+\phi\Phi^n\bar{x}-\phi\bar{x})]$$

We claim that this last integral vanishes unless $n = 0$. This is because, if $n \neq 0$,

$$\int d\bar{x}\, \psi'(J^n \bar{y} + \phi\Phi^n \bar{x} - \phi\bar{x}) = 0$$

Indeed, $\phi\Phi^n\bar{x} - \phi\bar{x}$ is a linear combination with integer coefficients of the components $\bar{x}_1, \ldots, \bar{x}_m$ of $\bar{x}$, and the coefficients do not all vanish because $\phi\Phi^n = \phi$ is impossible (Φ is hyperbolic and $\phi \neq 0$). Integrating the derivative ψ' with respect to a variable $\bar{x}_j$ really occuring in $\phi\Phi^\ell\bar{x} - \phi\bar{x}$ gives zero as announced.

Returning to (5) we have thus

$$\lambda_a^s + \lambda_o^u = \frac{a^2}{2} \int d\xi\, (\mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)))^2$$

$$= \frac{a^2}{2} \int d\bar{y} ((\nabla\phi)\psi'^s(\bar{y}))^2$$

which is the formula given for λ_0^s in Theorem 12. And according to Section 13 this completes our proof. □

15. Final remarks.

(a) Shub and Wilkinson [10] showed that close to a diffeomorphism (hyperbolic automorphism Φ of $\mathbf{T}^2$) $\times$ (identity on $\mathbf{T}$) there is a C^1 open set of ergodic volume preserving C^2 diffeomorphisms of $\mathbf{T}^3$ with central Lyapunov exponent $\lambda^c > 0$. They remark that their result extends to the situation where Φ is a hyperbolic automorphism of $\mathbf{T}^m$ with one-dimensional expanding eigenspace. More generally, if Φ is any hyperbolic automorphism of $\mathbf{T}^m$, Theorem 12 gives close to (Φ, rotation of $\mathbf{T}$) in $\mathrm{C}^2(\mathbf{T}^{m+1})$ a diffeomorphism F with $\lambda^c > 0$. Since λ^c is given by an integral over the volume of a local "central" stretching exponent, we have $\lambda^c > 0$ in a C^1 neighborhood of F. But by a result of Dolgopyat and Wilkinson [5] (Corollary 0.5), stable ergodicity is here C^1 open and dense in the C^2 volume preserving diffeomorphisms: we have center bunching and stable dynamical coherence because we consider perturbations of (Φ, rotation of $\mathbf{T}$) for which the center foliation is C^1, see [6], [7]. In conclusion, close to (hyperbolic automorphism Φ of $\mathbf{T}^m$) $\times$ (rotation on $\mathbf{T}$) there is a C^1 open set V of ergodic volume preserving C^2 diffeomorphisms of $\mathbf{T}^{m+1}$ with central Lyapunov exponent $\lambda^c > 0$ (or also with $\lambda^c < 0$). In particular, if $F \in V$, the conditional measures of the volume on the circles $\pi^{-1}\{x\}$ are atomic, as discussed in [9].

(b) The coefficient L in Proposition 9 is ≥ 0. Consider indeed the unitary operator U defined by $U\psi = \psi \circ F$ on $L^2(\mathbf{T}^{m+1}, \text{volume})$, and let $E(.)$ be the corresponding spectral measure, so that

$$U = \int_{\mathbf{T}} e^{2\pi i\theta} E(d\theta)$$

If $\psi(\xi) = \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi))$ we have a measure $\nu \geq 0$ on $\mathbf{T}$ defined by $\nu(d\theta) = (\psi, E(d\theta)\psi)$ and the Fourier coefficients

$$c_n = \int e^{2\pi n i\theta} \nu(d\theta) = \int d\xi\, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi))(D_0^{-1} D'(F_0^n \xi))$$

of this measure tend to zero exponentially. Therefore $\nu(d\theta) = \rho(\theta)d\theta$ has a smooth density ρ and

$$L = \frac{1}{2}\sum_{n=-\infty}^{\infty} c_n = \frac{1}{2}\rho(0) \geq 0$$

(c) Suppose now that F is not necessarily a volume preserving perturbation of F_0. We may still hope that F has an SRB measure ρ_a. If F_0 were hyperbolic, we would have an expansion

$$\rho_a = \rho_0 + a\delta + o(a)$$

(see [8]) with ρ_0 = Lebesgue measure and δ a distribution. For smooth Ψ, $\delta(\Psi)$ is given (because ρ_0 is Lebesgue measure) by the simple formula (see [8])

$$\delta(\Psi) = -\sum_{0}^{\infty} \rho_0((\Psi \circ F_0^n).\mathrm{div}(F' \circ F_0^{-1})$$

Similarly (replacing F by F^{-1}, hence F_0, $D_0^{-1}D'(\xi)$ by F_0^{-1}, $-D'(F_0^{-1}\xi)D_0^{-1}$ we see that the anti-SRB state has an expansion

$$\bar{\rho}_a = \rho_0 + a\bar{\delta} + o(a)$$

with

$$\bar{\delta}(\Psi) = \sum_{n=1}^{\infty}\int d\xi\, \Psi(F_0^{-n}\xi)\mathrm{Tr}_{\mathbf{R}^{m+1}}(D'(F_0^{-1}\xi)D_0^{-1})$$

$$= \sum_{n=0}^{\infty}\int d\xi\, \Psi(F_0^{-n}\xi)\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(\xi))$$

We can now estimate the Lyapunov exponents for (F, ρ_a) to second order in a even though we are not sure of the existence of the SRB measure ρ_a. We simply assume that we can use the formula for $\delta(\Psi)$. Going through the proof of Proposition 9 we have to replace $\int d\xi\, \log N(\xi, a)$ by $\rho_a(\log N(., a))$ and (to second order in a) this adds to the right-hand side of (4) a term

$$-a^2\sum_{n=1}^{\infty}\int d\xi\, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(\xi))$$

Taking into account the integrations by part we obtain now instead of (5) the formula

$$L_a - (m_1\lambda_1 + \ldots + m_r\lambda_r) = \frac{a^2}{2}\sum_{n=-\infty}^{\infty}\int d\xi\, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n\xi))$$

$$-a^2\sum_{n=-\infty}^{\infty}\int d\xi\, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(F_0^n\xi)) \qquad (6)$$

Let a^2L^s, a^2L^u, a^2L^c be the a^2 contributions to the sum of the noncentral negative, noncentral positive, and the central Lyapunov exponents for the SRB measure. We obtain a^2L^s from (6) when $n_r = n^s$. A similar calculation gives a^2L^u (it is convenient here to work via the anti-SRB measure, then replace F by F^{-1}). Estimating the average expansion coefficient gives $a^2(L^s + L^u + L^c) = \rho_a(\log\det(D_0 + aD'(.))$, hence $L^s + L^u + L^c$, hence L^c. The results are

$$L^s = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\xi\,\mathrm{Tr}^s(D_0^{-1}D'(\xi))\mathrm{Tr}^s(D_0^{-1}D'(F_0^n\xi))$$

$$-\sum_{n=-\infty}^{\infty}\int d\xi\,\mathrm{Tr}^s(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(F_0^n\xi))$$

$$L^u = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\xi\,\mathrm{Tr}^u(D_0^{-1}D'(\xi))\mathrm{Tr}^u(D_0^{-1}D'(F_0^n\xi))$$

$$L^c = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\xi\,\mathrm{Tr}^c(D_0^{-1}D'(\xi))\mathrm{Tr}^c(D_0^{-1}D'(F_0^n\xi))$$

$$-\sum_{n=-\infty}^{\infty}\int d\xi\,\mathrm{Tr}^c(D_0^{-1}D'(\xi))\mathrm{Tr}^u(D_0^{-1}D'(F_0^n\xi))$$

$$L^s + L^u + L^c = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\xi\,\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(F_0^n\xi))$$

which can be rewritten variously.

The existence of second order expansions for the Lyapunov exponents gives added interest to the question whether the SRB measure ρ_a really exists for (small) finite a.

(d) The author has not looked seriously into possible extensions of the results presented here. Generalizations are thus left for the reader to formulate, and to prove.

References.

[1] A. Avila and J. Bochi. "A formula with some applications to the theory of Lyapunov exponents." Preprint

[2] C. Bonatti, X. Gómez-Mont and M. Viana. "Généricité d'expoants de Lyapunov non-nuls des produits déterministes de matrices." Preprint

[3] K. Burns and A. Wilkinson. "Stable ergodicity of skew products." Ann. Sci. Ecole Norm. Sup. **32**,859-889(1999).

[4] D. Dolgopyat. "On differentiability of SRB states." Preprint.

[5] D. Dolgopyat and A. Wilkinson. "Stable accessibility is C^1 dense." Preprint

[6] M. Hirsch, C.C. Pugh, and M. Shub. *Invariant manifolds.* Lect. Notes in Math. **583** Springer, Berlin, 1977.

[7] C. Pugh, and M. Shub. "Stable ergodicity and julienne quasi-conformality." J. Eur. Math. Soc. **2**,1-52(2000).

[8] D. Ruelle. "Differentiation of SRB states." Commun. Math. Phys. **187**,227-241(1997).

[9] D. Ruelle and A. Wilkinson. "Absolutely singular dynamical foliations." Commun. Math. Phys., to appear.

[10] M. Shub and A. Wilkinson. "Pathological foliations and removable exponents." Inventiones Math. **139**,495-508(2000).