On the coexistence of absolutely continuous and singular continuous components of the spectral measure for some Sturm-Liouville operators with square summable potential.

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Abstract

We prove that there exist some Sturm-Liouville operators with square summable potentials such that the singular continuous component of the spectral measure lies on the positive half-line.

In [11], B.Simon states 15 open problems, one of which says: does $q(x)$ exist so that $|q(x)| < C/(1+x)^{1/2+\varepsilon}$, where $C, \varepsilon > 0$, and operator on the positive half-line $Hu = -u'' + q(x)u$ has singular continuous spectrum for some boundary condition at zero?

In the current paper, we construct Sturm-Liouville operator with smooth, square summable potential such that the singular continuous spectrum exists. Our technique is based on the relation between some Sturm-Liouville operators and so-called Krein systems.

The problem considered in this article has its analogs in the theory of polynomials orthogonal on the unit circle and in the theory of Krein systems. For polynomials, the answer is well known: Szegö condition on the measure $\int\limits_0^{2\pi} \ln \sigma'(\lambda) d\lambda > -\infty$ is equivalent to $a_n \in \ell^2$, where a_n are Geronimus coefficients [4]. That means even for ℓ^2 Geronimus coefficients, the singular continuous component of the measure can coexist with the absolutely continuous one. For the Krein systems, the criterion for "Szegö condition" was given in terms of coefficient $A(x)$, provided that it is uniformly square summable [3].

The Krein system is continuous analog of difference equations that determine polynomials orthogonal on the unit circle [1, 4]. Let us recall some basic facts from the theory of Krein systems [5, 10]. Consider $H(t) = \overline{H(-t)}$ - function summable on each segment $(-r, r)$.

Proposition. (M.G.Krein, [5]) *If for any continuous $\varphi(t)$ the following inequality holds*

$$\int\limits_0^r |\varphi(s)|^2 ds + \int\limits_0^r \int\limits_0^r H(t-s)\varphi(t)\overline{\varphi(s)} dt ds \geq 0 \tag{1}$$

for each $r > 0$, then, and in this case only, there exists the non-decreasing function $\sigma(\lambda)$ ($\lambda \in R$, $\sigma(0) = 0$, $\sigma(\lambda - 0) = \sigma(\lambda)$), such that

$$\int\limits_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{1+\lambda^2} < \infty,$$

$$\int_0^t (t-s)H(s)ds = \int_{-\infty}^{\infty} (1+\frac{i\lambda t}{1+\lambda^2} - e^{i\lambda t})\frac{d\sigma(\lambda)}{\lambda^2} + (i\gamma - \frac{sign(t)}{2})t,$$

where γ is real constant.

If in addition we presume that the equality in (1) is possible for $\varphi = 0$ only, then the Hermit kernel $H(t-s)$, $(0 \le t, s \le r)$ has Hermit resolvent $\Gamma_r(s,t) = \overline{\Gamma_r(t,s)}$ that satisfies the relation

$$\Gamma_r(t,s) + \int_0^r H(t-u)\Gamma_r(u,s)du = H(t-s), \ (0 \le s,t \le r). \tag{2}$$

The continuous analogs of polynomials orthogonal on the unit circle are defined by the formulas

$$\begin{aligned} P(r,\lambda) &= e^{i\lambda r}(1 - \int_0^r \Gamma_r(s,0)e^{-i\lambda s}ds), \\ P_*(r,\lambda) &= 1 - \int_0^r \Gamma_r(0,s)e^{i\lambda s}ds, \ r \ge 0. \end{aligned}$$

Using the well known properties of resolvents we obtain the following system, which is called Krein system

$$\begin{cases} \frac{dP(x,\lambda)}{dx} &= i\lambda P(x,\lambda) - \overline{A(x)}P_*(x,\lambda), \ P(0,\lambda) = 1; \\ \frac{dP_*(x,\lambda)}{dx} &= -A(x)P(x,\lambda), \ P_*(0,\lambda) = 1, \end{cases} \tag{3}$$

where $A(x) = \Gamma_x(0,x)$. On the other hand, system (3) itself defines measure σ, etc. [9].

Coefficient $A(x)$ is continuous analog of Geronimus parameters a_n. These parameters enter the system of difference equations that determine polynomials orthogonal on the unit circle. Consider A to be real valued coefficient. From [5], we know that $e^{-i\lambda x}P(2x,\lambda) = \Phi(x,\lambda) + i\Psi(x,\lambda)$, where Φ and Ψ are generalized eigenfunctions of equations

$$\Psi'' - q(x)\Psi + \lambda^2\Psi = 0, \Psi(0,\lambda) = 0, \ \Psi'(0,\lambda) = \lambda,$$

$$\Phi'' - q_1(x)\Phi + \lambda^2\Phi = 0, \Phi(0,\lambda) = 1, \ \Phi'(0,\lambda) + a(0)\Phi(0,\lambda) = 0,$$

where $q = a^2 + a'$, $q_1 = a^2 - a'$, and $a(x) = 2A(2x)$.

It is known [2] that the spectral measure ς of operator $Hu = -u'' + qu$, $u(0) = 0$ is related to the spectral measure $\sigma(t)$ of system (3) by the formula

$$\varsigma(t) = 4\int_0^{\sqrt{t}} \alpha^2 d\sigma(\alpha). \tag{4}$$

Assume that σ satisfies the following condition $t^2 \ln\left(2\pi\sigma'(t)\right) \in L^1(R)$.

Lemma. *If $A(x)$ is such that*

$$A(0) = A'(0) = 0 \tag{5}$$

and $A^{(j)}(x) \in L^1(R^+),\ j = 0, \ldots 3,$ *then*

$$-8A''(0) - \int_0^\infty q^2 dx = \frac{4}{\pi} \int_{-\infty}^{\infty} t^2 \ln\left(2\pi\sigma'(t)\right) dt. \tag{6}$$

Proof. The idea is quite simple. It is known [5, 10], that

$$\Pi(\lambda) = \exp\left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1 + t\lambda}{\lambda - t} \frac{\ln\left(2\pi\sigma'(t)\right)}{1 + t^2} dt \right\} \tag{7}$$

where $\Pi(\lambda) = \lim_{x\to\infty} P_*(x,\lambda)$ for $\Im\lambda > 0$, provided that this limit exists.

We let $\lambda = id, d \to +\infty$. Look at the asymptotics of the integral in the right hand side of (7). It is as follows

$$\begin{aligned}
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1 + idt}{id - t} \frac{\ln\left(2\pi\sigma'(t)\right)}{1 + t^2} dt &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln\left(2\pi\sigma'(t)\right) \left[\frac{1}{d} - \frac{t^2}{d^3} + \frac{t^4}{d^3(t^2 + d^2)} \right] dt = \\
&- \frac{1}{2\pi d} \int_{-\infty}^{\infty} \ln\left(2\pi\sigma'(t)\right) dt + \frac{1}{2\pi d^3} \int_{-\infty}^{\infty} t^2 \ln\left(2\pi\sigma'(t)\right) dt + \overline{o}(d^{-3}).
\end{aligned}$$

To get $\overline{o}(d^{-3})$, we assumed that $t^2 \ln\left(2\pi\sigma'(t)\right) \in L^1(R)$. This assumption will be true throughout the paper.

Consequently,

$$\begin{aligned}
\Pi(id) &= 1 - \frac{1}{2\pi d} \int_{-\infty}^{\infty} \ln\left(2\pi\sigma'(t)\right) dt + \frac{1}{2\pi d^3} \int_{-\infty}^{\infty} t^2 \ln\left(2\pi\sigma'(t)\right) dt + \\
&\frac{1}{8\pi^2 d^2} \left[\int_{-\infty}^{\infty} \ln\left(2\pi\sigma'(t)\right) dt \right]^2 - \frac{1}{48\pi^3 d^3} \left[\int_{-\infty}^{\infty} \ln\left(2\pi\sigma'(t)\right) dt \right]^3 + \overline{o}(d^{-3}).
\end{aligned}$$

On the other hand, we can compute the asymptotics of $\Pi(id)$ using equations (3) directly. Indeed, we have the system

$$\begin{aligned}
P' &= -dP - AP_*,\ P(0) = 1; \\
P_*' &= -AP,\ P_*(0) = 1.
\end{aligned}$$

Introduce the function $Q = e^{dx}P$. So,

$$\begin{aligned}
Q' &= -Ae^{dx}P_*,\ Q(0) = 1; \\
P_*' &= -Ae^{-dx}Q,\ P_*(0) = 1.
\end{aligned}$$

For P_*, we have integral equation

$$P_*(x,id) = 1 - \int\limits_0^x A(s)e^{-ds}ds + \int\limits_0^x P_*(s,id)A(s)e^{ds}\int\limits_s^x A(\tau)e^{-d\tau}d\tau ds. \qquad (8)$$

The Gronwall Lemma yields $P_*(x,id) \to 1$ as $d \to \infty$ uniformly in $x \in R^+$. Let us iterate (8). We have

$$\begin{aligned} P_*(x,id) &= 1 - \int\limits_0^x A(s)e^{-ds}ds + \int\limits_0^x \left[A(s)e^{ds}\int\limits_s^x A(\tau)e^{-d\tau}d\tau\right]\left[1 - \int\limits_0^s A(s_1)e^{-ds_1}ds_1\right] ds + \\ & \int\limits_0^x \left[A(s)e^{ds}\int\limits_s^x A(\tau)e^{-d\tau}d\tau\right]\left[\int\limits_0^s P_*(s_1,id)A(s_1)e^{ds_1}\int\limits_{s_1}^s A(\tau_1)e^{-d\tau_1}d\tau_1 ds_1\right] ds. \end{aligned}$$

Consequently, integrating by parts and using conditions $A(0) = A^{'}(0) = 0$, we have

$$\Pi(id) = 1 + \frac{1}{d}\int\limits_0^\infty A^2(s)ds + \frac{1}{2d^2}\left[\int\limits_0^\infty A^2(s)ds\right]^2 - \frac{1}{d^3}A^{''}(0) - \frac{1}{d^3}\int\limits_0^\infty A^{'}(s)^2 ds -$$

$$\frac{1}{d^3}\int\limits_0^\infty A^4(s)ds + \frac{1}{6d^3}\left[\int\limits_0^\infty A^2(s)ds\right]^3 + \overline{o}(d^{-3}).$$

Comparing the coefficients for d^{-1} and d^{-3}, we get

$$\begin{aligned} -\frac{1}{2\pi}\int\limits_{-\infty}^\infty \ln\left(2\pi\sigma'(t)\right) dt &= \int\limits_0^\infty A^2(s)ds, \\ -\frac{1}{2\pi}\int\limits_{-\infty}^\infty t^2 \ln\left(2\pi\sigma'(t)\right) dt &= A^{''}(0) + \int\limits_0^\infty A^{'}(s)^2 ds + \int\limits_0^\infty A^4(s)ds. \end{aligned}$$

Recall that $a(x) = 2A(2x),\ q = a^{'} + a^2$. Therefore, we have

$$\begin{aligned} -\frac{1}{\pi}\int\limits_{-\infty}^\infty \ln\left(2\pi\sigma'(t)\right) dt &= \int\limits_0^\infty q(s)ds, \\ -8A^{''}(0) - \frac{4}{\pi}\int\limits_{-\infty}^\infty t^2 \ln\left(2\pi\sigma'(t)\right) dt &= \int\limits_0^\infty q^2(s)ds. \end{aligned}$$

□

The further construction is as follows. Consider odd function $\sigma(t) = (2\pi)^{-1} t$ on $|t| \le 1$ and $|t| \ge 2$. On the segment $[1,2]$ it has absolutely continuous component, that satisfies "local Szegö condition", and singular continuous component. Construct the corresponding potential $q(x)$ using the Krein approach [5]. So, formally, using lemma, we may hope that $q \in L^2(R^+)$. But the problem is that we obtained (6) using smoothness of $A(x)$ and conditions (5). Now we will concentrate on making these arguments precise.

Theorem. *There exists smooth, square summable function q, such that the spectrum of operator $Hu = -u'' + qu$, $u(0) = 0$ contains the singular continuous component.*

Proof. Let $\sigma(t) = (2\pi)^{-1} + \varepsilon(t-1) + \sigma_{sc}(t)$ on the interval $[1, 2]$. Here ε is some positive constant smaller than $(2\pi)^{-1}$. $\sigma_{sc}(t)$ can be chosen as standard Cantor ladder such that $\sigma_{sc}(1) = 0$, $\sigma_{sc}(2) = (2\pi)^{-1} - \varepsilon$. Together with $\sigma(t)$, we consider its infinitely smooth approximations $\sigma_n(t)$. To be more specific, we choose $\sigma_n(t)$ in such a way that

$$\sigma_n(t) = (2\pi)^{-1} t \quad \text{for} \quad |t| > 5/2 \quad \text{and} \quad |t| < 1/2,$$

$$\sigma_n'(t) > \varepsilon/2,$$

and

$$\|\sigma_n(t) - \sigma(t)\|_{C(R)} \to 0 \quad \text{as} \quad n \to \infty$$

The existence of these approximations is obvious.

For each $\sigma_n(t)$ and $\sigma(t)$, we can construct the corresponding accelerants $H_n(x) = \int\limits_{-\infty}^{\infty} \cos(xt) d(\sigma_n(t) - (2\pi)^{-1} t) = 2 \int\limits_{1/2}^{5/2} \cos(xt)(\sigma_n'(t) - (2\pi)^{-1}) dt$, $H(x) = \int\limits_{-\infty}^{\infty} \cos(xt) d(\sigma(t) - (2\pi)^{-1} t)$.

$H_n(x)$ is from Schwartz space and $H_n(0) = 0$. It is due to the infinite smoothness of σ_n and equality $\sigma_n(t) = (2\pi)^{-1} t$ for $|t| > 5/2$. Consider $\rho_n(t) = 1 + 2\int\limits_0^{\infty} H_n(x) \cos(xt)\, dx = 1 + 2\pi(\sigma_n'(t) - (2\pi)^{-1}) = 2\pi\sigma_n'(t) > \pi\varepsilon$ for all t. Then, recover $A_n(x)$ and $A(x)$ that are related to σ_n and σ. To do that we should solve the following integral equations first

$$\Gamma_r^{(n)}(x) + \int\limits_0^r H_n(x-s)\Gamma_r^{(n)}(s) ds = H_n(x), \ 0 < x < r,$$

$$\Gamma_r(x) + \int\limits_0^r H(x-s)\Gamma_r(s) ds = H(x), \ 0 < x < r,$$

where $\Gamma_r(x) = \Gamma_r(0, x)$, $\Gamma_r^{(n)}(x) = \Gamma_r^{(n)}(0, x)$. From the Krein results [5, 8, 7], we infer that these two equations are solvable and $\Gamma_r^{(n)}(x), \Gamma_r(x)-$ continuous in x and r. Let $A_n(x) = \Gamma_x^{(n)}(x)$, $A(x) = \Gamma_x(x)$. We want to show that Lemma 1 is applicable to systems with coefficients $A_n(x)$. Together with $\Gamma_r^{(n)}(x)$, consider $\Gamma^{(n)}(x)$ which is the solution of the following equation

$$\Gamma^{(n)}(x) + \int\limits_0^{\infty} H_n(x-s)\Gamma^{(n)}(s) ds = H_n(x), \ 0 < x < \infty.$$

It is so-called Wiener-Hopf equation. Consider operator $\Theta^{(n)} : \Theta^{(n)} f = f(x) + \int\limits_0^{\infty} H_n(x-s) f(s) ds$. Because even function $H_n(x) \in L^1(R)$ and $\rho_n(t) > \pi\varepsilon$, operator $\Theta^{(n)}$ is invertible in the classes $L^1(R^+)$ and $L^{\infty}(R^+)$ [6]. In addition, for $\Gamma^{(n)}(x)$, we have an estimate

$$\left|\Gamma^{(n)}(x)\right| \leq \frac{C_{n,j}}{(1+x)^j},\ j=1,2,\dots\ . \tag{9}$$

Indeed, for $\widehat{\Gamma}^{(n)}(\omega) = \int\limits_0^\infty \Gamma^{(n)}(x)e^{i\omega x}dx$, we have the formula

$$\widehat{\Gamma}^{(n)} = \frac{\xi^{(n)}}{\xi^{(n)}+1} \tag{10}$$

where $\xi^{(n)}(\omega) = \exp\left[\int\limits_0^\infty \gamma^{(n)}(x)e^{i\omega x}dx\right] - 1$, and $\gamma^{(n)}(x)$ is defined by the relation $\ln\rho_n(\omega) = \int\limits_{-\infty}^\infty \gamma^{(n)}(x)e^{i\omega x}dx$. Function $\gamma^{(n)}(x)$ is from Schwartz space. Therefore, $\xi^{(n)}(\omega)$ has derivatives of any order summable on the whole line and $\xi^{(n)}(\pm\infty)$. From (10), we infer that all derivatives of $\widehat{\Gamma}^{(n)}(\omega)$ are summable on R and $\widehat{\Gamma}^{(n)}(\pm\infty) = 0$. $\Gamma^{(n)}(x)$ is bounded because $\Theta^{(n)}$ is invertible in $L^\infty(R)$. Therefore, we have (9).

Consider $\Theta^{(n)}$ as the operator acting in $L^\infty(R^+)$. It is invertible [6]. Consider also the function

$$\delta_r^{(n)}(x) = \begin{cases} \Gamma^{(n)}(x) - \Gamma_r^{(n)}(x), & 0<x<r, \\ 0, & x>r. \end{cases}$$

On the interval $[0,r]$, we have

$$\delta_r^{(n)}(x) + \int\limits_0^r H_n(x-s)\delta_r^{(n)}(s)ds = \delta_r^{(n)}(x) + \int\limits_0^\infty H_n(x-s)\delta_r^{(n)}(s)ds = -\int\limits_r^\infty H_n(x-s)\Gamma^{(n)}(s)ds. \tag{11}$$

$||\cdot||_{L^2[0,r]}$-norm of the right hand side is less then $\frac{C_{n,j}}{(1+r)^j}$ for all integer j. Because $\inf spec\left[\int\limits_0^r H_n(x-s)f(s)ds, f\in L^2(0,r)\right] \geq \inf spec\left[\int\limits_0^\infty H_n(x-s)f(s)ds, f\in L^2(R^+)\right] > -1$, we have $\left\|\delta_r^{(n)}(x)\right\|_{L^2(0,r)} \leq \frac{C_{n,j}}{1+r^j}$. We can also write (11) as $\Theta^{(n)}\delta_r^{(n)}(x) = -\int\limits_r^\infty H_n(x-s)\Gamma^{(n)}(s)ds$ on $[0,r]$. On the interval $x\in[r,\infty)$, we have

$$\Theta^{(n)}\delta_r^{(n)}(x) = \int\limits_0^r H_n(x-s)\delta_r^{(n)}(s)ds.$$

$\left\|\int\limits_r^\infty H_n(x-s)\Gamma^{(n)}(s)ds\right\|_{L^\infty(0,r)} \leq \frac{C_{n,j}}{(1+r)^j}$, and $\int\limits_0^r |H_n(x-s)|\left|\delta_r^{(n)}(s)\right|ds \leq ||H_n||_{L^2(R)}||\delta_r^{(n)}||_{L^2[0,r]} \leq \frac{C_{n,j}}{(1+r)^j}$. Consequently, $\left\|\delta_r^{(n)}(x)\right\|_\infty \leq \frac{C_{n,j}}{(1+r)^j}$. $\Gamma_r^{(n)}(x)$ is continuous in x and r on $0\leq x\leq r<+\infty$. Therefore, $A^{(n)}(r) = \Gamma_r^{(n)}(r)$ is continuous, and $\left|A^{(n)}(r)\right| \leq \frac{C_{n,j}}{(1+r)^j}$. So, $A^{(n)}(r)\in L^1(R^+)$. Notice that $H_n(0)=0$. Therefore, $A^{(n)}(0)=0$. To see that $\frac{dA^{(n)}(r)}{dr}$, $\frac{d^2A^{(n)}(r)}{dr^2}$, $\frac{d^3A^{(n)}(r)}{dr^3} \in L^1(R^+)$, it is enough to differentiate equality

$$A^{(n)}(r) + \int\limits_0^r \Gamma_r^{(n)}(s)H_n(r-s)ds = H_n(r)$$

successively three times. Then, one should use estimates on $\delta_r^{(n)}(x)$, $\Gamma^{(n)}(x)$, $H_n(r)$ and its derivatives. For example, using identity $\frac{\partial \Gamma_r(x)}{\partial r} = -\Gamma_r(r)\Gamma_r(r-x)$ [5], we get

$$\frac{dA^{(n)}(r)}{dr} + \int_0^r \Gamma_r^{(n)}(s) H_n'(r-s)ds - A(r) \int_0^r H_n(s)\Gamma_r^{(n)}(s)ds = \frac{dH_n(r)}{dr}. \qquad (12)$$

Trivial analysis shows that $\frac{dA^{(n)}(r)}{dr} \in L^1(R^+)$. For example, to prove that $\int_0^r \Gamma_r^{(n)}(s)H_n'(r-s)ds \in L^1(R^+)$, one can write this integral in the following way

$$\int_0^r \Gamma^{(n)}(s)H_n'(r-s)ds - \int_0^r \delta_r^{(n)}(s)H_n'(r-s)ds.$$

From the Young inequality for convolutions, we see that the first term is summable on R^+. For the second one, we can use an estimate on $\delta_r^{(n)}(s)$ to get

$$\left| \int_0^r \delta_r^{(n)}(s)H_n'(r-s)ds \right| \le \frac{C_{n,j}}{(1+r)^j}.$$

In the same way, one can prove that $\frac{d^2A^{(n)}(r)}{dr^2}$, $\frac{d^3A^{(n)}(r)}{dr^3} \in L^1(R^+)$. Because $H_n(r)$ is even, $\frac{dH_n}{dr}(0) = 0$. Therefore, $\frac{dA^{(n)}}{dr}(0) = 0$. Thus, Lemma 1 is applicable. Notice that $\frac{d^2A^{(n)}}{dr^2}(0) = \frac{d^2H_n}{dr^2}(0)$. Consequently, $\left|\frac{d^2A^{(n)}}{dr^2}(0)\right|$ is bounded in n.

Let q_n, q be potentials corresponding to coefficients A_n, A respectively. Notice that $H(x)$ is infinitely smooth, therefore, $A(x)$ and $q(x)$ are infinitely smooth as well. It follows from the Fredholm formulas for resolvent of the integral equation. From Lemma 1, we see that $||q_n||_2$ is bounded in n. It is because we have the lower bound from zero for $\sigma_n'(t)$.

Our goal is to prove that q is from $L^2(R^+)$. Consider the difference $H_n(x) - H(x) = 2\int_{1/2}^{5/2} \cos(tx)\, d\left[\sigma_n(t) - \sigma(t)\right]$. Integrating by parts, we have

$$H_n(x) - H(x) = 2\int_{1/2}^{5/2} x\sin(tx)\left[\sigma_n(t) - \sigma(t)\right] dt.$$

$\|\sigma_n(t) - \sigma(t)\|_{C(R)} \to 0$ as $n \to \infty$ so, $H_n(x) \to H(x)$ uniformly on each finite segment. In the same way, one can show that $H_n'(x) \to H'(x)$ on each finite segment. Fredholm formulas for the resolvent of integral equation implies that $\Gamma_r^{(n)}(s,x) \to \Gamma_r(s,x)$ uniformly in $0 \le x, s \le r \le M$ for each fixed M as $n \to \infty$. Consequently, $A_n \to A$ uniformly on each finite segment. From (12), we infer that $A_n' \to A'$ (and so $q_n \to q$) uniformly on each finite interval as well. Boundedness of $||q_n||_2$ leads to inclusion $q \in L^2(R^+)$. Therefore, due to (4), we constructed potential $q \in L^2(R^+)$, such that the corresponding Sturm-Liouville operator with Dirichlet boundary condition at zero has singular continuous component that coexists with absolutely continuous one.

Remark. The construction of potential, that admits the estimate suggested by B.Simon, is one step further in this direction. It requires very detailed study of the integral equation (2) that determines the resolvent kernel for each $r > 0$. We should also mention that though potential q from the Theorem is smooth and square summable, generally speaking, it might not be decaying at the infinity. To prove that decaying potentials from $L^2(R^+)$ class can admit coexistence of s.c. and a.c. components of the spectrum, one should use some identities that involve first derivative of q also. Then, using more or less the same technique, it might be possible to prove that properly chosen measure leads to potential from $W^{1,2}(R^+)$ class. That means $q(x)$ decays at the infinity.

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