Branched Polymers and Dimensional Reduction

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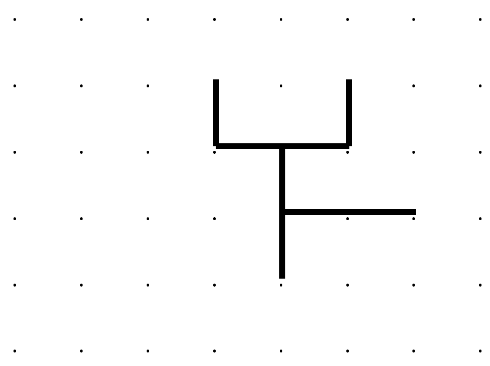
Abstract

We establish an exact isomorphism between self-avoiding branched polymers in $D+2$ continuum dimensions and the hard-core continuum gas at negative activity. We review conjectures and results on critical exponents for $D+2 = 2, 3, 4$ and show that they are corollaries of our result. We explain the connection (first proposed by Parisi and Sourlas) between branched polymers in $D+2$ dimensions and the Yang-Lee edge singularity in D dimensions.

1 Introduction

The picture shows an unlabeled self-avoiding branched polymer with $N = 9$ vertices on a two dimensional lattice.



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Following [Frö86, page 850], the number c_N of branched polymers modulo translations can be defined as follows: Let T be a tree graph on N vertices labeled $1, \ldots, N$ and let $y = (y_1, \ldots, y_N)$ be a sequence of distinct points in a simple cubic lattice $\mathbb{Z}^{D+2}$ with $y_1 = 0$. We say y *embeds* T if $y_{ij} := y_i - y_j$ has length one for all *bonds* ij in the tree T:

$$c_N = \frac{1}{N!} \sum_T \sum_y 1_{y \text{ embeds } T}.$$

The division by $N!$ removes the labels on the polymer.

c_N is expected to have an asymptotic law of the form

$$c_N \sim N^{-\theta} \left(\frac{1}{z_c} \right)^N,$$

where the *critical exponent* θ is believed to be *universal*. Universal means that this exponent should be independent of the local structure of lattice. For example, it should be the same on a triangular lattice, or in the continuum model to be considered in this paper. This is not the case for z_c.

In 1981 Parisi and Sourlas [PS81] conjectured exact values of θ and other critical exponents for self-avoiding branched polymers in $D+2$ dimensions by relating them to the Yang-Lee singularity of an Ising model in D dimensions. Various authors [Dha83, LF95, PF99] have also argued that the exponents of the Yang-Lee singularity are related in exact and simple ways to exponents for the hard-core gas at the negative value of activity which is the closest singularity to the origin in the pressure. In this paper we consider these models in the continuum and show that there is an exact isomorphism between the hard-core gas in D dimensions and branched polymers in $D+2$ dimensions. We prove that the Mayer expansion for the pressure of the hard-core gas is exactly equal to the generating function for branched polymers.

2 Background and Relation to Earlier Work

In this section we consider theoretical physics issues raised by our results. Precise statements of the results appear in Section 3, and the mathematical development follows.

Three classes of models will be of interest. The first model, *branched polymers*, may be defined either in the lattice, as above, or in the continuum. In the continuum case, there is an appropriate integral over configurations of N monomers which cannot overlap and which are stuck together in accordance with a loop-free graph (see Section 3). See [Frö86] for some background and basic results. The upper critical dimension is presumed to be 8. In addition to the counting exponent, θ, one can define a susceptibility exponent, γ, by working with the generating function $Z_{BP}(z)$, which is the sum of all equivalence classes of branched polymers, multiplied by z^N, where N is the number of monomers (see Section 3). Then $\left(z \frac{d}{dz}\right)^2 Z_{BP}(z) = \sum_x G_{0x}(z)$ should diverge as $(z - z_c)^{-\gamma}$

as $z \nearrow z_c$ with $\gamma = 3 - \theta$. Here $G_{0x}(z)$ is a sum of polymers containing 0 and x. Its rate of decay defines a length ξ whose divergence as $z \nearrow z_c$ defines the exponent ν. It has been proven that exponents take on mean-field values in dimension $d > 8$ (at least for spread-out models) [HS90, HS92]. Thus, for $d > 8$, $\theta = \frac{5}{2}$, $\gamma = \frac{1}{2}$, and $\nu = \frac{1}{4}$.

For the second model, consider the *Yang-Lee edge* $h_\sigma(T)$, defined for the Ising model above the critical temperature as the first occurrence of Lee-Yang zeroes [YL52] on the imaginary magnetic field axis. The density of zeroes is expected to exhibit a power-law singularity $g(h) \sim |h - h_\sigma(T)|^\sigma$ for $|\mathrm{Im}\ h| > |\mathrm{Im}\ h_\sigma(T)|$ [KG71]. This should lead to a branch cut in the magnetization, a singular part of the same form, and a free energy singularity of the form $(h - h_\sigma(T))^{\sigma+1}$. Fisher related this singularity to a field theory with a φ^3 interaction with imaginary coupling [Fis78]. For this model, the upper critical dimension should be 6. Above this dimension, σ should take on the mean-field value, $\frac{1}{2}$. For a mean-field model of this critical point, one can take the standard interaction potential

$$V(\varphi) = \frac{1}{2} r\varphi^2 + u\varphi^4 + h\varphi,$$

and let h move down the imaginary axis. The point φ_h where $V'(\varphi_h) = 0$ moves up from the origin, and when h reaches the Yang-Lee edge $h_\sigma(r, u)$, one finds a critical point with $V'(\varphi_h) = V''(\varphi_h) = 0$. The expansion of $V(\varphi + \varphi_h)$ then begins with a φ^3 term with purely imaginary coefficient. In zero and one dimension, the Ising model in a field is solvable and one obtains $\sigma(0) = -1$, $\sigma(1) = -\frac{1}{2}$ [Fis80].

For the third model, we consider the *universal repulsive-core singularity*, which is the free-energy singularity found for repulsive lattice and continuum gases at negative fugacity. Poland [Pol84] proposed that the exponent ϕ characterizing the singularity should be universal, depending only on the dimension. Baram and Luban [BL87] extended the class of models to include nonspherical particles and soft-core repulsions.

Branched polymers and the Yang-Lee edge, I. Parisi and Sourlas [PS81] connected branched polymers in d dimensions with the Yang-Lee edge in $d-2$ dimensions (see also [Frö86]). Working with the $n \to 0$ limit of a $\boldsymbol{\varphi}^3$ model, the leading diagrams are the same as those of a φ^3 model in imaginary random magnetic field. Dimensional reduction [PS79] relates this to the Yang-Lee edge interaction in two fewer dimensions. The free-energy singularities should coincide, so $2 - \gamma(d) = 1 + \sigma(d-2)$, therefore $\theta(d) = 3 - \gamma(d) = 2 + \sigma(d-2)$. A potential flaw in this argument is the fact that a similar dimensional reduction argument for the random-field Ising model leads to value of 3 for the lower critical dimension [PS79, KW81], in contradiction with the proof of LRO in $d = 3$ [Imb84, Imb85]. See [BDD98] for a discussion of these issues. We shall see, however, that our isomorphism between branched polymers and repulsive gases (Theorems 3.1 and 3.2), combined with the solid connection between repulsive gases and the Yang-Lee edge, leaves little room to doubt the result of Parisi and Sourlas.

The universal repulsive-core singularity and the Yang-Lee edge. This connection goes back to two articles: Cardy [Car82] related the Yang-Lee edge in D dimensions to directed animals in $D+1$ dimensions, and Dhar [Dha83] related directed animals in $D+1$ dimensions to hard-core lattice gases in D dimensions. Dhar (and later Baram and Luban [BL87]) also used Baxter's solution [Bax82] to the hard hexagon model in $D = 2$ (continued to negative activity) to determine the free-energy exponent $\phi(2) = \frac{5}{6}$. Hence $\sigma(2) = \phi(2) - 1 = -\frac{1}{6}$. This was confirmed by Cardy [Car85] with conformal field theory methods.

More recently, Lai and Fisher [LF95] and Park and Fisher [PF99] assembled additional evidence for the proposition that the hard-core repulsive singularity is of the Yang-Lee class. In the latter article, a model with hard cores and additional attractive and repulsive terms was translated into field theory by means of a sine-Gordon transformation. When the repulsive terms dominate, a saddle point analysis leads to the $i\varphi^3$ field theory. We can simplify this picture by considering an interaction potential $w(x-y)$ with $\hat{w}(k) > 0$, $\int d^D k\, \hat{w}(k) < \infty$. Then the sine-Gordon transformation leads to an interaction $U(\varphi) = -\tilde{z}e^{i\varphi}$, where φ is a Gaussian field with covariance βw and $\tilde{z} = ze^{\beta w(0)/2}$. In a mean-field analysis, φ is assumed to be constant, and with $r = (\beta\hat{w}(0))^{-1}$ we obtain a potential

$$V(\varphi) = -\tilde{z}e^{i\varphi} + \frac{1}{2}r\varphi^2.$$

If we put $\varphi = i\chi$, the saddle-point equation is

$$\frac{\tilde{z}}{r} = \chi e^{\chi},$$

which has two solutions for $-e^{-1} < \frac{\tilde{z}}{r} < 0$. When $\tilde{z} = \tilde{z}_c = -\frac{r}{e}$, the two critical points coincide at φ_c such that $V'(\varphi_c) = V''(\varphi_c) = 0$. Expanding about this point gives an $i\varphi^3$ field theory, plus higher-order terms. Complex interactions play an essential role here, since for real models, stability considerations prevent one from finding a critical theory by causing two critical points to coincide—normally at least three are needed, as for φ^4 theory.

Branched polymers and the Yang-Lee edge, II. Continuing with the mean-field picture, we observe that for $z - z_c$ small and positive, the critical point φ_z satisfies $\varphi_z - \varphi_{z_c} \sim (z - z_c)^{\frac{1}{2}}$. As $dV(\varphi_z)/d\tilde{z} = -e^{i\varphi_z}$, $V(\varphi_z)$ has to have a singular part behaving as $(z - z_c)^{\frac{3}{2}}$. If we assume this correctly describes the free-energy singularity above six dimensions, then Theorem 3.2 relates this to the singularity in Z_{BP}, the sum (mod translations) of branched polymers in two more dimensions. Differentiating twice, we obtain the susceptibility $\gamma = \frac{1}{2}$, and hence $\theta = \frac{5}{2}$ above eight dimensions.

Below the upper critical dimension, there should be corrections to these mean-field exponents. The Yang-Lee edge exponent appears as $\varphi - \varphi_c \sim (z - z_c)^{\sigma}$, and the free energy should have a singular part of the form $(z - z_c)^{1+\sigma}$. Assuming this to be the case, Theorem 3.1 implies the same behavior for Z_{BP}.

We can deduce as before that $1+\sigma(d-2) = 2-\gamma(d) = \theta(d)-1$, and so

$$\theta(d) = 2+\sigma(d-2), \tag{2.1}$$

confirming the dimensional reduction formula of [PS81].

3 Main Results

The Hard-Core Gas. Suppose we have "Particles" at positions $x_1,\dots,x_N$ in box $\Lambda \subset \mathbb{R}^D$. Let $x_{ij} = x_i - x_j$ and define the *Hard-Core Constraint*:

$$J(\{1,\dots,N\},\mathbf{x}) = \begin{cases} 1 \text{ if all } |x_{ij}| \geq 1 \\ 0 \text{ otherwise} \end{cases} .$$

By definition, the *Partition Function* for the *Hard-Core Gas* is the following power series in z:

$$Z_{\mathrm{HC}}(z) = \sum_{N\geq 0} \frac{z^N}{N!} \int d^N x\, J(\{1,\dots,N\},\mathbf{x}).$$

Branched Polymers in the Continuum. A *Labeled Branched Polymer* is a tree graph T on vertices $\{1,\dots,N\}$ together with an embedding into $\mathbb{R}^{D+2}$, i.e. positions $y_i \in \mathbb{R}^{D+2}$ for each $i = 1,\dots,N$, such that

(1) If $ij \in T$ then $|y_{ij}| = 1$;

(2) If $ij \notin T$ then $|y_{ij}| \geq 1$.

Define the weight of a tree as

$$W(T) := \int \prod_{ij\in T} \underbrace{d\Omega(y_{ij})}_{\substack{\text{surface measure} \\ \text{on unit ball}}} \prod_{ij\notin T} \mathbb{1}_{\{|y_{ij}|\geq 1\}}.$$

Then the generating function for branched polymers is

$$Z_{\mathrm{BP}}(z) = \sum \frac{z^N}{N!} \sum_{T \text{ on } \{1,\dots,N\}} W(T).$$

It is evident that $Z_{BP}(z)$ is a sum/integral over branched polymers mod translations and labels.

Theorem 3.1. *For all z such that the right-hand side converges absolutely,*

$$\lim_{\Lambda\to\mathbb{R}^D} \frac{1}{|\Lambda|} \log Z_{HC}(z) = -2\pi Z_{BP}\left(\frac{-z}{2\pi}\right).$$

Critical Exponents. For $D = 0$ the left-hand side is $\log(1+z)$ so we find that the critical activity for branched polymers in $d = D+2$ dimensions is exactly $\frac{1}{2\pi}$. Furthermore,

$$Z_{BP}(z) = -\frac{1}{2\pi}\log(1-2\pi z) = \sum_{N=1}^{\infty} \frac{1}{2\pi N}(2\pi z)^N,$$

and we have an exact formula for the weight of polymers of size N:

$$\frac{1}{N!}\sum_{T \text{ on } \{1,\ldots,N\}} W(T) = N^{-1}(2\pi)^{N-1}.$$

Thus $\theta = 1$. One can easily check from the definitions above that the phase space available to dimers and trimers is indeed π, $4\pi^2/3$, respectively.

For $D = 1$, the free energy of the hard-core gas is also computable (see [HH63], for example). Hence $-2\pi Z_{BP}\left(-\frac{z}{2\pi}\right)$ is the largest solution to $\chi e^{\chi} = z$ for $z > z_c = -e^{-1}$. Note that for $z - z_c$ small and positive, $\chi - \chi_c \sim (z-z_c)^{\frac{1}{2}}$, and so $\gamma = \frac{3}{2}$ and $\theta = 3 - \gamma = \frac{3}{2}$. One can also compute the leading behavior of $G_{0x}(z) \sim cx^{-(d-2+\eta)}e^{-x/\xi}$ from the two-point function of the hard-core gas, and one finds $\eta = -1$, $\nu = \frac{1}{2}$ in dimension $d = 3$.

For $D = 2$, the free energy of a gas of hard disks is not known, but if we assume the singularity of negative fugacity is in the same universality class as that of Baxter's model of hard hexagons on a lattice [Bax82], then one finds that the free energy has a leading singularity of the form $(z-z_c)^{\frac{5}{6}}$ [Dha83, BL87], so that $\gamma = \frac{7}{6}$, $\theta = \frac{11}{6}$. Assuming the hard-hexagon model is in the Yang-Lee class, one has $\sigma + 1 = \frac{5}{6}$, so $\sigma = -\frac{1}{6}$. Thus $\theta(4) = 2 + \sigma(2)$. As discussed above, the same line of reasoning gives (2.1) in any dimension. This relation, and the exponents given above for branched polymers in $d = 2, 3$, agree with the Parisi-Sourlas conjectures [PS81] and various scaling relations.

Of course, the exponents are expected to be universal, so one should find the same values for other models of branched polymers (*e.g.*, lattice trees) and also for animals.

A Generalization: Soft Polymers and the Yukawa Gas. We define

$$Z_v(z) = \sum_{N\geq 0} \frac{z^N}{N!}\int d^N x \exp\left(-\beta \sum_{1\leq i<j\leq N} v(|x_{ij}|^2)\right),$$

where $x_i \in \Lambda \subset \mathbb{R}^D$ and $v(r^2)$ is a differentiable, rapidly decaying, spherically symmetric two-particle potential. With $w(x) \equiv v(|x|^2)$, let us assume $\hat{w}(k) > 0$ for a repulsive interaction. Then there is a corresponding branched polymer model in $D+2$ dimensions with

$$W_v(T) := \int \prod_{ij\in T}\left[-\beta v'(|y_{ij}|^2)d^{D+2}y_{ij}\right] \prod_{1\leq i<j\leq N} e^{-\beta v(|y_{ij}|^2)}. \tag{3.1}$$

Note that by assumption, $v'(r^2)$ is rapidly decaying, so the monomers are stuck together along the branches of a tree. The polymers are softly self-avoiding, with the same weighting factor as for the Yukawa gas, albeit in two more dimensions. Defining, as before,

$$Z_{BP,v} = \sum_{N\geq 1} \frac{z^N}{N!} \sum_{T \text{ on } \{1,\dots,N\}} W(T),$$

we will prove

Theorem 3.2. *For all z such that the right-hand side converges absolutely,*

$$\lim_{\Lambda\nearrow\mathbb{R}^D} \frac{1}{|\Lambda|} \log Z_v(z) = -2\pi Z_{BP,v}\left(-\frac{z}{2\pi}\right).$$

Note that by the sine-Gordon transformation,

$$Z_v(z) = \int \exp\left(\int dx\, \tilde{z} e^{i\varphi(x)}\right) d\mu_{\beta W}(\varphi),$$

where $d\mu_{\beta W}$ is the Gaussian measure with covariance βW, and $\tilde{z} = ze^{\beta v(0)/2}$. Thus Theorem 3.2 provides a direct connection between a branched polymer model and the $-\tilde{z}e^{i\varphi}$ field theory. As discussed in Section 2), an expansion of $-\tilde{z}e^{i\varphi}$ about the critical point reveals an $i\varphi^3$ term (along with higher order terms), so it is reasonable to assume that this theory is in the Yang-Lee class.

4 A Fundamental Theorem of Calculus

Suppose $f(\mathbf{t})$ is a smooth function of compact support of a collection $\mathbf{t} = (t_{ij}), (t_i)$ of variables

$$\underbrace{(t_{ij})_{1\leq i<j\leq N}}_{\text{bond variables}} \quad \text{and} \quad \underbrace{(t_i)_{1\leq i\leq N}}_{\text{vertex variables}}.$$

A subset F of bonds $\{ij|1 \leq i < j \leq N\}$ is called a *Graph.* The labels $\{1,\dots,N\}$ are called *vertices.* A subset R of vertices is called a set of *roots. Forests* are graphs that have no loops. Note that the empty graph is a forest by this definition. The connected components of a forest are *trees*, provided we declare that the graph with no bonds and just one vertex is also a tree.

$f^{(F,R)}(\mathbf{t})$ denotes the derivative with respect to the variables t_{ij} with $ij \in F$ and t_i with $i \in R$. Let $z_1,\dots,z_N$ be complex numbers, $z_{ij} = z_i - z_j$ and set

$$t_{ij} = |z_{ij}|^2, \qquad t_i = |z_i|^2. \tag{4.1}$$

Theorem 4.1. *(Forest-Root Formula)*

$$f(\mathbf{0}) = \sum_{(F,R)} \int_{\mathbb{C}^N} f^{(F,R)}(\mathbf{t}) \left(\frac{d^2z}{-\pi}\right)^N, \tag{4.2}$$

where F, R is summed over all forests F and all sets R of roots constrained by the condition that each tree in F contain exactly one root from R. $d^2z = du\,dv$ where $z = u + iv$.

This result is a generalization of Theorem 3.1 in [BW88]. That paper and this one rely on an idea which is common to all of the papers [PS79, PS80, Lut83, AB84]. The proof will be given in Section 7). The assumption of compact support simplifies our discussion here. But having proved the theorem in this case it holds, by taking limits, for any function which decays to zero at infinity and whose first derivatives are continuous and integrable.

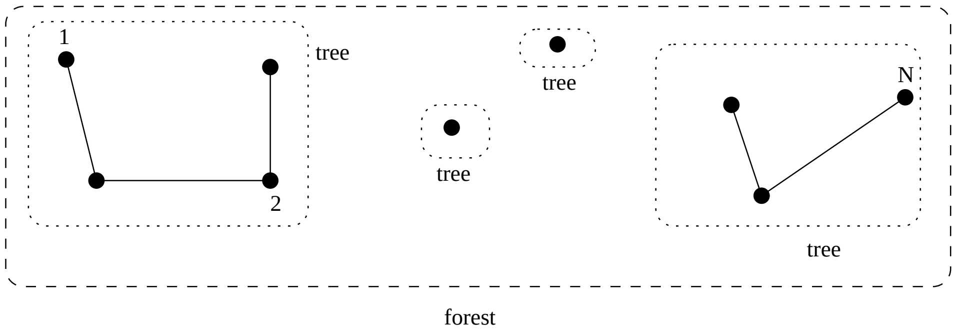

Figure 1: Example of a forest

5 A Tree Formula for Connected Parts

Let J be a function on finite subsets X of $\{1, 2, \ldots\}$. The *connected part* of J is a new function J_c on finite subsets, uniquely defined by solving

$$J(X) = \sum_{\{X_1,\ldots X_n\},\ \text{a partition of } X} J_c(X_1)\cdots J_c(X_n) \tag{5.1}$$

recursively, starting with $J_c(X) = J(X)$ if $|X| = 1$.

Corollary 5.1. *Let $J(X) = J(X, \mathbf{t})$ depend on auxiliary parameters $t_{ij} \geq 0$ for each unordered pair $\{i, j\} \subset X$, $i \neq j$. Assume that*

$$J(X \cup Y) = J(X)J(Y) \tag{5.2}$$

for disjoint sets X, Y whenever t_{ij} is sufficiently large for all $i \in X$, $j \in Y$, or vice versa. Then

$$J_c(X, \mathbf{0}) = \sum_{T \text{ on } X} \left(-\frac{1}{\pi}\right)^{N-1} \int_{\mathbb{C}^N/\mathbb{C}} J^{(T)}(X, \mathbf{t}), \tag{5.3}$$

where $N = |X|$ denotes the number of vertices in X. The integral is over $z_i \in \mathbb{C}$, $i = 1, \ldots, N$ with simultaneous translations $z_i \to z_i + c$ of all vertices factored out, and $t_{ij} = |z_i - z_j|^2$.

Remark. This result was first proven for two-body interactions in [BW88]. A simpler proof based on the Forest-Root formula will be given here for arbitrary $J(X)$.

Proof: Replace the labels $\{1, \ldots, N\}$ in the Forest-Root formula by the elements of X. Let g be a smooth, decreasing compactly supported function with $g(0) = 1$. Apply the Forest-Root Formula (4.2) to

$$f\big((t_{ij}), (t_i)\big) = J(X, (t_{ij})) \prod_i g(\varepsilon t_i),$$

and let $\varepsilon > 0$ tend to zero. Then Corollary 5.1 is proved by the following considerations:

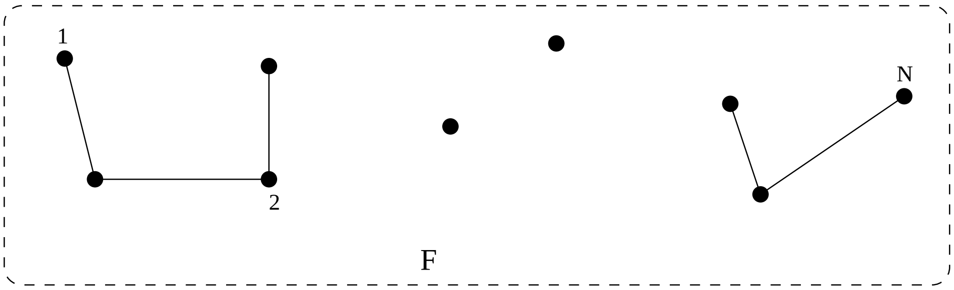

Figure 2: Partition on X defined by a forest F

1. A forest F on a set of vertices X uniquely determines a partition of X, each subset being the vertices in each tree. Therefore,

$$\sum_F (\cdots) = \sum_{\{X_1, \ldots, X_n\},\ \text{a partition of } X} \ \sum_{F,\ \text{compatible with } \{X_1, \ldots, X_n\}} (\cdots).$$

2. Consider any tree T of F, and let r be its root. There is a factor $\epsilon g'(\epsilon t_r)$ from the root derivative at r. Each of the other factors $g(\epsilon t_i)$ for $i \neq r$ can be replaced by $g(\epsilon t_r)$ because hypothesis (5.2) makes any t_{ij}-derivative vanish for $t_{ij} \geq$ const. This forces all z_i, with i a vertex in T, to be equal to within $O(1)$, and all $g(\epsilon t_i)$ to be equal to within $O(\epsilon)$.

3. From the last item, and the sum over r in T (which comes from the sum over R), there arises a factor $(-\varepsilon N(T)/\pi) g'(\varepsilon t_r) g^{N(T)-1}(\varepsilon t_r)$ for each tree. This is a very "flat" probability density on $\mathbb{C}$. The trees are distributed in z-space according to the product of these probability densities.

4. As $\varepsilon \to 0$ the probability that any pair of trees are within distance $o(\varepsilon^{-1})$ tends to zero. Thus, except for a set of vanishingly small measure, $J(X,t)$ factors into a product of terms, one for each tree on an X_i.

5. In the limit $\varepsilon \to 0$, $\sum_R \int_{\mathbb{C}^N} f^{(F,R)}(d^2z/(-\pi))^X$ equals the product over trees $T \subset F$ of factors

$$I(T) := \Big(-\frac{1}{\pi}\Big)^T \int_{\mathbb{C}^{N(T)}/\mathbb{C}} J^{(T)}(X_T, \mathbf{t}),$$

where X_T is the set of vertices in T.

6. The sum over forests factors into independent sums over trees on each of the X_i. It follows that $\Sigma_{T \text{ on } X} I(T)$ solves the recursion (5.1); therefore it must be $J_c(X, \mathbf{0})$.

□

6 Proof of the Main Results

We prove Theorem 3.1 (relation between the hard-core gas and branched polymers) by applying the tree formula for the connected parts to the Mayer expansion:

Theorem 6.1. *[May40] The formal power series for the logarithm of the partition function is given by*

$$\log Z_{HC}(z) = \sum_{N \geq 1} \frac{z^N}{N!} \int d^N x J_c(\{1, \ldots, N\}, \mathbf{x}).$$

The Hard-Core Constraint for particles with labels in X can be written as

$$J(X, \mathbf{x}) = \prod_{ij \in X} 1\!\!1_{\{|x_{ij}|^2 \geq 1\}}.$$

Let

$$J(X, \mathbf{x}, \mathbf{t}) = \prod_{ij \in X} \underbrace{1\!\!1_{\{|x_{ij}|^2 + t_{ij} \geq 1\}}}_{\text{smoothed in } t}.$$

Apply Corollary 5.1, noting that

$$1\!\!1_{\{|x_{ij}|^2 + z_{ij}\bar{z}_{ij} \geq 1\}}$$

is a hard-core condition in $D+2$ dimensions, and each t_{ij}-derivative becomes $\frac{1}{2}$ surface measure when the smoothing is removed. If we put $y_i = (x_i, z_i)$, a

$(D+2)$-dimensional vector, then

$$\lim_{\Lambda\nearrow\mathbb{R}^D} \frac{1}{\Lambda} \log Z_{HC}(z)$$

$$= \sum_{N\geq 1} \frac{z^N}{N!} \sum_{T \text{ on } \{1,\dots,N\}} \left(-\frac{1}{\pi}\right)^{N-1} \int \prod_{ij\in T} \left[\frac{1}{2} d\Omega(y_{ij})\right] \prod_{ij\notin T} \mathbb{1}_{\{|y_{ij}|\geq 1\}}$$

$$= -2\pi Z_{BP}\left(-\frac{z}{2\pi}\right),$$

provided the right-hand side converges absolutely. □

A similar argument can be used to prove Theorem 3.2. We put

$$J(X, \mathbf{x}, \mathbf{t}) = \prod_{ij\in X} e^{-\beta v(|x_{ij}|^2+t_{ij})} = \prod_{ij\in X} e^{-\beta v(|y_{ij}|^2)},$$

and then each t_{ij}-derivative leads to a factor $-\beta v'(|y_{ij}|^2)$. This leads to the form (3.1) for the tree weights $W_v(T)$. □

7 Proof of the Forest-Root Formula

Define the differential forms

$$\tau_{ij} = z_{ij}\bar{z}_{ij} + dz_{ij}d\bar{z}_{ij}/(2\pi i),$$
$$\tau_i = z_i\bar{z}_i + dz_i d\bar{z}_i/(2\pi i).$$

Forms are multiplied by the wedge product. Suppose $g(t_1)$ is a smooth function on the real line. Then we define a new form by the Taylor series

$$g(\tau_1) = g(z_1\bar{z}_1) + g'(z_1\bar{z}_1)dz_1d\bar{z}_1/(2\pi i),$$

which terminates after one term because all higher powers of $dz_1d\bar{z}_1$ vanish. More generally, given any smooth function of the variables $(t_i), (t_{ij})$, we define $g(\tau)$ by the analogous multivariable Taylor expansion. By *definition*, integration over $\mathbb{C}^N$ of forms is zero on all forms of degree not equal to $2N$.

Proposition 7.1. *For f smooth and compactly supported,* $\int_{\mathbb{C}^N} f(\tau) = f(0)$.

Let G be any graph on vertices $\{1, 2, \dots, N\}$. Define

$$(dzd\bar{z})^G = \prod_{ij\in G} dz_{ij}d\bar{z}_{ij},$$

and analogously, for R any subset of vertices

$$(dzd\bar{z})^R = \prod_{i\in R} dz_i d\bar{z}_i.$$

The Taylor series that defines $f(\tau)$ can be written in the form

$$f(\tau) = \sum_{G,R} f^{(G,R)}(z\bar{z}) \left(\frac{dzd\bar{z}}{2\pi i}\right)^G \left(\frac{dzd\bar{z}}{2\pi i}\right)^R,$$

where G is summed over all graphs and R is summed over all subsets of vertices.

- $(dzd\bar{z})^G = 0$ if G contains a loop L because $\sum_{ij\in L} z_{ij} = 0$. Therefore G must be a forest.
- $(dzd\bar{z})^F(dzd\bar{z})^R$ has degree $2N$ iff R has the same number of vertices as there are trees in F. This is because a tree on m vertices has $m-1$ lines.
- Each tree contains exactly one vertex from R, because, if T is a tree which include two vertices a, b from R then $(dzd\bar{z})^T(dzd\bar{z})^R = 0$ since $z_a - z_b$ is a sum of z_{ij} over ij in the path in T joining a to b.

By these considerations Theorem 4.1 is reduced to

Lemma 7.2. $\left(\frac{dzd\bar{z}}{2\pi i}\right)^F \left(\frac{dzd\bar{z}}{2\pi i}\right)^R = \frac{d^2z_1}{-\pi} \cdots \frac{d^2z_N}{-\pi}$.

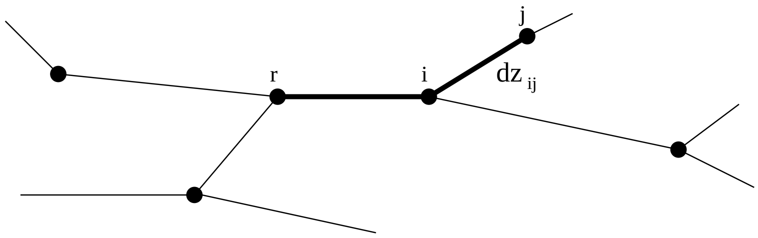

Figure 3: Unique path property

Proof: Suppose, by changing the labels if necessary, that vertices are labeled in such a way that as one traverses any path in F starting at a root r, the vertices one encounters have increasing labels. *E.g.*, in the figure, $r < i < j$. Let $j = N$. Then z_{ij} may be replaced by z_j because

$$dz_j = \sum_{kl \text{ in path}} dz_{kl} + dz_{\text{root}},$$

and $(dzd\bar{z})^F(dzd\bar{z})^R$ already contains dz_{root} and the other terms in path. This argument may be repeated for j decreasing through $N-1, N-2, \ldots 1$. The lemma then follows from the fact that if $z = x + iy$, then $dzd\bar{z} = -2i\, dx\, dy$. $\square$

8 Equivariant Flows and Dimensional Reduction

Proof of Proposition 7.1. We prove the identity $\int_{\mathbb{C}^N} f(\tau) = f(0)$ first in the special case $N = 1$ so that $f(\tau) = f(\tau_1)$:

$$\int_{\mathbb{C}} f(\tau_1) = \underbrace{\int_{\mathbb{C}} f(z_1\bar{z}_1)}_{0 \text{ by definition}} + \int_{\mathbb{C}} f'(z_1\bar{z}_1)dz_1 d\bar{z}_1/(2\pi i)$$

$$= -2\int_0^\infty f'(r^2)rdr = f(0).$$

Note that this proof for $N = 1$ generalizes to the case where f depends only on vertex variables $\tau_1, \ldots, \tau_N$. The remaining argument is a reduction to this case borrowing ideas from the proof of Duistermaat-Heckmann theorem in [AB84].

There is a flow on $\mathbb{C}^N$: $z_j \longmapsto e^{-2\pi i\theta}z_j$. Let V be the associated vector field and let i_V be the associated interior product which is an anti-derivation on forms. By definition $i_V dz_i = -2\pi iz$ and $i_V d\bar{z}_i = 2\pi i\bar{z}$. Let $\mathcal{L}_V$ be the associated Lie derivative on forms. It is a derivation on forms,

$$\mathcal{L}_V dz_i = \frac{d}{d\theta}\bigg|_{\theta=0} d(e^{-2\pi i\theta}z_i) = -2\pi i dz_i$$

and $\mathcal{L}_V d\bar{z}_i = 2\pi i d\bar{z}_i$. Define the antiderivation $Q = d + i_V$ and note that $Q^2 = di_V + i_V d = \mathcal{L}_V$ by Cartan's formula for $\mathcal{L}_V$.

1. $\tau_{ij} = Qu_{ij}$ with $u_{ij} = z_{ij}d\bar{z}_{ij}/(2\pi i)$.
2. $Q\tau_{ij} = Q^2u_{ij} = \mathcal{L}_V u_{ij} = 0$ because u_{ij} is invariant under the flow.
3. For any smooth function g, $Qg(\tau) = \sum g^{(ij)}(\tau)\, Q\tau_{ij} = 0$.
4. Fix any bond ij and define $f(\beta, \tau)$ by replacing τ_{ij} in $f(\tau)$ by $\beta\tau_{ij}$. Then

   $$\frac{d}{d\beta}f(\beta, \tau) = d \text{ (some form)} + \text{(a form of degree} < 2N),$$

   because the β-derivative is $f^{(ij)}(\beta, \tau)\tau_{ij}$ which equals

   $$f^{(ij)}(\beta, \tau)Qu = Q(f^{(ij)}(\beta, \tau)u),$$

   and $Q = d + i_V$ and i_V lowers the degree by one.
5. $\frac{d}{d\beta}\int_{\mathbb{C}^N} f(\beta, \tau) = 0$ because the integral annihilates the part of lower degree and also annihilates the part in the image of d by Stokes' theorem.

Thus every τ_{ij} in f can be deformed to 0 and Proposition 7.1 is reduced to the case where f is a function only of $(\tau_1, \tau_2, \ldots, \tau_N)$. □

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