The threshold regime of finite volume bootstrap percolation.

R. Cerf, F. Manzo. July 6, 2001

Abstract. We prove that the threshold regime for bootstrap percolation in a d-dimensional box of diameter L with parameters p and ℓ, where $3 \leq \ell \leq d$, is $L \sim \exp^{\circ(\ell-1)}(Cp^{-1/(d-\ell+1)})$, where $\exp^{\circ(\ell-1)}$ is the exponential iterated $\ell-1$ times and C is bounded from above and from below by two positive constants depending on d, ℓ only.

1. Introduction.

We consider the *bootstrap percolation* model, with initial occupation density p and parameter ℓ, in a finite set $\Gamma \subset \mathbb{Z}^d$. More precisely, each site x of $\Gamma \subset \mathbb{Z}^d$ is initially independently occupied with probability p and empty with probability $1-p$. Afterwards, we increase deterministically the set of occupied sites in Γ with the help of the following rule, until exhaustion: any site with at least ℓ occupied nearest neighbors in Γ is occupied. For a discussion on the physical relevance of this model, we refer to [**AL**]; for a nice review paper on bootstrap percolation, see [**Ad**].
We say that a set Γ is *internally spanned* if all its sites are occupied in the final configuration. The basic question we are interested in is whether or not $\Lambda^d(L)$ is internally spanned, where $\Lambda^d(L)$ is the d-dimensional cubic box of diameter L.
Let us denote the probability of this event by

$$R(L,p,d,\ell) := \mathbb{P}\begin{pmatrix}\text{the box } \Lambda^d(L) \text{ is internally spanned by the bootstrap} \\ \text{percolation process in } \Lambda^d(L) \text{ with parameters } p \text{ and } \ell\end{pmatrix}$$

Université Paris Sud, Orsay, France.

Université Paris Sud, Orsay, France and Technische Universität, Berlin, Germany.

Work partially supported by the European network “Stochastic Analysis and Its Applications” ERB-FMRX-CT96-0075

We focus on the behavior of $R(L, p, d, \ell)$ when L goes to infinity and p goes to zero. In the case $\ell > d$ we have that

$$\lim_{(L,p)\to(\infty,0)} R(L, p, d, \ell) = 0.$$

Indeed, the presence of a small empty cubic region in the initial configuration precludes the complete filling of $\Lambda^d(L)$. More interesting is the case $\ell \leq d$. Obviously, for L fixed and p very small the initial configuration will be completely empty with high probability, hence $\lim_{p\to 0} R(L, p, d, \ell) = 0$. On the other hand, from the much less obvious results of van Enter [**vE**] and Schonmann [**S**], we know that for p fixed and $\ell \leq d$, $\lim_{L\to\infty} R(L, p, d, \ell) = 1$. Therefore, we see that $\lim_{L\to\infty} \lim_{p\to 0} R(L, p, d, \ell) = 0$, while $\lim_{p\to 0} \lim_{L\to\infty} R(L, p, d, \ell) = 1$.

These different limiting behaviors indicate the occurrence of an interesting phenomenon: if we send simultaneously $L \to \infty$ and $p \to 0$, the limit of $R(L, p, d, \ell)$ will depend on the relative speeds of these convergences, i.e. if p goes extremely quickly (respectively slowly) to 0 compared to the way L goes to ∞, then $R(L, p, d, \ell)$ will converge to 0 (respectively 1). A natural problem is to describe precisely each regime and the threshold between them. In [**AL**], Aizenman and Lebowitz handled the case $\ell = 2$, $d \geq 2$. The threshold regime is

$$L \sim \exp\left(\text{const } p^{-\frac{1}{d-1}}\right).$$

In [**CeCi**], Cerf and Cirillo analyzed the case $d = \ell = 3$, for which the threshold regime turned out to be

$$L \sim \exp\exp\left(\text{const } p^{-1}\right).$$

We deal here with the general case $2 < \ell \leq d$. While the proof of the upper bound on L derives directly from an idea of [**ADE**] and the results of [**S**], the proof of the lower bound is obtained by using induction on the parameters (ℓ, d): by using the technique introduced in [**CeCi**], we reduce the estimate of the spanning probability for the model (ℓ, d) to the spanning probability for the model $(\ell - 1, d - 1)$. The base of the induction is the Aizenman-Lebowitz case $d \geq \ell = 2$.

A very challenging and interesting open problem is to decide whether a sharp constant can be put in the exponentials to separate the two regimes. Similar interesting questions can be raised in anisotropic models, as considered for instance in [**M**].

2. Basic notation.

For $t \in \mathbb{R}$, $n \in \mathbb{N}$, we denote by $\exp^{\circ n}(t)$ the exponential *iterated n times* of t: we set $\exp^{\circ 0}(t) := t$ and $\exp^{\circ(n+1)}(t) := \exp(\exp^{\circ n}(t))$. By $\Lambda^d(l)$ we denote the d-dimensional hypercube with diameter l centered at 0.

Let us give some definitions related to *site percolation* (see [**G**]). On a finite set $\Gamma \subset \mathbb{Z}^d$, let us consider a random configuration $\omega \in \{0, 1\}^\Gamma$ obtained by occupying (namely, by setting $\omega(x) = 1$) the sites with the product probability measure $\mathbb{P}_d^p$

with density p. We denote by $\mathbb{P}_d^p(\mathcal{E})$ the probability of the event $\mathcal{E}$ ($\mathcal{E}$ is a set of configurations in $\{0,1\}^\Gamma$).
We say that a configuration ω is *larger* than a configuration ω' if the set of the occupied sites in the former contains the set of the occupied sites in the latter.
An event $\mathcal{E}$ is called *increasing* if for any configuration $\omega \in \mathcal{E}$, all configurations $\omega' > \omega$ are in $\mathcal{E}$.
Our main object of investigation is the following *bootstrap* process, defined as a function of a site percolation configuration. On a finite set $\Gamma \subset \mathbb{Z}^d$, let us consider a random initial configuration obtained by occupying the sites with a product measure with probability p. We update this initial configuration by using iteratively the following deterministic rule:

1. we occupy every empty site with at least ℓ occupied nearest neighbors.
2. we leave all other sites unchanged.

Since Γ is finite, and the updating procedure cannot empty occupied sites, this procedure stops after a finite number of steps. We denote by $X_\Gamma^{d,\ell}$ the final configuration of the d-dimensional bootstrap process in the set Γ. Thus $X_\Gamma^{d,\ell}$ is a random map from Γ to $\{0,1\}$ and for $x \in \Gamma$, $X_\Gamma^{d,\ell}(x) = 1$ if x is occupied and 0 otherwise.
We will use the following basic facts:

a) The final configuration of the bootstrap process is a monotonic increasing function of the initial configuration.
b) The updating procedure gives the same final configuration if applied to any configuration larger than the initial configuration and lower than the final one.

In particular, b) implies that the updating order does not affect the final configuration. We say that a finite set $\Gamma \subset \mathbb{Z}^d$ is *internally spanned* if $X_\Gamma^{d,\ell}(x) = 1$ for any $x \in \Gamma$. We focus our attention on the behavior of the following probability:

$$R(L,p,d,\ell) := \mathbb{P}_p^d\left(\forall x \in \Lambda^d(L) \quad X_{\Lambda^d(L)}^{d,\ell}(x) = 1\right)$$

We call Γ-*clusters* the maximal connected sets of occupied sites in $X_\Gamma^{d,\ell}$. Notice that all clusters are internally spanned. We say that x *is connected to* y *in* Γ if there exists a Γ-cluster $\mathcal{C}$ such that $\{x,y\} \subset \mathcal{C} \subset \Gamma$; we denote this event by $\left\{x \stackrel{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Gamma\right\}$.
We will use the symbols c, C and γ for positive constants (possibly depending on the parameters of the bootstrap percolation model).

3. Main result.

The following theorem describes the threshold regime of finite volume bootstrap percolation for all values $3 \leq \ell \leq d$.

Theorem 3.1. *For $2 < \ell \le d$, there exist 2 constants $0 < \alpha_-(d,\ell) \le \alpha_+(d,\ell) < \infty$, independent of p, such that if*

$$L_\pm(d,\ell,p) := \exp^{\circ(\ell-1)}\left(\alpha_\pm\, p^{-\frac{1}{d-\ell+1}}\right), \tag{3.1}$$

then

a) $\mathbb{P}_p^{d,\ell}\left(\forall\, x \in \Lambda^d(L) \quad X^{d,\ell}_{\Lambda^d(L)}(x) = 1\right) \to 1$ *if* $(p,L) \to (0,\infty)$ *with* $L \ge L_+(d,\ell,p)$

b) $\mathbb{P}_p^{d,\ell}\left(\forall\, x \in \Lambda^d(L) \quad X^{d,\ell}_{\Lambda^d(L)}(x) = 1\right) \to 0$ *if* $(p,L) \to (0,\infty)$ *with* $L \le L_-(d,\ell,p)$

Remark: The case $2 = \ell \le d$ is handled in [**AL**]. The result in the case $\ell = d$ was a conjecture proposed in [**ADE**]. The specific case $\ell = d = 3$ was solved in [**CeCi**].

It looks like the phenomenon hidden behind this behavior is linked with the notion of "critical droplet".Indeed, the spanning probability has the same asymptotic behavior as the probability of finding in the volume $\Lambda^d(L)$ a suitably large internally-spanned cluster.

To prove this result, we use an inductive procedure. This is very natural for the estimate of the lower bound a). Indeed, the problem of the filling of a face of an hypercube once the hypercube is full is a bootstrap percolation problem with parameters $(d-1,\ell-1)$. By far less immediate is to see how to use induction in the proof of case b). We use there a natural generalization of the construction of Cerf and Cirillo, relating in this way a bootstrap percolation model with parameters (d,ℓ,p) with a bootstrap percolation model with parameters $(d-1,\ell-1,2p-p^2)$.

4. Proof of case a)

This is the easiest part of the proof. The argument is nothing new. In fact the idea of the argument is already present in [**ADE**]. To estimate from below the spanning probability, we use iteratively Straley's argument and the renormalization procedure introduced in [**AL**].
First, we use the renormalization scheme introduced in [**AL**] and [**S**] to prove that if $R(L^*,p,d,\ell) \ge (e(2d-1))^{-1}$ for some L^*, then $R(L,p,d,\ell) \ge 1 - Ce^{-L/L^*}$ for all $L > L^*$. Assume (for simplicity's sake) that L is an integer multiple of L^*. We tile $\Lambda^d(L)$ with the translates of $\Lambda^d(L^*)$. As initial condition for the bootstrap percolation process on the renormalized lattice we use the indicator functions of the events

$$\{L^*x + \Lambda^d(L^*) \text{ is internally spanned}\}, \quad x \in \mathbb{Z}^d.$$

It is clear that if the bootstrap process defined on the renormalized lattice spans the volume, so does the process on the original lattice. Hence,

$$R(L,p,d,\ell) \ge R\left(\frac{L}{L^*}, R(L^*,p,d,\ell), d, \ell\right) \tag{4.1}$$

If a box is not spanned then in the initial condition there must exist a cluster of empty sites that crosses the box. A standard site-percolation estimate, based on a Peierls type argument, gives

$$1 - R(N, q, d, \ell) \leq \sum_{l=N}^{\infty} q^l 2d(2d-1)^{l-1} = \frac{2d}{2d-1} \, \frac{((2d-1)q)^{N-1}}{1-(2d-1)q} \tag{4.2}$$

By (4.1) and by (4.2) with $N = L/L^*$ and $q = R(L^*, p, d, \ell) \geq (e(2d-1))^{-1}$, we get

$$R(L, p, d, \ell) \geq 1 - Ce^{-L/L^*}. \tag{4.3}$$

Next, we prove by induction the following property: there exists $\beta_+(d, \ell) > 0$ such that if we set

$$m_+(d, \ell, p) := \exp^{\circ(\ell-2)} \left(\beta_+(d, \ell) p^{-\frac{1}{d-\ell+1}} \right)$$

then

$$\forall L > m_+(d, \ell, p) \qquad R(L, p, d, \ell) \geq 1 - \exp\left(- \frac{L}{L_+(d, \ell, p)} \right) \tag{4.4}$$

For $\ell = 1$ and $\alpha_+(d, 1) = 1$, (4.4) is immediate, since a single occupied site in the initial condition is sufficient to span the entire volume. In the case $\ell = 2$, (4.4) has been proven in [**AL**] (see (1.5) therein). We end our induction proof by showing that if (4.4) holds for $(d-1, \ell-1)$ then $R(L^*, p, d, \ell) \geq (e(2d-1))^{-1}$ for $L^* \geq L_+(d, \ell, p)$. Let $m_+ := m_+(d, \ell, p)$ (we drop the dependency of m_+ on d, ℓ, p to lighten the notation). In the case $2 < \ell \leq d$, for $\beta_+(d, \ell) > \alpha_+(d-1, \ell-1)$ and sufficiently small p, we have that

$$m_+ \geq L_+^2(d-1, \ell-1, p). \tag{4.5}$$

To estimate from below the probability that $\Lambda^d(L)$ is spanned, we consider the event
1) at time 0 there exists in $\Lambda^d(L)$ a box $x\, m_+ + \Lambda^d(m_+)$ completely occupied and
2) for every $m_+ \leq k \leq L$ the $(d-1)$-dimensional bootstrap percolation models with parameters $(p, \ell-1)$ restricted to the faces of the boxes $x\, m_+ + \Lambda^d(k)$ are internally spanned.
The idea is that once a box $x\, m_+ + \Lambda^d(k)$ is occupied, the sites on a face of the box have an occupied neighbor in the box and therefore need only $\ell - 1$ neighbors in the face to become occupied. This procedure can be iterated to fill the whole $\Lambda^d(L)$.[1]
Thus,

$$R(L, p, d, \ell) \geq \left(1 - \left(1 - p^{m_+^d} \right)^{\left(\frac{L}{m_+}\right)^d} \right) \prod_{k=m_++1}^{L} R(k, p, d-1, \ell-1)^{2d} \geq$$

[1]We warn the reader that this argument is sligthly oversimplified since we are not considering the edges. We refer to [**S**] for the full construction.

$$\left(1 - e^{-p^{m_+^d}\left(\frac{L}{m_+}\right)^d}\right) \exp\left(-\sum_{k=m_++1}^{L} 2d\, e^{-\left(\frac{k}{\sqrt{m_+}}\right)}\right) \geq$$

$$\left(1 - e^{-p^{m_+^d}\left(\frac{L}{m_+}\right)^d}\right) \exp\left(-2d\sqrt{m_+}\, e^{-c\sqrt{m_+}}\right) \tag{4.6}$$

The factor $\sqrt{m_+}$ in the above formula (4.6) comes from (4.5). The last term in (4.6) clearly tends to one for $M \to \infty$. By using (3.1) we see that also the first term goes to 1 as $p \to 0$ for any $L \geq L_+(d, \ell, p)$: Indeed, for sufficiently small p,

$$m_+ p^{-\frac{m_+^d}{d}} \leq p^{-cm_+^d} = \exp\left(c \ln p^{-1}\left(\exp\left(d \exp^{\circ(\ell-3)}\left(\beta_+(d,\ell) p^{-\frac{1}{d-\ell+1}}\right)\right)\right)\right)$$
$$\leq \exp^{\circ(\ell-1)}\left(\alpha p^{-\frac{1}{d-\ell+1}}\right) \tag{4.7}$$

for sufficiently large $\alpha_+(d, \ell)$, r.h.s. of (4.7) tends to infinity slower than L_+ and for $L \geq L_+$, r.h.s. of (4.6) can be bounded by $(e(2d-1))^{-1}$. We can then use the bound in (4.3) and get (4.4).

5. Proof of case b)

In order to prove part b) of the Theorem, we give a bound on the probability that two points are in the same cluster for the bootstrap percolation process in a box with diameter of the order of the critical droplet. Since we choose the box with a sufficiently small diameter, the final configuration is "sub-critical" and looks like subcritical site percolation. This is the content of our key estimate. Let us set

$$m_-(d, \ell, p) := \exp^{\circ(\ell-2)}\left(\beta_-(d, \ell) p^{-\frac{1}{d-\ell+1}}\right), \tag{5.1}$$

where $\beta_-(d, \ell)$ is a constant independent from p.

Lemma 5.1. *Let $d \geq 1$ be fixed.*
For $\ell = 1$:

$$\forall m \in \mathbb{N} \qquad \mathbb{P}_p^d\left(x \stackrel{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m)\right) = 1 - (1-p)^{m^d} \tag{5.2}$$

For $d \geq \ell = 2$: there exist $\beta_-(d, 2) > 0$, $C > 0$ and $p(d, 2) > 0$ such that:
$\forall p < p(d, 2) \quad \forall m < m_-(d, 2, p) \quad \forall x, y \in \Lambda^d(m)$

$$\mathbb{P}_p^d\left(x \stackrel{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m)\right) \leq \left(C||x-y||_\infty^{d-1} p\right)^{||x-y||_\infty/2} \tag{5.3}$$

For $3 \leq \ell \leq d$: there exist $\beta_-(d, l) > 0$, $\gamma(d, \ell) > 0$, $p(d, \ell) > 0$ such that:
$\forall p < p(d, \ell) \quad \forall m < m_-(d, \ell, p) \quad \forall x, y \in \Lambda^d(m)$

$$\mathbb{P}_p^d\left(x \stackrel{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m)\right) \leq p^{-\gamma\ ||x-y||_\infty} \tag{5.4}$$

Next, for a box of diameter l to be internally spanned there must exist internally spanned regions of all intermediate diameters [**AL**] (more precisely, for any $k \leq l/2$ there must exist an internally spanned region of diameter between k and $2k+1$). We choose this intermediate diameter of the order of $m_-(d,\ell,p)$ to get the desired bound.

Lemma 5.2. [**AL**] *If $\Gamma \subset \mathbb{Z}^d$ is internally spanned then for all $k < (\mathrm{diam}\Gamma - 1)/2$ there exists $\Gamma_1 \subset \Gamma$ internally spanned with $k \leq \mathrm{diam}\Gamma_1 \leq 2k+1$.*

Sketch of proof. The proof can be found in [**AL**]. The idea is to realize the bootstrap percolation by an iterative algorithm which occupies at most one new site at each step. The new cluster obtained in this way has diameter not larger than $2k+1$, if k is the maximal diameter of the clusters that coalesce with the newly occupied site.

We now finish the proof of the case b) of the Theorem with the help of the two previous lemmata. Suppose Γ is a region which is internally spanned. Let Λ^d be a box such that $\Gamma \subset \Lambda^d$, diam $\Gamma =$ diam Λ^d and let $x, y \in \Gamma$ be such that $||x-y||_\infty =$ diam Γ. Then the event $\{x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d\}$ occurs. Therefore, by lemma 5.2 we have, for any L, κ such that $2\kappa + 1 < L$

$$R(L,p,d,\ell) \leq \mathbb{P}_p^d \begin{pmatrix} \exists\, m \quad \kappa \leq m < 2\kappa+1 \quad \exists z \in \Lambda^d(L) \quad z + \Lambda^d(m) \subset \Lambda^d(L) \\ \exists\, x,y \in z+\Lambda^d(m) \quad ||x-y||_\infty = m, \quad x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } z+\Lambda^d(m) \end{pmatrix}$$

$$\leq (\kappa+1)L^d(2\kappa+1)^{2d} \max_{\kappa \leq m < 2\kappa+1} \max_{\substack{z\in\Lambda^d(L) \\ z+\Lambda^d(m)\subset\Lambda^d(L)}} \max_{\substack{x,y\in z+\Lambda^d(m) \\ ||x-y||_\infty = m}} \mathbb{P}_p^d \left(x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } z+\Lambda^d(m) \right)$$

We distinguish two cases in order to bound $R(L,p,d,\ell)$. If $L < m_-(d,\ell,p)$, where $m_-(d,\ell,p)$ is defined in (5.1), then we choose $\kappa = L/3$ and we apply Lemma 5.1 in the case $\ell > 2$ to get

$$R(L,p,d,\ell) \leq L^{3d+1} p^{\gamma \frac{L}{3}}.$$

If $L \geq m_-(d,\ell,p)$, we choose $\kappa = m_-(d,\ell,p)/3$ and we get

$$R(L,p,d,\ell) \leq L^{3d+1} p^{\gamma m_-(d,\ell,p)/3}.$$

From these inequalities, we see that there exists $\alpha_-(d,\ell) > 0$ such that if

$$L \leq \exp^{\circ(\ell-1)} \left(\alpha_- \; p^{-\frac{1}{d-\ell+1}} \right),$$

then $R(L,p,d,\ell)$ goes to 0 in the limit where $(L,p) \to (\infty, 0)$.

Proof of Lemma 5.2. The result for $\ell = 1$ is immediate, since a single occupied site in the initial configuration is sufficient to span the entire volume.
For the case $\ell = 2$, we use a procedure introduced by Aizenman and Lebowitz in [**AL**]. We consider an integer

$$m \leq m_-(d,2,p) = \beta_-(d,2)\; p^{-\frac{1}{d-1}},$$

and we set $q := 2p - p^2$.
Let $x \in \mathbb{Z}^d$ be a d-dimensional vector; we denote by $\underline{x}$ its first $d-1$ coordinates and by $\overline{x}$ the last one. We write $x = (\underline{x}, \overline{x})$.
By symmetry, we can suppose that $(\underline{x}, \overline{x})$ and $(\underline{y}, \overline{y})$ in $\mathbb{Z}^d$ are such that $\overline{y} - \overline{x} = ||(\underline{x}, \overline{x}) - (\underline{y}, \overline{y})||_\infty$, namely that the distance along the d-th direction is larger than or equal to the distance in the other directions. We consider the slices

$$T_i := \big\{ (\underline{x}, \overline{x}) \in \Lambda^d(m) \ ; \ \overline{x} \in \{2i, 2i+1\} \big\} \, , \ i \in \mathbb{Z}.$$

Suppose $\{x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m)\}$ occurs. Let $\mathcal{C}$ be the $\Lambda^d(m)$-cluster that contains x and y and let A and B the first and the last indices of the slices intersecting $\mathcal{C}$. It is immediate to see that in all the slices T_i for $i \in [A, B]$ there exists at least one occupied site $(\underline{x}', \overline{x}',)$ such that $||\underline{x} - \underline{x}'||_\infty \le ||x - y||_\infty$. Let us set $l := ||x - y||_\infty$. By (5.2),

$$\mathbb{P}_p^d \left(x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m) \right) \le d \left(1 - (1-q)^{(2l+1)^{d-1}} \right)^{l/2} , \tag{5.5}$$

where the factor d comes from the possible directions where $||x - y||_\infty$ is realized. By using the fact that $\ln t \le t - 1$, we bound the r.h.s. of (5.5) by

$$d \left(- \ln (1-q)^{(2l+1)^{d-1}} \right)^{\frac{l}{2}} ;$$

for small q, $\ln(1-q) \ge -2q$ and hence we get bound (5.3).
For $\ell \ge 3$, we use an induction on the dimension d and on the parameter ℓ.
Following [**CeCi**], we define an auxiliary map $Z^d(x)$ on $\mathbb{Z}^d$ in the following way. In every slice T_i, we increase the initial configuration by occupying a site $(\underline{x}, 2i)$ (resp $(\underline{x}, 2i+1)$) if the corresponding site $(\underline{x}, 2i+1)$ (resp. $(\underline{x}, 2i)$) belonging to the other hyper-plane in the same slice is occupied. We build a configuration $Y^d((\underline{x}, \overline{x}))$ by updating on each slice this initial configuration according to the bootstrap percolation process in T_i with the neighboring slices occupied; more precisely, we occupy $\Lambda^d(m) \setminus T_i$, we run $X^{d,\ell}_{\Lambda^d(m)}$ under this initial condition and we define $Y^d(\underline{x}, 2i)) = Y^d(\underline{x}, 2i+1))$ as the restriction to T_i of this bootstrap percolation process.
The monotonicity properties of bootstrap percolation with respect to the initial configuration imply that $Y^d((\underline{x}, \overline{x})) \ge X^{d,\ell}_{\Lambda^d(m)}((\underline{x}, \overline{x}))$. The interesting point is that the set $\{\underline{x} \in \Lambda^{d-1}(m) \ ; \ Y^d((\underline{x}, 2i)) = 1\}$ is equal to the $(d-1)$-dimensional bootstrap of the $(d-1)$-dimensional configuration where the site $\underline{x} \in \Lambda^d(m)$ is occupied if either $(\underline{x}, 2i)$ or $(\underline{x}, 2i+1)$ was initially occupied. Thus, $Y^d((\underline{x}, \overline{x}))$ is a stack of $(d-1)$-dimensional bootstraps with parameters $\ell - 1$ and $q = 2p - p^2$. We set

$$n := \exp^{\circ(\ell-3)} \left(\varphi(d, \ell) p^{-\frac{1}{d-\ell+1}} \right) ,$$

for a suitable constant $\varphi(d, \ell)$ which will be chosen later on (see before (5.11)). We finally define our process as $Z^d((\underline{x}, \overline{x})) := Y^d((\underline{x}, \overline{x}))$ if the slice containing $(\underline{x}, \overline{x})$ does not contain any cluster larger than n; otherwise, we set $Z^d((\underline{x},\overline{x})) := 1$ for all the sites $(\underline{x},\overline{x})$ in the slice. For $\Gamma \in \Lambda^d(m)$, we denote by $\left\{\underline{x} \overset{Z}{\longleftrightarrow} \underline{y} \text{ in } \Gamma\right\}$ the event

$$\exists\, \mathcal{C} \subset T_i, \quad \mathcal{C} \text{ is connected}, \quad \{(\underline{x}, 2i), (\underline{y}, 2i)\} \subset \mathcal{C}, \quad \forall\, (\underline{y}', \overline{y}') \in \mathcal{C} \quad Z^d((\underline{y},\overline{y})) = 1\,.$$

By construction, we get for all $i \in [A, B]$

$$\mathbb{P}_p^d\left(\left\{\underline{x} \overset{Z}{\longleftrightarrow} \underline{y} \text{ in } T_i\right\} \cap \left\{\,\forall\, y \in T_i \quad Z^d(y) = 1\right\}^c\right) \le \mathbb{P}_q^{d-1}\left(\underline{x} \overset{X^{d-1,\ell-1}}{\longleftrightarrow} \underline{y} \text{ in } \Lambda^{d-1}(m)\right).$$

Let $\{I_k\}_{k\le U}$ be the ordered set of indices, between A and B of the slices that are completely full in Z^d i.e. $\left\{k; Z^d((\underline{x}, 2I_k)) = 1 \ \forall\ \underline{x} \in \Lambda^{d-1}(m)\right\}$. We set $I_0 := A$, $I_U := B$

We decompose our event $\left\{x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m)\right\}$ according to the possible values of A, B, U and $\{I_k\}_{k\le U}$. We have

$$\mathbb{P}_p^d\left(x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m)\right) \le d \sum_{a=1}^{m} \sum_{b=\overline{y}-\overline{x}+a}^{m} \sum_{u=0}^{b-a} \sum_{i_1<\cdots<i_u} \mathbb{P}_p^d\Big(\left\{x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m)\right\} \cap$$

$$\{A = a\} \cap \{B = b\} \cap \{U = u\} \cap \{I_1 = i_1, \ldots, I_u = i_u\}\Big) \tag{5.6}$$

where the factor d comes from the possible choices of the direction where $||x - y||_\infty$ is realized. We get

$$\mathbb{P}_p^d\left(\left\{x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m)\right\} \cap \{A = a\} \cap \{B = b\} \cap \{U = u\} \cap \{I_1 = i_1, \ldots, I_u = i_u\}\right)$$

$$\le \prod_{v=1}^{u} \mathbb{P}_p^d\left(\forall\, \underline{x}' \in \Lambda^{d-1}(m)\ Z^d(\underline{x}', 2i_v) = 1\right) \prod_{v=1}^{u+1} \mathbb{P}_p^d\left(\mathcal{E}(i_{v-1}+1, i_v - 1)\right), \tag{5.7}$$

where, for $i < j$, $\mathcal{E}(i,j)$ is the event:

$$\left\{[i,j] \cap \{I_k\}_{k\le U} = \emptyset,\ \exists\, x(i) \in T_i,\ y(j) \in T_j;\ \left\{x(i) \overset{Z}{\longleftrightarrow} y(j) \text{ in } \bigcup_{h=i}^{j} T_h\right\}\right\}$$

that is, in the process Z^d, none of the slices between T_i and T_j is full and there exists an occupied path between the slices T_i and T_j.

The occurrence of $\mathcal{E}(i,j)$ implies that there is a path $\{(\underline{x}_h, 2j_h)\}_{h\le R}$ (where R is a random value larger than or equal to $j - i$) such that

- $j_0 = i$, $j_R = j$,
- $\underline{x}_h \overset{Z}{\longleftrightarrow} \underline{x}_{h+1}$ in T_{j_h}
- if $j_h = j_{h'}$ then the event $\left\{\underline{x}_h \overset{Z}{\longleftrightarrow} \underline{x}_{h'} \text{ in } T_{j_h}\right\}$ does not occur.

We estimate $\mathbb{P}_p^d\left(\mathcal{E}(i,j)\right)$ with the help of the d-dimensional bootstrap percolation process with parameters $d-1$, $\ell-1$ and $2p-p^2$ by

$$\mathbb{P}_p^d\left(\mathcal{E}(i,j)\right) \le$$

$$\sum_{r\ge j-i}\ \sum_{\substack{\underline{x}_0,\dots,\underline{x}_r \ s.t.\\ ||\underline{x}_{h+1}-\underline{x}_h||_\infty\le n}} 2^r\ \mathbb{P}_p^d\Big(\bigcap_{h=1}^{r}\Big\{\{\underline{x}_h \overset{Z}{\longleftrightarrow} \underline{x}_{h+1} \text{ in } T_{i_h}\}\ \bigcap_{\substack{h':h<h'\le r\\ \text{s.t. } i_h=i_{h'}}}\ \{\underline{x}_h \overset{Z}{\longleftrightarrow} \underline{x}_{h'} \text{ in } T_{i_h}\}^c\Big\}\Big)$$

Now,

$$\mathbb{P}_p^d\Big(\bigcap_{h=1}^{r}\Big\{\Big\{\underline{x}_h \overset{Z}{\longleftrightarrow} \underline{x}_{h+1} \text{ in } T_{i_h}\Big\}\ \bigcap_{\substack{h'\in(h,r]\\ \text{s.t. } i_h=i_{h'}}}\ \Big\{\underline{x}_h \overset{Z}{\longleftrightarrow} \underline{x}_{h'} \text{ in } T_{i_h}\Big\}^c\Big\}\Big) \le$$

$$\mathbb{P}_p^d\left(\Big\{\underline{x}_1 \overset{Z}{\longleftrightarrow} \underline{x}_2 \text{ in } T_{i_1}\Big\} \circ \Big\{\underline{x}_2 \overset{Z}{\longleftrightarrow} \underline{x}_3 \text{ in } T_{i_2}\Big\} \circ \cdots \circ \Big\{\underline{x}_{r-1} \overset{Z}{\longleftrightarrow} \underline{x}_r \text{ in } T_{i_{r+1}}\Big\}\right), \tag{5.8}$$

where the symbol $\circ$ denotes the disjoint occurrence of the events (see [**G**] §2.3 p. 37 for the definition of disjoint events in percolation). Roughly speaking, given two increasing events $\mathcal{E}$, $\mathcal{E}'$, we define their *disjoint occurrence* $\mathcal{E}\circ\mathcal{E}'$ as the set of all configurations ω which contain two disjoint sets of occupied sites such that the first set implies $\mathcal{E}$ and the second set implies $\mathcal{E}'$.
Notice that the events we are considering are increasing events. We can use the van den Berg-Kesten inequality to bound r.h.s. of (5.8) by

$$\prod_{h=0}^{r-1}\mathbb{P}_q^{d-1}\left(\underline{x}_{h+1} \overset{X^{d,\ell}}{\longleftrightarrow} \underline{x}_h \text{ in } \Lambda^{d-1}(n)\right) \tag{5.9}$$

Remark: In fact, to get the bound (5.9) we use an inequality which is simpler than the van den Berg-Kesten inequality and that can be proved as follows (in the case of two events):

$$\mathbb{P}_p^d\left(\Big\{x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Gamma\Big\} \cap \Big\{w \overset{X^{d,\ell}}{\longleftrightarrow} z \text{ in } \Gamma\Big\} \cap \Big\{x \overset{X^{d,\ell}}{\longleftrightarrow} w \text{ in } \Gamma\Big\}^c\right)$$

$$= \sum_{\substack{\mathcal{C}\subset\Gamma\\ \{x,y\}\in\mathcal{C}}} \mathbb{P}_p^d\left(\mathcal{C} \text{ is a } \Gamma\text{-cluster }\right) \mathbb{P}_p^d\left(\Big\{w \overset{X^{d,\ell}}{\longleftrightarrow} z \text{ in } \Gamma\setminus(\mathcal{C}\cup\partial\mathcal{C})\Big\} \Big|\, \mathcal{C} \text{ is a } \Gamma\text{-cluster }\right).$$

However, the event $\{w \overset{X^{d,\ell}}{\longleftrightarrow} z \text{ in } \Gamma'\}$ does not depend on the configuration outside Γ'; moreover, its probability is non decreasing with the set Γ'. Thus, we can bound the second term of the product in the above sum by $\mathbb{P}_p^d\left(w \overset{X^{d,\ell}}{\longleftrightarrow} z \text{ in } \Gamma\right)$ and the whole sum by $\mathbb{P}_p^d\left(\Big\{x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Gamma\Big\}\right)\mathbb{P}_p^d\left(\Big\{w \overset{X^{d,\ell}}{\longleftrightarrow} z \text{ in } \Gamma\Big\}\right)$. The analogous result for a finite number of connections is proved by induction.

We use (5.9) to bound (5.8)

$$\leq \sum_{r\geq j-i} \sum_{\substack{\underline{x}_0,\dots,\underline{x}_r \ s.t. \\ ||\underline{x}_{h+1}-\underline{x}_h||_\infty\leq n}} 2^r \prod_{h=1}^{H} \mathbb{P}_q^{d-1}\left(\underline{x}_{h+1} \stackrel{X^{d,\ell}}{\longleftrightarrow} \underline{x}_h \text{ in } \Lambda^{d-1}(n)\right), \tag{5.10}$$

where the factor 2^r comes from the number of choices for the sequence of slices with $R = r$. Let $\rho_h := ||\underline{x}_h - \underline{x}_{h-1}||_\infty$ be the distance between consecutive points in the path. Once chosen the point $\underline{x}_{h-1}$ on the slice $T_{i_{h-1}}$, there are

$$|\Lambda^{d-1}(2\rho_h+1)\setminus\Lambda^{d-1}(2\rho_h-1)| = (2\rho_h+1)^{d-1} - (2\rho_h-1)^{d-1} \leq C\rho_h^{d-2}$$

ways to choose the point $\underline{x}_h$ at a distance ρ_h from $\underline{x}_{h-1}$.
In the case $\ell = 3$, we use (5.3) and bound r.h.s. of (5.10) by

$$\mathbb{P}_p^{d,3}\left(\mathcal{E}(i,j)\right) \leq m^{d-1} \sum_{r\geq j-i} \left(\sum_{\rho=1}^{n} 2C\rho^{d-2}\left(C_1\rho^{d-2}q\right)^{\frac{\rho}{2}}\right)^r$$

For $\ell = 3$ we have $n = \varphi(d,\ell)p^{-\frac{1}{d-2}}$. We estimate the inner sum of the above term:

$$\begin{aligned}\sum_{\rho=1}^{n} 2C\rho^{d-2}\left(C_1\rho^{d-2}q\right)^{\frac{\rho}{2}} &= \sum_{\rho=1}^{8} 2C\rho^{d-2}\left(C_1\rho^{d-2}q\right)^{\frac{\rho}{2}} + \sum_{\rho=9}^{n} 2C\rho^{d-2}\left(C_1\rho^{d-2}q\right)^{\frac{\rho}{2}} \\ &\leq 2C8^{d-2}\left(C_18^{d-2}q\right)^{\frac{1}{2}} + 2Cn^{d-1}\max_{9\leq\rho\leq n}\left(C_1\rho^{d-2}q\right)^{\frac{\rho}{2}}.\end{aligned}$$

Let $f(\rho) := \left(C_1\rho^{d-2}q\right)^{\frac{\rho}{2}}$. For $\varphi(d,\ell)$ small enough $f(\rho)$ is decreasing on $[9,n]$, whence

$$\begin{aligned}&\sum_{\rho=1}^{n} 2C\rho^{d-2}\left(C_1\rho^{d-2}q\right)^{\frac{\rho}{2}} \leq \\ &\quad 2C8^{d-2}\left(C_18^{d-2}q\right)^{\frac{1}{2}} + \ 2C\left(\varphi(d,\ell)p^{-\frac{1}{d-2}}\right)^{d-1}\left(C_19^{d-2}q\right)^{\frac{9}{2}} \leq C_2\sqrt{q}\end{aligned}$$

for some C_2 depending on $\varphi(d,\ell)$. For p small enough so that $C_2\sqrt{q}\leq\frac{1}{2}$,

$$\mathbb{P}_p^{d,3}\left(\mathcal{E}(i,j)\right) \leq 2m^{d-1}\left(C_2\sqrt{2p}\right)^{j-i} \leq m^{d-1}p^{\gamma(i-j)}. \tag{5.11}$$

In the case $\ell > 3$, we can use the inductive hypothesis. For sufficiently small p, r.h.s of (5.10) can be bounded by

$$m^{d-1}\sum_{r\geq j-i}\sum_{\rho_1,\dots,\rho_r\leq n} 2^r \prod_{h=0}^{r-1} C\rho_h^{d-2}p^{\gamma\rho_h} = m^{d-1}\sum_{r\geq j-i}\left(\sum_{\rho=1}^{n} C'\rho^{d-2}p^{\gamma\rho}\right)^r \tag{5.12}$$

Since, for sufficiently small p, $\rho^{d-2}p^{\gamma\rho}$ is decreasing with ρ we can get the bound:

$$\sum_{\rho=1}^{n} C'\rho^{d-2}p^{\gamma\rho} \le C'p^{\gamma}\left(1+\int_0^{\infty}(\rho+1)^{d-2}p^{\gamma\rho}\mathrm{d}\rho\right) \le p^{\gamma_1}, \tag{5.13}$$

where γ_1 is a positive constant smaller than γ. By putting (5.13) into (5.12), we get

$$\mathbb{P}_p^d\left(\mathcal{E}(i,j)\right) \le m^{d-1}\sum_{r\ge j-i} p^{\gamma_1 r} \le m^{d-1}p^{\gamma_1(j-i)},$$

formally, the same bound given by (5.11) for the case $\ell = 3$.
For $\ell \ge 3$, coming back to (5.7), we get

$$\mathbb{P}_p^d\left(\left\{x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m)\right\}\cap\{M=k\}\cap\{I_1=i_1,\dots,I_k=i_k\}\right) \le$$
$$\left(m^{d-1}p^{\gamma_1 n}\right)^k\left(m^{k(d-1)}p^{\gamma_2(b-a)}\right) = \left(m^{2(d-1)}p^{\gamma_1 n}\right)^k\left(p^{\gamma_2(b-a)}\right) \le \left(p^{\gamma_3}\right)^k\left(p^{\gamma_2(b-a)}\right),$$

where, in the last inequality we used the definitions of n and m and where γ_2, γ_3 are positive constants. Finally, by plugging this inequality in (5.6), we get

$$\mathbb{P}_p^d\left(x \overset{X^{d,\ell}}{\longleftrightarrow} y \text{ in } \Lambda^d(m)\right) \le dm^2\sum_{u=0}^{m} p^{\gamma_4 u} \le p^{\gamma\|x-y\|_\infty}$$

which is the desired estimate.

References

[Ad] J. Adler (1991). Bootstrap percolation. *Physica A* **171**, 453-470.

[ADE] J. Adler, J.A.M.S. Duarte, A.C.D. van Enter (1990). Finite–size effects for some bootstrap percolation models. *J. Statist. Phys.* **60**, 323–32.

[AL] M. Aizenman, J.L. Lebowitz (1988). Metastability effects in bootstrap percolation. *J. Phys. A: Math. Gen.* **21**, 3801.

[CeCi] R. Cerf, E.N.M. Cirillo (1999) Finite size scaling in three–dimensional bootstrap percolation *Ann. Probab.* **27**, No.4, 1837-1850

[vE] A.C.D. van Enter (1987). Proof of Straley's argument for Bootstrap Percolation. *J. Statist. Phys.* **48**, 943.

[G] G. Grimmett (1999) Percolation. 2nd ed. (English) Berlin: Springer.

[M] T.S. Mountford (1995). Critical length for semi–oriented bootstrap percolation. *Stochastic Process. Appl.* **56**, 185–205.

[S] R.H. Schonmann (1992). On the behavior of some cellular automata related to bootstrap percolation. *Ann. Probab.* **20**, 174.