SINGULAR CONTINUOUS SPECTRUM FOR A CLASS OF NONPRIMITIVE SUBSTITUTION SCHRÖDINGER OPERATORS

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ABSTRACT. We present a class of discrete Schrödinger operators, with potentials derived from nonprimitive substitutions, that has purely singular continuous spectrum. We give sufficient conditions on the substitution rule assuring singular continuous spectrum, either for a generic set in the hull of the potential or for a set of total invariant measure.

1. INTRODUCTION AND RESULTS

From the topological point of view the "building blocks" of dynamical systems are the minimal ones, i.e., those for which all orbits are dense; from the measure theoretical point of view such building blocks are the uniquely ergodic systems, i.e., those with just one probability invariant measure. In some cases these two properties are simultaneously verified and such dynamical systems are called *strictly ergodic*, a property that may present deep consequences. An important recent application of this concept is on the spectral study of a class of (discrete) Schrödinger operators, whose potentials take finite many values, generated by primitive substitution rules, with the predominance of pure singular continuous spectrum [1–16]. In this work we give new examples of Schrödinger operators with substitution potentials with purely singular continuous spectrum, but here we show that many techniques used to investigate the primitive case (including strict ergodicity) can be adapted to a class of nonprimitive substitutions; it expands and generalizes results for a particular substitution presented in our previous paper [17]. We also mention that, from the physical point of view, the substitution Schrödinger operators are quantum one-dimensional models of quasicrystals, whose potentials are neither random nor periodic, but exhibit self-similarities.

Let's clarify what is called a substitution potential and primitivity. Since we consider only potentials taking two values we shall restrict our discussion to substitutions $\xi : \mathcal{A} \mapsto \mathcal{A}^*$ on an alphabet $\mathcal{A} = \{a, b\}$; $\mathcal{A}^*$ is the set of all words of finite length on the alphabet. If we denote by $\mathcal{A}^{\mathbb{N}}$ the set of all sequences of letters in $\mathcal{A}$, the substitution ξ have natural extensions, by concatenation, to $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}}$; for example, $\xi(ab) = \xi(a)\xi(b)$. A *substitution sequence* is a fixed point $\bar{\eta}$ of ξ in $\mathcal{A}^{\mathbb{N}}$, i.e., $\xi(\bar{\eta}) = \bar{\eta}$. We consider on $\mathcal{A}^{\mathbb{N}}$ ($\mathcal{A}^{\mathbb{Z}}$) the point convergence topology generated by the metric

$$d(u, v) = \sum_n \frac{|u_n - v_n|}{2^{|n|}}, \quad u = (u_n), v = (v_n),$$

with $n \in \mathbb{N}$ ($n \in \mathbb{Z}$). Then, we take a as a seed, suppose that $\xi(a)$ begins with a and that the length of $\xi^k(a)$ goes to infinity as $k \mapsto \infty$ (ξ^k denotes the k-th iterate of ξ), assuring the existence of a fixed point for ξ (see [19], Proposition V.1).

Given a substitution sequence $\bar{\eta}$, consider the periodic sequences in $\mathcal{A}^{\mathbb{Z}}$

$$\eta_n = \cdots \xi^n(a)\xi^n(a).\xi^n(a)\xi^n(a) \cdots,$$

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with the dot indicating the position of the term of index zero. If $\xi(a)$ begins and ends with a, then (η_n) is a Cauchy sequence and one gets a well-defined limit sequence

$$\eta = \lim_{n\to\infty} \eta_n$$

in $\mathcal{A}^{\mathbb{Z}}$, called the bilateral substitution sequence generated by ξ. Such conditions will be satisfied by the substitution class ζ_N that will be considered in this work.

Let $T : \mathcal{A}^{\mathbb{Z}} \mapsto \mathcal{A}^{\mathbb{Z}}$ be the left shift $(Tu)_n = u_{n+1}$; the *hull* of η in $\mathcal{A}^{\mathbb{Z}}$ is defined as

$$\Omega = \Omega(\eta) = \text{closure}\{T^n\eta : \ n \in \mathbb{N}\}.$$

We notice that for a recurrent sequence η its hull Ω is a compact and T-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ with $T\Omega(\eta) = \Omega(\eta)$ ([19], Chapter V), so that the dynamical system $(\Omega(\eta), T)$ is well-defined.

Given a bijective real function $f : \mathcal{A} \mapsto \mathbb{R}$, we associate a sequence in $l^\infty(\mathbb{Z})$ to each $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \Omega$ by $(f(\omega_n))_{n\in\mathbb{Z}}$, i.e., we just replace letters by numbers, which we again denote by ω and call it a *substitution potential.* Therefore, every point $\omega \in \Omega$ defines a bounded self-adjoint operator H_ω on $l^2(\mathbb{Z})$ by

$$(H_\omega u)_n = u_{n+1} + u_{n-1} + \omega_n u_n. \tag{1}$$

Our goal is the investigation of the spectral properties of the Schrödinger operator (1) for a special class of substitutions defined below.

Let M be the matrix of the substitution ξ whose entries M_{ab} are given by the number of occurrences of the symbol a in $\xi(b)$, for $a, b \in \mathcal{A}$. Recall that a matrix M is primitive if all entries in a power of M are strictly positive. A substitution rule ξ is called primitive if its corresponding substitution matrix is primitive. It is known that for primitive substitution the subshift dynamical system (Ω, T) is strictly ergodic (see [19], Chapter V).

Here we consider a subclass of the following collection of substitutions $\zeta = \zeta_N$, for each $N \geq 1$, on $\mathcal{A} = \{a, b\}$, given by

$$\zeta(a) = \underbrace{a...a}_{A_1}\underbrace{b...b}_{B_1}\underbrace{a...a}_{A_2}\cdots\underbrace{b...b}_{B_N}\,\underbrace{a...a}_{A_{N+1}}, \qquad \zeta(b) = b, \tag{2}$$

with $A_j, B_j \geq 1, \ j = 1, \cdots, N$ and $A_{N+1} \geq 1$, being the number of factors in each block. Such substitution is clearly nonprimitive, for any positive power of its matrix of substitution has the form

$$\begin{bmatrix} A & 0 \\ B & 1 \end{bmatrix}$$

for some positive integers A, B.

It is possible to generate periodic sequences with appropriate choices of ζ in (2); for instance, the substitution $\zeta(a) = abba$, $\zeta(b) = b$. The aperiodicity of the substitution sequences is significant for our spectral results; the following lemma presents a very simple and rather general condition on ζ assuring that η is not periodic.

Lemma 1. *If $A_j \geq 2$ for some $1 \leq j \leq (N + 1)$, then the resulting substitution sequence η is not ultimately periodic.*

Proof. It is possible to restrict the argument for strictly positive index of η, i.e., to $\bar{\eta}$. For notational convenience we suppose that $A_1 = 2$; the other cases can be handled in a similar way. Notice first that all (constant) blocks of $a's$ (and $b's$) in η are finite. Suppose that η is periodic with minimal period $\tau \geq 2$ (η is clearly nonconstant) and denote the first minimal period block, represented by the first τ

letters of $\bar{\eta}$, by B. Let n_c be the unique integer such that ($|Q|$ denotes the length of $Q \in \mathcal{A}^*$; e.g., $|B| = \tau$. It is also supposed that $\zeta^0(a) = a$.)

$$|\zeta^{n_c-1}(a)| < \tau \leq |\zeta^{n_c}(a)|.$$

The first entries of $\bar{\eta}$ are

$$\zeta^{n_c+1}(a) = \zeta^{n_c}(a)\zeta^{n_c}(a)b\ldots b\zeta^{n_c}(a)\ldots\zeta^{n_c}(a)\cdots b\ldots b\zeta^{n_c}(a)\ldots\zeta^{n_c}(a).$$

The choice of n_c entails that, besides at position 1, a new period block B starts at position $(|\zeta^{n_c}(a)| + 1)$. Since the period block B can not appear shifted by any amount different from an integer multiple of τ (this would force a constant sequence $\bar{\eta}$), one sees that $|\zeta^{n_c}(a)|$ is an integer multiple of τ. Therefore, a new period block B should start at position $(2|\zeta^{n_c}(a)| + 1)$, but this can not occur since B starts with the letter a and at position $(2|\zeta^{n_c}(a)| + 1)$ of $\bar{\eta}$ there is a b. Therefore $\bar{\eta}$ (and consequently η) is not periodic.

Suppose now that η is ultimately periodic, so that there is an integer $k \geq 1$ such that η is periodic, with period τ, after its kth position ($k \geq 2$). Choose an integer m such that $|\zeta^m(a)| > k$ and $|\zeta^m(a)| > \tau$. The first entries of $\bar{\eta}$ are

$$\zeta^{m+1}(a)[\zeta^{m+1}(a)]b\ldots b\zeta^{m+1}(a)\ldots\zeta^{m+1}(a)\cdots b\ldots b\zeta^{m+1}(a)\ldots\zeta^{m+1}(a),$$

and writing out the above second block $\zeta^{m+1}(a)$ one gets, for the beginning of $\bar{\eta}$,

$$\zeta^{m+1}(a)[\zeta^m(a)\zeta^m(a)b\ldots b\zeta^m(a)\ldots\zeta^m(a)\cdots b\ldots b\zeta^m(a)\ldots\zeta^m(a)]b\ldots b.$$

The choice of m implies that a minimal period block B starts at position $(\zeta^{m+1}(a)+1)$ and also at $(|\zeta^{m+1}(a)| + |\zeta^m(a)| + 1)$, and with the letter a. The same argument used above implies that another period block B should start at position $(|\zeta^{m+1}(a)| + 2|\zeta^m(a)| + 1)$, i.e., B should start with a b. This contradiction shows that $\bar{\eta}$ and η are not ultimately periodic. □

Although the substitution ζ is not primitive, it will be possible to adapt some techniques used for the primitive case in order to prove spectral results for (1), with potentials along the hull of a subclass ζ of (2) (to be specified later), always assuming from now on that

(i) Ω_ζ denotes the hull of the substitution ζ defined by (2), so that (Ω_ζ, T) indicates the subshift dynamical system associated to ζ.
(ii) the function $f : \mathcal{A} \mapsto \mathbb{R}$ defining the substitution potential is bijective (in order to avoid constant potentials).

Some studies of the spectral properties of (1), with potentials (ω_n) generated by ζ_N satisfying (i) and (ii), will rely upon dynamical features of (Ω_ζ, T) and an adaptation of the so-called Gordon argument (see, for instance, [9] and Section 10.2 of [18]); these are the contents of Propositions 1, 2 and 4, whose proofs are postponed to other Sections of this work. However, they will be used below to present our spectral results, which are summarized in two theorems.

Propositon 1. *The subshift dynamical system (Ω_ζ, T) is strictly ergodic. We shall denote its unique probability ergodic measure by μ.*

From general arguments presented, for example, in Queffélec's book [19], Chapter V, it follows from the strict ergodicity of (Ω_ζ, T) that every word from $\mathcal{A}^*$ occurring in some $\omega \in \Omega_\zeta$ also occurs in every $\omega' \in \Omega_\zeta$, and with a well-defined and strictly positive frequency.

As usual, we shall indicate the spectrum of a self-adjoint operator H by $\sigma(H)$, and by $\sigma_{ac}(H)$ and $\sigma_p(H)$ the absolutely continuous and point spectrum of H, respectively. Recall that a *generic set* is a dense G_δ (countable intersections of open sets).

The above dynamical properties will imply the following spectral result

Propositon 2. *If $A_j \geq 2$ for some $1 \leq j \leq (N+1)$, then $\sigma_{ac}(H_\omega) = \emptyset$ for any $\omega \in \Omega_\zeta$.*

A curious property of some sequences is palindromicity, which plays an important role in the exclusion of eigenvalues for discrete Schrödinger operators (1). A word in $\mathcal{A}^*$ is called a palindrome [20, 12] if it is the same when read backwards, and an element of $\mathcal{A}^{\mathbb{Z}}$ is palindromic if it contains arbitrary long palindromes. A compact and T-invariant minimal subset of $\mathcal{A}^{\mathbb{Z}}$ is called palindromic if it contains one palindromic sequence.

Propositon 3. *If Ω_ζ is palindromic, or if $\zeta(a)$ is a palindrome, then there exists a generic set $\Upsilon \subset \Omega_\zeta$ such that $\sigma_p(H_\omega) = \emptyset$ for any $\omega \in \Upsilon$.*

Proof. Since the proof is essentially contained in [12] we just sketch it. If $\zeta(a)$ is a palindrome then $\zeta(w)$ is also a palindrome for any palindrome w, so that a ζ-fixed point η has palindromes of arbitrary long length and Ω_ζ is palindromic. Therefore the second alternative hypothesis in the proposition implies the first one which, by its turn, guarantees the presence of a dense set $\mathcal{S} \subset \Omega_\zeta$ of *strong palindromes* with $\sigma_p(H_\omega) = \emptyset$ for any $\omega \in \mathcal{S}$ (Proposition 2.1 and Theorem 5.1 of [12]). The generic property in the proposition follows by the set of results known as Wonderland Theorem [21] (see also Theorem 7.1 in [12]). □

Our motivating case was the particular substitution $\zeta^{(p)} : \mathcal{A} \to \mathcal{A}^*$, $\zeta^{(p)}(a) = aabaa$, $\zeta^{(p)}(b) = b$, discussed in [17]. According to Lemma 1 and Proposition 1, this is an example of a nonperiodic substitution sequence of the type (2) for which $\Omega_{\zeta^{(p)}}$ is palindromic, since $\zeta^{(p)}(a)$ is a palindrome.

Theorem 1. *Suppose $A_j \geq 2$ for some $1 \leq j \leq (N+1)$. If Ω_ζ is palindromic, or if $\zeta(a)$ is a palindrome, then there exists a generic set $\Upsilon \subset \Omega_\zeta$ such that for any $\omega \in \Upsilon$ the operator H_ω has purely singular continuous spectrum.*

Proof. Combine Propositions 2 and 3. □

Besides palindromicity, another way to exclude point spectrum is via the following adaptation of Gordon's argument [9, 14, 7, 5, 6, 18].

Propositon 4. *If either $(A_1 + A_{N+1}) \geq 4$ or $A_j \geq 4$ for some $1 \leq j \leq (N+1)$ (or both), then there exists a measurable set $\Xi \subset \Omega_\zeta$, with $\mu(\Xi) = 1$, such that $\sigma_p(H_\omega) = \emptyset$ for any $\omega \in \Xi$.*

Theorem 2. *If either $(A_1 + A_{N+1}) \geq 4$ or $A_j \geq 4$ for some $1 \leq j \leq (N+1)$, then there exists a measurable set $\Xi \subset \Omega_\zeta$, with $\mu(\Xi) = 1$, such that for any $\omega \in \Xi$ the operator H_ω has purely singular continuous spectrum.*

Proof. Combine Propositions 2 and 4. □

The above theorems present (very easy to verify) sufficient conditions for the occurrence of singular continuous spectra. For example, the substitutions $a \to abaaa$, $a \to abbaaaaba$, $a \to aaababababba$ ($b \to b$ in all cases) result in pure singular continuous spectrum of H_ω for ω in sets of total invariant measure in the respective hulls. The palindromic nonprimitive substitution $\zeta(a) = abaaba$ in (2) gives rise to singular continuous spectrum for the operator (1) in a generic set of potentials in its hull; we do not have any results on the measure theoretical side in this case. The operator H_ω, with potentials in the hull of the substitution $\zeta^{(p)}(a) = aabaa$, has pure singular continuous spectrum for both in a generic set and in a set of full invariant measure of its hull (notice that this improves results previously obtained in [17]).

The reminder of this work is organized as follows. In Section 2 we prove Propositions 1 and 2, and in Section 3 we give the proof of Proposition 4. In the final Section we present additional remarks.

2. Strict Ergodicity

In this section we prove Propositions 1 and 2. Before proceeding to the proof of Proposition 1, we recall that $|Q|$ is the length of $Q \in \mathcal{A}^*$ and say that such Q is called a k-block in $\omega \in \Omega_\zeta$ if Q is a word contained in ω and $|Q| = k$. If P and Q belong to $\mathcal{A}^*$, $\#_P Q$ denotes the number of occurrences of P in Q. An important point in Proposition 1 is to show that

$$\lim_{j\to\infty} \frac{\#_{\zeta^n(a)}(|\zeta^{n+j}(a)|\text{-block})}{|\zeta^{n+j}(a)|}$$

exists, is nonzero and independent of the $|\zeta^{n+j}(a)|$-block considered; this implies that (Ω_ζ, T) is uniquely ergodic ([23], Section 4.5). To perform the required estimates, we define $f_{N+1} = \sum_{i=1}^{N+1} A_i$ and $h_N = \sum_{i=1}^{N} B_i$; the jth power of f_{N+1} will be denoted by f_{N+1}^j, and the ratio

$$R_j = (f_{N+1}^j - 1)/(f_{N+1} - 1);$$

notice that $f_{N+1} \geq 2$.

Lemma 2. *The iterates of the substitution ζ satisfy*

(a) $|\zeta^j(a)| = f_{N+1}^j + h_N R_j$

(b) $\#_a(\zeta^j(a)) = f_{N+1}^j$.

Proof. (a) It is clear that $|\zeta(a)| = f_{N+1} + h_N$ and

$$\begin{aligned} |\zeta^2(a)| &= f_{N+1}(f_{N+1} + h_N) + h_N \\ &= f_{N+1}^2 + h_N(f_{N+1} + 1); \end{aligned}$$

an induction argument gives

$$\begin{aligned} |\zeta^j(a)| &= f_{N+1}^j + h_N[f_{N+1}^{j-1} + \cdots + 1] \\ &= f_{N+1}^j + h_N R_j. \end{aligned}$$

(b) It is a simple induction. □

Define C_A and C_{B_k} by

$$\#_{\zeta^n(a)}(\zeta^n(a)\zeta^n(a)) = 2 + C_A$$

$$\#_{\zeta^n(a)}(\zeta^n(a)\underbrace{b...b}_{B_k}\zeta^n(a)) = 2 + C_{B_k}, \quad 1 \leq k \leq N.$$

Lemma 3. *If $C_A > 0$, then $C_{B_q} = 0$ $\forall 1 \leq q \leq N$. Conversely, if $C_{B_q} > 0$ for some q, then $C_A = 0$.*

Proof. If $C_A > 0$ there exists $2 \leq j \leq N$ such that

$$\underbrace{a...a}_{A_j - A_{N+1}}\underbrace{b...b}_{B_j}\underbrace{a...a}_{A_{j+1}}\cdots\underbrace{a...a}_{A_{N+1}+A_1}\cdots\underbrace{b...b}_{B_{j-1}}\underbrace{a...a}_{A_j - A_1} = \underbrace{a...a}_{A_1}\underbrace{b...b}_{B_1}\underbrace{a...a}_{A_2}\cdots\underbrace{b...b}_{B_N}\underbrace{a...a}_{A_{N+1}};$$

then

$$(3) \qquad A_k = \begin{cases} A_{k+j-1} & k+j-1 < N+1 \\ A_1 + A_{N+1} & k+j-1 = N+1 \\ A_{k+j-1-N} & k+j-1 > N+1 \end{cases} \qquad 2 \leq k \leq N,$$

and

$$B_m = \begin{cases} B_{j+m-1} & k+j-1 \le N \\ B_{j+m-1-N} & k+j-1 > N \end{cases} \qquad 1 \le m \le N. \tag{4}$$

If $C_{B_q} > 0$ for some q there exists $2 \le i \le N$ such that

$$\underbrace{a...a}_{A_i}\underbrace{b...b}_{B_i}\underbrace{a...a}_{A_{i+1}}\cdots\underbrace{a...a}_{A_{N+1}}\underbrace{b...b}_{B_q}\underbrace{a...a}_{A_1}\cdots\underbrace{b...b}_{B_{i-2}}\underbrace{a...a}_{A_{i-1}} = \underbrace{a...a}_{A_1}\underbrace{b...b}_{B_1}\underbrace{a...a}_{A_2}\cdots\underbrace{b...b}_{B_N}\underbrace{a...a}_{A_{N+1}}$$

then

$$A_l = \begin{cases} A_{l+i-1} & l+i-1 \le N+1 \\ A_{l+i-1-(N+1)} & l+i-1 > N+1 \end{cases} \qquad 1 \le l \le N, \tag{5}$$

and

$$B_m = \begin{cases} B_{m+i-1} & m+i-1 < N+1 \\ B_q & m+i-1 = N+1 \\ B_{m+i-1-(N+1)} & m+i-1 > N+1 \end{cases} \qquad 1 \le m \le N. \tag{6}$$

If the conditions (3) and (5) are simultaneously satisfied we have (the number over the equal sign indicates the equation used in each step)

$$(A_1 + A_{N+1}) \overset{(3)}{=} A_{N-j+2} \overset{(5)}{=} A_{N-j+i+1} \overset{(3)}{=} A_i \overset{(5)}{=} A_1$$

if $j \ge i$, and

$$(A_1 + A_{N+1}) \overset{(3)}{=} A_{N-j+2} \overset{(5)}{=} A_{i-j} \overset{(3)}{=} A_{i-1} \overset{(5)}{=} A_{N+1}$$

if $j < i$. Since $A_{N+1} > 0$ and $A_1 > 0$, we have got contradictions and the Lemma is proven.

We remark that the maximum possible values of C_A and C_{B_q} are $(N-1)$ and N, respectively (and each of them occurs for suitable choices of ζ_N). □

Proof. (*Proposition 1*) To begin with we show that (Ω_ζ, T) is uniquely ergodic. Due to Lemma 3 it is enough to consider two cases:

(i) $\#_{\zeta^n(a)}(\zeta^n(a)\zeta^n(a)) = 2 + C_A, \quad \#_{\zeta^n(a)}(\zeta^n(a)B_k\zeta^n(a)) = 2;$
(ii) $\#_{\zeta^n(a)}(\zeta^n(a)\zeta^n(a)) = 2, \quad \#_{\zeta^n(a)}(\zeta^n(a)B_k\zeta^n(a)) = 2 + C_{B_k}.$

Notice that both cases include $C_A = C_{B_k} = 0$, for any k, as particular instances.

In the case (i), by Lemma 2,

$$\#_{\zeta^n(a)}(\zeta^{n+j}(a)) = f_{N+1}^j + (f_{N+1} - (N+1))C_A R_j,$$

and by counting the occurrences of $\zeta^n(a)$ in $|\zeta^n(a)|$ − blocks we find

$$\begin{aligned} f_{N+1}^j + (f_{N+1} - (N+1))C_A R_j - C_A - 1 &\le \#_{\zeta^n(a)}(|\zeta^{n+j}(a)|\text{-block}) \\ &\le f_{N+1}^j + (f_{N+1} - (N+1))C_A R_j + C_A - 1, \end{aligned}$$

so that

$$\begin{aligned} \lim_{j\to\infty} \frac{\#_{\zeta^n(a)}(|\zeta^{n+j}(a)|\text{-block})}{|\zeta^{n+j}(a)|} &= \lim_{j\to\infty} \frac{f_{N+1}^j + (f_{N+1} - (N+1))C_A R_j}{f_{N+1}^{j+n} + h_N R_{j+n}} \\ &= \frac{f_{N+1} + (f_{N+1} - (N+1))C_A - 1}{f_{N+1}^n [f_{N+1} + h_N - 1]} > 0. \end{aligned}$$

In the case (ii) there is an integer $M_k \ge 0$, which depends on N, such that (M_k takes into account the existence of eventual $B_l = B_k$ for $k \ne l$)

$$\#_{\zeta^n(a)}(\zeta^{n+j}(a)) = f_{N+1}^j + M_k R_j.$$

Again by counting

$$\begin{aligned} f_{N+1}^j + M_k R_j - C_{B_k} - 1 &\le \#_{\zeta^n(a)}(|\zeta^{n+j}(a)|\text{-block}) \\ &\le f_{N+1}^j + M_k R_j + C_{B_k} - 1, \end{aligned}$$

so that

$$\lim_{j\to\infty} \frac{\#_{\zeta^n(a)}(|\zeta^{n+j}(a)|\text{-block})}{|\zeta^{n+j}(a)|} = \lim_{j\to\infty} \frac{f^j_{N+1} + M_k R_j}{f^{j+n}_{N+1} + h_N R_{j+n}} = \frac{f_{N+1} + M_k - 1}{f^n_{N+1}[f_{N+1} + h_N - 1]} > 0.$$

(If there are other $C_{B_q} \neq 0$, $B_q \neq B_l$ for $k \neq l$, just replace M_k by $\sum_q M_q$ in the above average, with the sum over q such that $B_q \neq B_l$.) Therefore (Ω_ζ, T) is uniquely ergodic.

Now we show that (Ω_ζ, T) is minimal. Given a word $B \in \mathcal{A}^*$ occurring in η, there is an $m \in \mathbb{N}$ such that $B \subset \zeta^m(a)$. Now, $\zeta^m(a)$ occurs in η with bounded gaps (with bound $\leq \max\{B_1, B_2, \cdots, B_N\} + |\zeta^m(a)|$); this implies that η is almost periodic, and since from the very definition of ζ it follows that η is also recurrent, one concludes that (Ω_ζ, T) is minimal ([19], Section IV.2). Therefore (Ω_ζ, T) is strictly ergodic. □

Proof. (Proposition 2) The conclusions of Proposition 2 will follow from Proposition 1 and two additional results. The first one is due to Last and Simon [22] which, in this setting, states that for strictly ergodic potentials the absolutely continuous spectrum is independent of $\omega \in \Omega_\zeta$. The second result we use is due to Kotani [13]: for nonperiodic ergodic potentials taking only a finite number of values the set of potentials with no absolutely continuous spectrum has full ergodic measure. By Proposition 1 our (nonperiodic) substitution potentials are strictly ergodic, so by Last-Simon $\sigma_{ac}(H_\omega)$ is constant on Ω_ζ; since by Kotani and Lemma 1 there are potentials with $\sigma_{ac}(H_\omega) = \emptyset$, it follows that $\sigma_{ac}(H_\omega) = \emptyset$ for any $\omega \in \Omega_\zeta$. □

3. A Gordon Type Argument

Now we will show that if $(A_1 + A_{N+1}) \geq 4$ or $A_j \geq 4$ for some $1 \leq j \leq N+1$, then $\sigma_p(H_\omega) = \emptyset$ for ω in a set of full invariant measure μ in the hull Ω_ζ, i.e., we prove Proposition 4. We shall follow a Gordon type argument similar to the one used in [9, 14, 7, 5, 6, 18].

Let's first sketch the idea of the proof. For a fixed $\omega \in \mathcal{A}^{\mathbb{Z}}$, if there is a sequence $m \to \infty$ such that, for each such m,

$$\omega_{k-m} = \omega_k = \omega_{k+m}, \quad 1 \leq k \leq m,$$

then the Schrödinger operator (1) has no eigenvalues; this is the Gordon argument (see the above references). It will be shown that, under the assumptions of Proposition 4, the measurable sets

$$B(n) \equiv \{\omega \in \Omega_\zeta : \omega_{(k-|\zeta^n(a)|)} = \omega_k = \omega_{(k+|\zeta^n(a)|)}, 1 \leq k \leq |\zeta^n(a)|\},$$

satisfy

$$\mu(\limsup_{n\to\infty} B(n)) > 0,$$

and since $\limsup_{n\to\infty} B(n)$ is contained in the shift invariant set $\{\omega \in \Omega_\zeta : \sigma_p(H_\omega) = \emptyset\}$, it will follow, by an argument in ergodic theory, that the latter set has full invariant measure.

If η is the bilateral substitution sequence generated by ζ, it is then natural to investigate the sets

$$J(n) \equiv \{\text{distinct words of } \eta \text{ of the form } vvv \text{ with } |v| = |\zeta^n(a)|\}$$

and, in particular, the values of their measures $\mu(J(n))$, which can be recovered by ergodic averages. The following Lemmas will be used to estimate such averages.

Lemma 4. *If $(A_1 + A_{N+1}) \geq 4$ or $A_j \geq 4$ for some $1 \leq j \leq N+1$, then for all $n \geq 1$ $\#J(n)$ (the cardinality of $J(n)$) satisfies*

$$B_1 + A_2|\zeta^{n-1}(a)| \leq \#J(n) \leq |\zeta^n(a)|.$$

Proof. The occurrence of $aaaa$ in η yields existence of blocks

$$\zeta^n(a)\zeta^n(a)\zeta^n(a)\zeta^n(a)$$

that contain all different blocks of the form vvv, with $|v| = |\zeta^n(a)|$. Thus we consider cyclic permutations of $\zeta^n(a)$ to determine the number of different blocks vvv. For each block $u_1u_2\cdots u_l$ let

$$P(u_1u_2\cdots u_l) = u_2u_3\cdots u_lu_1.$$

Thus

$$\#J(n) = \min\{i \geq 1 : \zeta^n(a) = P^i(\zeta^n(a))\} \leq |\zeta^n(a)|.$$

For a general substitution of the type (2), by successively applying P to $\zeta(a)$, the first opportunity to recover $\zeta(a)$ is (see the examples at the end of this proof)

$$\underbrace{a...a}_{A_2-A_{N+1}}\ \underbrace{b...b}_{B_2}\cdots\ \underbrace{a...a}_{A_1+A_{N+1}}\ \underbrace{b...b}_{B_1}\ \underbrace{a...a}_{A_2-A_1}\ .$$

Hence we have $B_1+A_2 \leq \#J(1) \leq |\zeta(a)|$. By the very definition of the substitution, this property generalizes for any positive integer n

$$B_1 + A_2|\zeta^{n-1}(a)| \leq \#J(n) \leq |\zeta^n(a)|.$$

Both extreme conditions can occur. For example, if all B_j are pairwise different or $A_j \neq A_1 + A_{N+1}, \forall 2 \leq j \leq N$, one gets $\#J(n) = |\zeta^n(a)|$; while the case in which all B_j are equal and $A_j = A_1 + A_{N+1}, \forall 2 \leq j \leq N$, one gets $\#J(n) = B_1 + A_2|\zeta^{n-1}(a)|$. □

Lemma 5. *If vvv occurs in η with $|v| = |\zeta^n(a)|$, then*

$$\begin{aligned}&\#_{vvv}\zeta^{n+j}(a) \geq \#_{aaaa}\zeta^j(a) = \\ &\qquad = f_{N+1}^{j-1}\textstyle\sum_{\{i:A_i\geq 4\}}(A_i-3) + R_{j-1}E_N\sum_{i=1}^{N+1}(A_i-1),\end{aligned}$$

where

$$E_N = \begin{cases} 0 & \text{if } A_1 + A_{N+1} < 4 \\ 1 & \text{if } A_1 = A_{N+1} = 2 \\ 3 & \text{if } \min\{A_1, A_{N+1}\} \geq 3 \\ \min\{A_1, A_{N+1}\} & \text{otherwise} \end{cases}$$

Proof. By the construction of the blocks vvv with $|v| = |\zeta^n(a)|$, in the proof of Lemma 4, we have

$$\#_{vvv}\zeta^{n+j}(a) \geq \#_{\zeta^n(aaaa)}\zeta^{n+j}(a) = \#_{aaaa}\zeta^j(a).$$

Direct computation gives $\#_{aaaa}\zeta(a) = \sum_{\{i:A_i\geq 4\}}(A_i-3)$, and

$$\#_{aaaa}\zeta^2(a) = f_{N+1}\sum_{\{i:A_i\geq 4\}}(A_i-3) + E_N\sum_{i=1}^{N+1}(A_i-1).$$

Then, by induction one concludes that

$$\begin{aligned}\#_{aaaa}\zeta^j(a) &= \\ &= f_{N+1}^{j-1}\sum_{\{i:A_i\geq 4\}}(A_i-3) + [f_{N+1}^{j-2} + \cdots + 1]E_N\sum_{i=1}^{N+1}(A_i-1) \\ &= f_{N+1}^{j-1}\sum_{\{i:A_i\geq 4\}}(A_i-3) + R_{j-1}E_N\sum_{i=1}^{N+1}(A_i-1).\end{aligned}$$

□

The Borel σ-algebra in (Ω_ζ, T) is generated by the cylinders

$$[u_0 \cdots u_{l-1}]_{[m,\cdots,m+l-1]} \equiv \{\omega \in \Omega_\zeta : \omega_{m+i} = u_i, 0 \le i \le l-1\}.$$

Since Ω_ζ is strictly ergodic, it is know that ([19], Section V.4) the unique invariant probability measure μ satisfies

$$d(u_0 \cdots u_{l-1}) \equiv \mu([u_0 \cdots u_{l-1}]_{[m,\cdots,m+l-1]}) = \lim_{n\to\infty} \frac{\#_{u_0\cdots u_{l-1}}\zeta^n(a)}{|\zeta^n(a)|}.$$

Proof. (Proposition 4) By Lemma 5, for vvv occurring in η with $|v| = |\zeta^n(a)|$

$$\begin{aligned}
d(vvv) &= \lim_{j\to\infty} \frac{\#_{vvv}\zeta^{n+j}(a)}{|\zeta^{n+j}(a)|} \\
&\ge \lim_{j\to\infty} \frac{\#_{aaaa}\zeta^{j}(a)}{|\zeta^{n+j}(a)|} \\
&= \lim_{j\to\infty} \frac{f_{N+1}^{j-1}\sum_{\{i:A_i\ge 4\}}(A_i-3) + R_{j-1}E_N\sum_{i=1}^{N+1}(A_i-1)}{f_{N+1}^{n+j} + R_{n+j}h_N} \\
&= \frac{d_N}{f_{N+1}^{n+1}(f_{N+1}+h_N-1)},
\end{aligned}$$

with $d_N = [f_{N+1}-1]\sum_{\{i:A_i\ge 4\}}(A_i-3) + E_N\sum_{i=1}^{N+1}(A_i-1)$. Since the set $B(n)$ is the finite union of disjoint cylinder sets $[vvv]_{I_n}$, with $vvv \in J(n)$, where $I_n = [1-|\zeta^n(a)|, 2|\zeta^n(a)|]$,

$$\begin{aligned}
\mu(B(n)) &= \sum_{vvv\in J(n)} \mu([vvv]_{I_n}) \\
&= \sum_{vvv\in J(n)} d(vvv) \\
&\ge [B_1 + A_2(f_{N+1}^{n-1} + h_N R_{n-1})]d(vvv)
\end{aligned}$$

By writing out $d(vvv)$ one gets

$$\lim_{n\to\infty} \mu(B(n)) \ge \frac{A_2(f_{N+1}+h_N-1)d_N}{f_{N+1}^4 - 2f_{N+1}^3 + f_{N+1}^2 + (f_{N+1}^2 h_N)(f_{N+1}-1)} > 0.$$

Noticing that [7, 9]

$$\limsup B(n) \subseteq C_p \equiv \{\omega \in \Omega_\zeta : \sigma_p(H_\omega) = \emptyset\},$$

C_p is subshift invariant, μ is ergodic, and

$$\mu(C_p) \ge \mu(\limsup B(n)) \ge \lim \mu(B(n)) > 0,$$

one concludes that $\sigma_p(H_\omega) = \emptyset$ μ-a.e. $\omega \in \Omega_\zeta$, since subshift invariants sets in (Ω_ζ, T) have either null or full ergodic measure. □

4. Final Remarks

1. Notice that the integer 1 is always an eigenvalue of the matrix of substitution $M(\zeta)$, and the other eigenvalue is the positive integer $\sum_{k=1}^{N+1} A_k > 1$, clearly a non-Pisot number. Notice also that besides nonprimitive ζ is not irreducible.
2. We remark that Propositions 3 and 4, used to exclude point spectrum, can be adapted for ultimately periodic sequences. The nonperiodicity were important here for the exclusion of the absolutely continuous spectrum in Proposition 2.

3. The generalization of the above results for the case $\zeta(b) = bb...b$ in (2), with more than one factor, is not direct, since in this case there are words composed only of b's of arbitrary long length and (Ω_ζ, T) is not strictly ergodic.
4. As a direct corollary of our results it follows that for palindromic substitutions ζ, with $A_1 + A_{N+1} \geq 4$ or $A_j \geq 4$ for some $1 \leq j \leq (N+1)$, the spectrum $\sigma(H_\omega)$ is pure singular continuous for ω in a generic set as well as in a set of full invariant measure.
5. Another possibility to exclude point spectrum directly for the fixed-point sequence η of the substitution ζ is to apply the techniques presented in references [14, 5].
6. Given a particular substitution ζ_N (under the hypotheses of Proposition 4) it is possible to find the value of $\lim_{n\to\infty} \mu(B(n))$. E.g., if all B_j are pairwise different or $A_j \neq A_1 + A_{N+1}, \forall j$, then this limit is given by
$$\frac{(f_{N+1} + h_N - 1)d_N}{f_{N+1}^3 - 2f_{N+1}^2 + f_{N+1} + (f_{N+1}h_N)(f_{N+1} - 1)}.$$
7. An open problem is to characterize the spectral type [19] of the autocorrelation measure for the nonperiodic substitution sequences generated by ζ_N. How does it depend on $A_1, \cdots, A_{N+1}, B_1, \cdots, B_N$?

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