Linear Boltzmann equation as the long time dynamics of an electron weakly coupled to a phonon field

László Erdős *
School of Mathematics, GeorgiaTech
Atlanta, GA 30332, USA
lerdos@math.gatech.edu

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Dedicated to Domokos Szász on the occassion of his 60^{th} *birthday*

Abstract

We consider the long time evolution of a quantum particle weakly interacting with a phonon field. We show that in the weak coupling limit the Wigner distribution of the electron density matrix converges to the solution of the linear Boltzmann equation globally in time. The collision kernel is identified as the sum of an emission and an absorption term that depend on the equilibrium distribution of the free phonon modes.

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Contents

1 Introduction

Quantum evolution of a few particles can effectively be computed by using the Schrödinger equation. Applying the same first principles to macroscopic systems with many degrees of freedom, however, is practically impossible. Macroscopic theories have to be developed which retain the relevant features of the original problems but are simple enough to be computationally feasible.

Due to thermal energy, lattice vibrations in metals can result in random deviations from the periodic background potential. This vibration is usually modelled by many independent harmonic oscillators. These are called phonons, and they can be considered bosonic particles.

The interaction between the phonons and the electron is a collision process in which a phonon can be emitted or absorbed, subject to momentum and energy conservation. The number of phonons is not conserved, hence they are described by second quantization.

Our goal is to show that the long time Schrödinger evolution of a quantum particle (electron) in a phonon field can be described on large scales by the Boltzmann equation in $d \geq 3$ dimensions. We use two sets of variables; (x, t) stand for the microscopic space and time, and let

$$(X, T) := (\varepsilon x, \varepsilon t)$$

be the macroscopic variables. Here ε is the scaling parameter separating microscopic and macroscopic scales and we will consider the $\varepsilon \to 0$ *scaling limit*. The velocity is unscaled. The quantum dynamics is given in microscopic variables. The resulting Boltzmann equation gives the evolution of the time dependent phase space density, $F_T(X, V)$, in macroscopic variables, hence it contains only the macroscopic features of the evolution. We recall that the linear Boltzmann equation with dispersion relation $e(k)$ and collision kernel $\sigma(V, U)$ is

$$\begin{aligned} \partial_T F_T(X,V) &+ \nabla e(V) \cdot \nabla_X F_T(X,V) \\ &= \int \Big[\sigma(V,U)F_T(X,U) - \sigma(U,V)F_T(X,V)\Big]\mathrm{d}U \qquad (1.1) \\ &= \int \sigma(V,U)F_T(X,U)\mathrm{d}U - \sigma_0(V)\, F_T(X,V) \qquad (1.2) \end{aligned}$$

with the total cross section $\sigma_0(V) := \int \sigma(U, V)\mathrm{d}U$.

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1.1 Definitions

For convenience, we fix a convention to avoid carrying factors of 2π along the Fourier transforms. We *define* the measure $\mathrm{d}x$ on $\mathbf{R}^d$ to be the Lebesgue measure *divided* by $(2\pi)^{d/2}$. With this convention the d-dimensional Fourier transform (usually denoted by hat) is defined as

$$\widehat{f}(p) = \mathcal{F}f(p) := \int_{\mathbf{R}^d} f(x)e^{-ip\cdot x}\mathrm{d}x$$

and its inverse

$$f(x) = \mathcal{F}^{-1}\widehat{f}(x) = \int_{\mathbf{R}^d} \widehat{f}(p)e^{ip\cdot x}\mathrm{d}p\ .$$

In particular, $\int_{\mathbf{R}^d} e^{ip\cdot x}\mathrm{d}x = \delta(p)$, where $\delta(p)$ is understood with respect to the $\mathrm{d}p$ measure. This convention applies only to d-dimensional integrals, in one dimension $\mathrm{d}\alpha$ still denotes the usual Lebesgue measure and $\int_{-\infty}^{\infty} e^{i\alpha t}\mathrm{d}\alpha = 2\pi\delta(t)$. For functions $\psi \in C^1(\mathbf{R}^d)$ we also define the measure $\delta(\psi(p))\mathrm{d}p$ supported on the level set $\{\psi = 0\}$ by

$$\int F(p)\delta(\psi(p))\mathrm{d}p := \int_{\psi^{-1}(0)} \frac{F(p)}{|\nabla\psi(p)|}\mathrm{d}\sigma(p) , \qquad F \in C(\mathbf{R}^d) ,$$

where $\mathrm{d}\sigma$ is the $(d-1)$-dimensional surface measure inherited from $\mathrm{d}p$.

The configuration space is a big box $\Lambda = \Lambda_L := [-\sqrt{2\pi}L/2, \sqrt{2\pi}L/2]^d \subset \mathbf{R}^d$ in $d \geq 3$ dimensions with $\int_\Lambda \mathrm{d}x = L^d$. Its dual is $\Lambda^* = \Lambda_L^{dual} := (\sqrt{2\pi}L^{-1}\mathbf{Z})^d \subset \mathbf{R}^d$. We use the shorter integral notation for the normalized sum on the dual lattice

$$\int_{\Lambda^*} \mathrm{d}k := \frac{1}{L^d}\sum_{k\in\Lambda^*} .$$

With this notation $\int_\Lambda e^{ik\cdot x}\mathrm{d}x = \delta(k)$ and $\int_{\Lambda^*} e^{ik\cdot x}\mathrm{d}k = \delta(x)$ where the delta functions are with respect to the measures $\mathrm{d}k$ and $\mathrm{d}x$. In particular, $\delta(k)$ is the lattice delta function, i.e., $\delta(k) = L^d$ if $k = 0$ and is zero otherwise.

Convention. Letters x, y, z will usually denote configuration variables in $\Lambda \subset \mathbf{R}^d$; k, p, q, r, u, v, ξ stand for momentum variables in $\Lambda^* \subset \mathbf{R}^d$; $s, t, \alpha, \beta, \varepsilon, \nu, \eta, \tau, \theta, \omega$ are real scalars and i, j, κ, ℓ, m, n are integers. The same convention applies to capital letters.

For simplicity, we will omit the domains of integration in the notations. In general, the notation $\int$ is used for d-dimensional integration with respect to the modified Lebesgue measure on $\mathbf{R}^d$ if the $L \to \infty$ limit has already been taken. Otherwise, the domain of integration is Λ or Λ^*, depending on the integration variable. For one dimensional integrations the limits will always be indicated.

We will take the thermodynamic limit $L \to \infty$ before any other limit and our result will be uniform in L. The compact configuration space is a mere convenience in order to define certain expressions rigorously; for most purposes the reader can always think of $\mathbf{R}^d$ instead of Λ and Λ^*.

The electron is considered as a spinless nonrelativistic particle with state space $\mathcal{H}_e := L^2_{per}(\Lambda_L)$. We denote the electron dispersion relation by $e(k)$, $k \in \mathbf{R}^d$, i.e., the electron Hamiltonian, H_e, is multiplication by $e(k)$ in momentum space ($k \in \Lambda^*$).

The state space of m independent phonons is $\mathcal{H}_p^m := \mathcal{S}\Big[\otimes_{j=1}^m L^2_{per}(\Lambda_L)\Big]$, where $\mathcal{S}$ is the symmetrization operator. Their Hamiltonian, in momentum representation, is a multiplication operator by

$$H_{ph}^m(k_1, k_2, \ldots, k_m) = \sum_{j=1}^m \omega(k_j) , \qquad k_j \in \Lambda^* ,$$

where $\omega(k)$ is the phonon dispersion law defined for $k \in \mathbf{R}^d$.

The Fock space of the phonons is $\mathcal{H}_{ph} := \oplus_{m=0}^{\infty} \mathcal{H}_{ph}^m$, and the phonon Hamiltonian is $H_{ph} = \oplus_m H_{ph}^m$. The total Hilbert space, including the electron, is $\mathcal{H}_{tot} := \mathcal{H}_e \otimes \mathcal{H}_{ph}$. The m-phonon sector consists of wavefunctions $\Psi^m = \Psi^m(x; k_1, k_2, \ldots, k_m) \in \mathcal{H}_e \otimes \mathcal{H}_{ph}^m$, where x is the electron coordinate. The elements of $\mathcal{H}_{tot}$ are denoted by $\Psi = (\Psi^m)_{m=0}^{\infty}$. The total noninteracting Hamiltonian of the electron-phonon system is $H_e \otimes 1 + 1 \otimes H_{ph}$, which we shall write as $H_e + H_{ph}$ for brevity.

To define the interaction, we need to define the phonon creation and annilihation operators $c_k^\dagger$ and c_k as

$$
\begin{aligned}
(c_k \Psi)^m(x; k_1, k_2, \ldots, k_m) \quad &:= \quad \sqrt{m+1}\, \Psi^{m+1}(x; k, k_1, k_2, \ldots, k_m) && (1.3)\\
(c_k^\dagger \Psi)^m(x; k_1, k_2, \ldots, k_m) \quad &:= \quad \frac{1}{\sqrt{m}} \sum_{j=1}^{m} \Psi^{m-1}(x; k_1, \ldots, \widehat{k}_j, \ldots, k_m)\delta(k_j - k) && (1.4)
\end{aligned}
$$

(hat denotes omission). These operators satisfy the standard commutation relations, i.e.,

$$
[c_k, c_{k'}^\dagger] = \delta(k - k')
$$

and any other commutator is zero. In terms of these operators $H_{ph} = \int \omega(k) N_k \mathrm{d}k$, where $N_k := c_k^\dagger c_k$ is the number operator.

The equilibrium state of the phonons is the Gibbs state

$$
\gamma_{ph} := Z^{-1} \exp\Big(- \beta H_{ph} + \mu \int N_k \mathrm{d}k \Big) = Z^{-1} \exp\left(\int \Big[- \beta\omega(k) + \mu \Big] N_k \mathrm{d}k \right) \tag{1.5}
$$

with inverse temperature $\beta > 0$ and chemical potential μ. Here $Z = Z_\Lambda$ is the normalization, i.e.,

$$
Z := \mathrm{Tr}_{\mathcal{H}_{ph}} \exp\left(\int \Big[- \beta\omega(k) + \mu \Big] N_k \mathrm{d}k \right) \tag{1.6}
$$

such that $\mathrm{Tr}_{\mathcal{H}_{ph}} \gamma_{ph} = 1$. We define

$$
\mathcal{N}(k) := \mathrm{Tr}_{\mathcal{H}_{ph}} \Big(\gamma_{ph} N_k \Big) = \frac{e^{-\beta\omega(k)+\mu}}{1 - e^{-\beta\omega(k)+\mu}} \tag{1.7}
$$

to be the expected number of phonons in the mode k.

The interaction Hamiltonian is given by

$$
H_{e-p} := i\lambda \int_{\Lambda^*} Q(k) \Big[e^{-ik\cdot x} c_k^\dagger - e^{ik\cdot x} c_k \Big] \mathrm{d}k, \tag{1.8}
$$

as a multiplication operator on $\mathcal{H}_e$, where $Q(k)$ models the details of the electron-phonon interaction and λ is the coupling constant. We will choose $\lambda = \sqrt{\varepsilon}$ and this is the weakest coupling that yields a nontrivial (non-free) macroscopic evolution on the time scale $t = T/\varepsilon$.

The full dynamics is given by the Schrödinger equation

$$i\partial_t \Psi_t = H\Psi_t, \tag{1.9}$$

where

$$H = H_e + H_{ph} + H_{e-p}$$

is the full Hamiltonian. In particular, for $Q(k) = |k|^{-1}$ we obtain the well-known Fröhlich Hamiltonian describing the polaron.

The Schrödinger equation can be written as an evolution equation for density matrices Γ_t on $\mathcal{H}_{tot}$:

$$i\partial_t \Gamma_t = [H, \Gamma_t]. \tag{1.10}$$

For example, (1.10) and (1.9) are equivalent if Γ_t is the projection operator onto the state Ψ_t. The density matrix formalism is more general, and is appropriate to describe thermal states.

We let $\gamma_e = \gamma_{e,0}$, Tr $\gamma_e = 1$, be the initial electron density matrix; if we start from a pure electron state ψ_0, then $\gamma_e = \gamma_{e,0}$ is the projection onto ψ_0. The total initial density matrix is

$$\Gamma_0 := \gamma_e \otimes \gamma_{ph} \ . \tag{1.11}$$

The solution of (1.10) is $\Gamma_t = e^{-itH}\Gamma_0 e^{itH}$.

We are interested only in the evolution of the electron, i.e., we want to compute $\mathrm{Tr}_{\mathcal{H}}\Big(\Gamma_t \mathcal{O}\Big)$, where $\mathcal{O}$ is an observable acting on $\mathcal{H}_e$. We can first take partial trace $\mathrm{Tr}_{\mathcal{H}_{ph}} =: \mathrm{Tr}_{ph}$ to integrate out the phonon modes to obtain a density matrix on $\mathcal{H}_e$. The phase space properties of such density matrix are described by the Wigner distribution.

The Wigner distribution W_γ of any density matrix γ on $\mathcal{H}_e$ is defined as

$$W_\gamma(x, v) := \int_\Lambda e^{-iv\cdot y}\gamma\Big(x + \frac{y}{2}, x - \frac{y}{2}\Big)\mathrm{d}y \ , \tag{1.12}$$

or in momentum space

$$W_\gamma(x, v) := \int_\Lambda e^{iu\cdot x}\widehat{\gamma}\Big(v + \frac{u}{2}, v - \frac{u}{2}\Big)\mathrm{d}y \ ,$$

where hat on density matrices denotes Fourier transform in both variables. We use the convention that the hat on functions defined on the phase space $L^2(\Lambda \times \Lambda^*)$ means Fourier transform in the first variable only. In particular, the Fourier transform of the Wigner distribution is

$$\widehat{W}_\gamma(\xi, v) := \int_\Lambda e^{-i\xi\cdot x} W_\gamma(x,v)\mathrm{d}x = \widehat{\gamma}\Big(v + \frac{\xi}{2}, v - \frac{\xi}{2}\Big) \, . \tag{1.13}$$

For pure states, $\gamma = |\psi\rangle\langle\psi|$, we have $\gamma(x,y) = \psi(x)\overline{\psi}(y)$ and W_γ is equal to W_ψ, the Wigner transform of the wavefunction ψ, in accordance with [EY2].

We also define the rescaled Wigner distribution to describe large scale (macroscopic) properties of the density matrix

$$W_\gamma^\varepsilon(X,V) := \varepsilon^{-d} W_\gamma\Big(\frac{X}{\varepsilon}, V\Big) \, . \tag{1.14}$$

1.2 Assumptions

We introduce the notation $\langle x\rangle := (x^2+1)^{1/2}$. We use indexed constants $C_1, C_2, \ldots$ to quantify estimates in our assumptions.

For the electron and phonon dispersion relations, $e(k)$, $\omega(k) \geq 0$, we assume symmetry, $e(k) = e(-k)$, $\omega(k) = \omega(-k)$ and

$$|\nabla^\ell e(k)| \leq C_1(1 + \langle k\rangle^{2-\ell}) \qquad \ell = 0, 1, \ldots, 2d \, , \tag{1.15}$$

$$|\nabla^\ell \omega(k)| \leq C_2(1 + \langle k\rangle^{2-\ell}) \qquad \ell = 0, 1, \ldots, 2d \, . \tag{1.16}$$

We also assume that the functions $k \mapsto \Phi_\pm(p,k) := e(k+p) \pm \omega(k)$ satisfy

$$\lim_{k\to\infty} \Phi_\pm(p,k) = \infty \, , \tag{1.17}$$

and

$$0 < C_3 \leq \mathrm{Hess}_k \Phi_\pm(p,k) \leq C_4 \tag{1.18}$$

(the upper bound follows from (1.15) and (1.16)). By degree theory, these conditions imply that $k \mapsto \Phi_\pm(p,k)$ has only one critical point.

Let

$$E_\pm(p,\theta,\delta) := \Big\{ k \; : \; |\Phi_\pm(p,k) - \theta| \leq \delta \Big\} \tag{1.19}$$

be a small neighborhood of the level set $k \mapsto \Phi_\pm(p,k) = \theta$.

We remark that conditions (1.15)–(1.18) also imply the existence of two constants $\widetilde{\varrho}, \widetilde{C} > 0$ depending on $C_1, \ldots, C_4$ such that

$$\sup_{p,q,\theta} \Big| \, E_{\pm}(p, \theta, \delta) \cap B(q, \varrho) \, \Big| \leq \widetilde{C} \delta \varrho^{d-1} \tag{1.20}$$

whenever $\delta, \varrho \leq \widetilde{\varrho}$. Here $| \cdot |$ denotes the Lebesgue measure of a set, and $B(q, \varrho)$ is the ball of radius ϱ about $q \in \mathbf{R}^d$. We also need a certain transversality condition on two such sets.

Transversality condition: There exist positive constants $\widetilde{\varrho}, C_5$ such that for any $\delta_1, \delta_2, \varrho \leq \widetilde{\varrho}$, any $p_1, p_2 \in \mathbf{R}^d$ and any $\theta_1, \theta_2 \in \mathbf{R}$

$$\sup_{q} \Big| \, E_{\pm}(p_1, \theta_1, \delta_1) \cap E_{\pm}(p_2, \theta_2, \delta_2) \cap B(q, \varrho) \, \Big| \leq \frac{C_5 \delta_1 \delta_2 \varrho^{d-2}}{|p_1 - p_2|} \; . \tag{1.21}$$

In particular, elementary calculation shows that all these conditions are satisfied in $d \geq 3$ if $e(k) = \frac{1}{2}k^2$ and $\|\nabla^2 \omega\|_\infty$ is sufficiently small. This is our primary example.

To ensure that the statistical operator is trace class, we always assume that

$$\inf_k \omega(k) - \mu \beta^{-1} \geq C_6 > 0 \; . \tag{1.22}$$

Note that the number density function $\mathcal{N}$ is symmetric and it belongs to $C^{2d}(\mathbf{R}^d)$.

We assume that $Q(k) \in C^{2d}(\mathbf{R}^d)$ is real (since H_{e-p} is self-adjoint) and symmetric, i.e.,

$$Q(k) = Q(-k) = \overline{Q(k)} \; , \tag{1.23}$$

and it has a fast decay up to $2d$-derivatives

$$\max_{\ell = 0, \ldots, 2d} |\nabla_k^\ell Q(k)| \leq C_7 \langle k \rangle^{-2d-12} \; . \tag{1.24}$$

The initial electron density matrix $\gamma_e = \gamma_e^\varepsilon$ will depend on ε so that it has a macroscopic profile in the scaling limit. We assume that the limit

$$F_0(X, V) := \lim_{\varepsilon \to 0} W^\varepsilon_{\gamma_e^\varepsilon}(X, V) \tag{1.25}$$

exists weakly in $\mathcal{S}(\mathbf{R}^{2d})$. In addition, we assume that

$$\limsup_{\varepsilon \to 0} \int \langle p \rangle^{3d+12} \widehat{\gamma}_e^\varepsilon(p, p) \mathrm{d}p \leq C_8 < \infty \; . \tag{1.26}$$

For example, γ_e^ε can be a pure state, $\gamma_e^\varepsilon := |\psi^\varepsilon\rangle\langle\psi^\varepsilon|$, with a WKB wavefunction

$$\psi^\varepsilon(x) = \varepsilon^{d/2} A(\varepsilon x) e^{iS(\varepsilon x)/\varepsilon} \; ,$$

where $A, S \in \mathcal{S}(\mathbf{R}^d)$.

1.3 Main Theorem

Theorem 1.1 *Let $\lambda = \sqrt{\varepsilon}$ and let Γ_t^ε solve the Schrödinger equation (1.10) with initial condition $\Gamma_0^\varepsilon = \gamma_e^\varepsilon \otimes \gamma_{ph}$, where the initial electron density matrix satisfies (1.25) and (1.26). We let $\gamma_t^\varepsilon := Tr_{ph} \Gamma_t^\varepsilon$ be the electron density matrix at time t.*

We assume that the electron and phonon dispersion relations satisfy (1.15)–(1.18), (1.21) and (1.22), while the interaction function $Q(k)$ satisfies (1.23) and (1.24). Then for any $T > 0$

$$\lim_{\varepsilon \to 0} \lim_{L \to \infty} W^\varepsilon_{\gamma^\varepsilon_{T/\varepsilon}}(X, V) = F_T(X, V)$$

weakly in $\mathcal{S}(\mathbf{R}^d \times \mathbf{R}^d)$, and F_T satisfies the Boltzmann equation (1.2) with initial condition F_0 and collision kernel

$$\begin{aligned} \sigma(V, U) : \;&= \; 2\pi |Q(U - V)|^2 \Big\{ \Big(\mathcal{N}(U - V) + 1\Big) \delta\Big(e(V) - e(U) + \omega(U - V)\Big) \\ &\quad + \mathcal{N}(U - V) \delta\Big(e(V) - e(U) - \omega(V - U)\Big) \Big\} . \end{aligned} \tag{1.27}$$

These two terms correspond to phonon emission and absorption, respectively.

Remark 1. The Boltzmann equation is irreversible, while the Schrödinger dynamics is reversible. This is not controversal, since the time evolved quantum state cannot be reconstructed from the solution of the Boltzmann equation. The macroscopic equation gives only a robust information on the evolution: it contains neither the phonons nor the microscopic details of the electron state. The former was neglected when taking partial trace, the latter is lost in the scaling (weak) limit.

Remark 2. Scaling limit of the dynamics of a quantum particle in a weakly coupled random potential was obtained for short time in [S], [HLW] and for long time in [EY2]. A similar result was obtained in [EY1] in a different limiting regime, in the so-called low density limit. Despite these similarities, the current proof differs substantially from [EY2] (see also a short announcement [E]). We replace the idea of the partial time integration by a more effective classification of the indirect Feynman graphs. This enables us to expand the Duhamel formula further than in [EY2] and still control the error terms (so-called *indirect* and *recollision* terms). In particular, for any κ we will be able to gain an extra ε^κ factor relative to the size of the main term with the exception of $C(\kappa)^n$ pairings in the n-th order terms of the Duhamel expansion (see Propositions 7.4, and A.6).

Remark 3. The evolution problem for the electron density matrix $\gamma_t = \mathrm{Tr}_{ph}\Gamma_t$ is formally equivalent to the Schrödinger equation

$$i\partial_t \gamma_t = \Big[H_e + \lambda V_\omega, \gamma_t\Big]$$

with a time dependent Gaussian random potential $V_\omega(x,t)$ with covariance (in Fourier space)

$$\mathbf{E}\overline{\widehat{V}_\omega(p,t)}\widehat{V}_\omega(q,s) = \delta(p-q)|Q(p)|^2\Big[\Big(\mathcal{N}(p)+1\Big)e^{i(t-s)\omega(p)} + \mathcal{N}(p)e^{-i(t-s)\omega(p)}\Big]. \tag{1.28}$$

This means that the formal perturbation expansions of these two problems coincide term by term. However, this connection is only formal and in our rigorous proof we cannot and do not make use of it.

Remark 4. Long time evolution of a *microscopically localized* electron weakly coupled to a phonon bath was studied in [CEFM] in the dipole approximation (see also references therein). The limiting equation is diffusive (Fokker-Planck) already on the first nontrivial time scale (in the van Hove limit). In this case there is no Boltzmann equation before diffusion emerges. The diffusion mechanism is quite different; it is a resonance effect between certain phonon modes and the eigenfrequencies of the confinement.

Remark 5. In a more realistic transport model, electron-electron interactions should be included. This is a genuine many-body problem and is much more difficult. In classical dynamics an analogous result has been proven by O. Lanford [L] in the low density limit. There is no rigorous result in the quantum case.

Convention. Throughout the paper we use the letter C to denote various constants that depend only on the dimension d and the constants $C_1, C_2 \ldots$ quantifying the assumptions in Section 1.2. For brevity, we usually neglect certain variables in the formulas if it does not cause confusion. In particular, we will omit the superscript ε from Γ_t^ε and γ_t^ε.

2 Structure of the proof

2.1 Duhamel expansion

We use the Duhamel formula (2.1) for $H = H_0 + H_{e-p}$ with $H_0 = H_e + H_{ph} = e(-i\nabla) + H_{ph}$. Recall that H_e and H_{ph} commute.

We are interested in the partial trace of $\Gamma_t = e^{-itH}\Gamma_0 e^{itH}$ with respect to the phonon variables. In principle, we could fully expand e^{-itH} by Duhamel on both sides. But this expansion does not converge for large $T = t\varepsilon$, $\varepsilon := \lambda^2$, unless we give a fairly detailed classification of Feynman graphs. For each graph of order n the best generally valid theoretical bound is $(C\lambda^2 t)^n/n!$ (although we essentially prove only $(C_a\lambda^2 t)^n/[n!]^a$, $0 \le a < 1$, in Lemma 5.1), but then $n!$ is lost to combinatorics (number of graphs is $\sim C^n n!$) and the series does not converge for $t \ge C^{-1}\lambda^{-2}$, i.e., for $T \ge C^{-1}$. Even if we gain an extra $t^{-\kappa}$ from most pairings (due to reabsorption and crossing), with some fixed κ, it does not make the series converge.

So the idea is to stop the Duhamel expansion once the expanded part is small enough. This means that either there are many absorptions-emissions or at least one reabsorption. However, immediate reabsorptions do contribute to the main term, and they have to be resummed (see later in (10.14)). For simplicity, we use the word collision for absorption or emission.

The Duhamel formula for $H = H_0 + H_{e-p}$ states that for any $N_0 \ge 1$

$$\begin{aligned} e^{-itH} &= \sum_{N=0}^{N_0-1}(-i)^N \int_0^{t*}[\mathrm{d}s_j]_0^N e^{-is_0H_0}H_{e-p}e^{-is_1H_0}H_{e-p}\dots H_{e-p}e^{-is_NH_0} \\ &\quad +(-i)^{N_0}\int_0^{t*}[\mathrm{d}s_j]_0^{N_0} e^{-is_0H}H_{e-p}e^{-is_1H_0}H_{e-p}\dots H_{e-p}e^{-is_{N_0}H_0} \ . \end{aligned} \tag{2.1}$$

For brevity, we introduced the following notation

$$\int_0^{t*}[\mathrm{d}s_j]_0^N := \int_0^t \dots \int_0^t \Big(\prod_{j=0}^N \mathrm{d}s_j\Big)\delta\Big(t - \sum_{j=0}^N s_j\Big) \ , \tag{2.2}$$

where the star refers to the constraint $t = \sum_{j=0}^N s_j$. Notice that the upper integration limits on the right hand side of (2.2) could be omitted because the lower limits and the delta function together guarantee them.

For the threshold, we shall choose $N_0 = N_0(t)$ such that $t^2 \ll N_0! \ll t^{2.2}$ (see (3.34)).

Remark: This is *not* the choice in our earlier paper [EY2], where $N_0! \ll t$, and we used partial time integration. Here we rather expand further to avoid partial time integration. This will require more effective control on classically irrelevant interference terms.

Since we have to distinguish a few recollisions, the Duhamel formula has to be written more carefully. Write H_{e-p} (1.8) as follows

$$H_{e-p} = i\lambda\int Q(k)\Big[e^{-ik\cdot x}c_k^\dagger - e^{ik\cdot x}c_k\Big]\mathrm{d}k = i\lambda\int Q(k)e^{-ik\cdot x}b_k\mathrm{d}k \tag{2.3}$$

with

$$b_k := c_k^\dagger - c_{-k}, \qquad \text{hence} \qquad b_k^* = c_k - c_{-k}^\dagger = -b_{-k} \ ,$$

and we used the symmetry of Q. Here star means the adjoint, and b_k, b_k^* do *not* satify the canonical commutation relations, in fact

$$[b_k^{(*)}, b_m^{(*)}] = 0 \qquad \text{for any} \ \ k, m \ .$$

We also introduce the notation

$$\prod_{j=1}^{n} A_j := A_1 A_2 \ldots A_n \qquad \text{and} \qquad \prod_{j=n}^{1} A_j := A_n A_{n-1} \ldots A_2 A_1 \tag{2.4}$$

for the ordered product of noncommuting objects (operators) A_j.

With these notations we rewrite (2.1) in the Fourier space of $\mathcal{H}_e$ as

$$\begin{aligned} e^{-itH} \ = \ & \sum_{N=0}^{N_0-1} \lambda^N \int_0^{t*} [\mathrm{d}s_j]_0^N \int \Big(\prod_{j=1}^{N} \mathrm{d}k_j \Big) e^{-is_0 H_0} \Big(\prod_{j=1}^{N} Q(k_j) b_{k_j} e^{-is_j H_0} \Big) \\ & + \lambda^{N_0} \int_0^{t*} [\mathrm{d}s_j]_0^{N_0} \int \Big(\prod_{j=1}^{N_0} \mathrm{d}k_j \Big) e^{-is_0 H} \Big(\prod_{j=1}^{N_0} Q(k_j) b_{k_j} e^{-is_j H_0} \Big) \ . \end{aligned} \tag{2.5}$$

2.2 Stopping rule for the Duhamel expansion

The key observation is that the Duhamel formula (2.5) is too rigid. There is no need to decide apriori how long we expand each term. Once a new step of the expansion is made, and a factor H_{e-p} added, it contains an integration over a new variable k_j which is actually a big summation (2.3). This means that each term ramifies into a sum of several new terms. We can decide individually for each term whether we want to continue the Duhamel expansion or stop. The stopping rule must depend only on the already expanded terms. We can stop the expansion for those terms which already had enough number of recollisions (it turns out that one is enough) or which already had many collisions.

By *recollision* or *reabsorption* we mean $k_j + k_\ell = 0$ for some $j \neq \ell$ in (2.5). It contains both physically relevant (e.g. $c_k^\dagger c_k$ and $c_{-k} c_{-k}^\dagger$) and irrelevant (e.g., $c_k^\dagger c_{-k}^\dagger$ and $c_{-k} c_k$) interaction terms. The contribution of the irrelevant terms will vanish in the $L \to \infty$ limit. The recollision $k_j + k_\ell = 0$ is called *immediate* if $|j - \ell| = 1$, otherwise it is called *genuine*. The reason for this distinction is that immediate reabsorptions do occur without rendering the corresponding term small as $\varepsilon \to 0$.

Hence we will stop the Duhamel expansion if we either see N_0 collisions or if we see one genuine reabsorption. If we did not stop, it means *fully expanded* terms, which then has no reabsorption.

We need several notations to define these terms rigorously. Let $\chi(v)$, $v \in \Lambda^*$, be the characteristic function at 0, i.e., $\chi(v) = 1$ if $v = 0$ and $\chi(v) = 0$ otherwise.

Definition 2.1 *Let $\mathcal{M}(n, N)$ be the set of $(n+1)$-tuple of nonnegative integer numbers $\underline{m} := (m_0, m_1, \ldots, m_n)$ with the property that $n + 2|\underline{m}| = N$, $|\underline{m}| := \sum_j m_j$. Such an $(n+1)$-tuple is equivalent to an increasing subsequence $\mu = (\mu(1), \mu(2), \ldots, \mu(n))$ of the numbers $\{1, 2, \ldots, N\}$ such that all differences $\mu(j+1) - \mu(j)$ are odd for all $j = 0, 1, \ldots, N$. For convenience we set $\mu(0) := 0$ and $\mu(n+1) := N + 1$. The identification between $\underline{m}$ and μ is given by the relations $\mu(j+1) - \mu(j) = 2m_j + 1$ for all $j = 0, 1, \ldots, n$. We will use both representations simultaneously. Let*

$$I := I(\underline{m}) := \{\mu(j) \ : \ j = 1, 2, \ldots, n\} = \Big\{ j + 2\sum_{i=0}^{j-1} m_i \ : \ j = 1, 2, \ldots, n \Big\} .$$

For each $j = 0, 1, \ldots, n$ let

$$I_j := I_j(\underline{m}) := \{\mu(j) + 1, \mu(j) + 3, \ldots, \mu(j) + 2m_j - 1\}, \qquad (\textit{with } \ |I_j| = m_j) ,$$

and

$$I_j^c := I_j^c(\underline{m}) := \{\mu(j), \mu(j) + 2, \ldots, \mu(j) + 2m_j\}, \qquad (\textit{with } \ |I_j^c| = m_j + 1) ,$$

be its complement between two consecutive elements of the subsequence μ. Finally let

$$J := J(\underline{m}) := \bigcup_{j=0}^{n} I_j , \qquad J^c := \bigcup_{j=0}^{n} I_j^c ,$$

and clearly $\{0, 1, \ldots, N\} = J \cup J^c$ with $J \cap J^c = \emptyset$, $|J| = |\underline{m}|$ and $|J^c| = n + |\underline{m}| + 1$.

The increasing subsequence μ associated with $\underline{m} \in \mathcal{M}(n, N)$ will determine the indices of those phonon interactions which are not parts of immediate reabsorptions, i.e., $\underline{m}$ encodes the immediate reabsorption pattern. Here N refers to the total number of phonon interactions and n of them are not part of immediate reabsorptions. In the sequel, these two numbers always have the same parity without further remark. The number m_j denotes the number of immediate recollision pairs between $\mu(j)$ and $\mu(j+1)$.

For any $\underline{m} \in \mathcal{M}(n,N)$ we define

$$\Xi(\underline{k},\underline{m}) := \prod_{b\in J(\underline{m})} \chi(k_b + k_{b+1}) \tag{2.6}$$

to be the restriction onto momenta that respect the immediate reabsorption pattern given by $\underline{m}$. Here $\underline{k} := (k_1, \ldots, k_N)$ stands for the collection of the phonon variables. The phonon momenta indexed by the subsequence μ are called *external*, the rest are *internal*. The number of external momenta is n if $\underline{m} \in \mathcal{M}(n,N)$.

We then define the measure

$$\int^{(\underline{m})} \mathrm{d}\underline{k} := \int \mathrm{d}\underline{k}\; \Xi(\underline{k},\underline{m}) \prod_{b\neq b'\in I(\underline{m})} \Big(1 - \chi(k_b + k_{b'})\Big) , \tag{2.7}$$

where $\mathrm{d}\underline{k} := \prod_{j=1}^{N} \mathrm{d}k_j$ for simplicity. This measure excludes pairing among external momenta, while the internal momenta are consecutively paired. Notice that these restricted integrations make sense only for finite Λ. We also set

$$\int^{\#(n,N)} \mathrm{d}\underline{k} := \sum_{\underline{m}\in\mathcal{M}(n,N)} \int^{(\underline{m})} \mathrm{d}\underline{k} , \tag{2.8}$$

i.e., in this measure we exclude all recollisions, which are not immediate. In the sequel, genuine recollision will be simply called recollision.

Now we define the one recollision terms. These terms will always be amputated, i.e., the last free propagator $e^{-is_0H_0}$ will not be present.

For any $a \in \{2,3,\ldots,n\}$, we let $\mathcal{M}_a(n,N) \subset \mathcal{M}(n,N)$ be the set of those $\underline{m}$'s such that $m_0 = 0$, $\mu(a) \geq 3$ and if $a = 2$, we additionally require that $m_1 \geq 1$. The momenta with index $\mu(1) = 1$ and $\mu(a)$ will form the (genuine) recollision. We also let $I_a = I_a(\underline{m}) := I(\underline{m}) \setminus \{1, \mu(a)\}$ be the set of the indices of the remaining external momenta.

For $\underline{m} \in \mathcal{M}_a(n,N)$ we let

$$\int^{*(\underline{m},a)} \mathrm{d}\underline{k} := \int \mathrm{d}\underline{k}\; \chi(k_1 + k_{\mu(a)})\Xi(\underline{k},\underline{m}) \prod_{b\neq b'\in I_a} \Big(1 - \chi(k_b + k_{b'})\Big) \tag{2.9}$$

be the measure where the immediate reabsorption pattern given by $\underline{m}$ and a single genuine recollision between the first and the $\mu(a)$-th phonon is enforced. Let

$$\int^{*(n,N)} \mathrm{d}\underline{k} := \sum_{a=2}^{N} \sum_{\underline{m}\in\mathcal{M}_a(n,N)} \int^{*(\underline{m},a)} \mathrm{d}\underline{k} . \tag{2.10}$$

This measure expresses that there is no recollision of external momenta apart from the designated one between 1 and $\mu(a)$.

Now we define the (amputated) operator acting on $\mathcal{H}_{tot}$

$$\mathcal{A}(\tau, \underline{k}, N) := \lambda^N \int_0^{\tau *} [\mathrm{d}s_j]_1^N \Big(\prod_{j=1}^N Q(k_j) b_{k_j} e^{-is_j H_0} \Big) . \tag{2.11}$$

We can integrate out this operator on different sets of $\underline{k}$ corresponding to various reabsorption patterns (superscript denotes the number of reabsorptions):

$$\mathcal{D}^0_{n,N}(\tau) := \int^{\#(n,N)} \mathrm{d}\underline{k} \; \mathcal{A}(\tau, \underline{k}, N), \qquad \mathcal{D}^0_N(\tau) := \sum_{n=0}^N \mathcal{D}^0_{n,N}(\tau) \; , \tag{2.12}$$

$$\mathcal{D}^1_{n,N}(\tau) := \int^{*(n,N)} \mathrm{d}\underline{k} \; \mathcal{A}(\tau, \underline{k}, N), \qquad \mathcal{D}^1_N(\tau) := \sum_{n=2}^N \mathcal{D}^1_{n,N}(\tau) \; . \tag{2.13}$$

These are amputated objects. Depending on whether a term is fully expanded or not, we define non-amputated terms by letting e^{-isH_0} or e^{-isH} act on it. Terms with a recollision are always amputated.

We define

$$\begin{aligned} \mathcal{B}(t, \underline{k}, N) : &= \lambda^N \int_0^{t*} [\mathrm{d}s_j]_0^N e^{-is_0 H_0} \Big(\prod_{j=1}^N Q(k_j) b_{k_j} e^{-is_j H_0} \Big) \\ &= (-i) \int_0^t \mathrm{d}s_0 \; e^{-is_0 H_0} \mathcal{A}(t - s_0, \underline{k}, N) \; . \end{aligned} \tag{2.14}$$

The fully expanded terms are

$$\mathcal{E}^0_{n,N}(t) := (-i) \int_0^t e^{-is_0 H_0} \mathcal{D}^0_{n,N}(t - s_0) \mathrm{d}s_0 = \int^{\#(n,N)} \mathrm{d}\underline{k} \; \mathcal{B}(t, \underline{k}, N) \; , \tag{2.15}$$

and we let

$$\mathcal{E}^0_N(t) := \sum_{n=0}^N \mathcal{E}^0_{n,N}(t) \; .$$

The non-fully expanded terms are

$$\mathcal{H}^0_{n,N}(t) := (-i) \int_0^t e^{-is_0 H} \mathcal{D}^0_{n,N}(t - s_0) \mathrm{d}s_0 \; , \qquad \mathcal{H}^0_N(t) := \sum_{n=0}^N \mathcal{H}^0_{n,N}(t) \; ,$$

$$\mathcal{H}^1_{n,N}(t) := (-i) \int_0^t e^{-is_0 H} \mathcal{D}^1_{n,N}(t - s_0) \mathrm{d}s_0 \; , \qquad \mathcal{H}^1_N(t) := \sum_{n=2}^N \mathcal{H}^1_{n,N}(t) \; . \tag{2.16}$$

We will use the following Duhamel expansion:

Lemma 2.2 *For any fixed $N_0 \geq 1$ we have*

$$e^{-itH} = \mathcal{H}^0_{N_0}(t) + \sum_{N=3}^{N_0} \mathcal{H}^1_N(t) + \sum_{N=0}^{N_0-1} \mathcal{E}^0_N(t) \ . \tag{2.17}$$

Proof. When expanding the Duhamel formula, at each step we generate a new fully expanded term and a term with one more interaction H_{e-p} and a full propagator. Since H_{e-p} contains a sum over all momenta (2.3), we obtain $|\Lambda^*|$ new terms. Hence the expansion can be represented by a successively growing rooted tree-graph, where the vertices correspond to interaction terms from the expanded H_{e-p}'s. Each vertex ramifies into a terminal branch (free propagator) plus $|\Lambda^*|$ new branches whose endpoints are labelled by the momenta of the newly expanded interaction term. In this graph there is a unique path to the root from each vertex; the vertices on this path are called the *predecessors* of this vertex. In particular, every vertex (apart from the root) has an immediate predecessor, called its *father*.

We say that a vertex forms a *bond* (immediate recollision) with its father if they have the same momentum label and if the father does not already form a bond with its own father. The notion of the bond is defined successively as the tree grows along further expansion. Vertices that are not part of a bond are called *independent*.

We stop the expansion at any vertex if

(i) it has $N_0 - 1$ predecessor; or

(ii) its momentum label coincides with the label of any of its independent predecessor different from its father (genuine recollision).

These terms are included in $\mathcal{H}^0_{N_0}$ and $\mathcal{H}^1_N$, respectively, while the fully expanded terms (corresponding to terminal branches) are contained in the last term of (2.17). □

Therefore we can write for any $K > 1$ integer

$$\gamma_t = \mathrm{Tr}_{ph} e^{-itH}\Gamma_0 e^{itH} = \gamma_K^{main}(t) + \gamma_K^{err}(t) \ ,$$

where

$$\gamma_K^{main}(t) = \sum_{N,\widetilde{N}=0}^{K-1} \mathrm{Tr}_{ph}\mathcal{E}^0_N(t)\Gamma_0\Big[\mathcal{E}^0_{\widetilde{N}}(t)\Big]^* \tag{2.18}$$

corresponds to the main term containing no reabsorption and less than K collisions, and

$$\gamma_K^{err} := \sum_{\widetilde{N}=0}^{N_0-1}\sum_{N=K}^{N_0-1} \mathrm{Tr}_{ph}\mathcal{E}^0_N\Gamma_0\Big[\mathcal{E}^0_{\widetilde{N}}\Big]^* + \sum_{\widetilde{N}=K}^{N_0-1}\sum_{N=0}^{K-1} \mathrm{Tr}_{ph}\mathcal{E}^0_N\Gamma_0\Big[\mathcal{E}^0_{\widetilde{N}}\Big]^* + \mathrm{Tr}_{ph}\mathcal{H}^0_{N_0}\Gamma_0\Big[\mathcal{H}^0_{N_0}\Big]^* \tag{2.19}$$

$$+\sum_{N,\widetilde{N}=3}^{N_0}\mathrm{Tr}_{ph}\mathcal{H}^1_N\Gamma_0\Big[\mathcal{H}^1_{\widetilde{N}}\Big]^*+\sum_{N=0}^{N_0-1}\mathrm{Tr}_{ph}\bigg(\mathcal{E}^0_N\Gamma_0\Big[\mathcal{H}^0_{N_0}+\sum_{\widetilde{N}=3}^{N_0}\mathcal{H}^1_{\widetilde{N}}\Big]^*+\Big[\mathcal{H}^0_{N_0}+\sum_{\widetilde{N}=3}^{N_0}\mathcal{H}^1_{\widetilde{N}}\Big]\Gamma_0\Big[\mathcal{E}^0_N\Big]^*\bigg)\ .$$

2.3 Observables and Wigner transform

Eventually we want to compute $\mathrm{Tr}_e\ \gamma_t\mathcal{O}$ with some electron observable $\mathcal{O}$ acting on $\mathcal{H}_e$. For example, to determine the weak limit of $W^\varepsilon_{\gamma_t}$ in $\mathcal{S}(\mathbf{R}^d\times\mathbf{R}^d)$, we have to test the macroscopic Wigner distribution (1.14) against a smooth function $J(X,V)\in\mathcal{S}(\mathbf{R}^d\times\mathbf{R}^d)$. Let $J_\varepsilon(x,v):=\varepsilon^d J(x\varepsilon,v)$, hence $\widehat{J}_\varepsilon(\xi,v):=\varepsilon^{-d}\widehat{J}(\xi\varepsilon^{-1},v)$, recalling that hat means Fourier transform in the first variable.

A simple calculation shows that

$$\langle J,W^\varepsilon_{\gamma_t}\rangle:=\int\overline{J(X,V)}W^\varepsilon_{\gamma_t}(X,V)dXdV=\mathrm{Tr}_e\ \gamma_t\mathcal{O}_\varepsilon\ ,$$

where the observable $\mathcal{O}_\varepsilon$ is given by the kernel

$$\mathcal{O}_\varepsilon(u,v):=\widehat{J}_\varepsilon\Big(u-v,\frac{u+v}{2}\Big)\ . \tag{2.20}$$

To control $\mathrm{Tr}_e\ \gamma_t\mathcal{O}_\varepsilon$ we will need that $\mathcal{O}_\varepsilon\mathcal{O}^*_\varepsilon$ is a uniformly bounded operator:

$$\sup_\varepsilon\|\mathcal{O}_\varepsilon\mathcal{O}^*_\varepsilon\|<\infty\ . \tag{2.21}$$

In our case it is enough if J_ε satisfies

$$\|J\|:=\sup_\varepsilon\int\sup_v|\widehat{J}_\varepsilon(\xi,v)|\mathrm{d}\xi<\infty \tag{2.22}$$

since $\|\mathcal{O}_\varepsilon\mathcal{O}^*_\varepsilon\|\le\|J\|^2$ by an easy calculation. The estimate (2.22) is satisfied for $\widehat{J}_\varepsilon$ if

$$\int\sup_v|\widehat{J}(\xi,v)|\mathrm{d}\xi<\infty\ , \tag{2.23}$$

in particular if $J\in\mathcal{S}(\mathbf{R}^d\times\mathbf{R}^d)$.

In Sections 3–9 we will show that for any $T\ge 0$

$$\limsup_{K\to\infty}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\Big|\mathrm{Tr}_e\gamma^{err}_K(t)\mathcal{O}_\varepsilon\Big|=0 \tag{2.24}$$

with $t=T/\varepsilon$. In Section 10 we will check that the weak limit of $W^\varepsilon_{\gamma^{main}_K(t)}$ satisfies the Boltzmann equation as $L\to\infty$, then $\varepsilon\to 0$ and finally $K\to\infty$. This will complete the proof of the Theorem.

3 Estimating the error terms: statement of the Main Lemma

All error terms in $\mathrm{Tr}_e\gamma_K^{err}(t)\mathcal{O}$ have the structure

$$\mathrm{Tr}_{e+ph}\mathcal{G}(t)\Gamma_0\Big[\mathcal{G}(t)\Big]^*\mathcal{O}$$

with $\mathcal{G}=\mathcal{E}$ or $=\mathcal{H}$ with appropriate indices (see (2.19)). We use that $\Gamma_0\geq 0$ and a simple Schwarz inequality of prototype

$$\begin{aligned}
\Big|\mathrm{Tr}\; A\Gamma_0B^*\mathcal{O}\Big| &\leq \mathrm{Tr} A\;\Gamma_0A^* + \mathrm{Tr}\;\mathcal{O}^*B\Gamma_0B^*\mathcal{O}\\
&= \mathrm{Tr}\; A\Gamma_0A^* + \mathrm{Tr}\;\Gamma_0^{1/2}B^*\mathcal{O}\mathcal{O}^*B\Gamma_0^{1/2}\\
&\leq \mathrm{Tr}\; A\Gamma_0A^* + \|\mathcal{O}\mathcal{O}^*\|\;\mathrm{Tr}\;\Gamma_0^{1/2}B^*B\Gamma_0^{1/2}\\
&= \mathrm{Tr}\; A\Gamma_0A^* + \|\mathcal{O}\mathcal{O}^*\|\;\mathrm{Tr}\;B\Gamma_0B^*\;.
\end{aligned}$$

Hence after eliminating the "cross-terms" in $\gamma^{err}(t)$ (see (2.19)) by Schwarz inequalities, we have the estimate for any $0<\nu<1$, $K\leq N_0$

$$\begin{aligned}
\Big|\mathrm{Tr}_e\gamma_K^{err}(t)\mathcal{O}_\varepsilon\Big| \;\leq\; & C_{\mathcal{O}}\mathrm{Tr}_{e+ph}\Bigg\{\nu\sum_{N=0}^{K-1}\langle N\rangle^2\mathcal{E}_N^0(t)\Gamma_0\Big[\mathcal{E}_N^0(t)\Big]^* + \nu^{-1}\sum_{N=K}^{N_0-1}\langle N\rangle^2\mathcal{E}_N^0(t)\Gamma_0\Big[\mathcal{E}_N^0(t)\Big]^*\\
&+\nu^{-1}N_0\mathcal{H}_{N_0}^0(t)\Gamma_0\Big[\mathcal{H}_{N_0}^0(t)\Big]^* + \nu^{-1}N_0\sum_{N=3}^{N_0}\mathcal{H}_N^1(t)\Gamma_0\Big[\mathcal{H}_N^1(t)\Big]^*\Bigg\}\;, \qquad (3.25)
\end{aligned}$$

with $C_{\mathcal{O}}:=C(1+\sup_\varepsilon\|\mathcal{O}_\varepsilon\mathcal{O}_\varepsilon^*\|)<\infty$.

The terms in the first two summations are fully expanded, their estimates are easier. The third and fourth terms still contain the full propagator. We estimate them in terms of amputated fully expanded terms by using unitarity and paying a price of an extra factor t. The prototype of such term is

$$\widetilde{\gamma}(t) := \mathrm{Tr}_{ph}\int_0^t \mathrm{d}s_0\int_0^t \mathrm{d}\tilde{s}_0\; e^{-is_0H}A(t-s_0)\Gamma_0B^*(t-\tilde{s}_0)e^{i\tilde{s}_0H}\;, \qquad (3.26)$$

where A, B denote one of the amputated $\mathcal{D}_{n,N}^b$ for $b=0$ or $b=1$ and $n\leq N\leq N_0$ (see (2.12)–(2.13)). By a simple Schwarz estimate inside the time integrations in (3.26) we obtain

$$\mathrm{Tr}_e\widetilde{\gamma}(t)\leq t\cdot\mathrm{Tr}_e\Bigg[\int_0^t \mathrm{d}s_0\mathrm{Tr}_{ph}A(t-s_0)\Gamma_0A^*(t-s_0)+\int_0^t\mathrm{d}\tilde{s}_0\mathrm{Tr}_{ph}B(t-\tilde{s}_0)\Gamma_0B^*(t-\tilde{s}_0)\Bigg]\;. \quad (3.27)$$

Therefore we can continue (3.25) using the definition of $\mathcal{E}_N^0, \mathcal{D}_N^{0,1}$ and a Schwarz inequality:

$$\begin{aligned}
&\Big|\mathrm{Tr}_e \gamma_K^{err}(t)\mathcal{O}_\varepsilon\Big| \qquad\qquad (3.28)\\
\leq\ & C_{\mathcal{O}} \mathrm{Tr}_{e+ph}\Bigg\{\nu \sum_{N=0}^{K-1} \langle N\rangle^3 \sum_{n=0}^{N} \mathcal{E}_{n,N}^0(t)\Gamma_0\Big[\mathcal{E}_{n,N}^0(t)\Big]^* + \nu^{-1}\sum_{N=K}^{N_0-1} \langle N\rangle^3 \sum_{n=0}^{N} \mathcal{E}_{n,N}^0(t)\Gamma_0\Big[\mathcal{E}_{n,N}^0(t)\Big]^*\\
&+\nu^{-1} t N_0^2 \sum_{n=0}^{N_0}\int_0^t \mathrm{d}s\, \mathcal{D}_{n,N_0}^0(s)\Gamma_0\Big[\mathcal{D}_{n,N_0}^0(s)\Big]^* + \nu^{-1} t N_0^2 \sum_{N=3}^{N_0}\sum_{n=2}^{N}\int_0^t \mathrm{d}s\, \mathcal{D}_{n,N}^1(s)\Gamma_0\Big[\mathcal{D}_{n,N}^1(s)\Big]^*\Bigg\}.
\end{aligned}$$

The main lemma is the following:

Lemma 3.1 *Let $0 \leq a < 1$ and $n \leq N$ with the same parity. For simplicity, we let $M := \frac{n+N}{2}$. For the fully expanded objects:*

$$\limsup_{L\to\infty} Tr_{e+ph}\left(\mathcal{E}_{n,N}^0(t)\Gamma_0\Big[\mathcal{E}_{n,N}^0(t)\Big]^*\right) \leq \frac{(C_a\lambda^2 t)^N}{[M!]^a} + (C\lambda^2 t)^N \frac{n!}{t^{1/2}}(\log^* t)^4 \qquad (3.29)$$

(this will be used for $N < K$);

$$\limsup_{L\to\infty} Tr_{e+ph}\left(\mathcal{E}_{n,N}^0(t)\Gamma_0\Big[\mathcal{E}_{n,N}^0(t)\Big]^*\right) \leq \frac{(C_a\lambda^2 t)^N}{[M!]^a} + (C\lambda^2 t)^N \frac{n!}{t^6}(\log^* t)^{n+10} \qquad (3.30)$$

(this will be used for $K \leq N < N_0$). For the amputated objects, $N \geq 3$,

$$\limsup_{L\to\infty} Tr_{e+ph}\left(\mathcal{D}_{n,N}^0(t)\Gamma_0\Big[\mathcal{D}_{n,N}^0(t)\Big]^*\right) \leq \frac{1}{t}\,\frac{(C_a\lambda^2 t)^N}{[M!]^a} + \frac{1}{t}\,(C\lambda^2 t)^N \frac{n!}{t^6}(\log^* t)^{n+10}\chi(N\geq 7)\ ; \qquad (3.31)$$

$$\limsup_{L\to\infty} Tr_{e+ph}\left(\mathcal{D}_{n,N}^1(t)\Gamma_0\Big[\mathcal{D}_{n,N}^1(t)\Big]^*\right) \leq \frac{1}{t}\,(C\lambda^2 t)^N \left[\frac{1}{t^2}(\log^* t)^6 + \frac{n!}{t^6}(\log^* t)^{n+10}\chi(N\geq 7)\right]. \qquad (3.32)$$

We introduced the notation $\log^* t := \max\{1, \log t\}$ and χ is the characteristic function.

Remark 1. Each estimate has two parts. The first terms include the contribution of the direct pairing term (*"ladder"* diagram) with possible immediate recollisions (*"one-loop renormalization"*). The second terms contain the non-classical indirect terms (so-called *"crossing"* terms).

Remark 2. From the indirect terms in (3.29) we gain only $t^{1/2}$ instead of t, but the power of $\log^* t$ is bounded. In (3.30) we gain full t-powers but we lose in the power of $\log^* t$. With more careful estimates it is possible to remove the logarithms and still gain full t-powers but we do not need it.

Remark 3. The basic idea in these estimates is that we do not gain factorials *together with* crossing or recollision gains. This is only a technical convenience. In fact, instead of (3.32), it is possible to prove

$$\limsup_{L\to\infty} \mathrm{Tr}_{e+ph}\Big(\mathcal{D}^1_{n,N}(t)\Gamma_0\big[\mathcal{D}^1_{n,N}(t)\big]^*\Big) \le \frac{1}{t}\,\frac{(C_a\lambda^2 t)^N}{[M!]^a}\,\frac{n!}{t^2} \tag{3.33}$$

which would be equally sufficient to estimate the recollision terms. In this estimate we would gain a factorial together with a recollision gain. We follow the first path since it is somewhat shorter.

We use Lemma 3.1 to estimate the terms in (3.28). We choose $\frac{1}{1.01} < a < 1$ and

$$N_0 := \frac{2.2\ \log t}{\log\log t}\ , \tag{3.34}$$

hence

$$t^{2.1} \ll N_0! \ll t^{2.2}\ , \quad (\log^* t)^{N_0} = t^{2.2}\ , \quad C^{N_0} \ll t^{\delta}$$

for any $\delta > 0$, $t \gg 1$.

First we let $\varepsilon \to 0$ then $K \to \infty$ and finally $\nu \to 0$ in (3.28). Recall that $\lambda^2 = \varepsilon$ and $t = T/\varepsilon$, i.e., $\lambda^2 t = T$. Easy calculation shows that

$$\lim_{\nu\to 0}\limsup_{K\to\infty}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\Big|\mathrm{Tr}_e\gamma_K^{err}(t)\mathcal{O}_\varepsilon\Big| = 0\ , \tag{3.35}$$

which proves (2.24).

4 Formulas

In this section is we derive expressions for the terms in Lemma 3.1.

4.1 Definition and estimates of basic functions

We define the following functions for $p, k \in \mathbf{R}^d$, $\sigma \in \{\pm\}$, $\eta \neq 0$, $\alpha, s \in \mathbf{R}$

$$\begin{aligned} M(k,\sigma) &:= |Q(k)|^2\Big(\mathcal{N}(k) + \frac{\sigma+1}{2}\Big)\ , && (4.1)\\ \Phi_\sigma(p,k) &:= e(k+p) + \sigma\omega(k)\ , && (4.2)\\ \Theta(s,p,\Omega) &:= \sum_{\sigma\in\{\pm\}}\int e^{-is[\Phi_\sigma(p,k)+\Omega]}M(k,\sigma)\mathrm{d}k\ , && (4.3)\\ \Upsilon_\eta(\alpha,p) &:= \sum_{\sigma\in\{\pm\}}\int \frac{M(k,\sigma)\mathrm{d}k}{\alpha - \Phi_\sigma(p,k) + i\eta}\ . && (4.4) \end{aligned}$$

We also define

$$M^*(k) := \max_{\sigma\in\{\pm\}} \max_{j=0,1,\ldots,2d} |\nabla_k^j M(k,\sigma)| \tag{4.5}$$

and from (1.24) and the properties of $\mathcal{N}$ from Section 1.2 we have

$$|M^*(k)| \leq C\langle k\rangle^{-4d-24} . \tag{4.6}$$

The following lemma estimates the functions Θ and Υ:

Lemma 4.1 *Under the conditions in Section 1.2 and (4.6) we have*

$$\sup_{p,\Omega} |\Theta(s,p,\Omega)| \leq \frac{C}{\langle s\rangle^{d/2}} ; \tag{4.7}$$

$$\sup_{p,\alpha,\eta} |\Upsilon_\eta(\alpha,p)| \leq C , \qquad \sup_{\eta} |\Upsilon_\eta(\alpha,p)| \leq \frac{C\langle p\rangle^d}{\langle\alpha - e(p)\rangle} ; \tag{4.8}$$

$$\sup_{p,\alpha,\eta} \Big(\Big|\nabla_p \Upsilon_\eta(\alpha,p)\Big| + \Big|\partial_\alpha \Upsilon_\eta(\alpha,p)\Big| + \Big|\partial_\eta \Upsilon_\eta(\alpha,p)\Big| \Big) \leq C\eta^{-1/2} . \tag{4.9}$$

For $\omega = (const.)$ we also have

$$\sup_{p,\alpha,\eta} |\nabla_p^\ell \Upsilon_\eta(\alpha,p)| \leq C , \qquad \sup_{\eta} |\nabla_p^\ell \Upsilon_\eta(\alpha,p)| \leq \frac{C\langle p\rangle^d}{\langle\alpha - e(p)\rangle} , \qquad \ell \leq d . \tag{4.10}$$

Moreover, the limit

$$\Upsilon_{0+}(\alpha,p) := \lim_{\eta\to 0+0} \Upsilon_\eta(\alpha,p) \tag{4.11}$$

exists, and

$$Im\, \Upsilon_{0+}(\alpha,p) = -\pi \sum_{\sigma\in\{\pm\}} \int |Q(k)|^2 \Big(\mathcal{N}(k) + \frac{\sigma+1}{2}\Big)\, \delta\Big(\alpha - \Phi_\sigma(p,k)\Big)\mathrm{d}k . \tag{4.12}$$

Proof. Apart from the proof of (4.12), we fix σ and for simplicity we omit it from the notation, i.e., we assume that Θ and Υ are defined without the summation over σ. The inequality (4.7) for $|s| \leq 1$ is trivial. For large $|s|$ it follows from the stationary phase formula

$$\Big| \int e^{-is\Phi(p,k)} M(k)\mathrm{d}k \Big| \leq \frac{C}{s^{d/2}} \Big(\|\langle\nabla\rangle^d M\|_{L^2} + \|\langle\nabla\rangle^d M\|_{L^1} \Big) \tag{4.13}$$

The prototypes of (4.13) are the linear and quadratic cases; $\Phi(p,k) = u\cdot k$ or $\Phi(p,k) = k^2$. The linear phase factor is trivial:

$$\Big| \int e^{-isu\cdot k} M(k)\mathrm{d}k \Big| \leq \frac{C}{\langle s|u|\rangle^d} \|\langle\nabla\rangle^d M\|_{L^1} . \tag{4.14}$$

For the quadratic case, by standard estimate on the Schrödinger evolution kernel

$$\Big|\int e^{-isk^2} M(k)\mathrm{d}k\Big| \le \frac{C}{s^{d/2}}\|\widehat{M}\|_{L^1} \tag{4.15}$$

and $\|\widehat{M}\|_{L^1} \le C\|\langle\nabla\rangle^d M\|_{L^2}$. Similar estimates are valid for other purely quadratic phase factors.

The proof of (4.13) for a general phase function uses a partition of unity separating the isolated critical point. In a small neighborhood of the nondegerate critical point, a smooth change of variables transforms the phase into a purely quadratic function and (4.15) applies. Away from the critical point the gradient of Φ is separated away from zero, i.e., (4.14) applies with $|u| \ge C > 0$. The size of the neighborhood and the Jacobians of the change of variable transformations are controlled uniformly in p using (1.18). The details of this standard technique are left to the reader (see also Sec. VIII.2 of [St]).

For the first estimate in (4.8) we use the formula

$$\int \frac{M(k)\mathrm{d}k}{\alpha - \Phi(p,k) + i\eta} = i\int_0^\infty \mathrm{d}s\; e^{is(\alpha+i\eta)}\int e^{-is\Phi(p,k)}M(k)\mathrm{d}k \tag{4.16}$$

for any $\eta > 0$. We estimate $\int e^{-is\Phi(p,k)}M(k)\mathrm{d}k$ from (4.7) and use that $d \ge 3$ hence $\langle s\rangle^{-d/2}$ is integrable. This inequality is a extension of Lemma 3.10 of [EY2].

To obtain the factor decaying in $\langle \alpha - e(p)\rangle$, we insert a smooth cutoff function $k \mapsto \theta_p(k)$ supported on the set $\{k\ :\ |\alpha - \Phi(p,k)| \le 2\}$, so that $1-\theta_p$ is supported on $\{k\ :\ |\alpha-\Phi(p,k)| \ge 1\}$. Using (1.15) and (1.16), we see that $|\nabla^\ell\theta_p(k)| \le C_\ell(\langle p\rangle^\ell + \langle k\rangle^\ell)$ for any ℓ. On the support of θ_p we repeat the argument above and use (4.13) with $\big\|\langle\nabla\rangle^d(M\theta_p)\big\|_{L^2} \le C\langle p\rangle^d$. On the support of $1-\theta_p$ the denominator can be estimated as

$$\frac{1}{|\alpha - \Phi(p,k) + i\eta|} \le \frac{C}{\langle \alpha - e(p+k) \pm \omega(k)\rangle} \le \frac{C\langle k\rangle\big(\langle p\rangle + \langle k\rangle\big)}{\langle \alpha - e(p)\rangle}\ ,$$

again by (1.15), and then the k integration is finite.

We will sketch the estimate of the first term in (4.9), the other two are similar. We have from (4.16)

$$\Big|\nabla_p \Upsilon_\eta(\alpha,p)\Big| \le \int_0^\infty \; s\; e^{-\eta s}\Big|\int e^{-is\Phi(p,k)}\Big[\nabla_p\Phi(p,k)\Big]M(k)\mathrm{d}k\Big|\;\mathrm{d}s\ .$$

Using stationary phase (4.13), the smoothness and decay properties of Φ and M, we estimate the term in absolute value by $C\langle s\rangle^{-d/2}$, which gives (4.9) for $d \ge 3$.

If $\omega = (const.)$, then after a change of variables

$$\nabla_p^\ell \Upsilon_\eta(\alpha, p) = \int \frac{\nabla^\ell M(k-p)\, \mathrm{d}k}{\alpha - e(k) \pm \omega + i\eta} \, . \tag{4.17}$$

Applying the same argument as in the proof of (4.8), using that $\nabla^\ell M$ is d-times differentiable with decaying derivatives, we obtain (4.10).

Finally, the limit (4.11) exists by the estimate of the last term in (4.9). The formula (4.12) is a straighforward calculation using the smoothness of $e(k), \omega(k), \mathcal{N}(k), Q(k)$. $\square$

4.2 Graphical representation of collision histories

We will write $\mathcal{E}^0_{n,N}$ (2.15) in momentum representation; p_j stand for electron momenta, and k_j denote phonon momenta (see Section 4.4 later for more details). The history of the electron-phonon collisions will be represented by a graph, where the electron lines are solid and the phonon lines are dashed. The bullets represent collisions. We fix an orientation of the edges of the graph (Fig. 1). The momentum label on each edge expresses the momentum flow in the direction of the oriented edge.

At each collision the adjacent three momenta are subject to momentum conservation. Fig. 1 represents a history with N collisions with initial momentum p_N and final momentum p_0.

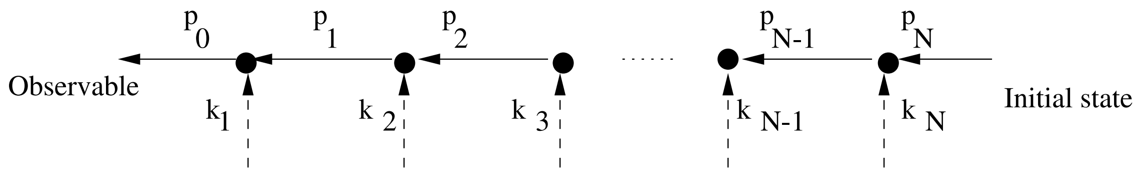

Fig. 1. Electron (p) and phonon (k) momenta.

When we compute $\mathrm{Tr}_{ph}\mathcal{E}^0_{n,N}\Gamma_0\left[\mathcal{E}^0_{n,N}\right]^*$ or similar quadratic expressions, we need another copy of the electron history and we will pair the $2N$ phonon momenta lines using the analogue of the Wick theorem. This will be represented by another copy of the graph on Fig. 1 but all the arrows reversed. The corresponding momenta will be distinguished by tilde. *Pairing* means joining the phonon lines and identifying the associated phonon momenta. More precisely, pairing of k_a and k_b means identification $k_a = -k_b$ and similarly for two tilde variables. If k_a and $\tilde{k}_b$ are paired, then the identification is $k_a = \tilde{k}_b$. The graph consisting of two copies of

Fig.1 with a pairing of the phonon lines will be called the *pairing graph.* The orientation is neglected on the figures.

Immediate reabsorption occurs when neighboring phonon momenta are paired and these momenta will be integrated out separately. Therefore we will mainly focus on the pairing structure of the n external phonon lines.

Definition 4.2 *A pairing line in the graph is called an* **internal** *(immediate recollision) line if it pairs* (k_a, k_{a+1}) *or* $(\tilde{k}_a, \tilde{k}_{a+1})$ *for some* $a \leq N-1$*. Otherwise it is called an* **external** *(genuine pairing) line. The* **skeleton** *of a graph is defined by removing all internal lines together with their vertices (Fig.2).*

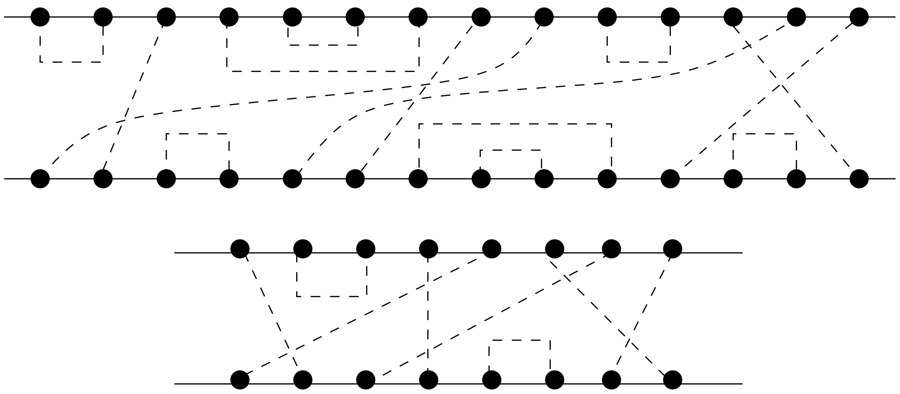

Fig.2. A pairing graph and its skeleton (N=14, n=8).

For most of our estimates only the skeleton will play a role and only the identity of the external lines should be kept in the notation. Hence we will relabel the indices of the phonon momenta (Fig. 3). Moreover, due to momentum conservation, the electron momenta between successive immediate recollisions are the same so they can also be relabelled.

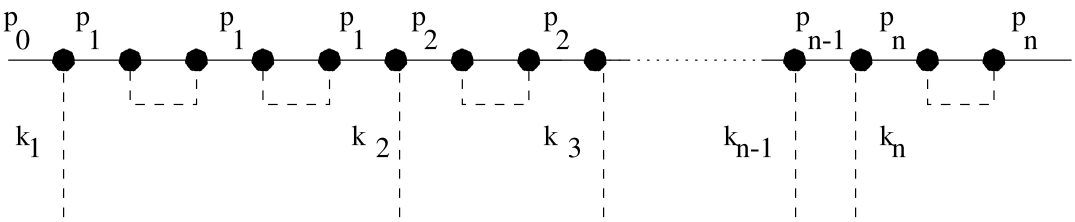

Fig. 3. Immediate recollisions with relabelled momenta

The size of the contribution of a specific pairing to $\mathrm{Tr}_{ph}\mathcal{E}^0_{n,N}\Gamma_0\left[\mathcal{E}^0_{n,N}\right]^*$ is determined by the skeleton of the pairing. There is no recollision line, i.e., all external lines join an "upper" bullet with a "lower" one. The main (physical) contribution comes from the so-called **direct** pairing (or "ladder graph"). A pairing and the corresponding graph is called **indirect** or **crossing** if some $(k_a, \tilde{k}_b)$ and $(k_{a'}, \tilde{k}_{b'})$ are paired with $a < b$, $a' > b'$. This notion is independent of relabelling. Fig.4 shows a direct and an indirect graph.

Fig.4. Skeleton of a direct pairing and part of an indirect pairing with a crossing.

A graph is said to contain a *genuine reabsorption* if some k_a is paired with k_b, $|a - b| \geq 2$, or the same happens with some $\tilde{k}_a$ and $\tilde{k}_b$. This concept is defined according to the original labelling, but notice that it depends only on one copy of the electron history. Fig.5 shows graphs with genuine reabsorption. Such graphs represent the trace of recollision terms, $\mathrm{Tr}\ \mathcal{D}^1_{n,N}\Gamma_0\left[\mathcal{D}^1_{n,N}\right]^*$ (2.13).

Notice that neighboring momenta may be paired in a skeleton if they have corresponded to a genuine recollision in the original graph and all the momenta in between formed immediate pairs (Fig.2).

Fig.5. Pairings with genuine recollisions (only part of the pairing lines are shown).

4.3 The main representation formula

Let Π_n denote the set of permutations on $\{1, 2, \ldots, n\}$.

Proposition 4.3 *For any $n \leq N$, $N-n$ even, we have*

$$\limsup_{L\to\infty}\left| Tr_{e+ph}\left(\mathcal{E}^0_{n,N}(t)\Gamma_0\left[\mathcal{E}^0_{n,N}(t)\right]^*\right) - \sum_{\underline{m},\underline{\widetilde{m}}\in\mathcal{M}(n,N)}\sum_{\pi\in\Pi_n} C_{\underline{m},\underline{\widetilde{m}},\pi}(t)\right| = 0 \tag{4.18}$$

with

$$C_{\underline{m},\underline{\widetilde{m}},\pi}(t) := \lambda^{2N}\sum_{\substack{\sigma_j\in\{\pm\}\\ j=1,\dots,n}}\int \mathrm{d}\nu_\pi(p_n,\tilde{p}_n,\underline{k},\underline{\tilde{k}},\underline{\sigma})\; Y_{\underline{m},\underline{\widetilde{m}},\pi}(t;p_n,\tilde{p}_n,\underline{k},\underline{\tilde{k}},\underline{\sigma})\ , \tag{4.19}$$

where we define the measure

$$\mathrm{d}\nu_\pi(p_n,\tilde{p}_n,\underline{k},\underline{\tilde{k}},\underline{\sigma}) := \Big(\prod_{j=1}^n M(k_j,\sigma_j)\delta(k_j-\tilde{k}_{\pi(j)})\mathrm{d}k_j\mathrm{d}\tilde{k}_j\Big)\mathrm{d}p_n\mathrm{d}\tilde{p}_n\delta(p_n-\tilde{p}_n)\widehat{\gamma}_e(p_n,\tilde{p}_n) \tag{4.20}$$

on $\Lambda^*\times\Lambda^*\times(\Lambda^*)^n\times(\Lambda^*)^n\times\{\pm\}\subset\mathbf{R}^d\times\mathbf{R}^d\times(\mathbf{R}^d)^n\times(\mathbf{R}^d)^n\times\{\pm\}^n$, and we let

$$\begin{aligned} Y_{\underline{m},\underline{\widetilde{m}},\pi}(t;p_n,\tilde{p}_n,\underline{k},\underline{\tilde{k}},\underline{\sigma}) :\ &= \int_0^{t*}[\mathrm{d}s_b]_{b=0}^N\prod_{j=0}^n\Bigg[\prod_{b\in I_j^c}e^{-is_b[e(p_j)+\Omega_j]}\prod_{b\in I_j}\Theta(s_b,p_j,\Omega_j)\Bigg] &(4.21)\\ &\quad\times\int_0^{t*}[\mathrm{d}\tilde{s}_{\tilde{b}}]_{\tilde{b}=0}^N\prod_{j=0}^n\Bigg[\prod_{\tilde{b}\in\tilde{I}_j^c}e^{i\tilde{s}_{\tilde{b}}[e(\tilde{p}_j)+\widetilde{\Omega}_j]}\prod_{\tilde{b}\in\tilde{I}_j}\overline{\Theta}(\tilde{s}_{\tilde{b}},\tilde{p}_j,\widetilde{\Omega}_j)\Bigg] \\ &= e^{2t\eta}\int_{-\infty}^{\infty}\mathrm{d}\alpha\, e^{-it\alpha}\prod_{j=0}^n R_j^{m_j+1}\Big[\Upsilon_\eta(\alpha-\Omega_j,p_j)\Big]^{m_j} &(4.22)\\ &\quad\times\int_{-\infty}^{\infty}\mathrm{d}\tilde{\alpha}\, e^{it\tilde{\alpha}}\prod_{j=0}^n \widetilde{R}_j^{m_j+1}\Big[\overline{\Upsilon}_\eta(\tilde{\alpha}-\widetilde{\Omega}_j,\tilde{p}_j)\Big]^{\widetilde{m}_j}\ , \end{aligned}$$

where $I_j,\widetilde{I}_j$ etc. depend on $\underline{m},\underline{\widetilde{m}}$ (see Definition 2.1) and we define

$$R_j := R_j(\alpha,p_j,\Omega_j,\eta) = \frac{1}{\alpha-e(p_j)-\Omega_j+i\eta}\ , \tag{4.23}$$

$$\widetilde{R}_j := R_j(\tilde{\alpha},\tilde{p}_j,\widetilde{\Omega}_j,\eta) = \frac{1}{\tilde{\alpha}-e(\tilde{p}_j)-\widetilde{\Omega}_j-i\eta}\ .$$

In these formulas, p_j's are functions of p_n and $\underline{k}=(k_1,\dots,k_n)$, and similarly for the tilde variables:

$$p_j := p_n + \sum_{\ell=j+1}^n k_\ell,\qquad \tilde{p}_j := \tilde{p}_n + \sum_{\ell=j+1}^n \tilde{k}_\ell, \tag{4.24}$$

and

$$\Omega_j = \Omega_j(\underline{k},\underline{\sigma}) := \sum_{\ell=j+1}^n \sigma_\ell\omega(k_\ell),\qquad \widetilde{\Omega}_j := \Omega_j(\underline{\tilde{k}},\underline{\sigma}\circ\pi^{-1}) = \sum_{\ell=j+1}^n \sigma_{\pi^{-1}(\ell)}\omega(\tilde{k}_\ell) \tag{4.25}$$

with $\underline{\sigma}=(\sigma_1,\dots,\sigma_n)$.

Remark. This proposition shows that only those pairings are relevant that respect the no-recollision rule. In particular, every pairing can be identified with a permutation on the (relabelled) indices of the external lines. In the sequel we use this identification freely.

The following subsections contain the proof of Proposition 4.3.

4.4 Duhamel formula in momentum space

We express $\mathcal{E}^0_{n,N}$ (2.15) in momentum space. Given $\underline{k} := (k_1, k_2, \dots, k_N)$ and p_N, we define variables $p_0, p_1, \dots, p_{N-1}$ as follows (Fig.1)

$$p_j = p_N + \sum_{\ell=j+1}^{N} k_\ell, \qquad j = 0, 1, \dots, N-1 \ . \tag{4.26}$$

Sometimes these relations will be expressed as

$$\int \Big(\prod_{j=0}^{N-1} \mathrm{d}p_j \Big) \Delta(\underline{p}, \underline{k})$$

with

$$\Delta(\underline{p}, \underline{k}) := \prod_{j=0}^{N-1} \delta\Big(p_j - p_N - \sum_{\ell=j+1}^{N} k_\ell \Big) \ , \tag{4.27}$$

but mostly we consider $p_0, p_1, \dots, p_{N-1}$ as functions of $k_1, \dots, k_N$ and p_N.

For expressions that are quadratic in $\mathcal{E}^0_{n,N}$, e.g. (4.18), we need another set of variables distinguished by tilde. Given $\underline{\tilde{k}} := (\tilde{k}_1, \tilde{k}_2, \dots, \tilde{k}_N)$ and $\tilde{p}_N$, we define, for $j = 0, 1, 2, \dots, N-1$,

$$\tilde{p}_j := \tilde{p}_N + \sum_{\ell=j+1}^{N} \tilde{k}_\ell \ . \tag{4.28}$$

Note that the $(k_1, k_2, \dots, k_n)$, and $(p_0, p_1, \dots, p_n)$ variables in Proposition 4.3 will be only a subset of $(k_1, \dots, k_N)$ and $(p_0, \dots, p_N)$ after relabelling them (see later).

Using this notation, we can rewrite the kernel $\mathcal{B}(t, \underline{k}, N; p_0, p_N)$ of the operator $\mathcal{B}(t, \underline{k}, N)$ (2.14) in the Fourier space of $\mathcal{H}_e$

$$\begin{aligned} &\mathcal{B}(t, \underline{k}, N; p_0, p_N) : \\ &= \lambda^N \int \Big(\prod_{j=1}^{N-1} \mathrm{d}p_j \Big) \Delta(\underline{p}, \underline{k}) \int_0^{t*} [\mathrm{d}s_j]_0^N \ e^{-is_0[e(p_0) + H_{ph}]} \Big(\prod_{j=1}^{N} Q(k_j) b_{k_j} e^{-is_j[e(p_j) + H_{ph}]} \Big) \ . \end{aligned} \tag{4.29}$$

This is an operator acting on $\mathcal{H}_{ph}$. Notice that p_0 is not integrated out, i.e., there is a delta function $\delta\big(p_0 - p_N - \sum_{\ell=1}^{N} k_\ell\big)$ on the right hand side showing that $\underline{k}, p_0, p_N$ are not independent.

Equivalently, we can write

$$\mathcal{B}\Big(t,\underline{k},N;p_N+\sum_{j=1}^N k_j,p_N\Big) := \lambda^N \int_0^{t*} [\mathrm{d}s_j]_0^N \; e^{-is_0[e(p_0)+H_{ph}]}\Big(\prod_{j=1}^N Q(k_j) b_{k_j} e^{-is_j[e(p_j)+H_{ph}]}\Big) \quad (4.30)$$

if we consider p_j (for $j=0,1,\ldots,N-1$) as functions of $k_1,\ldots,k_N$ and p_N given by (4.26).

We collect the terms depending on the phonon operators. Recall that

$$e^{-isH_{ph}} c_k e^{isH_{ph}} = e^{is\omega(k)} c_k, \qquad e^{-isH_{ph}} c_k^\dagger e^{isH_{ph}} = e^{-is\omega(k)} c_k^\dagger \; .$$

Define

$$b_k(s) := e^{-isH_{ph}} b_k e^{isH_{ph}} = e^{-is\omega(k)} c_k^\dagger - e^{is\omega(k)} c_{-k} \; ,$$

then its adjoint is $b_k^*(s) = -b_{-k}(s)$ and

$$[b_k(\tau), b_m(s)] = \delta(k+m)\Big[e^{-i(\tau-s)\omega(k)} - e^{i(\tau-s)\omega(k)}\Big] \; . \quad (4.31)$$

Let

$$G^\#(u,\tau) := \Big\{e^{-i\tau\omega(u)}\mathcal{N}(u) + e^{i\tau\omega(u)}(\mathcal{N}(u)+1)\Big\} \; , \quad (4.32)$$

then

$$\mathrm{Tr}_{ph}\gamma_{ph} b_u(\tau) b_v(s) = \mathrm{Tr}_{ph}\gamma_{ph} b_u^*(\tau) b_v^*(s) = -G^\#(u,\tau-s)\delta(u+v) \; , \quad (4.33)$$

$$\mathrm{Tr}_{ph}\gamma_{ph} b_u(\tau) b_v^*(s) = \mathrm{Tr}_{ph}\gamma_{ph} b_u^*(\tau) b_v(s) = G^\#(u,\tau-s)\delta(u-v) \; .$$

We define

$$\tau_j := s_0 + s_1 + \ldots + s_{j-1} \; , \qquad \tilde\tau_j := \tilde s_0 + \tilde s_1 + \ldots + \tilde s_{j-1} \quad (4.34)$$

for $j \geq 1$ and $\tau_0 = \tilde\tau_0 := 0$ and let $\underline{\tau} := (\tau_1,\ldots,\tau_N)$, $\underline{\tilde\tau} := (\tilde\tau_1,\ldots,\tilde\tau_N)$.

The terms involving phonon operators in (4.30) are:

$$e^{-is_0H_{ph}}\left(\prod_{j=1}^N b_{k_j} e^{-is_jH_{ph}}\right) = \Big(\prod_{j=1}^N b_{k_j}(\tau_j)\Big) e^{-itH_{ph}} \quad (4.35)$$

recalling the convention (2.4). We define

$$\begin{aligned} T(\underline{k},\underline{\tilde k},\underline{\tau},\underline{\tilde\tau}) : &= \mathrm{Tr}_{ph}\Big[\Big(\prod_{j=1}^N b_{k_j}(\tau_j)\Big) e^{-itH_{ph}}\gamma_{ph} e^{itH_{ph}}\Big(\prod_{j=N}^1 b_{\tilde k_j}^*(\tilde\tau_j)\Big)\Big] \quad (4.36) \\ &= \mathrm{Tr}_{ph}\Big[\gamma_{ph} b_{\tilde k_N}^*(\tilde\tau_N)\ldots b_{\tilde k_2}^*(\tilde\tau_2) b_{\tilde k_1}^*(\tilde\tau_1) b_{k_1}(\tau_1) b_{k_2}(\tau_2)\ldots b_{k_N}(\tau_N)\Big] \; . \end{aligned}$$

Combining (2.15), (4.29), (4.35) and (4.36) we obtain an integral formula for the fully expanded term with no reabsorptions:

$$
\begin{aligned}
&\mathrm{Tr}_{e+ph}\Big(\mathcal{E}^0_{n,N}(t)\Gamma_0\big[\mathcal{E}^0_{n,N}(t)\big]^*\Big) \qquad (4.37)\\
&= \lambda^{2N}\int^{\#(n,N)}\mathrm{d}\underline{k}\mathrm{d}\underline{\tilde{k}}\int_0^{t*}[\mathrm{d}s_j]_0^N[\mathrm{d}\tilde{s}_j]_0^N T(\underline{k},\underline{\tilde{k}},\underline{\tau},\underline{\tilde{\tau}})\int \mathrm{d}p_N\mathrm{d}\tilde{p}_N\delta(p_N-\tilde{p}_N)\widehat{\gamma}_e(p_N,\tilde{p}_N)\\
&\quad\times\Big[\int\Big(\prod_{j=0}^{N-1}\mathrm{d}p_j\Big)\Delta(\underline{p},\underline{k})e^{-is_0e(p_0)}\Big(\prod_{j=1}^{N}e^{-is_je(p_j)}Q(k_j)\Big)\Big]\\
&\quad\times\Big[\int\Big(\prod_{j=0}^{N-1}\mathrm{d}\tilde{p}_j\Big)\Delta(\underline{\tilde{p}},\underline{\tilde{k}})e^{i\tilde{s}_0e(\tilde{p}_0)}\Big(\prod_{j=1}^{N}e^{i\tilde{s}_je(\tilde{p}_j)}Q(\tilde{k}_j)\Big)\Big]\,.
\end{aligned}
$$

Here we used that $Q(k)$ is real and that $\delta(p_0-\tilde{p}_0)$ from Tr_e is equivalent to $\delta(p_N-\tilde{p}_N)$.

4.5 Computing the phonon trace and the thermodynamic limit

We use Wick's theorem and the definition of $G^{\#}$ (4.32) to compute the phonon trace (4.36) appearing in (4.37). We have to consider all pairings within the indices of the set $\{\underline{k},\underline{\tilde{k}}\}:=\{k_1,\ldots,k_N,\tilde{k}_1,\ldots,\tilde{k}_N\}$. We recall the definition of $\int^{\#(n,N)}$ from (2.8), and similarly we have

$$\int^{\#(n,N)}\mathrm{d}\underline{\tilde{k}}=\sum_{\underline{\widetilde{m}}\in\mathcal{M}(n,N)}\int^{(\underline{\widetilde{m}})}\mathrm{d}\underline{\tilde{k}}\,.$$

We also recall that $\underline{m}\in\mathcal{M}(n,N)$ is equivalent to a subsequence μ, which is viewed as a monotonic map $\mu:\{1,\ldots,n\}\mapsto\{1,\ldots,N\}$. The sets I,I_j,J depend on $\underline{m}$ and we use tilde for the sets $\tilde{I}:=I(\underline{\widetilde{m}})$, $\tilde{I}_j:=I_j(\underline{\widetilde{m}})$ and $\tilde{J}:=J(\underline{\widetilde{m}})$.

Certain pairs are already prepared by imposing the immediate reabsorptions in the measures via $\underline{m},\underline{\widetilde{m}}$. We say that a Wick-pairing of $\{\underline{k},\underline{\tilde{k}}\}$ *respects* a given pair $\underline{m},\underline{\widetilde{m}}\in\mathcal{M}(n,N)$, if b is paired with $b+1$ for all $b\in J$ and similarly for $\tilde{b}\in\tilde{J}$ (see (2.6)).

If a Wick-pairing respects $(\underline{m},\underline{\widetilde{m}})$, then only the indices from I and $\tilde{I}$ will be paired freely. Moreover, since the momenta with indices from I cannot be paired (see (2.7)), and so are the momenta within $\tilde{I}$, the pairing occurs between the momenta k_b ($b\in I$) and $\tilde{k}_{\tilde{b}}$ ($\tilde{b}\in\tilde{I}$). Hence any Wick-pairing that respects $(\underline{m},\underline{\widetilde{m}})$ can be uniquely identified with a map $\pi^*\in\mathcal{P}_n:=\tilde{\mu}\Pi_n\mu^{-1}$ between the sets I and $\tilde{I}$. Here $\mu,\tilde{\mu}$ are the maps associated with $\underline{m},\underline{\widetilde{m}}$, and $\mathcal{P}_n=\tilde{\mu}\Pi_n\mu^{-1}$ is the set of maps of the form $\tilde{\mu}\circ\pi\circ\mu^{-1}:I\mapsto\tilde{I}$ with $\pi\in\Pi_n$.

The following lemma states that only those Wick-pairings are relevant that respect a given $(\underline{m},\underline{\widetilde{m}})$, the rest are negligible in the thermodynamic limit.

Lemma 4.4 *Let $\underline{m}, \underline{\widetilde{m}} \in \mathcal{M}(n,N)$ and let $\underline{\tau}, \underline{\tilde{\tau}}$ be fixed times. For any function F of the momenta $\underline{k}, \underline{\tilde{k}}$ and times $\underline{\tau}, \underline{\tilde{\tau}}$ we have*

$$\int^{(\underline{m})} \mathrm{d}\underline{k} \int^{(\underline{\widetilde{m}})} \mathrm{d}\underline{\tilde{k}}\; F(\underline{k}, \underline{\tilde{k}}, \underline{\tau}, \underline{\tilde{\tau}}) T(\underline{k}, \underline{\tilde{k}}, \underline{\tau}, \underline{\tilde{\tau}}) = \int \mathrm{d}\underline{k} \int \mathrm{d}\underline{\tilde{k}}\; F(\underline{k}, \underline{\tilde{k}}, \underline{\tau}, \underline{\tilde{\tau}})$$
$$\times \sum_{\pi^* \in \mathcal{P}_n} \Big[\prod_{b \in I(\underline{m})} \delta\Big(k_b - \tilde{k}_{\pi^*(b)}\Big) G^{\#}(k_b, \tilde{\tau}_{\pi^*(b)} - \tau_b) \Big] \overline{\mathcal{X}}(\underline{k}, \underline{\tau}, \underline{m})\; \mathcal{X}(\underline{\tilde{k}}, \underline{\tilde{\tau}}, \underline{\widetilde{m}}) + O\Big(\frac{\|F\|_\infty}{|\Lambda|}\Big) ,$$

where

$$\mathcal{X}(\underline{k}, \underline{\tau}, \underline{m}) := \prod_{b \in J(\underline{m})} \delta(k_b + k_{b+1}) G^{\#}(k_b, \tau_{b+1} - \tau_b) \tag{4.38}$$

expresses the trace of the immediate reabsorptions. Notice that the delta functions in $\mathcal{X}$ coincide with the characteristic functions of Ξ built into the measure $\int^{(\underline{m})}$ (2.6). The error depends only on N and on the maximum of the function $\mathcal{N}$.

Proof. Consider a factor $\chi(k_b + k_{b+1})$ in the definition of $\Xi(\underline{k}, \underline{m})$. Suppose that the Wick pairing did not pair k_b with k_{b+1}. Then one can integrate out all the other $k, \tilde{k}$ variables in such order that eventually all delta functions disappear and we are left with an integration

$$\Big| \int \mathrm{d}k_b \mathrm{d}k_{b+1} \chi(k_b + k_{b+1}) H(k_b, k_{b+1}, \underline{\tau}, \underline{\tilde{\tau}}) \Big| \le \frac{\|H\|_\infty}{|\Lambda|} , \tag{4.39}$$

where H is the result of all the other integrations and clearly $\|H\|_\infty \le C(N)\|F\|_\infty$.

Therefore only those Wick-pairings are relevant that respect the prepared immediate reabsorptions built into the measures via $\underline{m}, \underline{\widetilde{m}}$ and they are described by a map $\pi^* \in \mathcal{P}_n$. For such pairing one can remove all further restrictions from the measure (imposed by $(\underline{m}), (\underline{\widetilde{m}})$ superscripts in (2.7)) since $\Xi(\underline{k}, \underline{m})$ is imposed by the delta functions in $\mathcal{X}$, while the complement of the condition, that $k_b \neq -k_{b'}$ for $b \neq b' \in I$ (see (2.7)), is small by a volume factor using an estimate similar to (4.39). $\square$

Remark: Similar argument is valid for amputated and recollision terms, $\mathcal{D}^{0,1}_{n,N}$. Since we always take the thermodynamic limit $L \to \infty$ first, we can always assume that the prepared immediate reabsorptions are respected by the Wick pairings, modulo negligible errors.

Using this Lemma and the fact that the expression in the big curly bracket in (4.37) has a volume independent bound, we obtain from (4.37)

$$\mathrm{Tr}_{e+ph} \bigg(\mathcal{E}^0_{n,N}(t) \Gamma_0 \Big[\mathcal{E}^0_{n,N}(t) \Big]^* \bigg) = \sum_{\underline{m}, \underline{\widetilde{m}} \in \mathcal{M}(n,N)} \sum_{\pi \in \Pi_n} C_{\underline{m}, \underline{\widetilde{m}}, \pi}(t) + O(|\Lambda|^{-1}) , \tag{4.40}$$

where for any $\underline{m}, \underline{\widetilde{m}} \in \mathcal{M}(n, N)$ and for any $\pi \in \Pi_n$ we define $\pi^* := \tilde{\mu} \circ \pi \circ \mu^{-1}$ and

$$
\begin{aligned}
C_{\underline{m},\underline{\widetilde{m}},\pi}(t) := \lambda^{2N} & \sum_{\substack{\sigma_b \in \{\pm\} \\ b \in I \cup J \cup \widetilde{J}}} \int \mathrm{d}\underline{k}\, \mathrm{d}\underline{\tilde{k}} \prod_{b \in I} \delta\Big(k_b - \tilde{k}_{\pi^*(b)}\Big) \Bigg[\prod_{b \in J} \delta(k_b + k_{b+1}) \prod_{\tilde{b} \in \widetilde{J}} \delta(\tilde{k}_{\tilde{b}} + \tilde{k}_{\tilde{b}+1}) \Bigg] \\
& \times \int \mathrm{d}p_N \mathrm{d}\tilde{p}_N \delta(p_N - \tilde{p}_N) \widehat{\gamma}_e(p_N, \tilde{p}_N) \Big(\prod_{b \in I \cup J} M(k_b, \sigma_b) \prod_{\tilde{b} \in \widetilde{J}} M(\tilde{k}_{\tilde{b}}, \sigma_{\tilde{b}}) \Big) \\
& \times \int_0^{t*} [\mathrm{d}s_j]_0^N \int_0^{t*} [\mathrm{d}\tilde{s}_j]_0^N \Big(\prod_{b \in I} e^{i\sigma_b(\tilde{\tau}_{\pi^*(b)} - \tau_b)\omega(k_b)} \Big) \Bigg[\prod_{b \in J} e^{-i\sigma_b s_b \omega(k_b)} \prod_{\tilde{b} \in \widetilde{J}} e^{i\sigma_{\tilde{b}} \tilde{s}_{\tilde{b}} \omega(\tilde{k}_{\tilde{b}})} \Bigg] \\
& \times \Bigg[\int \Big(\prod_{j=0}^{N-1} \mathrm{d}p_j \Big) \Delta(\underline{p}, \underline{k}) \Big(\prod_{j=0}^{N} e^{-is_j e(p_j)} \Big) \Bigg] \Bigg[\int \Big(\prod_{j=0}^{N-1} \mathrm{d}\tilde{p}_j \Big) \Delta(\underline{\tilde{p}}, \underline{\tilde{k}}) \Big(\prod_{j=0}^{N} e^{i\tilde{s}_j e(\tilde{p}_j)} \Big) \Bigg] .
\end{aligned}
\tag{4.41}
$$

We used $\tau_{b+1} - \tau_b = s_b$ and the definition of M (4.1) to combine the $G^\#$ terms with the $Q(k)$ factors:

$$
Q(r)^2 G^\#(r, \zeta) = \sum_{\sigma = \pm 1} e^{i\sigma\zeta\omega(r)} M(r, \sigma) . \tag{4.42}
$$

4.6 Proof of the representation formula

Recall that $\{0, 1, \ldots, N\}$ is a disjoint union $J \cup J^c$. Let $b \mapsto \chi_b$ be the characteristic function of J, i.e., $\chi_b = 1$ if $b \in J$ and $\chi_b = 0$ otherwise. Similarly, $\tilde{\chi}_b$ is the characteristic function of $\widetilde{J}$.

Using the definition of $\tau_j, \tilde{\tau}_j$, we can rewrite the phonon phase factors in (4.41)

$$
\prod_{b \in I} e^{-i\sigma_b \tau_b \omega(k_b)} \prod_{b \in J} e^{-i\sigma_b s_b \omega(k_b)} = \prod_{j=0}^{n} \Big(\prod_{b \in I_j \cup I_j^c} e^{-is_b[\Omega_j(k, \underline{\sigma}) + \chi_b \sigma_b \omega(k_b)]} \Big)
$$

and

$$
\prod_{b \in I} e^{i\sigma_b \tilde{\tau}_{\pi^*(b)} \omega(k_b)} \prod_{\tilde{b} \in \widetilde{J}} e^{i\sigma_{\tilde{b}} \tilde{s}_{\tilde{b}} \omega(\tilde{k}_{\tilde{b}})} = \prod_{j=0}^{n} \Big(\prod_{\tilde{b} \in \tilde{I}_j \cup \tilde{I}_j^c} e^{i\tilde{s}_{\tilde{b}}[\widetilde{\Omega}_j(\tilde{k}, \underline{\sigma} \circ (\pi^*)^{-1}) + \tilde{\chi}_{\tilde{b}} \sigma_{\tilde{b}} \omega(\tilde{k}_{\tilde{b}})]} \Big)
$$

with

$$
\Omega_j := \Omega_j(k, \underline{\sigma}) := \sum_{b \in I\ ,\ b > \mu(j)} \sigma_b \omega(k_b) \tag{4.43}
$$

$$
\widetilde{\Omega}_j := \widetilde{\Omega}_j(\tilde{k}, \underline{\sigma} \circ (\pi^*)^{-1}) := \sum_{\tilde{b} \in \tilde{I}\ ,\ \tilde{b} > \tilde{\mu}(j)} \sigma_{(\pi^*)^{-1}(\tilde{b})} \omega(\tilde{k}_{\tilde{b}})
$$

(recall that $\mu, \tilde{\mu}$ are associated with $\underline{m}, \underline{\widetilde{m}}$). Here $\underline{\sigma} := \{\sigma_b \ :\ b \in I\}$ and usually we will omit the arguments.

Hence we have for $\underline{m}, \underline{\widetilde{m}} \in \mathcal{M}(n,N)$ that

$$
\begin{aligned}
C_{\underline{m},\underline{\widetilde{m}},\pi}(t) : \; = \; & \lambda^{2N} \sum_{\substack{\sigma_b \in \{\pm\} \\ b \in I \cup J \cup \widetilde{J}}} \int \Big(\prod_{b \in I \cup J} \mathrm{d}k_b \Big) \Big(\prod_{\tilde{b} \in \tilde{I} \cup \widetilde{J}} \mathrm{d}\tilde{k}_{\tilde{b}} \Big) \prod_{b \in I} \delta\Big(k_b - \tilde{k}_{\pi^*(b)} \Big) \\
& \times \int \mathrm{d}p_N \mathrm{d}\tilde{p}_N \delta(p_N - \tilde{p}_N) \widehat{\gamma}_e(p_N, \tilde{p}_N) \Big(\prod_{b \in I \cup J} M(k_b, \sigma_b) \prod_{\tilde{b} \in \widetilde{J}} M(\tilde{k}_{\tilde{b}}, \sigma_{\tilde{b}}) \Big) \\
& \times \int_0^{t*} [\mathrm{d}s_j]_0^N \left[\prod_{j=0}^{n} \prod_{b \in I_j \cup I_j^c} e^{-is_b [e(p_{\mu(j)} + \chi_b k_b) + \Omega_j + \chi_b \sigma_b \omega(k_b)]} \right] \\
& \times \int_0^{t*} [\mathrm{d}\tilde{s}_j]_0^N \left[\prod_{j=0}^{n} \prod_{\tilde{b} \in \tilde{I}_j \cup \tilde{I}_j^c} e^{i\tilde{s}_{\tilde{b}} [e(\tilde{p}_{\tilde{\mu}(j)} + \tilde{\chi}_{\tilde{b}} \tilde{k}_{\tilde{b}}) + \widetilde{\Omega}_j + \tilde{\chi}_{\tilde{b}} \sigma_{\tilde{b}} \omega(\tilde{k}_{\tilde{b}})]} \right] .
\end{aligned}
\tag{4.44}
$$

In this formula we considered $p_j, \tilde{p}_j$ as functions of $p_N = \tilde{p}_N$ and k's or $\tilde{k}$'s, respectively (see (4.26)), and we used that

$$
\begin{aligned}
p_b = \; & p_{\mu(j)} + k_b \quad \text{for} \quad b \in I_j \; , \\
p_b = \; & \quad p_{\mu(j)} \quad\;\; \text{for} \quad b \in I_j^c \; ,
\end{aligned}
$$

and similarly for tildes. We also freely integrated out $k_{b+1}, \tilde{k}_{\tilde{b}+1}$ for $b \in J$, $\tilde{b} \in \widetilde{J}$ since they do not appear any more in the formulas.

Let $p_j^* := p_{\mu(j)}$, $\tilde{p}_j^* := p_{\tilde{\mu}(j)}$, $k_j^* := k_{\mu(j)}$, $\tilde{k}_j^* := \tilde{k}_{\tilde{\mu}(j)}$, $\sigma_j^* := \sigma_{\mu(j)}$ for $j = 1, \ldots, n$ and $p_0^* := p_0$, $\tilde{p}_0^* := \tilde{p}_0$. These will be the relabelled momenta (see Figs. 1-3.), temporarily distinguished by star. Then by (4.26) we have $p_n^* = p_N$, $\tilde{p}_n^* = \tilde{p}_N$ and

$$
p_j^* = p_n^* + \sum_{\ell=j+1}^{n} k_\ell^*, \qquad \tilde{p}_j^* = \tilde{p}_n^* + \sum_{\ell=j+1}^{n} \tilde{k}_\ell^*
\tag{4.45}
$$

for $j = 0, 1, \ldots, n$. Also notice that

$$
\Omega_j = \sum_{\ell=j+1}^{n} \sigma_\ell^* \omega(k_\ell^*), \qquad \widetilde{\Omega}_j = \sum_{\ell=j+1}^{n} \sigma_{\pi^{-1}(\ell)}^* \omega(\tilde{k}_\ell^*) \; .
$$

Clearly $\Omega_j, \widetilde{\Omega}_j$ depend only on the star variables, i.e., with a slight abuse of notation we continue to denote them as

$$
\Omega_j = \Omega_j(\underline{k}^*, \underline{\sigma}^*) \; , \qquad \widetilde{\Omega}_j = \widetilde{\Omega}_j(\underline{\tilde{k}}^*, \underline{\sigma}^* \circ \pi^{-1}) \; .
\tag{4.46}
$$

With these variables and after separating the terms with immediate recollisions, we obtain from (4.44) that

$$
C_{\underline{m},\underline{\widetilde{m}},\pi}(t) := \lambda^{2N} \sum_{\substack{\sigma_j^* \in \{\pm\} \\ j=1,\ldots,n}} \int \mathrm{d}\nu_\pi(p_n^*, \tilde{p}_n^*, \underline{k}^*, \underline{\tilde{k}}^*, \underline{\sigma}^*) \; Y_{\underline{m},\underline{\widetilde{m}},\pi}(t; p_n^*, \tilde{p}_n^*, \underline{k}^*, \underline{\tilde{k}}^*, \underline{\sigma}^*)
\tag{4.47}
$$

with

$$Y_{\underline{m},\underline{\widetilde{m}},\pi}(t;p_n^*,\tilde{p}_n^*,\underline{k}^*,\underline{\tilde{k}}^*,\underline{\sigma}^*) := \sum_{\substack{\sigma_b\in\{\pm\}\\ b\in J\cup\widetilde{J}}} \int \Big(\prod_{b\in J} M(k_b,\sigma_b)\mathrm{d}k_b\Big)\Big(\prod_{\tilde{b}\in\widetilde{J}} M(\tilde{k}_{\tilde{b}},\sigma_{\tilde{b}})\mathrm{d}\tilde{k}_{\tilde{b}}\Big) \tag{4.48}$$
$$\times \int_0^{t*}[\mathrm{d}s_b]_{b=0}^N \prod_{j=0}^n \Bigg[\prod_{b\in I_j^c} e^{-is_b[e(p_j^*)+\Omega_j]} \prod_{b\in I_j} e^{-is_b[e(p_j^*+k_b)+\Omega_j+\sigma_b\omega(k_b)]}\Bigg]$$
$$\times \int_0^{t*}[\mathrm{d}\tilde{s}_{\tilde{b}}]_{\tilde{b}=0}^N \prod_{j=0}^n \Bigg[\prod_{\tilde{b}\in \tilde{I}_j^c} e^{i\tilde{s}_{\tilde{b}}[e(\tilde{p}_j^*)+\widetilde{\Omega}_j]} \prod_{\tilde{b}\in\tilde{I}_j} e^{i\tilde{s}_{\tilde{b}}[e(\tilde{p}_j^*+\tilde{k}_{\tilde{b}})+\widetilde{\Omega}_j+\sigma_{\tilde{b}}\omega(\tilde{k}_{\tilde{b}})]}\Bigg],$$

where $\underline{k}^*$ stands for the sequence $k_1^*,\dots,k_n^*$ and similarly for the other variables. We recall that p_j^*, $\tilde{p}_j^*$, Ω_j, $\widetilde{\Omega}_j$ are functions of π, $\underline{\sigma}^*$, $\underline{k}^*$, $\underline{\tilde{k}}^*$ and p_n^* ($\underline{m},\underline{\widetilde{m}}$ are considered given).

There are two different expressions for $Y_{\underline{m},\underline{\widetilde{m}},\pi}$ after integrating out the internal momenta for all immediate recollisions. Recalling the definition of Θ (4.3), we have

$$Y_{\underline{m},\underline{\widetilde{m}},\pi}(t;p_n^*,\tilde{p}_n^*,\underline{k}^*,\underline{\tilde{k}}^*,\underline{\sigma}^*) = \int_0^{t*}[\mathrm{d}s_b]_{b=0}^N \prod_{j=0}^n \Bigg[\prod_{b\in I_j^c} e^{-is_b[e(p_j^*)+\Omega_j]} \prod_{b\in I_j} \Theta(s_b,p_j^*,\Omega_j)\Bigg] \tag{4.49}$$
$$\times \int_0^{t*}[\mathrm{d}\tilde{s}_{\tilde{b}}]_{\tilde{b}=0}^N \prod_{j=0}^n \Bigg[\prod_{\tilde{b}\in \tilde{I}_j^c} e^{i\tilde{s}_{\tilde{b}}[e(\tilde{p}_j^*)+\widetilde{\Omega}_j]} \prod_{\tilde{b}\in\tilde{I}_j} \overline{\Theta}(\tilde{s}_{\tilde{b}},\tilde{p}_j^*,\widetilde{\Omega}_j)\Bigg].$$

Notice that all non-star variables have disappeared.

The other expression for $Y_{\underline{m},\underline{\widetilde{m}},\pi}$ relies on performing all time integrals by using

$$\delta\Big(t-\sum_{b=0}^N s_b\Big) = \int_{-\infty}^{\infty} e^{-i\alpha\left(t-\sum_{b=0}^N s_b\right)}\mathrm{d}\alpha\,,$$

and we can regularize the $\mathrm{d}s$ integrations by shifting $e(p_j)\to e(p_j)-i\eta$.

The result is

$$\int_0^{t*}[\mathrm{d}s_b]_{b=0}^N \prod_{j=0}^n \Bigg[\prod_{b\in I_j^c} e^{-is_b[e(p_j^*)+\Omega_j]} \prod_{b\in I_j} e^{-is_b[e(p_j^*+k_b)+\Omega_j+\sigma_b\omega(k_b)]}\Bigg] \tag{4.50}$$

$$= e^{t\eta}\int_{-\infty}^{\infty}\mathrm{d}\alpha e^{-it\alpha}\prod_{j=0}^n\Bigg[\bigg(\frac{1}{\alpha-e(p_j^*)-\Omega_j+i\eta}\bigg)^{m_j+1}\prod_{b\in I_j}\frac{1}{\alpha-e(p_j^*+k_b)-\Omega_j-\sigma_b\omega(k_b)+i\eta}\Bigg],$$

and we will always choose $\eta := t^{-1}$. We can use a similar identity for the $\mathrm{d}\tilde{s}_j$ integrals.

Again, we can integrate out k_b, $b \in J$, and similarly for the tilde variables. Recalling the definition of Υ (4.4) we obtain from (4.49)

$$
\begin{aligned}
Y_{\underline{m},\underline{\widetilde{m}},\pi} &= e^{2t\eta} \int_{-\infty}^{\infty} \mathrm{d}\alpha e^{-it\alpha} \prod_{j=0}^{n} \left(\frac{1}{\alpha - e(p_j^*) - \Omega_j + i\eta} \right)^{m_j+1} \Big[\Upsilon_\eta(\alpha - \Omega_j, p_j^*) \Big]^{m_j} \\
&\quad \times \int_{-\infty}^{\infty} \mathrm{d}\tilde{\alpha} e^{it\tilde{\alpha}} \prod_{j=0}^{n} \left(\frac{1}{\tilde{\alpha} - e(\tilde{p}_j^*) - \widetilde{\Omega}_j - i\eta} \right)^{\widetilde{m}_j+1} \Big[\overline{\Upsilon}_\eta(\tilde{\alpha} - \widetilde{\Omega}_j, \tilde{p}_j^*) \Big]^{\widetilde{m}_j} \qquad (4.51)
\end{aligned}
$$

and again, all non-star variables have disappeared.

Finally, Proposition 4.3 follows from (4.40), (4.47), (4.49) and (4.51) if we remove the stars from each variable. $\square$.

5 Apriori bound

We start with the following result that is weaker than (3.29) but we need it to estimate certain pairings later.

Lemma 5.1 *For any $0 \le a < 1$, $n \le N$, $\underline{m}, \underline{\widetilde{m}} \in \mathcal{M}(n, N)$, $\pi \in \Pi_n$ we have*

$$
|C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \le \frac{(C_a \lambda^2 t)^N}{[M!]^a} \qquad (5.1)
$$

(recall that $M := (n+N)/2$), hence

$$
\limsup_{L \to \infty} Tr_{e+ph} \Big(\mathcal{E}^0_{n,N}(t) \Gamma_0 \Big[\mathcal{E}^0_{n,N}(t) \Big]^* \Big) \le n! \times \frac{(C_a \lambda^2 t)^N}{[M!]^a} \ . \qquad (5.2)
$$

Remark. The estimate (5.2) is summable over n and N only for short macroscopic time $T = \lambda^2 t \le T_0$.

Proof. The second inequality (5.2) easily follows from (5.1) and (4.18) since the number of pairs $\underline{m}, \underline{\widetilde{m}} \in \mathcal{M}(n, N)$ is bounded by

$$
\binom{N}{n}^2 \le C^N \ . \qquad (5.3)
$$

The bound (5.1) is trivial if $N = 0$. Otherwise we split $|Y| = |Y|^{a/2} |Y|^{1-a/2}$ in (4.19) and we apply supremum bound in the first term. Using (4.21) and (4.7), we see that

$$
|Y_{\underline{m},\underline{\widetilde{m}},\pi}(t; p_n, \tilde{p}_n, \underline{k}, \underline{\tilde{k}}, \underline{\sigma})| \le C^N \int_0^{t*} [\mathrm{d}s_b]_{b=0}^N \prod_{b \in J} \frac{1}{\langle s_b \rangle^{d/2}} \int_0^{t*} [\mathrm{d}\tilde{s}_{\tilde{b}}]_{\tilde{b}=0}^N \prod_{\tilde{b} \in \widetilde{J}} \frac{1}{\langle \tilde{s}_{\tilde{b}} \rangle^{d/2}} \ .
$$

An easy calculation shows that ($d \geq 3$)

$$\int_0^{t*} [\mathrm{d}s_b]_{b=0}^N \prod_{b\in J} \frac{1}{\langle s_b\rangle^{d/2}} \leq C^N \frac{t^{|J^c|-1}}{(|J^c|-1)!} = C^N \frac{t^M}{M!} \tag{5.4}$$

(recall that $M := (n+N)/2 = n + |\underline{m}|$), hence

$$|Y_{\underline{m},\underline{\widetilde{m}},\pi}(t;p_n,\tilde{p}_n,\underline{k},\underline{\tilde{k}},\underline{\sigma})|^{a/2} \leq C^N \Big(\frac{t^M}{M!}\Big)^a . \tag{5.5}$$

For the $|Y|^{1-a/2}$ part we use a Schwarz inequality to separate the tilde variables. The result from (4.19), (4.22), (4.8) and (5.5) is

$$\begin{aligned} |C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \;\leq\;& C^N\lambda^{2N}\Big(\frac{t^M}{M!}\Big)^a \sup_{\underline{\sigma}} \int \mathrm{d}\nu^*_\pi(p_n,\tilde{p}_n,\underline{k},\underline{\tilde{k}}) \\ &\times\Big\{ \Big[\int_{-\infty}^\infty \mathrm{d}\alpha \prod_{j=0}^n |R_j|^{m_j+1}\Big]^{2-a} + \Big[\int_{-\infty}^\infty \mathrm{d}\tilde\alpha \prod_{j=0}^n |\widetilde{R}_j|^{\widetilde{m}_j+1}\Big]^{2-a}\Big\} \end{aligned} \tag{5.6}$$

with $\eta := t^{-1}$ and with a slight modification of the measure $\mathrm{d}\nu_\pi$:

$$\mathrm{d}\nu^*_\pi(p_n,\tilde{p}_n,\underline{k},\underline{\tilde{k}}) := \Big(\prod_{j=1}^n M^*(k_j)\mathrm{d}k_j\mathrm{d}\tilde{k}_j\Big)\Big(\prod_{j=1}^n \delta(k_j-\tilde{k}_{\pi(j)})\Big)\mathrm{d}p_n\mathrm{d}\tilde{p}_n\delta(p_n-\tilde{p}_n)\widehat{\gamma}_e(p_n,\tilde{p}_n) \tag{5.7}$$

and we recall the definition of M^* from (4.5). The estimate of the two terms in the last line of (5.6) are identical, so we consider only the first one.

The case $n=0$, $m_0\geq 1$ is trivial by integrating out $\mathrm{d}\alpha$, and collecting $t^{m_0(2-a)} = t^{\underline{m}(2-a)}$ using

$$\sup_{p_0}\int_{-\infty}^\infty \frac{\mathrm{d}\alpha}{|\alpha - e(p_0) + i\eta|^{m_0+1}} \leq (Ct)^{m_0} .$$

This gives (5.1) together with the $t^{Ma} = t^{\underline{m}a}$ factor from (5.5).

From now on we assume that $n\geq 1$. We can immediately integrate out $\tilde{k}_1,\tilde{k}_2,\ldots,\tilde{k}_n$ and $\tilde{p}_n$. Moreover, we use the trivial estimate

$$|R_j|^{m_j+1} \leq t^{m_j}|R_j| \tag{5.8}$$

for all j. So we obtain

$$|C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \leq C^N\lambda^{2N}\Big(\frac{t^M}{M!}\Big)^a t^{(2-a)|\underline{m}|} \sup_{\underline{\sigma}}\int \mathrm{d}\mu(\underline{k},p_n)\Big[\int_{-\infty}^\infty \mathrm{d}\alpha\prod_{j=0}^n |R_j|\Big]^{2-a} , \tag{5.9}$$

where

$$\mathrm{d}\mu(\underline{k},p_n) := \widehat{\gamma}_e(p_n,p_n)\Big(\prod_{j=1}^n M^*(k_j)\mathrm{d}k_j\Big)\mathrm{d}p_n \tag{5.10}$$

is a positive measure. The rest of the proof is similar to the proof of (3.16) in [EY2].

Fix all momentum variables, $\underline{p}, \underline{k}$'s, for a moment and let

$$G(\alpha) := \prod_{j=0}^{n-2} |R_j| \; . \tag{5.11}$$

For any given p_n, k_n, σ_n we denote by ϱ the probability measure on $\mathbf{R}$

$$\varrho(\mathrm{d}\alpha) = \varrho(\mathrm{d}\alpha; p_n, k_n) := \frac{\mathrm{d}\alpha}{Z(p_n,k_n)\Big|\alpha - A + i\eta\Big|\Big|\alpha - B + i\eta\Big|}$$

with $A := e(p_n)$, $B := e(p_{n-1}) + \sigma_n\omega(k_n) = e(p_n + k_n) + \sigma_n\omega(k_n)$ and

$$Z = Z(p_n, k_n) := \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{|\alpha - A + i\eta||\alpha - B + i\eta|} \tag{5.12}$$

(we neglect σ_n dependence in the notation). We have a simple estimate for Z:

$$Z \leq \frac{C}{|A - B + i\eta|}\Big[1 + \log_+ \Big|\frac{A-B}{\eta}\Big|\Big] \; . \tag{5.13}$$

Rewrite the $\mathrm{d}\alpha$ integration in (5.9) as

$$\int_{-\infty}^{\infty} \mathrm{d}\alpha \, \prod_{j=0}^{n} |R_j| = Z \int_{-\infty}^{\infty} \varrho(\mathrm{d}\alpha) G(\alpha)$$

and use Hölder (recall that $2 - a > 1$)

$$\Big(\int_{-\infty}^{\infty} \mathrm{d}\alpha \, \prod_{j=0}^{n} |R_j|\Big)^{2-a} \leq Z^{2-a} \int_{-\infty}^{\infty} \varrho(\mathrm{d}\alpha) G(\alpha)^{2-a} \; . \tag{5.14}$$

We then obtain

$$|C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \leq (C\lambda^2)^N \Big(\frac{t^M}{M!}\Big)^a t^{(2-a)|\underline{m}|} \sup_{\underline{\sigma}} \int \mathrm{d}p_n \mathrm{d}k_n [Z(p_n,k_n)]^{2-a} \widehat{\gamma}_e(p_n,p_n) M^*(k_n) \tag{5.15}$$

$$\times \int_{-\infty}^{\infty} \varrho(\mathrm{d}\alpha; p_n, k_n) \int \frac{M^*(k_{n-1})\mathrm{d}k_{n-1}}{\Big|\alpha - e(p_{n-2}) + i\eta - \sum_{\ell=n-1}^{n} \sigma_\ell \omega(k_\ell)\Big|^{2-a}}$$

$$\times \int \frac{M^*(k_{n-2})\mathrm{d}k_{n-2}}{\Big|\alpha - e(p_{n-3}) + i\eta - \sum_{\ell=n-2}^{n} \sigma_\ell \omega(k_\ell)\Big|^{2-a}} \times \ldots \times \int \frac{M^*(k_1)\mathrm{d}k_1}{\Big|\alpha - e(p_0) + i\eta - \sum_{\ell=1}^{n} \sigma_\ell \omega(k_\ell)\Big|^{2-a}} \; .$$

We have

$$\sup_{\alpha} \sup_{\underline{\sigma}} \sup_{p_n, k_{j+1}, k_{j+2}, \ldots, k_n} \int \frac{M^*(k_j)\mathrm{d}k_j}{\Big|\alpha - e(p_{j-1}) + i\eta - \sum_{\ell=j}^{n} \sigma_\ell \omega(k_\ell)\Big|^{2-a}} \leq C_a t^{1-a} \tag{5.16}$$

with $\eta := t^{-1}$. Recalling $p_{j-1} = p_j + k_j$, the prototype of this inequality for $b > 1$ is

$$\sup_{p,\theta} \int \frac{M^*(k)\mathrm{d}k}{|\theta - \Phi_\pm(p,k) + i\eta|^b} \le C_b t^{b-1} \ . \tag{5.17}$$

There will be several similar inequalities in the rest of the paper. They follow from the assumptions in Section 1.2 by elementary arguments. Their proofs use the same idea and the details will be omitted. Here we only sketch the proof of (5.17).

Proof of (5.17). We consider a tiling of $\mathbf{R}^d$ by identical cubes $\{Q_j\}$ of size $\widetilde{\varrho}$. We recall the quantity $\widetilde{\varrho}$ from (1.20). On each fixed cube $Q = Q_j$ we resolve the singularity on an exponential scale. We define the sets

$$S_\ell = S_\ell(p) := Q \cap \Big\{ k \ : \ 2^{-\ell-1}\widetilde{\varrho} < |\Phi_\pm(p,k) - \theta| \le 2^{-\ell}\widetilde{\varrho} \Big\} \tag{5.18}$$

for $1 \le \ell < \ell_0 := [\log^* t]$, $\ell \in \mathbf{N}$. We let

$$S_{\ell_0} := Q \cap \Big\{ k \ : \ |\Phi_\pm(p,k) - \theta| \le 2^{-\ell_0}\widetilde{\varrho} \Big\}$$

and $S_0 := Q \setminus \bigcup_\ell S_\ell$. The integrand on S_ℓ, $\ell = 0, 1, \ldots, \ell_0$ is bounded by $2^{b\ell} C \langle \mathrm{dist}(Q,0)\rangle^{-4d-24}$ using the decay of $M^*(k)$. The volume of S_ℓ is of order $2^{-\ell}$ by (1.20). We can sum up these estimates for ℓ to obtain $C_Q t^{b-1}$. Notice that for $b = 0$ one obtains $C_Q \log^* t$. Finally, the strong decay of the coefficients $C_Q \sim \langle \mathrm{dist}(Q,0)\rangle^{-4d-24}$ allows to sum up the contributions from each cube. Notice that only the following property of M^* was used in this proof

$$\sum_j \|M^*\|_{L^\infty(Q_j)} < \infty \ . \qquad \square \tag{5.19}$$

We continue the estimate of (5.15). Using (5.16), we integrate out $k_1, k_2, \ldots, k_{n-1}$ in this order, then we integrate out $\mathrm{d}\alpha$ using that $\varrho(\mathrm{d}\alpha)$ is a probability measure:

$$|C_{\underline{m},\widetilde{\underline{m}},\pi}(t)| \le C_a^N \lambda^{2N} \left(\frac{t^M}{M!}\right)^a t^{(2-a)|\underline{m}|+(1-a)(n-1)} \int \mathrm{d}p_n \mathrm{d}k_n [Z(p_n,k_n)]^{2-a} \widehat{\gamma}_e(p_n,p_n) M^*(k_n) \ . \tag{5.20}$$

We then use the estimate (5.13) and integrate out $\mathrm{d}k_n$ to collect one more factor $C_a t^{1-a}$ using (5.17). At the end, to do the $\mathrm{d}p_n$ integration, we use that $\int \widehat{\gamma}_e(p,p)\mathrm{d}p = \mathrm{Tr}\ \gamma_e = 1$. The total power of t is $Ma + (2-a)|\underline{m}| + (1-a)n = n + 2|\underline{m}| = N$ using $M = n + |\underline{m}|$. This completes the proof of Lemma 5.1. $\square$

6 Fully expanded terms with small n and no recollision

Here we prove (3.29). We use the representation (4.40). We can assume that $n \geq 4$, otherwise the apriori bound (5.2) applies.

Definition 6.1 *A pairing $\pi \in \Pi_n$ is called* **crossing** *if $\pi \neq id$. In particular, there exists $a < b$, with $\pi(b) < \pi(a)$. In this case the pair of indices (a, b) is called* **crossing pair**.

Remark. This definition differs from the one given in [EY2] (Definition 2.5). Here only pairing lines between tilde and non-tilde variables can form a crossing pair.

The only non-crossing pairing is the so-called *direct* pairing, $\pi = id$, it is estimated by (5.1) and it gives the first term on the right hand side of (3.29).

Lemma 6.2 *Let $n \leq N$, $\underline{m}, \underline{\widetilde{m}} \in \mathcal{M}(n, N)$ and let $\pi \in \Pi_n$ be a crossing pairing. Then*

$$\limsup_{L\to\infty} |C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \leq t^{-1/2}(C\lambda^2 t)^N (\log^* t)^4. \tag{6.1}$$

From this lemma and the combinatorial bound (5.3) the estimate (3.29) follows immediately.

Proof of Lemma 6.2. If (a, b) is a crossing pair, then for fixed b there could be many different a's which form a crossing pair with b. Consider the smallest one, i.e., $\pi(b) < \pi(a)$, but for all $c < a$, $\pi(c) < \pi(b)$. This property will be called the *minimality* of the crossing pair.

From (4.19), (4.22), (4.8) and the bound (5.8) we have

$$|C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \leq (C\lambda)^{2N} t^{2|\underline{m}|} \sup_{\underline{\sigma}} \int \mathrm{d}\nu^*_\pi(p_n, \tilde{p}_n, \underline{k}, \underline{\tilde{k}}) \int_{-\infty}^{\infty} \mathrm{d}\alpha \prod_{j=0}^{n} |R_j| \int_{-\infty}^{\infty} \mathrm{d}\tilde{\alpha} \prod_{j=0}^{n} |\widetilde{R}_j| \,. \tag{6.2}$$

Here we considered $p_j, \tilde{p}_j$ as functions of $p_n = \tilde{p}_n$ and $\underline{k}$ or $\underline{\tilde{k}}$, respectively, according to (4.24) and recall the definition of $R_j, \widetilde{R}_j$ from (4.23).

We need the following estimate to take care of the $\alpha, \tilde{\alpha}$ integrations. For any fixed index ℓ

$$\langle p_\ell \rangle \leq C^n \left(\prod_{j=1}^{n} \langle k_j \rangle \right) \langle p_n \rangle \tag{6.3}$$

and the same estimate is true for $\langle \tilde{p}_\ell \rangle$. This easily follows from $\langle a + b \rangle \leq C \langle a \rangle \langle b \rangle$.

We also define

$$L(k) := \langle k \rangle^{12} M^*(k) \tag{6.4}$$

and we use that

$$L(k) \leq C\langle k\rangle^{-4d-12} \tag{6.5}$$

by (4.6).

Let $A := \{a-1, a, n\}$ and $\widetilde{A} := \{\pi(b)-1, \pi(b), n\}$. Notice that these are sets of three distinct elements. We then perform a Schwarz estimate in (6.2) to obtain

$$\begin{aligned} |C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \;\; &\leq \;\; (C\lambda)^{2N} t^{2|\underline{m}|} \sup_{\underline{\sigma}} \int \mathrm{d}\nu_\pi^*(p_n, \tilde{p}_n, \underline{k}, \underline{\tilde{k}}) \int_{-\infty}^{\infty} \mathrm{d}\alpha \mathrm{d}\tilde{\alpha} \\ &\times \Big[t^{-1/2} \prod_{j\notin \widetilde{A}} |\widetilde{R}_j|^2 + t^{1/2} \prod_{j\notin A} |R_j|^2 \Big] \prod_{j\in A} |R_j| \prod_{j\in\widetilde{A}} |\widetilde{R}_j| =: (I.) + (II.)\;, \end{aligned} \tag{6.6}$$

where the decomposition $(I.)+(II.)$ is according to the summation in the big square bracket.

To estimate the term $(I.)$, we use $\tilde{p}_n$ and $\tilde{k}_j$, $j = 1, \ldots, n$, as variables and we simply estimate the terms R_a, R_{a-1} by Ct each. Moreover, we can gain an extra factor $\langle\alpha\rangle$ at the expense of $\langle p_n\rangle^2 \prod_{j=1}^n \langle k_j\rangle^4$. Using (6.3) and (1.16):

$$|R_a| \leq \frac{Ct}{\langle \alpha - e(p_a) - \Omega_a(k, \underline{\sigma})\rangle} \leq Ct \frac{\langle p_a\rangle^2 \langle \Omega_a(k,\underline{\sigma})\rangle}{\langle\alpha\rangle} \leq C^n t \frac{\langle p_n\rangle^2}{\langle\alpha\rangle} \prod_{j=1}^n \langle k_j\rangle^4\;. \tag{6.7}$$

We also need the estimate

$$\prod_{j=1}^n \langle \tilde{k}_j\rangle^{-2} \leq C^n \frac{\langle\tilde{p}_n\rangle^2}{\langle \tilde{p}_{\pi(b)}\rangle^2} \tag{6.8}$$

to insert a decay in $\tilde{p}_{\pi(b)}$ and we also insert an explicit extra $\prod_j \langle\tilde{k}_j\rangle^{-2}$ decay. Finally we estimate $M^*(k)$ by $L(k)\langle k\rangle^{-8}$, which makes up for the $\prod_j \langle k_j\rangle^8$ factors used in these estimates.

Then we can integrate out all k_j and p_n freely, set $p_n = \tilde{p}_n$ in R_n and we are left with

$$\begin{aligned} (I.) \;\; &\leq \;\; t^{-1/2} t^2 (C\lambda)^{2N} t^{2|\underline{m}|} \sup_{\underline{\sigma}} \int \Big(\prod_{j=1}^n L(\tilde{k}_j)\mathrm{d}\tilde{k}_j \Big) \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha\mathrm{d}\tilde{\alpha}}{\langle\alpha\rangle} \int \mathrm{d}\tilde{p}_n \langle\tilde{p}_n\rangle^4 \widehat{\gamma}_e(\tilde{p}_n, \tilde{p}_n) \\ &\times |R_n|\, |\widetilde{R}_n|\, |\widetilde{R}_{\pi(b)-1}|\, |\widetilde{R}_{\pi(b)}| \Big(\prod_{j\notin\widetilde{A}} |\widetilde{R}_j|^2 \Big) \frac{1}{\langle\tilde{p}_{\pi(b)}\rangle^2} \prod_{j=1}^n \frac{1}{\langle\tilde{k}_j\rangle^2}\;. \end{aligned} \tag{6.9}$$

Since $\widetilde{R}_j$ depends on $\tilde{k}_{j+1}, \tilde{k}_{j+2}, \ldots, \tilde{k}_n$ and $\tilde{p}_n$, the variables $\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_{n-1}, \tilde{k}_n$ can be integrated out successively in this order. The integrals of $\tilde{k}_j$ with $j \neq \pi(b), \pi(b)+1$ successively eliminate the factors $\widetilde{R}_{j-1}$ that depend on $\tilde{p}_{j-1} = \tilde{p}_j + \tilde{k}_j$, and they each give Ct by

$$\sup_{p,\theta} \int \frac{L(k)\mathrm{d}k}{\big|\theta - \Phi_\pm(p,k) + i\eta\big|^2} \leq Ct \tag{6.10}$$

(see (5.17)) with $k = \tilde{k}_j$, $p = \tilde{p}_j$ and $\theta = \tilde{\alpha} - \widetilde{\Omega}_j$, where we recall (4.46) that

$$\widetilde{\Omega}_j = \sum_{m=j+1}^{n} \sigma_{\pi^{-1}(m)} \omega(\tilde{k}_m)$$

and that $\tilde{p}_j = \tilde{p}_n + \sum_{m=j+1}^{n} \tilde{k}_m$. We recall that the function $L(k)$ also satisfies (5.19).

The integrals of $\tilde{k}_{\pi(b)}$ and $\tilde{k}_{\pi(b)+1}$ eliminate the factors $\widetilde{R}_{\pi(b)-1}$ and $\widetilde{R}_{\pi(b)}$ and they each give $C \log^* t$ by the estimate

$$\sup_{p,\theta} \int \frac{L(k)\mathrm{d}k}{\big|\theta - \Phi_{\pm}(p,k) + i\eta\big|} \leq C \log^* t\ . \tag{6.11}$$

Moreover, from the $\tilde{k}_{\pi(b)}$ integral we gain an additional $\langle\tilde{\alpha}\rangle$:

$$\int \frac{L(\tilde{k}_{\pi(b)})\mathrm{d}\tilde{k}_{\pi(b)}}{\big|\tilde{\alpha} - e(\tilde{p}_{\pi(b)} + \tilde{k}_{\pi(b)}) + i\eta \pm \omega(\tilde{k}_{\pi(b)}) - \widetilde{\Omega}_{\pi(b)}\big| \, \langle\tilde{p}_{\pi(b)}\rangle^2} \leq \frac{C^n (\log^* t) \prod_j \langle k_j\rangle^2}{\langle\tilde{\alpha}\rangle}\ .$$

The prototype of this inequality is

$$\sup_{p} \int \frac{L(k)\mathrm{d}k}{\big|\theta - \Phi_{\pm}(p,k) + i\eta\big| \, \langle p\rangle^2} \leq \frac{C \log^* t}{\langle\theta\rangle} \tag{6.12}$$

with $\theta = \tilde{\alpha} - \widetilde{\Omega}_{\pi(b)}$. The proofs of (6.11) and (6.12) follow the same route as (5.17) and we omit them. To gain $\langle\tilde{\alpha}\rangle$, we use (1.16)

$$\frac{1}{\langle\tilde{\alpha} - \widetilde{\Omega}\rangle} \leq \frac{C\langle\widetilde{\Omega}\rangle}{\langle\tilde{\alpha}\rangle} \leq \frac{C^n \prod_j \langle k_j\rangle^2}{\langle\tilde{\alpha}\rangle}\ . \tag{6.13}$$

We insert these estimates into (6.9); we have collected $(Ct)^{n-2}(C\log^* t)^2$ so far and we also gained $\langle\tilde{\alpha}\rangle$.

Finally we integrate out $\alpha, \tilde{\alpha}$ and $\tilde{p}_n$: we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha\mathrm{d}\tilde{\alpha}}{\langle\alpha\rangle\langle\tilde{\alpha}\rangle} \int \frac{\widehat{\gamma}_e(\tilde{p}_n,\tilde{p}_n)\langle\tilde{p}_n\rangle^4 \mathrm{d}\tilde{p}_n}{|\alpha - e(\tilde{p}_n) + i\eta| \, |\tilde{\alpha} - e(\tilde{p}_n) + i\eta|} \\ \leq \ (C\log^* t)^2 \int \widehat{\gamma}_e(\tilde{p}_n,\tilde{p}_n)\langle\tilde{p}_n\rangle^4 \mathrm{d}\tilde{p}_n \leq (C\log^* t)^2 \end{aligned} \tag{6.14}$$

using (1.26). Altogether the first term in (6.6) gives

$$(I.) \leq t^{-1/2}(C\lambda^2 t)^N (\log^* t)^4.$$

We now estimate the second term $(II.)$ in (6.6) after integrating out $\tilde{k}_j$'s and $\tilde{p}_n$

$$(II.) \quad = \quad t^{1/2}(C\lambda^2)^N t^{2|\underline{m}|} \sup_{\underline{\sigma}} \int \Big(\prod_{j=1}^n M^*(k_j)\mathrm{d}k_j \Big) \mathrm{d}p_n \widehat{\gamma}_e(p_n, p_n) \qquad (6.15)$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}\alpha \mathrm{d}\tilde{\alpha} \prod_{j \notin A} |R_j|^2 \prod_{j \in A} |R_j| \prod_{j \in \widetilde{A}} |\widetilde{R}_j| \; .$$

Recall that R_j depends on p_j's that are functions of the variables k_j and p_n. We express everything in terms of k_j, p_n variables by (4.24). Similarly,

$$\tilde{p}_j = p_n + \sum_{m \,:\, \pi(m) > j} k_m$$

by the pairing $k_j = \tilde{k}_{\pi(j)}$. In particular we have

$$p_{a-1} = p_a + k_a = v + k_a + k_b \qquad \text{with} \qquad v := p_n + \sum_{\substack{j \geq a+1 \\ j \neq b}} k_j \; ,$$
$$\tilde{p}_{\pi(b)} = u + k_a \qquad \text{with} \qquad u := p_n + \sum_{\substack{j \,:\, \pi(j) > \pi(b) \\ j \neq a}} k_j \; ,$$
$$\widetilde{\Omega}_{\pi(b)} = \widetilde{\Omega}^* + \sigma_a \omega(k_a) \qquad \text{with} \qquad \widetilde{\Omega}^* := \sum_{\substack{j \,:\, \pi(j) > \pi(b) \\ j \neq a}} \sigma_j \omega(k_j) \; .$$

Notice that v, u and $\widetilde{\Omega}^*$ are independent of $k_1, \ldots, k_a$ and k_b. This is clear for v, and it follows for u and $\widetilde{\Omega}^*$ from the minimality of (a, b).

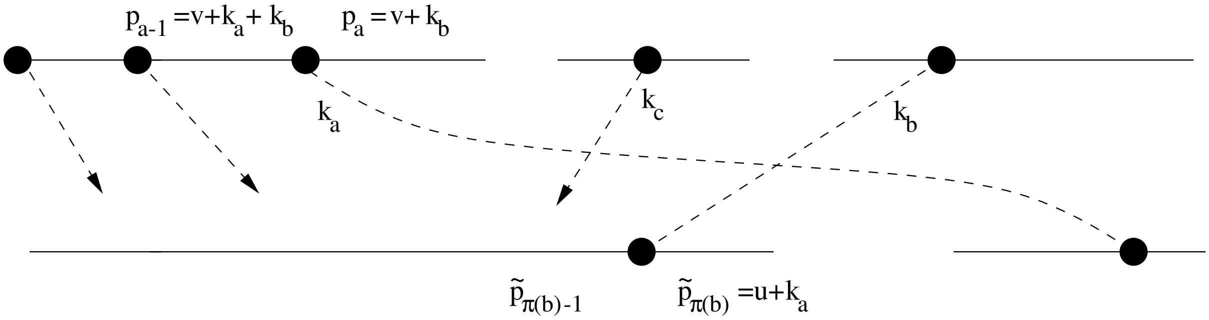

Fig.6. Basic crossing estimate.

Now we start estimating (6.15). First we estimate $\widetilde{R}_{\pi(b)-1}$ by Ct and we also gain a factor $\langle \tilde{\alpha} \rangle^{-1}$ using

$$|\widetilde{R}_{\pi(b)-1}| = \frac{1}{\left| \tilde{\alpha} - e(\tilde{p}_{\pi(b)-1}) + i\eta - \widetilde{\Omega}_{\pi(b)-1} \right|} \leq C^n t \frac{\langle p_n \rangle^2}{\langle \tilde{\alpha} \rangle} \prod_{j=1}^n \langle k_j \rangle^4 \qquad (6.16)$$

similarly to (6.7). We also need

$$\prod_{j=1}^{n} \langle k_j \rangle^{-2} \leq C^n \frac{\langle p_n \rangle^2}{\langle p_a \rangle^2}$$

analogously to (6.8). Hence we have

$$\begin{aligned} (II.) \;\; \leq \;\; & t^{3/2} (C\lambda^2)^N t^{2|\underline{m}|} \sup_{\underline{\sigma}} \int \Big(\prod_{j=1}^{n} L(k_j) \mathrm{d}k_j \Big) \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha \mathrm{d}\tilde{\alpha}}{\langle \tilde{\alpha} \rangle} && (6.17) \\ & \times \int \mathrm{d}p_n \langle p_n \rangle^4 \widehat{\gamma}_e(p_n, p_n) |\widetilde{R}_{\pi(b)}| \, |\widetilde{R}_n| \Bigg(\prod_{j \notin A} |R_j|^2 \prod_{j \in A} |R_j| \Bigg) \frac{1}{\langle p_a \rangle^2} \prod_{j=1}^{n} \frac{1}{\langle k_j \rangle^2} \, . \end{aligned}$$

We then integrate out $k_1, \ldots, k_{a-1}$ using (6.10) and collecting $(Ct)^{a-1}$. This eliminates $R_0, R_1, \ldots, R_{a-2}$. Notice that $\widetilde{R}_{\pi(b)}$ is independent of these variables.

Next we integrate out k_a. This occurs only in R_{a-1} and $\widetilde{R}_{\pi(b)}$

$$\int |R_{a-1}| \, |\widetilde{R}_{\pi(b)}| \langle p_a \rangle^{-2} L(k_a) \mathrm{d}k_a \leq \frac{(C \log^* t)^2}{|p_a - u| \langle \alpha - \Omega_a \rangle} \, . \tag{6.18}$$

Here we used the inequality

$$\sup_{\tilde{\theta}} \int \frac{L(k) \mathrm{d}k}{\Big|\theta - \Phi_\pm(p,k) + i\eta\Big| \, \Big|\tilde{\theta} - \Phi_\pm(u,k) + i\eta\Big| \langle p \rangle^2} \leq \frac{(C \log^* t)^2}{|p-u| \langle \theta \rangle} \tag{6.19}$$

with $k = k_a$, $p = p_a$, $\theta = \alpha - \Omega_a$, $\tilde{\theta} = \tilde{\alpha} - \widetilde{\Omega}^*$.

Proof of (6.19). We use the transversality condition (1.21) and a similar resolution of singularities as in the proof of (5.17). Again, we consider a tiling by cubes of size $\sim \widetilde{\varrho}$. We fix a cube Q and we recall the definition of the sets $S_\ell(p)$ (5.18). On the set $S_\ell(p) \cap \widetilde{S}_{\widetilde{\ell}}(u)$ the integrand is bounded by $2^{\ell + \widetilde{\ell}} C \langle \mathrm{dist}(Q,0) \rangle^{-4d-12} \langle p \rangle^{-2}$, the volume is bounded by $2^{-(\ell+\widetilde{\ell})} C |p - u|^{-1}$. To ensure the decay in θ for $k \in S_\ell(p)$, $\ell \geq 1$, we use $\langle p \rangle^{-2} \leq \langle \mathrm{dist}(Q,0) \rangle^2 \langle \theta \rangle^{-1}$ by (1.15) and (1.16). On $S_0(p)$ we have

$$\frac{1}{\Big|\theta - \Phi_\pm(p,k) + i\eta\Big| \langle p \rangle^2} \leq \frac{\langle \mathrm{dist}(Q,0) \rangle^2}{\langle \theta \rangle} \, .$$

After summing these bounds for $\ell, \widetilde{\ell} = 0, \ldots, [\log^* t]$, then summing up the result for all Q, we obtain (6.19). $\square$

Continuing (6.18), from $\langle \theta \rangle^{-1} = \langle \alpha - \Omega_a \rangle^{-1}$ we also gain a factor $\langle \alpha \rangle^{-1}$ similarly to (6.13) using the decaying factor $\prod_j \langle k_j \rangle^{-2}$ in (6.17). Hence we have

$$\begin{aligned} (II.) \;\; \leq \;\; & t^{3/2} (Ct)^{a-1} (C \log^* t)^2 (C\lambda^2)^N t^{2|\underline{m}|} \sup_{\underline{\sigma}} \int \Big(\prod_{j=a+1}^{n} L(k_j) \mathrm{d}k_j \Big) && (6.20) \\ & \times \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha \mathrm{d}\tilde{\alpha}}{\langle \alpha \rangle \langle \tilde{\alpha} \rangle} \int \mathrm{d}p_n \langle p_n \rangle^4 \widehat{\gamma}_e(p_n, p_n) \, |R_a| \, |R_n| \, |\widetilde{R}_n| \, \Bigg(\prod_{j=a+1}^{n-1} |R_j|^2 \Bigg) \frac{1}{|p_a - u|} \, . \end{aligned}$$

Let $c \geq a+1$ be the smallest index such that $\pi(c) \leq \pi(b)$ (see Fig. 6). Such index exists, if not else then $c = b$. By the definition of v, u, we have

$$p_a - u = v + k_b - u =: k_c + w$$

where w depends only on $k_{c+1}, \ldots, k_n, p_n$ by the minimality of c. Simply k_c is the momentum with the smallest index which appears in the difference

$$v - u = \sum_{\substack{j \geq a+1 \\ j \neq b}} k_j - \sum_{\substack{j \,:\, \pi(j) > \pi(b) \\ j \neq a}} k_j,$$

i.e., the first momentum which appears in v but not in u; or if the smallest such index is bigger than b, then $c := b$.

Now we can integrate out k_{a+1} in (6.20) to eliminate $|R_a|$. This gives only a $C \log^* t$ factor by (6.11) if $c > a+1$, and by

$$\int \frac{L(k_{a+1}) \mathrm{d}k_{a+1}}{\Big|\alpha - e(p_{a+1} + k_{a+1}) + i\eta \pm \omega(k_{a+1}) - \Omega_{a+1}\Big| \, |k_{a+1} + w|} \leq C \log^* t$$

if $c = a+1$. The prototype of this inequality is

$$\sup_{p,w,\theta} \int \frac{L(k) \mathrm{d}k}{\Big|\theta - \Phi_{\pm}(p,k) + i\eta\Big| \, |k + w|} \leq C \log^* t \tag{6.21}$$

and it is proven similarly as (5.17). Here the point singularity around w also has to be resolved on an exponential scale; we find that within each cube Q the integrand is bounded by $2^{\ell + \widetilde{\ell}}$ except for a set of measure $\sim 2^{-\ell - (d-1)\widetilde{\ell}}$ using (1.20). The details are omitted.

If $c = a+1$, then all the remaining factors $|R_j|^2$, $j = a+1, a+2, \ldots, n-1$, can be integrated out successively. The order of the integration is $\mathrm{d}k_{a+2}, \ldots, \mathrm{d}k_n$. This gives $(Ct)^{n-a-1}$, which makes altogether a t-power $N - 1/2$, taking the $t^{3/2}(Ct)^{a-1} t^{2|\underline{m}|}$ prefactor into account and recalling that $n + 2|\underline{m}| = N$.

If $c > a+1$, then we integrate out $k_{a+2} \ldots, k_{c-1}$ in this order (it is void if $c = a+2$). These integrations eliminate $|R_j|^2$, $j = a+1, \ldots, c-2$, they do not affect the denominator $|p_a - u| = |k_c + w|$, and they give $(Ct)^{c-a-2}$ by (6.10). Now we perform the k_c integral and eliminate R_{c-1}:

$$\int \frac{L(k_c) \mathrm{d}k_c}{\Big|\alpha - e(p_c + k_c) + i\eta \pm \omega(k_c) - \Omega_c\Big|^2 \, |k_c + w|} \leq Ct$$

uniformly in w by

$$\sup_{p,w,\theta} \int \frac{L(k)\mathrm{d}k}{\left|\theta - \Phi_\pm(p,k) + i\eta\right|^2 \, |k+w|} \leq Ct \, . \tag{6.22}$$

This inequality is obtained exactly as (6.21).

Then we do the $k_{c+1}, \ldots, k_n$ integrations using (6.10), eliminating all $|R_j|^2$ factors in (6.20). This also collects

$$t^{3/2}(Ct)^{a-1}(C\log^* t)^3 (Ct)^{c-a-2} Ct (Ct)^{n-c} t^{2|\underline{m}|} = (Ct)^{N-1/2}(\log^* t)^3 \, .$$

Finally, we finish the estimate of (6.20) with the $\alpha, \tilde{\alpha}, p_n$ integrations as in (6.14). This completes the proof of Lemma 6.2. $\square$

7 Fully expanded terms with big n and no recollision

Here we prove (3.30). The basic idea is that we consider a few disjoint crossing pairs (see Definition 6.1), which are well ordered (to make successive integration possible) and we plan to gain t^{-1} from each.

The key is the right definition of "well-ordered" crossing pairs. The main difficulty lies in the fact that it is not true that "more crossings" give better bound. In particular, the contribution of the fully crossing pairing, $\pi(j) = (n+1) - j$, is of order t^{-d+1} independently on the number of crossing pairs.

So we need a more refined definition, which actually excludes the full crossing, and lets crossing pairs alternate with noncrossing ones.

Definition 7.1 *Fix a pairing $\pi \in \Pi_n$. A number a is called* **peak** *if $\pi(a-1) < \pi(a) > \pi(a+1)$ (Fig. 7) and it is called* **valley** *if $\pi(a-1) > \pi(a) < \pi(a+1)$. The endpoints $a = 1$ and $a = n$ are not considered peak or a valley. The valleys and peaks clearly alternate. We define a complete ordering on the set of peaks according to the value of their height $\pi(a)$.*

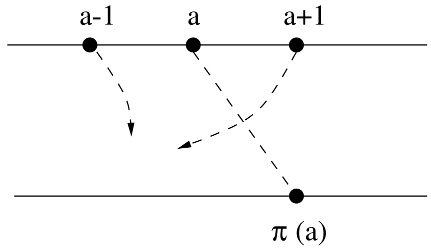

Fig 7. A peak

The number of peaks measures the complexity of the pairing. The proof of the following lemma is given in the Appendix B.

Lemma 7.2 *The number of pairings in* Π_n *with no more than* K *peaks is at most* $n^{4K+3}(2K+2)^n$.

We want to gain a factor t^{-1} from each peak. The idea is that the factor $\widetilde{R}_{\pi(a)-1}$ (together with R_{a-1}, R_a) will be eliminated by integrating out p_{a-1}, p_a at the expense of only $\log^* t$ factors instead of the trivial estimate $|\widetilde{R}_{\pi(a)-1}| \leq t$. In order to estimate this integral, we make sure that only these three denominators depend on p_{a-1}, p_a since we cannot estimate integrals with many denominators. This requires a certain combinatorial structure of the peaks.

There is a difference whether ω is constant or not. The $\omega = (const.)$ case is simpler to integrate out and we will be able to gain a factor t^{-1} from each element of a *monotonic* peak sequence.

In the $\omega \neq (const.)$ case the propagators $R_j, \widetilde{R}_j$ have more complicated momentum dependence (see Appendix A.2 for more details), hence we will need a stronger combinatorial structure for the peaks to be able to perform the integrations. The appropriate structure (*"monotonic staircase"*) is introduced in the Appendix A.1.

We explain the proof of the $\omega = (const.)$ case in details. The necessary modifications for the nonconstant case are sketched in the Appendix A.1, A.2.

7.1 Combinatorics for the case of the constant ω

The following theorem is a well known Ramsey-type theorem in combinatorics:

Lemma 7.3 *Any sequence of different numbers of $\alpha\beta + 1$ elements either contains an increasing subsequence of $(\alpha + 1)$ elements or it contains a decreasing subsequence of $(\beta + 1)$ elements.* $\square$

We will prove the following estimate.

Proposition 7.4 *Let $n \le N$, $\underline{m}, \underline{\widetilde{m}} \in \mathcal{M}(n, N)$. Suppose that ω is constant and that $\pi \in \Pi_n$ has either an increasing sequence of κ peaks or a decreasing sequence of κ peaks. Then*

$$|C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \le t^{-\kappa}(C\lambda^2 t)^N (\log^* t)^{n+\kappa+2} .$$

From these statements (3.30) follows for the ω constant case. Choose $\kappa = 6$, then all pairings which have a monotonic sequence of at least 6 peaks can be included into the second term in (3.30). Choosing $\alpha = \beta = 5$ and $K = 5 \cdot 5 + 1 = 26$, it is clear from Lemmas 7.2 and 7.3 that with the exception of at most $54^n n^{4\cdot 26+3} \le C^n$ pairings, we are in the situation of Proposition 7.4. The exceptional pairings are estimated by the apriori bound (5.1) to give the first term in (3.30).

Remark: In general, we get $t^{-\kappa}$ with the exception of $(C\kappa)^{2n}$ pairings.

7.2 Estimate for the constant ω case.

Proof of Proposition 7.4. Let $a_1 < a_2 < \ldots < a_\kappa$ be the locations of the monotonic peak-sequence. We start with the expression (6.2). We estimate all the $\widetilde{R}_j$'s trivially by Ct except $\widetilde{R}_{\pi(a_m)-1}$, with $m = 1, 2, \ldots, \kappa$, and except $\widetilde{R}_n$. This gives a factor $(Ct)^{n-\kappa}$. In fact, with the help of (6.16), we gain a $\langle\tilde{\alpha}\rangle^{-1}$ factor from one of these estimates, at the expense of collecting $\langle p_n\rangle^2 \prod_j \langle k_j\rangle^4$ factors. We also insert a factor

$$1 \le C^n \frac{1}{\langle p_{a_1}\rangle^2}\Big(\langle p_n\rangle^2 \prod_j \langle k_j\rangle^2\Big) , \tag{7.1}$$

this will help to secure a decay in $\langle\alpha\rangle$. All these factors will be incorporated into the integration measures as before at the expense of changing $M^*(k)$ to $L(k) = M^*(k)\langle k\rangle^{12}$.

Figs. 8–9 show the leftover $\widetilde{R}$'s represented by their electron momenta $\tilde{p}_{\pi(a_m)-1}$ (bold lines) in case of an increasing and in a decreasing sequence of peaks, respectively. We want to gain a factor of t from each bold line.

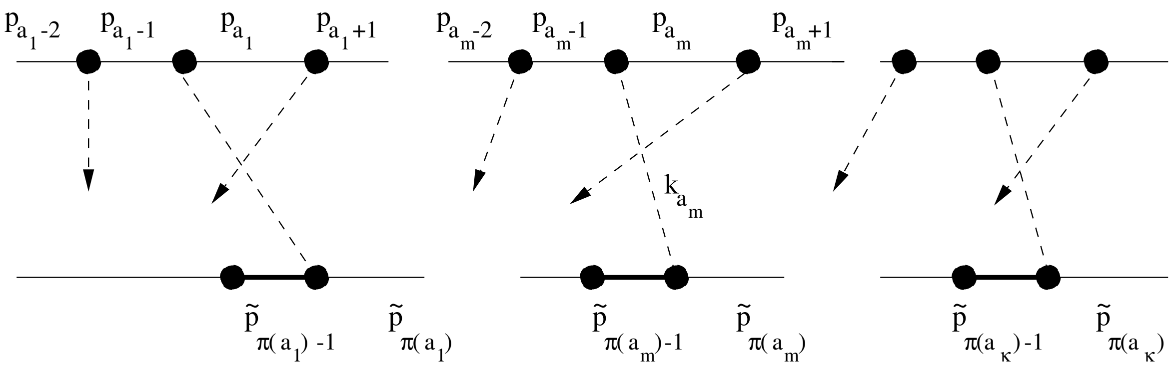

Fig. 8. Increasing sequence of peaks

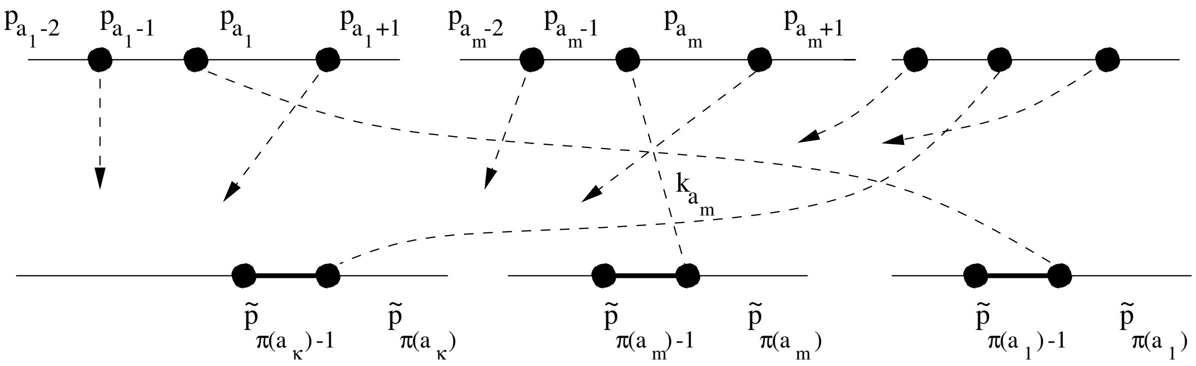

Fig. 9. Decreasing sequence of peaks

Now we express everything in terms of p_j, $j = 0, 1, \ldots, n$. Recall that

$$\widetilde{R}_{\pi(a_m)-1} = \frac{1}{\left|\tilde{\alpha} - e\Big(\tilde{p}_{\pi(a_m)} + p_{a_m-1} - p_{a_m}\Big) + i\eta - \widetilde{\Omega}_{\pi(a_m)-1}\right|}$$

where now

$$\widetilde{\Omega}_{\pi(a_m)-1} = \Big(\sum_{j\,:\,\pi(j)\geq\pi(a_m)} \sigma_j\Big)\omega \ ,$$

i.e., it is independent of the electron momenta. Similar formula is valid for Ω_{a_m}.

The important thing is that $\tilde{p}_{\pi(a_m)}$ is independent of p_{a_j-1} and p_{a_j} for all $j \leq m$ in the case of monotonically increasing peaks, and it is independent of p_{a_j-1} and p_{a_j} for all $j \geq m$ in the case of monotonically decreasing peaks. This can be seen from the expression

$$\tilde{p}_{\pi(a_m)} = p_n + \sum_{j\,:\,\pi(j)>\pi(a_m)} (p_{j-1} - p_j)$$

and from the structure of the peaks.

Hence we can integrate out p_{a_m} and p_{a_m-1}, $m = 1, 2, \ldots, \kappa$, variables in the following order

$$p_{a_1-1}, p_{a_1}, p_{a_2-1}, p_{a_2}, \ldots, p_{a_\kappa-1}, p_{a_\kappa},$$

if we have monotonically increasing peaks; and in the order

$$p_{a_\kappa-1}, p_{a_\kappa}, p_{a_{\kappa-1}-1}, p_{a_{\kappa-1}}, \ldots, p_{a_2-1}, p_{a_2}, p_{a_1-1}, p_{a_1} \tag{7.2}$$

if we have monotonically decreasing peaks. Notice that the $\mathrm{d}p_{a_m}$ and $\mathrm{d}p_{a_m-1}$ integrations involve only the factors R_{a_m-1}, R_{a_m} and $\widetilde{R}_{\pi(a_m)-1}$ assuming that all other p_{a_j-1}, p_{a_j} have already been integrated out for $j < m$ in case of increasing peaks, and for $j > m$ in case of decreasing peaks.

The result of each integration is a $C \log^* t$ factor by using the following inequality:

$$\begin{aligned} \int \frac{L(p_{a_m-2} - p_{a_m-1})L(p_{a_m-1} - p_{a_m})L(p_{a_m} - p_{a_m+1})\mathrm{d}p_{a_m-1}\mathrm{d}p_{a_m}}{\langle p_{a_m}\rangle^2 \big|\alpha - e(p_{a_m-1}) + i\eta - \Omega_{a_m-1}\big| \, \big|\alpha - e(p_{a_m}) + i\eta - \Omega_{a_m}\big|} \\ \times \frac{1}{\Big|\tilde{\alpha} - e\Big(\tilde{p}_{\pi(a_m)} + p_{a_m-1} - p_{a_m}\Big) + i\eta - \widetilde{\Omega}_{\pi(a_m)-1}\Big|} \;\; \le \;\; \frac{(C \log^* t)^3 \langle n\omega\rangle}{\langle\alpha\rangle\langle p_{a_m-2} - p_{a_m+1}\rangle^{d+1}} \, . \end{aligned} \tag{7.3}$$

The factor $\langle p_{a_m}\rangle^{-2}$ on the left hand side and the factor $\langle\alpha\rangle^{-1}$ on the right hand side are present only for $m = 1$. The prototype of (7.3) is the following estimate

$$\sup_{p,q,r,\theta_1,\tilde{\theta}} \int \frac{\langle p-u\rangle^{-d-1}\langle u-v\rangle^{-d-1}\langle v-q\rangle^{-d-1}\mathrm{d}u\mathrm{d}v}{\langle v\rangle^2 \big|\theta_1 - e(u) + i\eta\big| \, \big|\theta_2 - e(v) + i\eta\big| \, \big|\tilde{\theta} - e(r+u-v) + i\eta\big|} \le \frac{(C\log^* t)^3}{\langle\theta_2\rangle} \tag{7.4}$$

with the choice $p = p_{a_m-2}$, $u = p_{a_m-1}$, $v = p_{a_m}$, $q = p_{a_m+1}$ and $r = \tilde{p}_{\pi(a_m)}$. Again, if there is no factor $\langle v\rangle^{-2}$ on the left, then there is no factor $\langle\theta_2\rangle^{-1}$ on the right. Since $\theta_2 = \alpha - \Omega_{a_m}$, we can estimate

$$\frac{1}{\langle\theta_2\rangle} \le \frac{C\langle n\omega\rangle}{\langle\alpha\rangle} \, .$$

We used that $r = \tilde{p}_{\pi(a_m)}$ is independent of $u = p_{a_m-1}$ and $v = p_{a_m}$ so that we could take the supremum over $r = \tilde{p}_{\pi(a_m)}$ outside of the integration. We also kept a momentum decay relation from the three L-factors using that $L(k) \le C\langle k\rangle^{-d-1}\langle k\rangle^{-d-1}$ and

$$\frac{1}{\langle p_{a_m-2} - p_{a_m-1}\rangle\langle p_{a_m-1} - p_{a_m}\rangle\langle p_{a_m} - p_{a_m+1}\rangle} \le \frac{C}{\langle p_{a_m-2} - p_{a_m+1}\rangle} \, .$$

Only part of the decay in $L(k)$ was needed in (7.4).

For the proof of (7.4) we use (6.19) to perform the $\mathrm{d}v$ integration, then (6.21) for the $\mathrm{d}u$ integration.

After having eliminated all remaining $\widetilde{R}_j$ factors, the rest can be done by successively integrating out

$$p_0, p_1, \ldots, p_{a_1-2}, \widehat{p}_{a_1-1}, \widehat{p}_{a_1}, p_{a_1+1} \ldots, p_{a_2-2}, \widehat{p}_{a_2-1}, \widehat{p}_{a_2}, p_{a_2+1} \ldots, p_{n-1}$$

in this order (hat denotes missing variable). Notice that we saved a successive momentum decay for the remaining variables so we can use (see (6.11))

$$\sup_{c,q} \int \frac{\langle p-q\rangle^{-d-1} \mathrm{d}p}{|c-e(p)+i\eta|} \leq C \log^* t$$

with $c = \alpha - \Omega_{a_j}$, $p = p_j$, $q = p_{j+1}$ for all $j \notin \{a_m-2, a_m-1, a_m \ : \ m = 1, 2, \ldots, \kappa\}$, and simply q becomes p_{j+3} for $j = a_m - 2$. We collect $(C \log^* t)^{n-2\kappa}$.

At the end we are left with an integration identical to (6.14). This completes the proof of Proposition 7.4. $\square$

8 Amputated term without reabsorption

Here we indicate the proof of (3.31) which is very similar to that of (3.30). We explain the necessary modifications in the corresponding formulas leading to (3.30). One has to check only Sections 4.3, 5 and 7.

Our starting point is Proposition 4.3. Almost the same representation is valid for the amputated term $\mathrm{Tr}\mathcal{D}^0_{n,N}\Gamma_0\big[\mathcal{D}^0_{n,N}\big]^*$ except that in the set $\mathcal{M}(n, M)$ we additionally require $m_0 = 0$, $\mu(1) = 1$, i.e., $I_0 = \emptyset$ and we also set $I_0^c := \emptyset$. In particular the index 0 is not part of the set $J \cup J^c$ and $|J^c| := n + |\underline{m}|$ is reduced by one. This expresses the fact that there is no propagator $e^{is_0 e(p_0)}$ associated with the momentum p_0. The definition of $Y_{\underline{m},\underline{\widetilde{m}},\pi}$ (4.21), (4.22) is changed slightly: the products over j start from $j = 1$ and there is no $s_0, \tilde{s}_0$.

The additional gain $1/t$ is due to the fact that the factors R_0 and $\widetilde{R}_0$ are missing and that the set J^c has smaller cardinality.

In the proof of Lemma 5.1 one can easily see that there is one less factor of t^a in (5.5) due to the decreased size of $|J^c|$ in (5.4). The other t^{1-a} gain comes from the fact that the last integrand in (5.15) does not have a denominator, so the $\mathrm{d}k_1$ integration is bounded by a constant.

In the proof of Propositions 7.4 the gain comes from the fact that $\widetilde{R}_0$ is missing hence it does not have to be estimated by Ct at the very beginning. Notice that $\widetilde{R}_0$ has indeed been estimated by Ct since $\pi(a_m) \geq 2$ for all peaks. For nonconstant ω (see Appendix A) the argument is similar: in the proof of Proposition A.6 the first factor $\widetilde{S}_0$ is missing in (A.4). If $\pi(a_1) = 1$ and there are no $\widetilde{S}_j$ factors and $\tilde{\beta}$ integration, then we gain from the missing $\widetilde{R}_0$ term. This completes the proof of (3.31). $\square$

9 One reabsorption

In this section we prove (3.32). Recall that the measure $\int^{*(n,N)} \prod \mathrm{d}k_j$ in the definition of $\mathcal{D}^1_{n,N}$ (see (2.10) and (2.13)) contains a double summation over measures $\int^{*(\underline{m},a)} \prod \mathrm{d}k_j$, where $\underline{m} \in \mathcal{M}_a(n,N)$. Hence we can write

$$\mathcal{D}^1_{n,N}(t) = \sum_{a=2}^{N} \sum_{\underline{m}\in\mathcal{M}_a(n,N)} \mathcal{D}^{\underline{m},a}_{n,N}(t)$$

with

$$\mathcal{D}^{\underline{m},a}_{n,N}(t) = \int^{*(\underline{m},a)} \Big(\prod_{j=1}^{N} \mathrm{d}k_j\Big)\mathcal{A}(t,\underline{k},N) \; . \tag{9.1}$$

After a Schwarz inequality we can symmetrize

$$\mathrm{Tr}_{e+ph}\left(\mathcal{D}^1_{n,N}\Gamma_0\Big[\mathcal{D}^1_{n,N}\Big]^*\right) \leq C^N N \sum_{a=2}^{N} \sum_{\underline{m}\in\mathcal{M}_a(n,N)} \mathrm{Tr}_{e+ph}\left(\mathcal{D}^{\underline{m},a}_{n,N}\Gamma_0\Big[\mathcal{D}^{\underline{m},a}_{n,N}\Big]^*\right)$$

using that the cardinality of $\mathcal{M}_a(n,N)$ is bounded by C^N.

Therefore it is sufficient to show that

$$\limsup_{L\to\infty} \mathrm{Tr}_{e+ph}\left(\mathcal{D}^{\underline{m},a}_{n,N}\Gamma_0\Big[\mathcal{D}^{\underline{m},a}_{n,N}\Big]^*\right) \leq \frac{1}{t}(C\lambda^2 t)^N\left[\frac{(\log^* t)^6}{t^2} + \frac{n!}{t^6}(\log^* t)^{n+10}\chi(N\geq 7)\right] \tag{9.2}$$

for any $2 \leq a \leq N$ and $\underline{m} \in \mathcal{M}_a(n,N)$. From now on we fix a and $\underline{m}$ and recall that $I_a = I \setminus \{1, \mu(a)\}$. Let $I_a^c := \{1,\ldots,N\} \setminus I_a$.

We separate the measure of the reabsorpting phonon momenta in (9.1):

$$\int^{*(\underline{m},a)} \mathrm{d}\underline{k} = \int^{\#(\underline{m},a)} \Big(\prod_{b\in I_a} \mathrm{d}k_b\Big) \int \mathrm{d}k_1 \mathrm{d}k_{\mu(a)}\Big(\prod_{b\in J} \mathrm{d}k_b \mathrm{d}k_{b+1}\Big) F_a([k]_{I_a},[k]_{I_a^c})$$

with

$$\int^{\#(\underline{m},a)} \Big(\prod_{b\in I_a} \mathrm{d}k_b\Big) := \int \Big(\prod_{b\in I_a} \mathrm{d}k_b\Big) \prod_{b\neq b'\in I_a} \Big(1-\chi(k_b+k_{b'})\Big) \tag{9.3}$$

and

$$F_a([k]_{I_a},[k]_{I_a^c}) := \chi(k_1+k_{\mu(a)})\Xi(\underline{k},\underline{m}) \prod_{b\notin I_a} \Big(1-\chi(k_b+k_1)\Big)\Big(1-\chi(k_b+k_{\mu(a)})\Big) \; ,$$

where in the notation we split the variables $\underline{k}=\{k_1,\ldots,k_N\}$ into two sets: $\{\underline{k}\}=[k]_{I_a}\cup[k]_{I_a^c}$ and for any $S\subset\{1,2,\ldots,N\}$ we let $[k]_S := \Big\{k_b \;:\; b\in S\Big\}$. We let

$$\mathcal{F}\Big([k]_{I_a},p_N\Big) := \int \mathrm{d}k_1 \mathrm{d}k_{\mu(a)} \Big(\prod_{b\in J} \mathrm{d}k_b \mathrm{d}k_{b+1}\Big) F_a([k]_{I_a},[k]_{I_a^c}) \mathcal{A}\Big(t,\underline{k},N;p_N+\sum_{j=1}^N k_j,p_N\Big) \qquad (9.4)$$

be an operator in the phonon space, parametrized by $[k]_{I_a}$ and p_N. Here $\mathcal{A}\Big(t,\underline{k},N;p_0,p_N\Big)$ is the kernel of the operator $\mathcal{A}(t,\underline{k},N)$ in Fourier space of $\mathcal{H}_e$ similarly to (4.29) and (4.30). Using the delta function $\delta\Big(p_0-p_N-\sum_{j=1}^N k_j\Big)$ we have

$$\mathcal{A}\Big(t,\underline{k},N;p_N+\sum_{j=1}^N k_j,p_N\Big) := \lambda^N \int \Big(\prod_{j=1}^{N-1}\mathrm{d}p_j\Big)\Delta^*(\underline{p},\underline{k}) \int_0^{t*} [\mathrm{d}s_j]_1^N \Big(\prod_{j=1}^N Q(k_j) b_{k_j} e^{-is_j[e(p_j)+H_{ph}]}\Big), \qquad (9.5)$$

which is an operator acting on $\mathcal{H}_{ph}$. Here we define Δ^* by the relation $\Delta(\underline{p},\underline{k}) = \Delta^*(\underline{p},\underline{k})\delta\Big(p_0-p_N-\sum_{j=1}^N k_j\Big)$ (see (4.27)).

The same formulas are valid for the other copy of $\mathcal{D}^{\underline{m},a}_{n,N}$ in (9.2); the variables are denoted by tilde. The result is

$$\begin{aligned}\mathrm{Tr}_{e+ph}\Big(\mathcal{D}^{\underline{m},a}_{n,N}\Gamma_0\Big[\mathcal{D}^{\underline{m},a}_{n,N}\Big]^*\Big) &= \int^{\#(\underline{m},a)}\Big(\prod_{b\in I_a}\mathrm{d}k_b\Big)\int^{\#(\underline{m},a)}\Big(\prod_{b\in I_a}\mathrm{d}\tilde{k}_b\Big)\int \mathrm{d}p_N \widehat{\gamma}_e(p_N,p_N) \\ &\quad\times \mathrm{Tr}_{ph}\mathcal{F}\Big([k]_{I_a},p_N\Big)\gamma_{ph}\Big[\mathcal{F}\Big([\tilde{k}]_{I_a},p_N\Big)\Big]^* . \qquad (9.6)\end{aligned}$$

We again used that the $\delta(p_0-\tilde{p}_0)$ coming from taking the electron trace can be replaced with $\delta(p_N-\tilde{p}_N)$ and we integrated out $\tilde{p}_N$.

Similarly to Section 4.5, we again notice that the phonon trace in (9.6) is zero unless there is a complete pairing between all the involved $2N$ momenta $\underline{k},\underline{\tilde{k}}$. The pairing must respect the prepared immediate reabsorptions and the reabsorptions between $k_1,k_{\mu(a)}$ and $\tilde{k}_1,\tilde{k}_{\mu(a)}$, apart from an error term that is negligible in the thermodynamic limit (Lemma 4.4). Moreover, there is no pairing between k_b and $k_{b'}$, $b,b'\in I_a$ (and the same for the variables with tilde) by the no-reabsorption condition built into the measure $\int^{\#(\underline{m},a)}\prod \mathrm{d}k_b$ (9.3). Hence the possible pairings are parametrized by a permutation $\pi\in\Pi_n^a$, where Π_n^a is the set of all permutations on the set $\{2,\ldots,\widehat{a},\ldots,n\}$. The map $\pi^* := \mu\circ\pi\circ\mu^{-1} : I_a\to I_a$ gives the pairing of the indices of those k's and $\tilde{k}$'s that are not prescribed for immediate recollisions similarly to Section 4.5. This means that a factor

$$\sum_{\pi\in\Pi_n^a}\Big(\prod_{\substack{j=2\\ j\neq a}}^{n}\chi(k_{\mu(j)}-\tilde{k}_{\mu(\pi(j))})\Big)$$

can be freely inserted into (9.6) modulo a negligible error. We obtain

$$\mathrm{Tr}_{e+ph}\Big(\mathcal{D}^{\underline{m},a}_{n,N}(t)\Gamma_0\Big[\mathcal{D}^{\underline{m},a}_{n,N}(t)\Big]^*\Big)=\sum_{\pi\in\Pi_n^a}C^{\underline{m},a}_{n,N}(\pi;t)+O(|\Lambda|^{-1}) \tag{9.7}$$

with

$$\begin{aligned} C^{\underline{m},a}_{n,N}(\pi;t):\; = \;& \int^{\#(\underline{m},a)}\Big(\prod_{b\in I_a}\mathrm{d}k_b\Big)\int^{\#(\underline{m},a)}\Big(\prod_{b\in I_a}\mathrm{d}\tilde{k}_b\Big)\int \mathrm{d}p_N\widehat{\gamma}_e(p_N,p_N) \\ & \times\Big(\prod_{\substack{j=2\\ j\neq a}}^{n}\chi(k_{\mu(j)}-\tilde{k}_{\mu(\pi(j))})\Big)\;\mathrm{Tr}_{ph}\mathcal{F}\Big([k]_{I_a},p_N\Big)\gamma_{ph}\Big[\mathcal{F}\Big([\tilde{k}]_{I_a},p_N\Big)\Big]^*\,. \end{aligned} \tag{9.8}$$

Peaks and valleys of $\pi\in\Pi_n^a$ are defined exactly as before (Definition 7.1) and Lemma 7.2 remains valid if n is replaced with $n-2$. The estimate (9.2) follows from Lemma 7.2 and the lemma below.

Lemma 9.1 *Let $N\geq 3$, $a\geq 2$, $n\geq 0$ be integers and $\underline{m}\in\mathcal{M}_a(n,N)$ as before. For any $\pi\in\Pi_n^a$ we have*

$$C^{\underline{m},a}_{n,N}(\pi;t)\leq\frac{1}{t}\cdot(C\lambda^2t)^N\cdot\frac{1}{t^2}(\log^* t)^6\,. \tag{9.9}$$

Moreover, if π has $\kappa\geq 2$ peaks, then

$$C^{\underline{m},a}_{n,N}(\pi;t)\leq\frac{1}{t}\cdot(C\lambda^2t)^N\cdot\frac{1}{t^{\kappa-2}}(\log^* t)^{n+\kappa+2}\,. \tag{9.10}$$

In case of $\omega=$ (const.), the power $\kappa-2$ can be improved to κ.

The proof of (9.9) and (9.10) are given in Sections 9.1 and 9.2, respectively. These are the recollision analogues of the apriori bound (5.1) in Lemma 5.1 and Proposition 7.4.

9.1 General bound for recollision pairings

For the proof of (9.9), we start by symmetrizing the momenta in the definition (9.8). Using a Schwarz inequality within Tr_{ph} in (9.8)

$$\Big|\mathrm{Tr}_{ph}\mathcal{F}\Big([k]_{I_a},p_N\Big)\gamma_{ph}\Big[\mathcal{F}\Big([\tilde{k}]_{I_a},p_N\Big)\Big]^*\Big| \tag{9.11}$$

$$\le \frac{1}{2}\mathrm{Tr}_{ph}\mathcal{F}\Big([k]_{I_a},p_N\Big)\gamma_{ph}\Big[\mathcal{F}\Big([k]_{I_a},p_N\Big)\Big]^* + \frac{1}{2}\mathrm{Tr}_{ph}\mathcal{F}\Big([\tilde{k}]_{I_a},p_N\Big)\gamma_{ph}\Big[\mathcal{F}\Big([\tilde{k}]_{I_a},p_N\Big)\Big]^* .$$

We deal only with the first term, the second is identical. Hence

$$\begin{aligned} C^{\underline{m},a}_{n,N}(\pi;t) \;\le\; & \int^{\#(\underline{m},a)} \Big(\prod_{b\in I_a} \mathrm{d}k_b\Big) \int^{\#(\underline{m},a)} \Big(\prod_{b\in I_a} \mathrm{d}\tilde{k}_b\Big) \int \mathrm{d}p_N \widehat{\gamma}_e(p_N,p_N) \\ & \times \Bigg(\prod_{\substack{j=2\\ j\neq a}}^{n} \chi(k_{\mu(j)} - \tilde{k}_{\mu(\pi(j))}) \Bigg) \; \mathrm{Tr}_{ph}\mathcal{F}\Big([k]_{I_a},p_N\Big)\gamma_{ph}\Big[\mathcal{F}\Big([k]_{I_a},p_N\Big)\Big]^* , \end{aligned}$$

and we can freely integrate out all $\tilde{k}_b$, $b\in I_a$. Each $\tilde{k}_b$ integration gives a factor $\frac{1}{|\Lambda|}$, so after these integrations

$$C^{\underline{m},a}_{n,N}(\pi;t) \le \frac{1}{|\Lambda|^{n-2}} \int^{\#(\underline{m},a)} \Big(\prod_{b\in I_a} \mathrm{d}k_b\Big) \int \mathrm{d}p_N \widehat{\gamma}_e(p_N,p_N)\; \mathrm{Tr}_{ph}\mathcal{F}\Big([k]_{I_a},p_N\Big)\gamma_{ph}\Big[\mathcal{F}\Big([k]_{I_a},p_N\Big)\Big]^* .$$

Notice that this bound is independent of π.

Now we write out $\mathcal{F}$ explicitly using (9.4) and (9.5). We obtain

$$\begin{aligned} C^{\underline{m},a}_{n,N}(\pi;t) \le & \frac{\lambda^{2N}}{|\Lambda|^{n-2}} \int^{\#(\underline{m},a)} \Big(\prod_{b\in I_a} \mathrm{d}k_b\Big) \int \mathrm{d}p_N \widehat{\gamma}_e(p_N,p_N) \int \mathrm{d}k_1 \mathrm{d}k_{\mu(a)} \mathrm{d}\tilde{k}_1 \mathrm{d}\tilde{k}_{\mu(a)} \\ & \times \int \Big(\prod_{b\in J} \mathrm{d}k_b \mathrm{d}k_{b+1}\Big)\Big(\prod_{b\in J} \mathrm{d}\tilde{k}_b \mathrm{d}\tilde{k}_{b+1}\Big) F_a([k]_{I_a},[k]_{I_a^c}) F_a([k]_{I_a},[\tilde{k}]_{I_a^c}) \\ & \times \int \Big(\prod_{j=1}^{N-1} \mathrm{d}p_j\Big) \Delta^*(\underline{p},\underline{k}) \int \Big(\prod_{j=1}^{N-1} \mathrm{d}\tilde{p}_j\Big) \Delta^*(\underline{\tilde{p}},\underline{k}^*) \int_0^{t*} [\mathrm{d}s_j]_1^N [\mathrm{d}\tilde{s}_j]_1^N \Big(\prod_{j=1}^{N} Q(k_j) e^{-is_j e(p_j)}\Big) \\ & \times \mathrm{Tr}_{ph}\Bigg[\Big(\prod_{j=1}^{N} b_{k_j}(\tau_j)\Big) e^{-itH_{ph}} \gamma_{ph} e^{itH_{ph}} \Big(\prod_{j=1}^{N} b^*_{k_j^*}(\tilde{\tau}_j)\Big)\Bigg] \Big(\prod_{j=1}^{N} Q(k_j) e^{i\tilde{s}_j e(\tilde{p}_j)}\Big) , \end{aligned}$$

with $k_j^* := k_j$ if $j\in I_a$ and $k_j^* := \tilde{k}_j$ if $j\in I_a^c$, and we recall the definition of $\tau_j,\tilde{\tau}_j$ from (4.34).

Again, the pairing in the phonon trace must respect the recollisions prepared in the F_a factors and the prepared $b_{k_j} b^*_{k_j}$ pairing for $j\in I_a$ (modulo an error $O(|\Lambda|^{-1})$). These latter pairings yield a factor $|\Lambda|^{n-2}$ from the $n-2$ delta functions $\delta(k_j - k_j) = |\Lambda|$.

Now we proceed similarly to Section 4.6 by integrating out the internal momenta $k_b,\tilde{k}_b$ for $b\in J$, relabelling the external ones and expressing the relabelled electron momenta by the relabelled phonon momenta (4.24). We need a slight modification of the expression $\tilde{p}_j$ $(j=1,\ldots,n-1)$ compared to (4.24)

$$p_j := p_n + \sum_{m=j+1}^{n} k_m \; , \quad \tilde{p}_j := p_n + \chi(j\le a-1)\tilde{k}_a + \sum_{\substack{m=j+1\\ m\neq a}}^{n} k_m \; . \tag{9.12}$$

i.e., now these are considered functions of $k_2, \ldots, k_n$, $\tilde{k}_a$ and p_n. Similarly, we slightly modify the definition of $\widetilde{\Omega}_j$, $j = 1, 2, \ldots, n$, (compare with (4.25)):

$$\Omega_j := \sum_{m=j+1}^{n} \sigma_m \omega(k_m) \; , \qquad \widetilde{\Omega}_j := \chi(j \leq a-1)\tilde{\sigma}_a \omega(\tilde{k}_a) + \sum_{\substack{m=j+1 \\ m \neq a}}^{n} \sigma_m \omega(k_m) \; . \tag{9.13}$$

The exact dependence on the σ's are irrelevant. The important property is that Ω_j depends only on the momenta $k_{j+1}, k_{j+2} \ldots, k_n$ and the same for $\widetilde{\Omega}_j$ just k_a is replaced by $\tilde{k}_a$.

The result of mimicking the argument in Section 4.6 leading to Proposition 4.3 is

$$C^{\underline{m},a}_{n,N}(\pi; t) \leq \widetilde{C}^{\underline{m},a}_{n,N}(t) \tag{9.14}$$

with

$$\begin{aligned} \widetilde{C}^{\underline{m},a}_{n,N}(t) : \; &= \; \lambda^{2N} \sum_{\sigma_2, \ldots, \sigma_n; \tilde{\sigma}_a} \int \mathrm{d}p_n \widehat{\gamma}_e(p_n, p_n) \int \Big(\prod_{j=2}^{n} M(k_j, \sigma_j) \mathrm{d}k_j \Big) \\ &\quad \times \int M(\tilde{k}_a, \tilde{\sigma}_a) \mathrm{d}\tilde{k}_a \; Y_{\underline{m},\underline{m}}(t, p_n, p_n, \underline{k}, \underline{\tilde{k}}, \underline{\sigma} \cup \{\tilde{\sigma}_a\}) + O(|\Lambda|^{-1}) \; , \end{aligned} \tag{9.15}$$

where the definition of Y (4.22)–(4.23) is slightly modified as follows. There is no permutation π. The products over j in (4.21)–(4.22) start from $j = 1$ and there is no $s_0, \tilde{s}_0$. The sets of phonon momenta are $\underline{k} = [k]_{I_a} \cup \{k_a\}$ and $\underline{\tilde{k}} = [k]_{I_a} \cup \{\tilde{k}_a\}$. The electron momenta p_j's are functions of $\underline{k}$ as given in (9.12), similarly for the tilde variables. Finally, we use the definition of $\Omega_j, \widetilde{\Omega}_j$ (9.13), in particular Y depends on $\{\sigma_2, \ldots, \sigma_n, \tilde{\sigma}_a\}$ which replace the variables $\underline{\sigma}$ in the original definition of Y.

The estimate of $\widetilde{C}^{\underline{m},a}_{n,N}(t)$ is different for $a \geq 3$ and for $a = 2$; these cases will be discussed separately. The first case is similar to the crossing estimate in Section 3.3. [EY2]. The second case corresponds to the nested pairings. We recall the definition from Section 3.4 [EY2]:

Definition 9.2 *A pairing of the momenta* $\{k_1, \ldots, k_N\}, \{\tilde{k}_1, \ldots, \tilde{k}_N\}$ *is called* **nested** *if there exist indices* $j_1 < j_2 < j_3 < j_4$ *such that* (k_{j_1}, k_{j_4}) *and* (k_{j_2}, k_{j_3}) *are paired, or the same is true for the tilde variables. Notice that this notion is defined in the original graph and not in its skeleton.*

In particular, a nested recollision graph with $a = 2$ corresponds to $j_1 := 1$, $j_4 := \mu(2) \geq 4$ and (j_2, j_3) with $j_3 := j_2 + 1$ being the indices of any immediate recollisions in between $(1 < j_2 < \mu(2)$ and j is even).

9.1.1 Pairing with a distant recollision: Case $a \geq 3$

We assume that $a \geq 3$. In this case the recollision pairing line actually crosses at least another pairing line. In [EY2] we treated this case together with the crossing estimates. The proof we give here is similar although in this paper we do not call it crossing pairing.

To estimate Y in (9.15), we bound all Υ's by a constant (4.8) and use (5.8). We obtain

$$C^{\underline{m},a}_{n,N}(\pi;t) \leq \widetilde{C}^{\underline{m},a}_{n,N}(t) \;\leq\; (C\lambda)^{2N} t^{2|\underline{m}|} \sup_{\underline{\sigma},\tilde{\sigma}_a} \int \mathrm{d}p_n \widehat{\gamma}_e(p_n,p_n) \int_{-\infty}^{\infty} \mathrm{d}\alpha \mathrm{d}\tilde{\alpha} \\ \times \int \Big(\prod_{j=2}^n M^*(k_j)\mathrm{d}k_j \Big) M^*(\tilde{k}_a)\mathrm{d}\tilde{k}_a \prod_{j=1}^n |R_j|\,|\widetilde{R}_j| \,, \tag{9.16}$$

and we choose $\eta = t^{-1}$ in the definition of $R_j, \widetilde{R}_j$ (4.23).

For $j \geq 1$ we define

$$P_j := p_n + \sum_{\substack{m=j+1\\ m\neq a}}^n k_m, \qquad \Omega_j^{\#} := \sum_{\substack{m=j+1\\ m\neq a}}^n \sigma_m \omega(k_m) \;.$$

We start by integrating k_2, this involves $R_1, \widetilde{R}_1$, explicitly (see (6.19))

$$\int \frac{M^*(k_2)\mathrm{d}k_2}{\Big|\alpha - e(k_2+P_2+k_a) + i\eta - \Omega_2^{\#} - \sigma_2\omega(k_2) - \sigma_a\omega(k_a)\Big|} \\ \times \frac{1}{\Big|\tilde{\alpha} - e(k_2+P_2+\tilde{k}_a) + i\eta - \Omega_2^{\#} - \sigma_2\omega(k_2) - \tilde{\sigma}_a\omega(\tilde{k}_a)\Big|} \;\leq\; \frac{(C\log^* t)^2}{|k_a - \tilde{k}_a|} \,.$$

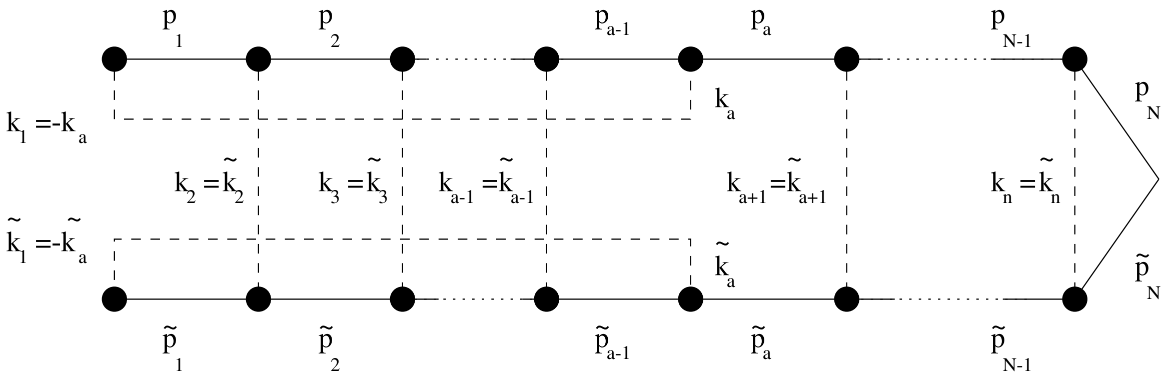

Fig. 10. One recollision after symmetrization and relabelling.

Now we integrate out $k_3, \ldots, k_{a-1}$ in this order if $a \geq 4$. At the j-th step, $3 \leq j \leq a-1$, we eliminate $R_{j-1}, \widetilde{R}_{j-1}$, using

$$\int \frac{M^*(k_j)\mathrm{d}k_j}{\Big|\alpha - e(k_j+P_j+k_a) + i\eta - \Omega_j^{\#} - \sigma_j\omega(k_j) - \sigma_a\omega(k_a)\Big|} \tag{9.17}$$

$$\times \frac{1}{\left|\tilde\alpha - e(k_j + P_j + \tilde k_a) + i\eta) - \Omega_j^\# - \sigma_j\omega(k_j) - \tilde\sigma_a\omega(\tilde k_a)\right|} \;\leq\; Ct\,,$$

and this gives $(Ct)^{a-3}$ (this is valid even if $a = 3$). This estimate follows from (6.10) after a Schwarz inequality.

Now we do the $\mathrm{d}k_a, \mathrm{d}\tilde k_a$ integrals to eliminate R_{a-1} and $\widetilde R_{a-1}$:

$$\begin{aligned}&\int \frac{M^*(k_a)\mathrm{d}k_a}{\left|\alpha - e(k_a + P_a) + i\eta - \sigma_a\omega(k_a) - \Omega_a^\#\right|} \\ &\times \int \frac{1}{|k_a - \tilde k_a|}\frac{M^*(\tilde k_a)\mathrm{d}\tilde k_a}{\left|\tilde\alpha - e(\tilde k_a + P_a) + i\eta - \tilde\sigma_a\omega(\tilde k_a) - \Omega_a^\#\right|} \;\leq\; \frac{(C\log^* t)^2}{\langle \alpha - \tilde\alpha\rangle}\,.\end{aligned} \tag{9.18}$$

The prototype of the $\mathrm{d}\tilde k_a$ integral is

$$\sup_{c,q} \int \frac{1}{|k-q|}\frac{M^*(k)\mathrm{d}k}{\left|c - e(k+P) + i\eta \pm \omega(k)\right|} \leq \frac{C\log^* t}{\langle c - e(P)\rangle}\,, \tag{9.19}$$

which is a straightforward strengthening of (6.21). The same estimate is valid without the $|k-q|^{-1}$ factor, and that is the prototype of the $\mathrm{d}k_a$ integration. Finally, we use

$$\sup_P \frac{1}{\langle c - e(P)\rangle\langle \tilde c - e(P)\rangle} \leq \frac{C}{\langle c - \tilde c\rangle} \tag{9.20}$$

to obtain the $\langle\alpha - \tilde\alpha\rangle^{-1}$ decay.

Now we do $\mathrm{d}k_{a+1}, \mathrm{d}k_{a+2}, \ldots, \mathrm{d}k_n$ integrals; each gives a factor Ct and altogether we collect $(Ct)^{n-a}$. In the j-th step ($a+1 \leq j \leq n$) we eliminate $R_{j-1}, \widetilde R_{j-1}$ by using a Schwarz estimate and (6.10) as in (9.17).

Finally we are left with

$$\int_{-\infty}^{\infty} \mathrm{d}\alpha\mathrm{d}\tilde\alpha \frac{1}{\langle\alpha - \tilde\alpha\rangle}\int \frac{\widehat\gamma_e(p_n,p_n)\mathrm{d}p_n}{|\alpha - e(p_n) + i\eta|\;|\tilde\alpha - e(p_n) + i\eta|} \leq (C\log^* t)^2\,. \tag{9.21}$$

Altogether we have

$$(C\log^* t)^2(Ct)^{a-3}(C\log^* t)^2(Ct)^{n-a}(C\log^* t)^2 = \frac{(\log^* t)^6}{t^3}(Ct)^n$$

for the integrals in (9.16), which finishes the proof of (9.9) in the $a \geq 3$ case.

9.1.2 Pairing with a nested recollision: Case $a = 2$

Here we prove (9.9) for the remaining case $a = 2$. In this case $m_1 \geq 1$ since we had a genuine recollision before relabelling. Therefore the original pairing was nested.

We start from (9.15) and perform explicitly the k_2 integration. This involves the following

$$A = A(\alpha, p_2, \Omega_2, \sigma_2) := \int M(k_2, \sigma_2)\Big[R_1(\alpha, p_1, \Omega_1, \eta)\Big]^{m_1+1}\Big[\Upsilon_\eta(\alpha - \Omega_1, p_1)\Big]^{m_1} \mathrm{d}k_2 \qquad (9.22)$$

using (4.22) and recalling that $p_1 = p_2 + k_2$, $\Omega_1 = \Omega_2 + \sigma_2\omega(k_2)$.

Lemma 9.3 *With the notation above and for $m_1 \geq 1$, $\eta \leq 1$ we have*

$$|A(\alpha, p_2, \Omega_2, \sigma_2)| \leq \frac{C^{m_1}\langle p_2\rangle^d \eta^{-m_1+1/2}}{\langle \alpha - e(p_2) - \Omega_2\rangle} \,. \qquad (9.23)$$

Proof of Lemma 9.3. We present the proof for $\omega = (const.)$ case here and explain the necessary modifications in Appendix A.3 for the nonconstant case. The proof of the constant ω case is a simple stationary phase calculation, while the nonconstant case is an integration by parts similar to Section 3.4 [EY2].

For the constant ω case we recall (4.23), (4.4) and we write

$$A = i^{m_1+1}\int_0^\infty s^{m_1} e^{is(\alpha - \Omega_2 - \sigma_2\omega + i\eta)}\int e^{-ise(p)}\Big[\Upsilon_\eta\Big(\alpha - \Omega_2 - \sigma_2\omega, p\Big)\Big]^{m_1} M(p - p_2, \sigma_2)\mathrm{d}p\mathrm{d}s$$

after a change of variables. Using (4.8), (4.10) and that M has decaying derivatives up to order d we obtain by stationary phase argument (see (4.13))

$$\Big|\int e^{-ise(p)}\Big[\Upsilon_\eta\Big(\alpha - \Omega, p\Big)\Big]^{m_1} M(p - p_2, \sigma_2)\mathrm{d}p\Big| \leq \frac{C^{m_1}\langle p_2\rangle^d \eta^{-m_1+1/2}}{\langle s\rangle^{d/2}\langle \alpha - e(p_2) - \Omega_2\rangle}$$

After performing the ds integration, we obtain (9.23) if $d \geq 3$. $\square$

The same estimate as (9.23) is valid for the $\tilde{k}_2$ integration which can be performed independently. We collected a factor $(Ct)^{2m_1-1}$ (with $\eta := t^{-1}$) and we gain a $\langle \alpha - \tilde{\alpha}\rangle^{-1}$ factor, similarly to (9.20). The extra $\langle p_2\rangle^{2d}$ factor can be compensated as before, using (6.3).

Then we estimate all the other R_j and Υ factors in absolute value in (9.15), we use (4.8) and (5.8) for $j \geq 2$, collecting $(Ct)^{2|\underline{m}|-2m_1}$. Then we follow the end of the proof in Section 9.1.1 by integrating d$k_3, \ldots,$ dk_n and further collecting $(Ct)^{n-2}$. Hence the total t power is $N - 3$, finishing the proof of (9.9) for the $a = 2$ case. $\square$

9.2 Bound for recollision pairings with many peaks

Here we prove (9.10) of Lemma 9.1. The argument is similar to the proof of Proposition 7.4; in fact the recollision will not be used.

We start from (9.8) and follow the argument of Section 9.1 except the Schwarz inequality (9.11). We obtain (compare with (9.15)):

$$C^{\underline{m},a}_{n,N}(\pi;t) = \lambda^{2N} \sum_{\sigma_2,\ldots,\sigma_n;\tilde\sigma_a} \int \mathrm{d}p_n \widehat\gamma_e(p_n,p_n) \int \Big(\prod_{\substack{j=2\\ j\neq a}}^n M(k_j,\sigma_j)\delta(k_j-\tilde k_{\pi(j)})\mathrm{d}k_j \mathrm{d}\tilde k_j\Big)$$

$$\times \int M(k_a,\sigma_a)\mathrm{d}k_a \int M(\tilde k_a,\tilde\sigma_a)\mathrm{d}\tilde k_a \; Y_{\underline{m},\underline{m},\pi}(t,p_n,p_n,\underline{k},\underline{\tilde k},\underline{\sigma}\cup\{\tilde\sigma_a\}) + O(|\Lambda|^{-1}) \; ,$$

where the definition of Y (see (4.22)) is modified as follows. The two products over j in (4.22) start from $j=1$ and there is no $s_0, \tilde s_0$. The electron momenta $p_j, \tilde p_j$, $j\geq 1$, are given in (4.24) and $\Omega_j, \widetilde\Omega_j$ are given in (4.25) with the modification that $\sigma_{\pi^{-1}(a)} := \tilde\sigma_a$ in the definition of $\widetilde\Omega_j$.

After the trivial estimates (4.8) and (5.8) we obtain (compare with (9.16))

$$\Big|C^{\underline{m},a}_{n,N}(\pi;t)\Big| \leq (C\lambda)^{2N} t^{2\underline{m}} \sup_{\underline{\sigma},\tilde\sigma_a} \int \mathrm{d}p_n \widehat\gamma_e(p_n,p_n) \int_{-\infty}^{\infty} \mathrm{d}\alpha \mathrm{d}\tilde\alpha$$

$$\times \int \Big(\prod_{\substack{j=2\\ j\neq a}}^n M^*(k_j)\delta(k_j-\tilde k_{\pi(j)})\mathrm{d}k_j\mathrm{d}\tilde k_j\Big) \int M^*(k_a)\mathrm{d}k_a M^*(\tilde k_a)\mathrm{d}\tilde k_a \; \prod_{j=1}^n |R_j|\,|\widetilde R_j| + O(|\Lambda|^{-1}) \; ,$$

where (4.23)–(4.25) are in effect.

This estimate is our starting point and it should be compared with (6.2) which was the starting point of the proof of Proposition 7.4.

For $\omega = (const.)$ we follow the proof of Proposition 7.4. Let $a_1 < a_2 < \ldots < a_\kappa \in I_a$ be the location of the monotonic peak-sequence as before. We estimate all $|\widetilde R_j|$'s trivially by Ct except $\widetilde R_{\pi(a_m)-1}$, $m = 1,2,\ldots,\kappa$. Of course the term $\widetilde R_0$ is missing, which gives the extra t^{-1} factor compared to the proof in Section 7.2.

The fact that $k_1 + k_a = \tilde k_1 + \tilde k_a = 0$ does not change the dependence of p_j and $\tilde p_j$ on $\underline{k}$, $\underline{\tilde k}$ $(j\geq 1)$. Since p_0 and $\tilde p_0$ does not appear in $R_j, \widetilde R_j$, $j\geq 1$, the only new momentum constraints due to recollision, $p_0 - p_1 + p_{a-1} - p_a = 0$ and $\tilde p_0 - \tilde p_1 + \tilde p_{a-1} - \tilde p_a = 0$, play no role. Hence the argument in Section 7.2 remains unchanged; the key point being that $\tilde p_{\pi(a_m)}$ was independent of p_{a_m-1}, p_{a_m} in (7.3). Therefore the $\mathrm{d}p_{a_m}, \mathrm{d}p_{a_m-1}$ integrations can be done successively; starting from a_1 or a_κ depending on the monotonicity of the peaks.

For $\omega \neq (const.)$ we follow the proof of Proposition A.6 in Appendix A.2; the proof is unchanged, just the $\mathrm{d}p_0, \mathrm{d}\tilde p_0$ integrations and the corresponding $R_0, \widetilde R_0$ factors are missing. $\square$

10 Computing the Wigner transform of the main term

To identify the weak limit of $W^{\varepsilon}_{\gamma_K^{main}(t)}(X,V)$, we test it against a function $J \in \mathcal{S}(\mathbf{R}^d \times \mathbf{R}^d)$. We fix $K \geq 5$. We recall the definitions of $\gamma_K^{main}(t)$ (2.18), $Y_{\underline{m},\underline{\widetilde{m}},\pi}$ (4.21)-(4.22) and $\widehat{J}_\varepsilon(\xi,v) = \varepsilon^{-d}\widehat{J}(\xi\varepsilon^{-1},v)$. Then we have the following proposition which can be proven exactly as Proposition 4.3.

Proposition 10.1 *For any fixed $K \geq 5$ and $J \in \mathcal{S}(\mathbf{R}^d \times \mathbf{R}^d)$*

$$\limsup_{L\to\infty}\Big|\Big\langle J, W^{\varepsilon}_{\gamma_K^{main}(t)}\Big\rangle - \sum_{N,\widetilde{N}=0}^{K-1}\sum_{n=0}^{\min\{N,\widetilde{N}\}}\sum_{\underline{m}\in\mathcal{M}(n,N)}\sum_{\underline{\widetilde{m}}\in\mathcal{M}(n,\widetilde{N})}\sum_{\pi\in\Pi_n} C^*_{\underline{m},\underline{\widetilde{m}},\pi}(t)\Big| = 0$$

with $\bar{N} := (N+\widetilde{N})/2$ and

$$C^*_{\underline{m},\underline{\widetilde{m}},\pi}(t) := \lambda^{2\bar{N}} \sum_{\substack{\sigma_j\in\{\pm\}\\ j=1,\dots,n}} \int \mathrm{d}\bar{\nu}_\pi(\xi,v,\underline{k},\underline{\tilde{k}},\underline{\sigma})\; Y_{\underline{m},\underline{\widetilde{m}},\pi}\Big(t; v+\tfrac{\xi}{2}, v-\tfrac{\xi}{2}, \underline{k}, \underline{\tilde{k}}, \underline{\sigma}\Big) \tag{10.1}$$

(compare with (4.19)). The difference between $C^*_{\underline{m},\underline{\widetilde{m}},\pi}$ and $C_{\underline{m},\underline{\widetilde{m}},\pi}$ from (4.19) is twofold. We replaced the measure $\mathrm{d}\nu_\pi(p_n,\tilde{p}_n,\underline{k},\underline{\tilde{k}},\underline{\sigma})$ (see (4.20)) with

$$\begin{aligned}\mathrm{d}\bar{\nu}_\pi(\xi,v,\underline{k},\underline{\tilde{k}},\underline{\sigma}) :&= \delta\Big(v_0 - v - \sum_{j=1}^n k_j\Big)\, \overline{\widehat{J}_\varepsilon(\xi,v_0)}\, \widehat{\gamma}_e\Big(v+\tfrac{\xi}{2}, v-\tfrac{\xi}{2}\Big) \\ &\quad\times\Big(\prod_{j=1}^n \delta(k_j - \tilde{k}_{\pi(j)}) M(k_j,\sigma_j)\mathrm{d}k_j\mathrm{d}\tilde{k}_j\Big)\,\mathrm{d}\xi\mathrm{d}v\,,\end{aligned} \tag{10.2}$$

and instead of (4.24) we considered p_j, $\tilde{p}_j$ as functions of $v,\xi,\underline{k},\underline{\tilde{k}}$ as follows

$$p_j = v + \tfrac{\xi}{2} + \sum_{\ell=j+1}^n k_\ell\,, \qquad \tilde{p}_j = v - \tfrac{\xi}{2} + \sum_{\ell=j+1}^n \tilde{k}_\ell\,. \tag{10.3}$$

With these notations, the definition of $R_j, \widetilde{R}_j$ (4.23) and $\Omega_j, \widetilde{\Omega}_j$ (4.25) are unchanged. $\square$

The next lemma gives an estimate on all $C^*_{\underline{m},\underline{\widetilde{m}},\pi}(t)$, and a stronger bound if π is crossing.

Lemma 10.2 *Let $K \geq 5$, $n \leq N \leq K$, $\tilde{n} \leq \widetilde{N} \leq K$, $\underline{m} \in \mathcal{M}(n,N)$, $\underline{\widetilde{m}} \in \mathcal{M}(n,\widetilde{N})$, then for any $\pi \in \Pi_n$*

$$\limsup_{L\to\infty} |C^*_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \leq \frac{(C_a\lambda^2 t)^{\bar{N}}}{(M!\,\widetilde{M}!)^{a/2}}\,, \tag{10.4}$$

where $M := (N+n)/2$, $\widetilde{M} := (\widetilde{N}+n)/2$ and $0 \leq a < 1$. Moreover, if $\pi \in \Pi_n$ is a crossing pairing ($\pi \neq id$), then

$$\limsup_{L\to\infty} |C^*_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \leq t^{-1/2}(C\lambda^2 t)^{\bar{N}}(\log^* t)^4\,. \tag{10.5}$$

Proof. For (10.4) we follow the proof of the analogous Lemma 5.1 starting from (10.1). We again split $|Y| = |Y|^{a/2}|Y|^{1-a/2}$. The first factor is estimated in supremum norm using $|Y| \leq C^{\bar N} t^{M+\widetilde M}/(M!\,\widetilde M!)$ (see (5.5)). The estimate (5.6) is modified as

$$|C^*_{\underline{m},\underline{\widetilde m},\pi}(t)| \leq (C\lambda)^{N+\widetilde N}\Big(\frac{t^{M+\widetilde M}}{M!\,\widetilde M!}\Big)^{a/2} \sup_{\underline{\sigma}} \int \mathrm{d}\bar\nu^*_\pi(\xi, v, \underline{k}, \underline{\tilde k})$$

$$\times\Big\{ t^{(|\underline{\widetilde m}|-|\underline{m}|)(1-a/2)}\Big[\int_{-\infty}^{\infty}\mathrm{d}\alpha \prod_{j=0}^n |R_j|^{m_j+1}\Big]^{2-a} + t^{(|\underline{m}|-|\underline{\widetilde m}|)(1-a/2)}\Big[\int_{-\infty}^{\infty}\mathrm{d}\tilde\alpha \prod_{j=0}^n |\widetilde R_j|^{\widetilde m_j+1}\Big]^{2-a}\Big\} .$$

with a modification of $\mathrm{d}\bar\nu_\pi$

$$\begin{aligned}\mathrm{d}\bar\nu^*_\pi(\xi, v, \underline{k}, \underline{\tilde k}) : \;&=\; \delta\Big(v_0 - v - \sum_{j=1}^n k_j\Big)\; |\widehat J_\varepsilon(\xi, v_0)|\, \Big|\widehat\gamma_e\Big(v+\tfrac{\xi}{2}, v-\tfrac{\xi}{2}\Big)\Big| \\ &\qquad \times\Big(\prod_{j=1}^n \delta(k_j - \tilde k_{\pi(j)}) M^*(k_j)\mathrm{d}k_j \mathrm{d}\tilde k_j\Big)\, \mathrm{d}\xi\mathrm{d}v .\end{aligned} \tag{10.6}$$

The rest of the proof goes through until (5.20), just p_n is replaced with $v+\frac{\xi}{2}$ and $\tilde p_n$ with $v-\frac{\xi}{2}$, i.e., (10.3) is used instead of (4.24), the t-powers are adjusted and $\mathrm{d}\mu$ is redefined as

$$\mathrm{d}\bar\mu(\xi, v, \underline{k}) := \;|\widehat J_\varepsilon(\xi, v_0)|\, \delta\Big(v_0 - v - \sum_{j=1}^n k_j\Big)\;\; \Big|\widehat\gamma_e\Big(v+\tfrac{\xi}{2}, v-\tfrac{\xi}{2}\Big)\Big|\Big(\prod_{j=1}^n M^*(k_j)\mathrm{d}k_j\Big)\, \mathrm{d}\xi\mathrm{d}v .$$

Let $J^*_\varepsilon(\xi) := \sup_w |\widehat J_\varepsilon(\xi, w)|$. After the $\mathrm{d}k_n$ integration in the analogue of (5.20), we arrive at

$$\begin{aligned}|C^*_{\underline{m},\underline{\widetilde m},\pi}(t)| \;&\leq\; (C\lambda)^{N+\widetilde N}\Big(\frac{t^{M+\widetilde M}}{M!\,\widetilde M!}\Big)^{a/2} t^{(|\underline{\widetilde m}|+|\underline{m}|)(1-a/2)+(1-a)n}\int \mathrm{d}\xi\mathrm{d}v\; J^*_\varepsilon(\xi)\,\Big|\widehat\gamma_e\Big(v+\tfrac{\xi}{2}, v-\tfrac{\xi}{2}\Big)\Big| \\ &\leq\; \frac{(C_a\lambda^2 t)^{\bar N}}{(M!\,\widetilde M!)^{a/2}}\int \mathrm{d}\xi\mathrm{d}v\; J^*_\varepsilon(\xi)\,\Big[\widehat\gamma_e\Big(v-\tfrac{\xi}{2}, v-\tfrac{\xi}{2}\Big) + \widehat\gamma_e\Big(v+\tfrac{\xi}{2}, v+\tfrac{\xi}{2}\Big)\Big] ,\end{aligned} \tag{10.7}$$

using $|\widehat\gamma_e(p,p')| \leq \frac{1}{2}\Big[\widehat\gamma_e(p,p) + \widehat\gamma_e(p',p')\Big]$ by $\gamma_e \geq 0$. Then (10.4) follows from (2.22) and (1.26).

The proof of (10.5) requires very similar modifications along the proof of the analogous Lemma 6.2. The estimate (6.2) is modified as

$$|C^*_{\underline{m},\underline{\widetilde m},\pi}(t)| \leq (C\lambda)^{N+\widetilde N} t^{|\underline{m}|+|\underline{\widetilde m}|} \sup_{\underline{\sigma}} \int \mathrm{d}\bar\nu^*_\pi(\xi, v, \underline{k}, \underline{\tilde k}) \int_{-\infty}^{\infty}\mathrm{d}\alpha \prod_{j=0}^n |R_j| \int_{-\infty}^{\infty}\mathrm{d}\tilde\alpha \prod_{j=0}^n |\widetilde R_j| ,$$

keeping in mind (10.3).

It is easy to check that all estimates in the proof of Lemma 6.2 go through, since they were always valid uniformly in the incoming electron momenta denoted by p in (6.10), (6.11), (6.12), (6.21) and (6.22), hence a shift $\pm\frac{\xi}{2}$ does not make a difference. Finally, instead of (6.14) we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha\mathrm{d}\tilde\alpha}{\langle\alpha\rangle\langle\tilde\alpha\rangle}\int \mathrm{d}\xi\mathrm{d}v\ J_\varepsilon^*(\xi)\frac{\Big|\widehat{\gamma}_e\Big(v+\frac{\xi}{2},v-\frac{\xi}{2}\Big)\Big|\Big\langle v+\frac{\xi}{2}\Big\rangle^2\Big\langle v-\frac{\xi}{2}\Big\rangle^2}{\Big|\alpha-e\Big(v+\frac{\xi}{2}\Big)+i\eta\Big|\ \Big|\tilde\alpha-e\Big(v-\frac{\xi}{2}\Big)+i\eta\Big|}\\
&\le\ (C\log^* t)^2\int \mathrm{d}\xi\mathrm{d}v\ J_\varepsilon^*(\xi)\Big[\widehat{\gamma}_e\Big(v-\tfrac{\xi}{2},v-\tfrac{\xi}{2}\Big)\langle v-\tfrac{\xi}{2}\rangle^4+\widehat{\gamma}_e\Big(v+\tfrac{\xi}{2},v+\tfrac{\xi}{2}\Big)\langle v+\tfrac{\xi}{2}\rangle^4\Big]\langle\xi\rangle^4\\
&\le\ (C\log^* t)^2\ .
\end{aligned}$$

We again used (1.26) and

$$\sup_{\varepsilon\le 1}\int J_\varepsilon^*(\xi)\langle\xi\rangle^4\mathrm{d}\xi=\sup_{\varepsilon\le 1}\int\sup_v|\widehat{J}_\varepsilon(\xi,v)|\langle\xi\rangle^4\mathrm{d}\xi<\infty\ .$$

This is an extension of (2.22), and it is clearly valid for $J\in\mathcal{S}(\mathbf{R}^d\times\mathbf{R}^d)$. □

From (10.5) we know that only the direct pairing counts. To compute $C^*_{\underline{m},\underline{\widetilde m},id}$, we use the second formula for $Y_{\underline{m},\underline{\widetilde m},id}$ (4.22) and that $\Omega_j=\widetilde\Omega_j$

$$\begin{aligned}
C^*_{\underline{m},\underline{\widetilde m},id}(t):\ =\ &\lambda^{2\bar N}\sum_{\substack{\sigma_j\in\{\pm\}\\ j=1,\dots,n}}\int \mathrm{d}\bar\nu_{id}(\xi,v,\underline{k},\underline{\tilde k},\underline{\sigma})e^{2t\eta}\int_{-\infty}^{\infty}\mathrm{d}\alpha\ e^{-it\alpha}\\
&\times\prod_{j=0}^n R_j^{m_j+1}\Big[\Upsilon_\eta(\alpha-\Omega_j,p_j)\Big]^{m_j}\int\mathrm{d}\tilde\alpha\ e^{it\tilde\alpha}\prod_{j=0}^n\widetilde R_j^{m_j+1}\Big[\overline{\Upsilon}_\eta(\tilde\alpha-\Omega_j,\tilde p_j)\Big]^{\widetilde m_j}\ .
\end{aligned}\tag{10.8}$$

Introduce

$$v_j:=v+\sum_{m=j+1}^n k_m\ ,\qquad m=1,2,\dots,n-1\tag{10.9}$$

and $v_n:=v$. Note that $p_j=v_j+\frac{\xi}{2}$. We show that $\Upsilon_\eta(\alpha-\Omega_j,p_j)$ and $\Upsilon_\eta(\tilde\alpha-\Omega_j,\tilde p_j)$ can be replaced with

$$\Psi_j:=\Upsilon_{0+}(e(v_j),v_j)\ ,\quad\text{and}\quad\overline{\Psi}_j=\overline{\Upsilon_{0+}(e(v_j),v_j)}$$

modulo a negligible error (recall (4.11)). Notice that Ψ_j depend on $v,\xi,k_{j+1},\dots,k_n$.

Lemma 10.3

$$C^*_{\underline{m},\underline{\widetilde m},id}(t)=C^{**}_{\underline{m},\underline{\widetilde m},id}(t)+O\Big((C\lambda^2t)^{\bar N}t^{-1/2}(\log^* t)^4\Big)\tag{10.10}$$

with

$$\begin{aligned}
C^{**}_{\underline{m},\underline{\widetilde m},id}(t):\ =\ &\lambda^{2\bar N}e^{2t\eta}\sum_{\substack{\sigma_j\in\{\pm\}\\ j=1,\dots,n}}\int \mathrm{d}\bar\nu_{id}(\xi,v,\underline{k},\underline{\tilde k},\underline{\sigma})\\
&\times\int_{-\infty}^{\infty}\mathrm{d}\alpha\ e^{-it\alpha}\prod_{j=0}^n R_j^{m_j+1}\Psi_j^{m_j}\int\mathrm{d}\tilde\alpha\ e^{it\tilde\alpha}\prod_{j=0}^n\widetilde R_j^{m_j+1}\overline{\Psi}_j^{\widetilde m_j}\ .
\end{aligned}\tag{10.11}$$

We also have

$$\limsup_{L\to\infty} |C^{**}_{\underline{m},\underline{\widetilde{m}},id}(t)| \le \frac{(C_a\lambda^2 t)^{\bar N}}{(M!\ \widetilde{M}!)^{a/2}} \; . \tag{10.12}$$

Proof of Lemma 10.3. The estimate (10.12) is proven exactly as (10.4). For (10.10), we compute the difference $C^* - C^{**}$. We obtain a similar formula as (10.8), just $\prod_j \Upsilon(\ldots)^{m_j}$ is replaced with $\prod_j \Upsilon(\ldots)^{m_j} - \prod_j \Psi_j^{m_j}$ and similarly for the factors with conjugate. We then estimate this expression exactly as in the proof of (10.4) with $a = 0$.

Let $\theta : \mathbf{R} \to \mathbf{R}$ be defined as $\theta(s) := |s|$ for $|s| \le 1$ and $\theta(s) \equiv 1$ for $|s| \ge 1$. Using (1.15), (4.8) and (4.9), we see by a telescopic estimate that

$$\left|\prod_{j=0}^n \Big[\Upsilon_\eta(\alpha - \Omega_j, p_j)\Big]^{m_j} - \prod_{j=0}^n \Psi_j^{m_j}\right| \le C^n \eta^{-1/2} \sum_{j=0}^n \Big[\theta\Big(\alpha - \Omega_j - e(p_j)\Big) + O\Big(|\xi|(\langle p_j\rangle + \langle\xi\rangle)\Big)\Big] \; . \tag{10.13}$$

For the first term in (10.13) we notice that

$$\theta\Big(\alpha - \Omega_j - e(p_j)\Big)\; |R_j| \le \frac{C}{\langle \alpha - \Omega_j - e(p_j)\rangle} \; ,$$

in other words, the net effect of a factor $\theta\Big(\alpha - \Omega_j - e(p_j)\Big)$ is that it neutralizes the singularity of an $|R_j|$ factor and still keeping its decay property. Hence, effectively, m_j is decreased by one (see (5.8)). We gain a factor t from this decrease (notice the dependence of the t power on $|\underline{m}|$ in (10.7)), but we lose $\eta^{-1/2} = t^{1/2}$ in (10.13). This gives the extra factor $t^{-1/2}$ in the error term of (10.10) relative to the robust estimate (10.4). The special case $m_j = 0$ require small modifications which we leave to the reader.

For the second term in (10.13) we use that $\int |\xi|\langle\xi\rangle \widehat{J}^*_\varepsilon(\xi)\mathrm{d}\xi = O(\varepsilon)$ to gain $\eta^{-1/2}\varepsilon = t^{-1/2}$. The factor $\langle p_j\rangle$ can be absorbed into the decay of M^* and γ_e as before. $\square$.

Combining Proposition 10.1, Lemma 10.2 and Lemma 10.3 we obtain

Proposition 10.4 *For $K \ge 5$,*

$$\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\left|\Big\langle J, W^\varepsilon_{\gamma_K^{main}(t)}\Big\rangle - \sum_{N,\widetilde{N}=0}^{K-1}\;\sum_{n=0}^{\min\{N,\widetilde{N}\}}\;\sum_{\underline{m}\in\mathcal{M}(n,N)}\;\sum_{\underline{\widetilde{m}}\in\mathcal{M}(n,\widetilde{N})} C^{**}_{\underline{m},\underline{\widetilde{m}},id}(t)\right| = 0 \; . \qquad \square$$

Using the factorials in the estimate (10.12), we can extend the summations over all $\underline{m}, \underline{\widetilde{m}}$; the error term goes to zero as $K \to \infty$:

$$\limsup_{K\to\infty}\limsup_{\varepsilon\to 0}\limsup_{L\to\infty}\left|\Big\langle J, W^\varepsilon_{\gamma_K^{main}(t)}\Big\rangle - \sum_{n=0}^{\infty} C(n,t)\right| = 0 \; ,$$

where

$$C(n,t) := \sum_{m_0,\ldots,m_n=0}^{\infty} \sum_{\widetilde{m}_0,\ldots,\widetilde{m}_n=0}^{\infty} C^{**}_{\underline{m},\underline{\widetilde{m}},id}(t) \; .$$

The summations over $m_j, \widetilde{m}_j$ give a geometric series in (10.11). We obtain a shift in the denominators of R_j (*"one loop renormalization"*):

$$\sum_{m_j=0}^{\infty} R_j^{m_j+1}(\lambda^2\Psi_j)^{m_j} = \frac{1}{\alpha - e(p_j) - \Omega_j - \lambda^2\Psi_j + i\eta} \; . \tag{10.14}$$

We can change back the dα and d$\tilde{\alpha}$ integrations into subsequent time integrations since Ψ's are independent of them (see the similar identity in (4.50)). We obtain

$$\begin{aligned} C(n,t) \;\; &= \;\; \lambda^{2n} \sum_{\substack{\sigma_j\in\{\pm\}\\ j=1,\ldots,n}} \int \mathrm{d}\bar{\nu}_{id}(\xi, v, \underline{k}, \underline{\tilde{k}}, \underline{\sigma}) \int_0^{t*} [\mathrm{d}s_j]_0^n \int_0^{t*} [\mathrm{d}\tilde{s}_j]_0^n \\ &\times \prod_{j=0}^n \exp\Big\{ - i\Big[s_j\Big(e(p_j) + \Omega_j + \lambda^2\Psi_j\Big) - \tilde{s}_j\Big(e(\tilde{p}_j) + \Omega_j + \lambda^2\overline{\Psi}_j\Big)\Big]\Big\} \; . \end{aligned} \tag{10.15}$$

The rest is similar to the end of Section 4 in [EY2] and we will skip a few technical details.

We introduce the new time variables $a_j := (s_j + \tilde{s}_j)/2$, $b_j := (s_j - \tilde{s}_j)/2$ for $j = 0, 1, \ldots, n$ and we have

$$\begin{aligned} C(n,t) = 2^n\lambda^{2n} &\sum_{\substack{\sigma_j\in\{\pm\}\\ j=1,\ldots,n}} \int \Big(\prod_{j=0}^n \mathrm{d}v_j\Big)\mathrm{d}\xi \; \overline{\widehat{J}_\varepsilon(\xi, v_0)} \int \Big(\prod_{j=0}^{n-1} M(v_j - v_{j+1}, \sigma_{j+1})\Big)\widehat{\gamma}_e\Big(v_n + \tfrac{\xi}{2}, v_n - \tfrac{\xi}{2}\Big) \\ &\times \int_0^t \Big(\prod_{j=0}^n \mathrm{d}a_j\Big)\delta\Big(t - \sum_{j=0}^n a_j\Big) \exp\Big\{ - i\xi\cdot\sum_{j=0}^n a_j\nabla e(v_j) + 2\sum_{j=0}^n a_j\Big(\lambda^2\mathrm{Im}\Psi_j + O(\xi^2)\Big)\Big\} \\ &\times \prod_{j=0}^n \Big(\int_{-a_j}^{a_j} \mathrm{d}b_j\Big)\delta\Big(\sum_{j=0}^n b_j\Big) \prod_{j=0}^n \exp\Big\{ - 2ib_j\Big[e(v_j) + \Omega_j + \lambda^2\mathrm{Re}\Psi_j + O(\xi^2)\Big]\Big\} \; . \end{aligned}$$

Here we used $p_j = v_j + \frac{\xi}{2}$, $\tilde{p}_j = v_j - \frac{\xi}{2}$ and the second order Taylor expansion of $e(v)$

$$e(v_j \pm \tfrac{\xi}{2}) = e(v_j) \pm \tfrac{\xi}{2}\cdot\nabla e(v_j) + O(\xi^2)$$

with a uniform error bound (see (1.15)).

We rescale all microscopic variables into macroscopic ones, i.e., $\alpha_j := \varepsilon a_j$, $\zeta = \varepsilon^{-1}\xi$, $t = \varepsilon^{-1}T$ and recall that $\varepsilon = \lambda^2$. We define

$$\chi_{\underline{a}}(\underline{b}) := \chi\Big(- a_0 \le \sum_{j=0}^{n-1} b_j \le a_0\Big) \prod_{j=0}^{n-1} \chi(-a_j \le b_j \le a_j)$$

for any $\underline{a} := (a_0, a_1, \ldots, a_n)$ and $\underline{b} := (b_0, b_1, \ldots, b_{n-1})$. Then

$$\begin{aligned} C(n,t) \;&=\; \sum_{\substack{\sigma_j\in\{\pm\}\\ j=1,\ldots,n}} \int \mathrm{d}\zeta \int \Big(\prod_{j=0}^{n} \mathrm{d}v_j\Big)\, \overline{\widehat{J}(\zeta, v_0)} \int \Big(\prod_{j=0}^{n-1} M(v_j - v_{j+1}, \sigma_{j+1})\Big) \\ &\times \int_0^T \Big(\prod_{j=0}^{n} \mathrm{d}\alpha_j\Big)\delta\Big(T - \sum_{j=0}^{n}\alpha_j\Big) \int_{-\infty}^{\infty} \mathrm{d}\underline{b}\; \chi_{\varepsilon^{-1}\underline{\alpha}}(\underline{b}) \\ &\times \prod_{j=0}^{n-1} 2\exp\Big\{ -2ib_j\Big[e(v_j) - e(v_n) + \Omega_j + \varepsilon(\mathrm{Re}\Psi_j - \mathrm{Re}\Psi_n) + O(\varepsilon^2\zeta^2)\Big]\Big\} \\ &\times\; \exp\Big\{ -i\zeta\cdot\sum_{j=0}^{n}\alpha_j\nabla e(v_j) + 2\sum_{j=0}^{n}\alpha_j\Big(\mathrm{Im}\Psi_j + O(\varepsilon\zeta^2)\Big)\Big\}\, \widehat{W}_{\gamma_\varepsilon}(\varepsilon\zeta, v_n)\;. \end{aligned}$$

It is easy to see that the error terms are negligible as $\varepsilon \to 0$ since T is fixed and ζ is essentially bounded by the decay of $\widehat{J}$. The $\mathrm{d}b_j$ integrations give $2\pi\delta\Big(e(v_j) - e(v_n) + \Omega_j\Big)$ delta functions on the energy shell as $\varepsilon \to 0$.

Following the analogous calculations on pp. 709-710 of [EY2], using the macroscopic profile of the initial state (1.25) and recalling the definition of the Boltzmann collision kernel (1.27)

$$\sigma(V,U) := 2\pi\sum_{\sigma=\pm} M(V-U,\sigma)\delta\Big(e(V) - e(U) + \sigma\omega(V-U)\Big)\;,$$

we obtain ($V_j \equiv v_j$)

$$\begin{aligned} \lim_{\varepsilon\to 0}\lim_{L\to\infty} C(n,\varepsilon^{-1}T) \;&=\; \int \mathrm{d}X\mathrm{d}V_0\ldots\mathrm{d}V_n\overline{J(X,V_0)} \int_0^T \left(\prod_{j=0}^{n} e^{2\alpha_j\mathrm{Im}\Psi_j}\mathrm{d}\alpha_j\right)\delta\left(T - \sum_{j=0}^{n}\alpha_j\right) \\ &\times\left(\prod_{j=0}^{n-1}\sigma(V_j, V_{j+1})\right) F_0\left(X - \sum_{j=0}^{n}\alpha_j\nabla e(V_j),\; V_n\right)\;. \end{aligned} \tag{10.16}$$

We also rearranged the energy conservation delta functions as

$$\prod_{j=0}^{n-1}\delta\Big(e(v_j) - e(v_n) + \Omega_j\Big) = \prod_{j=0}^{n-1}\delta\Big(e(v_j) - e(v_{j+1}) + \sigma_{j+1}\omega(v_j - v_{j+1})\Big)\;.$$

Noticing that

$$-2\mathrm{Im}\Psi_j = \int \sigma(U, V_j)\mathrm{d}U$$

by (4.12), we see that (10.16) is exactly the n-th order term in the Dyson series solution of (1.2). This completes the proof of the Main Theorem. $\square$

A Case of general ω

Here we show how to modify the proof of Lemma 3.1 if ω is not constant. The assumption that $\omega = (const.)$ was used only in Sections 7 and 9.

The proof of (3.30) for $\omega \neq (const.)$ will be given first. The necessary combinatorial structure is introduced in Section A.1 and the key analytic estimate (Proposition A.6) is proven in Section A.2. These sections show how to modify the argument in Section 7. The modifications for the proof of (3.31) were already explained in Section 8 both for constant and nonconstant ω.

Most of the proof of (3.32) in Section 9 is valid for arbitrary ω. The assumption $\omega = (const.)$ was used only when we estimated the nested pairing in Section 9.1.2 (proof of Lemma 9.3). The technical modifications for nonconstant ω are given in Appendix A.3.

A.1 Combinatorics for general ω

Since the momentum-dependence structure is more complicated, in order to perform the succesive integration of Section 7 we need more control on the structure of the pairing.

Definition A.1 *Fix a pairing $\pi \in \Pi_n$. A sequence of consecutive numbers $a, a+1, a+2, \ldots, a+h \subset \{1, 2, \ldots, n\}$ is called a* **down-stair** *of length h if $\pi(a) > \pi(a+1) > \ldots > \pi(a+h)$; and it is called an* **up-stair** *of length h if $\pi(a) < \pi(a+1) < \ldots < \pi(a+h)$. We always assume that $h \geq 1$.*

Notice that the elements in a stair must be consecutive, it is not sufficient if they just form a monotonic subsequence.

Definition A.2 *A set of κ up-stairs;*

$$a_1, a_1+1, \ldots, a_1+h_1 \;;$$

$$a_2, a_2+1, \ldots, a_2+h_2 \;;$$

$$\vdots$$

$$a_\kappa, a_\kappa+1, \ldots, a_\kappa+h_\kappa$$

is called an **increasing κ-staircase** *if it satisfies the following properties:*

(i) $a_j + h_j < a_{j+1}$ *for* $j = 1, 2, \ldots, \kappa - 1$*;*

(ii) Each $a_j + h_j$ *is a peak;*

(iii) The π*-images of the peaks are increasing:*

$$\pi(a_1 + h_1) < \pi(a_2 + h_2) < \ldots < \pi(a_\kappa + h_\kappa) ;$$

(iv) There is no peak p *with* $a_j < p < a_{j+1}$, $\pi(p) > \pi(a_j + h_j)$, $j = 1, 2, \ldots, \kappa - 1$*;*

(v) The images of the stairs minimally overlap, i.e.,

$$\pi(a_{j+1} + 1) > \pi(a_j + h_j) > \pi(a_{j+1}) \qquad j = 1, 2, \ldots, \kappa - 1 .$$

The numbers $a_1 + h_1, a_2 + h_2, \ldots, a_\kappa + h_\kappa$ *are called the* **tips** *of the stairs, the numbers* $a_1, a_2, \ldots, a_\kappa$ *are the* **bottoms** *of the stairs.*

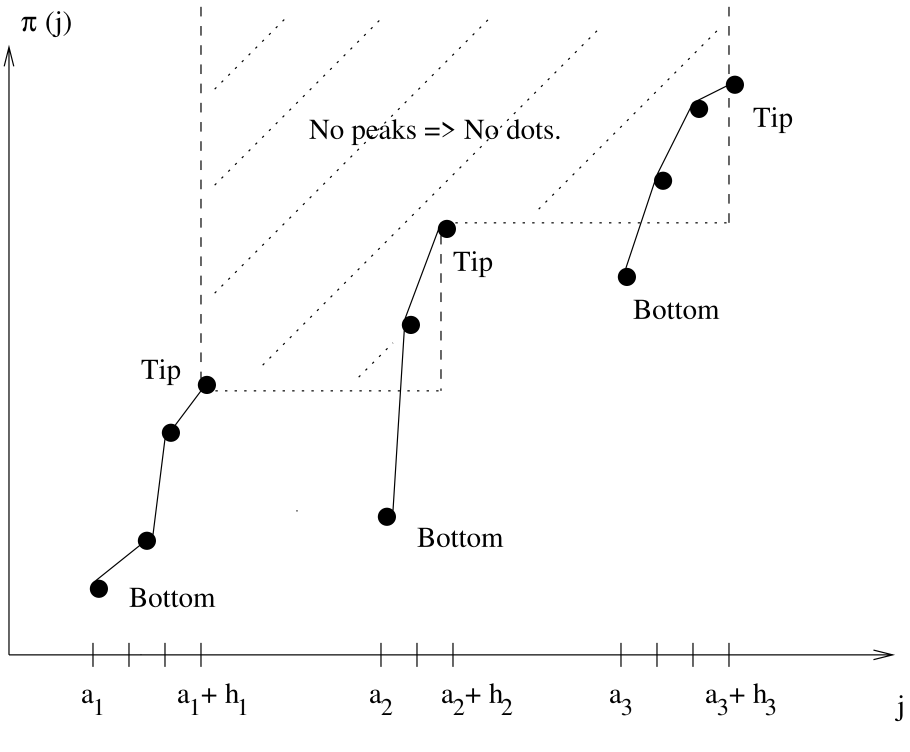

Fig 11. Graph of a permutation with an increasing κ=3 staircase.

In other words, an increasing κ-staircase consists of κ up-stairs with lengths $h_1, h_2, \ldots, h_\kappa$ in such a way that every up-stair ends just above where the next one starts, i.e., if we removed the bottom from each stair then the stairs would not overlap. (See Fig. 11). Moreover, there is no peak between two stairs which would be higher than the tip of the lower stair. This implies, in particular, that there is no dot at all in the shaded regime, i.e., there is no p with $a_j < p < a_{j+1}$, $\pi(p) > \pi(a_j+h_j)$, $j = 1, 2, \ldots, \kappa-1$. Note that the bottoms are not necessarily valleys.

Similarly we define:

Definition A.3 *A set of κ down-stairs;*

$$a_1, a_1+1, \ldots, a_1+h_1 \; ;$$

$$a_2, a_2+1, \ldots, a_2+h_2 \; ;$$

$$\vdots$$

$$a_\kappa, a_\kappa+1, \ldots, a_\kappa+h_\kappa$$

is called a **decreasing κ-staircase** *if it satisfies the following properties:*

(i) $a_j + h_j < a_{j+1}$ for $j = 1, 2, \ldots, \kappa-1$;

(ii) Each a_j is a peak;

(iii) The π-images of the peaks are decreasing:

$$\pi(a_1) > \pi(a_2) > \ldots > \pi(a_\kappa) \; ;$$

(iv) There is no peak p with $a_j < p < a_{j+1}$, $\pi(p) > \pi(a_{j+1})$, $j = 1, 2, \ldots, \kappa-1$;

(v) The images of the stairs minimally overlap, i.e.,

$$\pi(a_j+h_j-1) > \pi(a_{j+1}) > \pi(a_j+h_j) \; , \qquad j = 1, 2, \ldots, \kappa-1 \; .$$

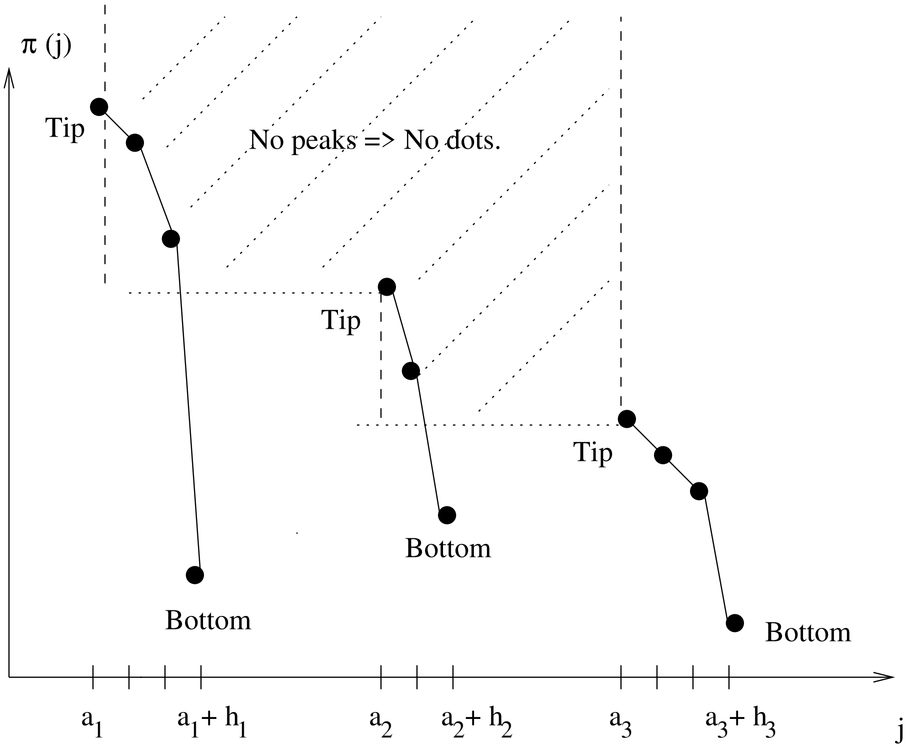

Fig 12. Graph of a permutation with a decreasing κ=3 staircase.

The following Proposition shows that essentially every permutation has either an increasing or a decreasing κ-staircase. This is again a Ramsey type theorem. However, the estimate is very bad in κ, so practically it can be used for finite κ only.

Proposition A.4 *For any fixed κ, with the exception of*

$$n^{4\cdot 4^{\kappa-1}+3}(2\cdot 4^{\kappa-1}+2)^n$$

pairings, a pairing $\pi \in \Pi_n$ either contains an increasing or a decreasing κ-staircase (or both).

Proof. We need the following lemma whose proof is given in the Appendix B.

Lemma A.5 *Suppose that a permutation has at least $\binom{\alpha+\beta}{\alpha}$ peaks. Then it has either an increasing $(\alpha+1)$-staircase or it has a decreasing $(\beta+1)$-staircase (or both).*

From this Lemma and Lemma 7.2 the proof of Proposition A.4 follows easily. Choose $\alpha = \beta = \kappa - 1$, and use that $\binom{2\kappa-2}{\kappa-1} < 4^{\kappa-1}$. $\square$

In the next section we will prove

Proposition A.6 *Let $n \leq N$, $\underline{m}, \underline{\widetilde{m}} \in \mathcal{M}(n, N)$. Let $\omega \neq$ (const.), satisfying the assumptions in Section 1.2, and assume that $\pi \in \Pi_n$ has either an increasing κ-staircase or a decreasing κ-staircase. Then*

$$|C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \leq t^{-\kappa+2}(C\lambda^2 t)^N (\log^* t)^{n+\kappa+2} .$$

From these statements (3.30) follows for the $\omega \neq$ (*const.*) case. Choose $\kappa = 8$, then all contributions which have a monotonic 8-staircase can be included into the second term in (3.30). The exceptional $n^{4\cdot 4^7+3}(2 \cdot 4^7 + 2)^n \leq C^n$ pairings can be estimated by the apriori bound from Lemma 5.1 and be included in the first term in (3.30).

Remark: In general, we gain $t^{-\kappa+2}$ with the exception of $n^{4\cdot 4^{\kappa-1}+3}(2 \cdot 4^{\kappa-1} + 2)^n$ pairings.

A.2 Estimate of the indirect term for the nonconstant ω case.

Proof of Proposition A.6. The difficulty is that there are momentum dependences within the argument of ω. This makes it impossible to eliminate all the designated $\widetilde{R}$ factors by integrating out the peak variables as in (7.2). In Section 7.2 when we integrated out p_{a_m-1} and p_{a_m}, we used that only three denominators (two R and one $\widetilde{R}$ factors) depended on these variables (see (7.3)). In case of nonconstant ω, still only these three denominators depend on p_{a_m-1} and p_{a_m} as far as the electron kinetic energy is concerned, but due to the cumulation of ω's in the form of $\pm\omega(p_{\ell+1} - p_{\ell+2}) \pm \omega(p_{\ell+2} - p_{\ell+3}) \ldots$ in R_ℓ, all R_j factors depend on p_{a_m-1} and p_{a_m} for $j < a_m$.

Hence we have to integrate out all p_j in consecutive order: $p_1, p_2, \ldots$. This means that the designated $\widetilde{R}_{\pi(a_m)-1}$ factors are eliminated in an alternating order with the regular R_j factors. One has to ensure that when we integrate out p_j with some $a_{m-1} < j < a_m - 1$, the designated $\widetilde{R}_{\pi(a_i)-1}$, $i \geq m$, denominators are not affected since we cannot integrate out a variable which appear in many $\widetilde{R}$ factors.

The staircase construction of Section A.1 is designed to ensure these restricted dependences so that the $p_1, p_2, \ldots$ integrations could be done in this order.

There is one additional difficulty: the staircase controls the ordering of the momenta between the first and the last stairs, but it does not control the ordering before and after the staircase. The natural idea to consider the "first" staircase does not work (the first staircase could be too short).

So we have to separate the staircase from the rest, unless $a_1 = 1$ or $\pi(a_1) = 1$. We use the semigroup property of the measure $\int^{t*}[\mathrm{d}s_j]_0^n$:

$$\int_0^{t*}[\mathrm{d}s_j]_0^{n+m} = \int_0^t \mathrm{d}s\Big(\int_0^{s*}[\mathrm{d}s_j]_0^n\Big)\Big(\int_0^{(t-s)*}[\mathrm{d}s_j]_{n+1}^m\Big)\ , \tag{A.1}$$

and we will lose a t-factor by estimating the outer integral trivially. This is why the estimate in Proposition A.6 is weaker by a t^2 factor than that in Proposition 7.4.

A.2.1 Estimate for increasing staircase

Assume that π has an increasing κ-staircase with a_j bottoms and $a_j + h_j$ tips for $1 \le j \le \kappa$ (see Definition A.2). Our starting point is (4.44). We use (A.1) for the time integrals in (4.44) as follows:

$$\begin{aligned}
&\int_0^{t*}[\mathrm{d}s_j]_0^N\left[\prod_{j=0}^n \prod_{b\in I_j\cup I_j^c} e^{-is_b[e(p_{\mu(j)}+\chi_b k_b)+\Omega_j+\chi_b\sigma_b\omega(k_b)]}\right] \\
&= \int_0^t \mathrm{d}s\left(\int_0^{s*}[\mathrm{d}s_j]_0^{\mu(a_1)-1}\left[\prod_{j=0}^{a_1-1}\prod_{b\in I_j\cup I_j^c} e^{-is_b[e(p_{\mu(j)}+\chi_b k_b)+\Omega_j-\Omega_{a_1-1}+\chi_b\sigma_b\omega(k_b)]}\right]\right)e^{-is\Omega_{a_1-1}} \\
&\quad\times\left(\int_0^{(t-s)*}[\mathrm{d}s_j]_{\mu(a_1)}^N\left[\prod_{j=a_1}^n\prod_{b\in I_j\cup I_j^c} e^{-is_b[e(p_{\mu(j)}+\chi_b k_b)+\Omega_j+\chi_b\sigma_b\omega(k_b)]}\right]\right)\ ,
\end{aligned} \tag{A.2}$$

i.e., we separated the $\mathrm{d}s_j$, $j < \mu(a_1)$, integrations from the rest by prescribing their total sum to be s. We also pulled out a factor $e^{-is\Omega_{a_1}-1}$ which shifted all Ω_j's in the first group of propagators by $\Omega_{a_1} - 1$. Notice that

$$\Omega_{a_1-1} - \Omega_j = \sum_{m=j+1}^{a_1-1} \sigma_m\omega(k_m)\ , \quad j \le a_1 - 1\ , \tag{A.3}$$

depends only on $k_{j+1}, \ldots, k_{a_1-1}$, and Ω_j for $j \ge a_1$ depends only on $k_{a_1+1}, \ldots, k_n$.

The decomposition of the $\mathrm{d}\tilde{s}_j$ integrals is similar; we separate the $\mathrm{d}\tilde{s}_j$, $j < \tilde{\mu}(\pi(a_1))$, integrations (their sum is $\tilde{s}$) and pull out a factor $e^{i\tilde{s}\widetilde{\Omega}_{\pi(a_1)-1}}$. We again see that the k-dependences of

$$\widetilde{\Omega}_j - \widetilde{\Omega}_{\pi(a_1)-1} = \sum_{\substack{m \\ j<\pi(m)<\pi(a_1)}} \sigma_m\omega(k_m)\ ,$$

for $j \le \pi(a_1) - 1$, and those of

$$\widetilde{\Omega}_j = \sum_{m\,:\,\pi(m)>j} \sigma_m\omega(k_m)\ ,$$

for $j \geq \pi(a_1)$, are separated.

Now we use (4.50) for each time integral in (A.2) separately, we integrate out k_b, $b \in J$, and similarly for $\tilde{k}_{\tilde{b}}, \tilde{b} \in \tilde{J}$ as in (4.51). We then estimate everything by absolute value, we use (4.8) and (5.8) and we estimate the two outside time integrals $(\mathrm{d}s, \mathrm{d}\tilde{s})$ trivially. In this last step we lose an extra t^2. We also integrate out $\tilde{p}_n$ and all $\tilde{k}_j$ and express everything as a function of k_j and p_n. The result, similarly to (6.2), is

$$
\begin{aligned}
|C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \;\leq\; & (C\lambda)^{2N} t^{2+2|\underline{m}|} \sup_{\underline{\sigma}} \int \mathrm{d}\nu^*_\pi(p_n, \tilde{p}_n, \underline{k}, \underline{\tilde{k}}) \\
& \times \int_{-\infty}^{\infty} \mathrm{d}\alpha \mathrm{d}\beta \prod_{j=0}^{a_1-1} |S_j| \prod_{j=a_1}^{n} |R_j| \int_{-\infty}^{\infty} \mathrm{d}\tilde{\alpha} \mathrm{d}\tilde{\beta} \prod_{j=0}^{\pi(a_1)-1} |\widetilde{S}_j| \prod_{j=\pi(a_1)}^{n} |\widetilde{R}_j| \,,
\end{aligned}
\tag{A.4}
$$

where in addition to recalling the definition of $R_j, \widetilde{R}_j$ (4.23), we define

$$
S_j := S_j(\beta, p_j, \Omega_j - \Omega_{a_1-1}, \eta) := \frac{1}{\beta - e(p_j) - (\Omega_j - \Omega_{a_1-1}) + i\eta} \,,
$$

$$
\widetilde{S}_j := \widetilde{S}_j(\tilde{\beta}, \tilde{p}_j, \widetilde{\Omega}_j - \widetilde{\Omega}_{\pi(a_1)-1}, \eta) := \frac{1}{\tilde{\beta} - e(\tilde{p}_j) - (\widetilde{\Omega}_j - \widetilde{\Omega}_{\pi(a_1)-1}) - i\eta} \,,
$$

and, as usual, $p_j, \tilde{p}_j$ are viewed as functions of p_n and $\underline{k}$ (4.24).

If $a_1 = 1$ or $\pi(a_1) = 1$ then there is no need for separation, i.e., there are no additional $\mathrm{d}\beta$ or $\mathrm{d}\tilde{\beta}$ integrations. In this case (6.2) can be used directly. We will not discuss this simpler case in detail.

We introduce

$$
b_j := a_j + h_j \,, \qquad 1 \leq j \leq \kappa
$$

for the location of the tips. All factors in (A.4) with tilde except $\widetilde{R}_n$ and $\widetilde{R}_{\pi(b_j)-1}$, $j = 1, 2, \ldots, \kappa$, are estimated trivially by (Ct), this gives $(Ct)^{n-\kappa}$. Moreover, we make sure that we gain a $\langle\tilde{\alpha}\rangle^{-1}\langle\tilde{\beta}\rangle^{-2}$ factor from these L^∞-estimates, exactly as in the constant ω case, using (6.16). Notice that we need to gain a second $\langle\tilde{\beta}\rangle^{-1}$-factor to make the $\mathrm{d}\tilde{\beta}$ integration finite. This is possible only if $\pi(a_1) > 1$, but there is no need for $\mathrm{d}\tilde{\beta}$ integration at all if $\pi(a_1) = 1$. We also insert an explicit extra $\prod_j \langle k_j \rangle^{-4}$ decay. Since all $\tilde{\beta}$ denominators are estimated in L^∞ norm, at the end we can perform the $\mathrm{d}\tilde{\beta}$ integration. We get

$$
\begin{aligned}
|C_{\underline{m},\underline{\widetilde{m}},\pi}(t)| \leq\; & t^{-\kappa+2} (C\lambda^2 t)^N \sup_{\underline{\sigma}} \int \Big(\prod_{j=1}^{n} \frac{L^*(k_j)\delta(k_j - \tilde{k}_{\pi(j)}) \mathrm{d}k_j \mathrm{d}\tilde{k}_j}{\langle k_j \rangle^4} \Big) \int \mathrm{d}p_n \langle p_n \rangle^{d+12} \widehat{\gamma}_e(p_n, p_n) \\
& \times \int_{-\infty}^{\infty} \mathrm{d}\alpha \mathrm{d}\beta \Big(\frac{1}{\langle p_{a_1-2} \rangle^2 \langle p_{a_1-1} \rangle^{d+1}} \prod_{j=0}^{a_1-1} |S_j| \Big) \Big(\frac{1}{\langle p_{b_1-1} \rangle^2} \prod_{j=a_1}^{n} |R_j| \Big) \Big(\int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{\alpha}}{\langle\tilde{\alpha}\rangle} \prod_{m=1}^{\kappa} |\widetilde{R}_{\pi(b_m)-1}| \Big)
\end{aligned}
$$

with

$$L^*(k) := M^*(k)\langle k\rangle^{2d+22} \tag{A.5}$$

so $L^*(k) \le C\langle k\rangle^{-2d-2}$. Notice that we also inserted $\langle p_{a_1-2}\rangle^{-2}\langle p_{a_1-1}\rangle^{-d-1}\langle p_{b_1-1}\rangle^{-2}$ using (6.8) to prepare for the decays in β and α at the expense of increasing the $\langle p_n\rangle$-power and using up a few $\prod_j\langle k_j\rangle$ power from M^*. If $a_1 = 1$, then there is no need for $\langle p_{a_1-2}\rangle^{-2}$ insertion.

We express the electron momentum in $\widetilde{R}_{\pi(b_m)-1}$ as $\tilde{p}_{\pi(b_m)-1} = \tilde{p}_{\pi(b_m)} + p_{b_m-1} - p_{b_m}$.

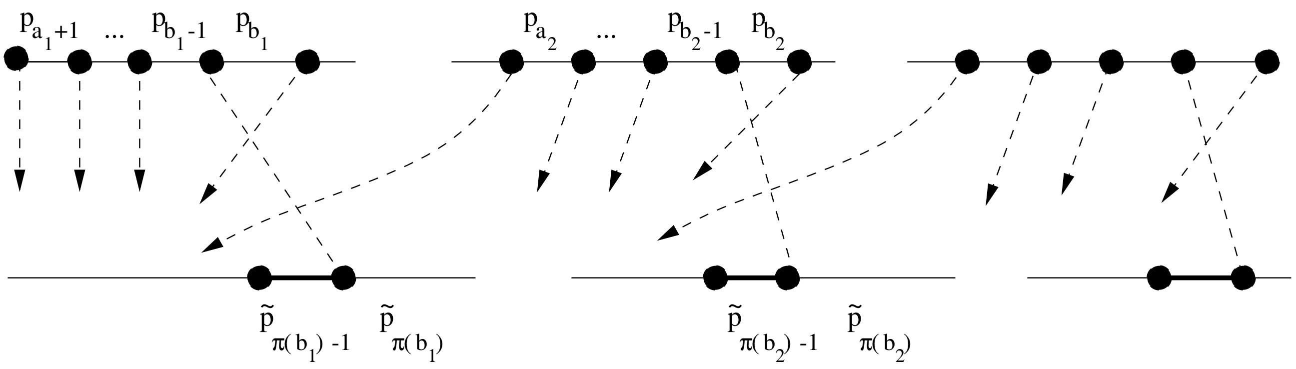

Fig. 13. Increasing staircase.

Fig. 13 shows an increasing staircase; the bold lines indicate the electron momenta in those $\widetilde{R}_j$'s which we kept ($j = \pi(b_m) - 1$, $m = 1, \ldots, \kappa$).

Now we change variables; we use $p_0, p_1, \ldots, p_{n-1}$ instead of $k_1, k_2, \ldots, k_n$. Notice that for each $m \le \kappa$ the factor $\widetilde{R}_{\pi(b_m)-1}$, i.e., the momentum $\tilde{p}_{\pi(b_m)-1}$ and $\widetilde{\Omega}_{\pi(b_m)-1}$ do not depend on p_j, $a_1 \le j < b_m - 1$. This follows from the staircase construction.

First we estimate the factor $L^*(p_{a_1-1} - p_{a_1})$ by a constant. In this way the p-dependence of the remaining L^* factors are separated. The inserted $\langle p_{a_1-1}\rangle^{-d-1}$ factor ensures that the integration of the momenta before the staircase is still finite.

We then integrate out $p_{a_1}, p_{a_1+1}, \ldots, p_{b_1-2}$, these variables do not appear in $\widetilde{R}_{\pi(b_m)-1}$, $m = 1, \ldots, \kappa$, or in S_j, $j \le a_1 - 1$ (see (A.3)). We get a $(C\log^* t)^{h_1-1}$ factor by (6.11) and we have eliminated R_j, $j = a_1, \ldots, b_1 - 2$. For the p_{b_1-1} and p_{b_1} integrations we are in a situation similar to (7.3). We explicitly write out the factors R_{b_1-1}, R_{b_1} and $\widetilde{R}_{\pi(b_1)-1}$ that are involved and we obtain

$$\int \frac{L^*(p_{b_1-1} - p_{b_1})L^*(p_{b_1} - p_{b_1+1})\mathrm{d}p_{b_1-1}\mathrm{d}p_{b_1}}{\langle p_{b_1-1}\rangle^2\Big|\alpha - e(p_{b_1-1}) + i\eta \pm \omega(p_{b_1-1} - p_{b_1}) \pm \omega(p_{b_1} - p_{b_1+1}) - \Omega_{b_1+1}\Big|} \tag{A.6}$$

$$
\times \frac{1}{\Big|\alpha - e(p_{b_1}) + i\eta \pm \omega(p_{b_1} - p_{b_1+1}) - \Omega_{b_1+1}\Big|}
$$

$$
\times \frac{1}{\Big|\tilde{\alpha} - e\Big(\tilde{p}_{\pi(b_1)} + p_{b_1-1} - p_{b_1}\Big) + i\eta \pm \omega(p_{b_1-1} - p_{b_1}) - \widetilde{\Omega}_{\pi(b_1)-1}\Big|} \leq \frac{(C \log^* t)^3}{\langle \alpha - \Omega_{b_1+1} \rangle} .
$$

Notice that $\tilde{p}_{\pi(b_1)}$ is independent of $p_{a_1}, p_{a_1+1}, \dots, p_{b_1}$, i.e., of all the variables we have integrated out so far. The prototype of this inequality is

$$
\sup_{q,r,\tilde{\theta}} \int \frac{\langle v-u \rangle^{-d-1} \langle u-q \rangle^{-d-1} \mathrm{d}v\mathrm{d}u}{\langle v \rangle^2 \Big|\theta - e(v) + i\eta \pm \omega(v-u) \pm \omega(u-q)\Big|} \tag{A.7}
$$

$$
\times \frac{1}{\Big|\theta - e(u) + i\eta \pm \omega(u-q)\Big|} \frac{1}{\Big|\tilde{\theta} - e(r+v-u) + i\eta \pm \omega(v-u)\Big|} \leq \frac{(C \log^* t)^3}{\langle \theta \rangle}
$$

(with $v = p_{b_1-1}$, $u = p_{b_1}$, $q = p_{b_1+1}$, $r = \tilde{p}_{\pi(b_1)}$, $\theta = \alpha - \Omega_{b_1+1}$, $\tilde{\theta} = \tilde{\alpha} - \widetilde{\Omega}_{\pi(b_1)-1}$) which is proven similarly to (7.4). First we perform the $\mathrm{d}v$ integration using (6.19), then (6.21) estimates the $\mathrm{d}u$ integration. We can change the decaying factor $\langle \alpha - \Omega_{b_1+1} \rangle^{-1}$ in (A.6) into $\langle \alpha \rangle^{-1}$ at the expense of the product of $\langle k_j \rangle^2 = \langle p_j - p_{j-1} \rangle^2$ factors ($j = b_1 - 1, \dots, n$) similarly to (6.13).

Next we integrate out $p_{b_1+1}, p_{b_1+2}, \dots, p_{a_2}, \dots, p_{a_2+h_2-2} = p_{b_2-2}$. By the staircase property (especially (iv) in Definition A.2), these variables do not appear in the remaining factors $\widetilde{R}_{\pi(b_j)-1}$, $j \geq 2$, hence they freely give $C \log^* t$ factors and they eliminate R_j, $j = b_1 + 1, \dots, b_2 - 2$. Now we integrate out p_{b_2-1} and p_{b_2} exactly as in (A.6), etc.

Once we are done with all p_{b_m-1} and p_{b_m} ($1 \leq m \leq \kappa$) integrations, then there are no more $\widetilde{R}$ factors left. So we integrate out the remaining variables, $p_0, p_1, \dots, p_{a_1-1}$, and $p_{b_m+1}, \dots, p_{n-1}$ in this order; it is easy to see that at each step only one S_j or R_j factors depends on these variables. At the p_{a_1-2} and p_{a_1-1} integration we use

$$
\int \frac{L^*(p_{a_1-2} - p_{a_1-1}) \mathrm{d}p_{a_1-2}}{\Big|\beta - e(p_{a_1-2}) + i\eta \pm \omega(p_{a_1-2} - p_{a_1-1})\Big| \langle p_{a_1-2} \rangle^2} \leq \frac{C \log^* t}{\langle \beta \rangle} \tag{A.8}
$$

and

$$
\int \frac{\mathrm{d}p_{a_1-1}}{\Big|\beta - e(p_{a_1-1}) + i\eta\Big| \langle p_{a_1-1} \rangle^{d+1}} \leq \frac{C \log^* t}{\langle \beta \rangle} \tag{A.9}
$$

that follow from (6.12) and (6.11). The extra $\langle \beta \rangle^{-2}$ factor makes the $\mathrm{d}\beta$ integration convergent.

Finally from the $\alpha, \tilde{\alpha}$ and p_n integration we obtain a factor $(C \log^* t)^2$ similarly to (6.14). This completes the proof of Proposition A.6 for the case of increasing κ-staircase.

A.2.2 Estimate for decreasing staircase

The proof for the decreasing staircase is very similar. Here we chop off the end of the expansion. Let $b_j := a_j + h_j$ as before, but now these are the bottoms of the stairs. Then instead of (A.2) we split the s_j variables into two groups: $j = 0, 1, \ldots, \mu(b_\kappa) - 1$ and $j = \mu(b_\kappa), \ldots, N$. The result, analogous to (A.4), is

$$
\begin{aligned}
|C_{\underline{m},\widetilde{\underline{m}},\pi}(t)| \;\; &\leq \;\; (C\lambda)^{2N} t^{2+2|\underline{m}|} \sup_{\underline{\sigma}} \int \mathrm{d}\nu^*_\pi(p_n, \tilde{p}_n, \underline{k}, \underline{\tilde{k}}) \\
&\quad \times \int_{-\infty}^{\infty} \mathrm{d}\alpha \mathrm{d}\beta \prod_{j=0}^{b_\kappa - 1} |S_j| \prod_{j=b_\kappa}^{n} |R_j| \int_{-\infty}^{\infty} \mathrm{d}\tilde{\alpha}\mathrm{d}\tilde{\beta} \prod_{j=0}^{\pi(b_\kappa)-1} |\widetilde{S}_j| \prod_{j=\pi(b_\kappa)}^{n} |\widetilde{R}_j| \,,
\end{aligned}
$$

where S_j and $\widetilde{S}_j$ have been redefined accordingly:

$$
S_j := \frac{1}{\beta - e(p_j) - (\Omega_j - \Omega_0) + i\eta}, \qquad \widetilde{S}_j := \frac{1}{\tilde{\beta} - e(\tilde{p}_j) - (\widetilde{\Omega}_j - \widetilde{\Omega}_{\pi(b_\kappa)-1}) - i\eta} \,.
$$

Notice that we pulled out the factor $e^{-is\Omega_0}$ and $e^{i\tilde{s}\widetilde{\Omega}_{\pi(b_\kappa)-1}}$ in the analogue of (A.2). In this way, S_j depends on $p_0, \ldots, p_j$ and $\widetilde{S}_j$ depends on $\tilde{p}_j, \ldots, \tilde{p}_{\pi(b_\kappa)-1}$.

All factors with tilde except $\widetilde{R}_n$ and $\widetilde{S}_{\pi(a_m)-1}$, $m = 1, 2, \ldots, \kappa$, are estimated trivially, and we can ensure the decaying $\langle\tilde{\alpha}\rangle^{-1}\langle\tilde{\beta}\rangle^{-2}$ terms by inserting approriate factors at the expense of $\langle p_n\rangle$ and $\langle k_j\rangle$ powers as before. We get

$$
\begin{aligned}
&|C_{\underline{m},\widetilde{\underline{m}},\pi}(t)| \leq t^{-\kappa+2}(C\lambda^2 t)^N \sup_{\underline{\sigma}} \int \Big(\prod_{j=1}^{n} \frac{L^*(k_j)\delta(k_j - \tilde{k}_{\pi(j)})\mathrm{d}k_j \mathrm{d}\tilde{k}_j}{\langle k_j\rangle^4} \Big) \int \mathrm{d}p_n \langle p_n\rangle^{2d+12} \widehat{\gamma}_e(p_n, p_n) \\
&\times \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha\mathrm{d}\beta\mathrm{d}\tilde{\alpha}\mathrm{d}\tilde{\beta}}{\langle\tilde{\alpha}\rangle\langle\tilde{\beta}\rangle^2 \, \langle p_{a_1-2}\rangle^{d+1}\langle p_{b_\kappa-1}\rangle^{d+1}\langle p_{a_\kappa-1}\rangle^2\langle p_{b_\kappa}\rangle^2} \Bigg(\prod_{j=0}^{b_\kappa-1} |S_j| \Bigg) \Bigg(\prod_{j=b_\kappa}^{n} |R_j| \Bigg) \, |\widetilde{R}_n| \prod_{m=1}^{\kappa} |\widetilde{S}_{\pi(a_m)-1}|
\end{aligned}
$$

and we will again use the variables $p_0, p_1, \ldots, p_{n-1}$ instead of $k_1, \ldots, k_n$.

We first estimate the factors $L^*(p_{a_1-2} - p_{a_1-1})$ and $L^*(p_{b_\kappa-1} - p_{b_\kappa})$ by a constant. Then the integrations of $p_{b_\kappa-1}, \ldots, p_{a_\kappa+1}$ can be done successively in this order; no $\widetilde{S}_j$ factor depends on these and we collect a $(C\log^* t)$ factor from eliminating each S_j. We then perform $\mathrm{d}p_{a_\kappa-1}, \mathrm{d}p_{a_\kappa}$ integrations, very similarly to (A.6):

$$
\begin{aligned}
&\int \frac{L^*(p_{a_\kappa-2} - p_{a_\kappa-1})L^*(p_{a_\kappa-1} - p_{a_\kappa})\mathrm{d}p_{a_\kappa-1}\mathrm{d}p_{a_\kappa}}{\langle p_{a_\kappa-1}\rangle^2 \big|\beta - e(p_{a_\kappa-1}) + i\eta \pm \omega(p_{a_\kappa+1} - p_{a_\kappa}) \pm \omega(p_{a_\kappa} - p_{a_\kappa-1}) - \Omega_{a_\kappa+1}\big|} \qquad \text{(A.10)} \\
&\qquad\qquad \times \frac{1}{\big|\beta - e(p_{a_\kappa}) + i\eta \pm \omega(p_{a_\kappa+1} - p_{a_\kappa}) - \Omega_{a_\kappa}\big|} \\
&\quad \times \frac{1}{\Big|\tilde{\alpha} - e\Big(\tilde{p}_{\pi(a_\kappa)} + p_{a_\kappa-1} - p_{a_\kappa}\Big) + i\eta \pm \omega(p_{a_\kappa} - p_{a_\kappa-1}) - \widetilde{\Omega}_{\pi(b_\kappa)-1}\Big|} \;\; \leq \;\; \frac{(C\log^* t)^3}{\langle\beta - \Omega_{a_k+1}\rangle} \,.
\end{aligned}
$$

Again, we used that $\tilde{p}_{\pi(a_\kappa)}$ does not depend on the momenta integrated out so far. We perform $p_{a_\kappa-2}, p_{a_\kappa-3}, \ldots, p_{b_{\kappa-1}}, \ldots, p_{a_{\kappa-1}+1}$ successively; no remaining $\widetilde{S}_j$ factors depend on them. We then perform $p_{a_{\kappa-1}}, p_{a_{\kappa-1}-1}$ exactly as in (A.10), etc. Once we are done with p_{a_1}, p_{a_1-1}, there are no $\widetilde{S}_j$ factors left, and the rest can be integrated out successively: $p_0, p_1, \ldots, p_{n-1}$. The $\mathrm{d}p_{a_1-2}$ integration uses the $\langle \mathrm{d}p_{a_1-2}\rangle^{-d-1}$ decay since there is no L^* factor left. We also gain a $\langle\beta\rangle^{-1}$ factor from this integration similarly to (A.9). The decay $\langle \beta - \Omega_{a_k+1}\rangle^{-1}$ is changed to $\langle\beta\rangle^{-1}$ using the extra decaying factors similarly to (6.13). This ensures the finiteness of the β integration. From the p_{b_κ} integration we gain an extra $\langle\alpha\rangle^{-1}$ because of the inserted $\langle p_{b_\kappa}\rangle^{-2}$.

Finally the $\alpha, \tilde{\alpha}, p_n$ integrations are done exactly as in (6.14), which completes the proof of Proposition A.6. $\square$

A.3 Estimate of the nested term for the nonconstant ω case.

In Section 9.1 we proved (9.9) for $a \geq 3$ and for $a = 2$ with $\omega = (const.)$. Here we outline the argument for the remaining case when $\omega \neq (const.)$, $a = 2$. Only the proof of Lemma 9.3 needs to be modified, the rest of the argument from Section 9.1 is unchanged.

Proof of Lemma 9.3 for $\omega \neq (const.)$. The proof is an extension of the argument in Section 3.4 of [EY2] and we only indicate the main steps. We write

$$A = i^{m+1} \int M(k) \int_0^\infty s^m e^{is(\alpha - \Phi(p,k) + i\eta)} \Big[\Upsilon_\eta(\alpha - \omega(k), p + k)\Big]^m \mathrm{d}s\mathrm{d}k \ , \tag{A.11}$$

where we omitted σ_2 and we let $m = m_1$, $p = p_2$, and $\Omega_2 = 0$ for simplicity. By a partition of unity we divide the space into cubes Q of size $\tilde{\varrho}$. Correspondingly, we can replace $M(k)$ in (A.11) by $M_Q(k)$ that is supported on a cube Q and $M = \sum_Q M_Q$. We can assume that M_Q is as smooth as M. Moreover, $\|M_Q\|_\infty \leq C\langle \mathrm{dist}(0, Q)\rangle^{-2d}$ and similar estimates are valid for its derivatives, since M_Q inherits the size of M on Q. We denote the corresponding expression by A_Q

$$A_Q = A_Q(\alpha, p) := i^{m+1} \int M_Q(k) \int_0^\infty s^m e^{is(\alpha - \Phi(p,k) + i\eta)} \Big[\Upsilon_\eta(\alpha - \omega(k), p + k)\Big]^m \mathrm{d}s\mathrm{d}k \ ,$$

and we neglect the p variable which we consider fixed.

After a diffeomorphic change of the k variable we arrive at one of the following normal forms depending on whether Q contains a critical point or not:

$$A_Q = \int E_Q(\tilde{\alpha}, k) \int_0^\infty s^m e^{is(\tilde{\alpha} - k^2 + i\eta)} \mathrm{d}s\mathrm{d}k \ ; \tag{A.12}$$

or

$$A_Q = \int E_Q(\tilde{\alpha}, k) \int_0^\infty s^m e^{is(\tilde{\alpha} - u\cdot k + i\eta)} \mathrm{d}s \mathrm{d}k \ ,$$

where $u = u_Q$ is a constant unit vector, $\tilde{\alpha} = \tilde{\alpha}_Q \in \mathbf{R}$, and we defined

$$E_Q(\tilde{\alpha}, k) := i^{m+1} M_Q(\psi(k)) \Big[\Upsilon_\eta(\tilde{\alpha} - \omega(\psi(k)), \psi(k) + p) \Big]^m J(k) \ .$$

Here the function ψ expresses the change of variable, J is the corresponding Jacobian. These are as smooth functions as Φ and ω and they depend on Q and p with uniform bounds.

We will discuss the more complicated first case, the other case is similar. We rewrite the $\mathrm{d}k$ integration as follows

$$\int e^{-isk^2} E_Q(\tilde{\alpha}, k) \mathrm{d}k = (2\pi)^{-d/2} \int_0^\infty e^{-is\vartheta} \bigg(\int_{k^2=\vartheta} E_Q(\tilde{\alpha}, k) d\sigma(k) \bigg) \vartheta^{d/2-1} \mathrm{d}\vartheta \ ,$$

where $d\sigma(k)$ is the normalized surface measure on the sphere $\{k \ : \ k^2 = \vartheta\}$. Let

$$\begin{aligned} H_Q(\alpha, k) : \ &= \ -i\vartheta^{-d/2+1} \frac{\partial}{\partial\vartheta} \bigg(\vartheta^{d/2-1} \int_{\tilde{k}^2=\vartheta} E_Q(\tilde{\alpha}, \tilde{k}) d\sigma(\tilde{k}) \bigg) \bigg|_{\vartheta := k^2} \qquad \text{(A.13)} \\ &= \ -i(\tfrac{d}{2} - 1)k^{-2} \int_{\tilde{k}^2=k^2} E_Q(\tilde{\alpha}, \tilde{k}) d\sigma(\tilde{k}) - i \frac{\partial}{\partial\vartheta} \bigg(\int_{\tilde{k}^2=\vartheta} E_Q(\tilde{\alpha}, \tilde{k}) d\sigma(\tilde{k}) \bigg) \bigg|_{\vartheta := k^2} . \end{aligned}$$

Using integration by parts

$$\int e^{-isk^2} E_Q(\tilde{\alpha}, k) \mathrm{d}k = s^{-1} \int e^{-isk^2} H_Q(\tilde{\alpha}, k) \mathrm{d}k \ ,$$

hence after undoing the $\mathrm{d}s$ integration in (A.12),

$$A_Q = i^m \int \frac{H_Q(\tilde{\alpha}, k) \mathrm{d}k}{(\tilde{\alpha} - k^2 + i\eta)^m} \ .$$

We show that

$$|H_Q(\tilde{\alpha}, k)| \le C_Q(\eta^{-1/2} + |k|^{-2}) \ . \qquad \text{(A.14)}$$

H_Q consists of two terms (A.13), the first one is easy since $E_Q(\tilde{\alpha}, k)$ is uniformly bounded from (4.8). For the second term, we use the bound on the derivative of $\Upsilon_\eta(\alpha, p)$ in p and α (4.9). All the other factors in E_Q have derivatives bounded by a constant.

Since $k \mapsto H_Q(\tilde{\alpha}, k)$ is compactly supported, (A.14) gives an estimate of order $C_Q t^{m-1/2}$ for A_Q (with $\eta := t^{-1}$), where the constant C_Q depends on the cube Q and it behaves as $C_Q \le C\langle \mathrm{dist}(0, Q) \rangle^{-2d}$. In particular, these estimates are summable over all the cubes Q of the partition of unity. This will complete the proof of Lemma 9.3. $\square$

B Proofs of the combinatorial Lemmas

Proof of Lemma 7.2. We consider a pairing $\pi \in \Pi_n$ that has exactly K peaks at the locations $a_1, a_2, \ldots, a_K$. Such pairing has $K-1, K$ or $K+1$ valleys at $b_0, b_1, \ldots, b_K$ such that

$$b_0 < a_1 < b_1 < a_2 < \ldots < b_{K-1} < a_K < b_K \ ,$$

where b_0 and b_K might not be present.

The number of possible peak-and-valley configurations, including their heights is at most n^{4K+2}, since one only has to prescribe the values $a_j, \pi(a_j)$ and $b_j, \pi(b_j)$.

Once the peak-and-valleys are fixed, we consider the set

$$S = \{1, 2, \ldots, n\} \setminus \bigcup_j \Big(\pi(a_j) \cup \pi(b_j) \Big) \ .$$

We define the following sequences:

$$\begin{aligned} D_j : &= \Big(\pi(a_j+1), \pi(a_j+2), \ldots, \pi(b_j-1) \Big) \\ U_j : &= \Big(\pi(b_j+1), \pi(b_j+2), \ldots, \pi(a_{j+1}-1) \Big) \end{aligned}$$

for $j = 0, 1, \ldots, K$ and $a_0 = 0$, $a_{K+1} = n+1$ for definiteness. Clearly $S = \bigcup_j \Big(U_j \cup D_j \Big)$ is a partition; and U_j (D_j) is a monotonically increasing (decreasing) subsequence. This partition consists of at most $2K+2$ non-empty sets, we label them with distinct labels. For any permutation π with fixed peak-and-valleys, we assign to every element in S the label of its set in the partition. This can be done by at most $(2K+2)^{|S|} \leq (2K+2)^n$ ways. Notice that once the peak-and-valleys and this assigment are fixed, the permutation is uniquely determined, since the exact order within U_j and D_j are determined by the monotonicity. Hence the number of such permutations is not more than $n^{4K+2}(2K+2)^n$. If we consider pairings with at most K peaks, then we have to add these numbers up for $K = 1, 2, \ldots$, which gives a total number at most $n^{4K+3}(2K+2)^n$ $\square$

Proof of Lemma A.5. Let $f(\alpha, \beta)$ be the smallest number such that any permutation with at least $f(\alpha, \beta)$ peaks has either an increasing $(\alpha+1)$-staircase or a decreasing $(\beta+1)$-staircase. We show the recursion

$$f(\alpha, \beta+1) \leq f(\alpha-1, \beta+1) + f(\alpha, \beta) \tag{B.1}$$

and the relations

$$f(\alpha,1)=\alpha+1, \qquad f(1,\beta)=\beta+1\ . \tag{B.2}$$

From these recursive relations we easily obtain the estimate $f(\alpha,\beta)\le\binom{\alpha+\beta}{\alpha}$. We first notice the following fact:

Observation: If we have an increasing κ-staircase with tips at $a_1<a_2<\ldots<a_\kappa$, and if there is a peak $a>a_\kappa$ which is higher than the highest tip, i.e., $\pi(a_\kappa)<\pi(a)$, then we also have an increasing $(\kappa+1)$-staircase. Similarly, if we have a decreasing κ-staircase with tips at $a_1>a_2>\ldots>a_\kappa$, and if there is a peak $a<a_1$ which is higher than highest tip, i.e., $\pi(a_1)<\pi(a)$, then we also have an decreasing $(\kappa+1)$-staircase.

For the proof of this observation in the increasing case, suppose that we have a peak $a>a_\kappa$ with $\pi(a_\kappa)<\pi(a)$. Then, we simply choose the smallest number $b>a_\kappa$, such that b is a peak and $\pi(b)>\pi(a_\kappa)$. Let b be the tip of the $(\kappa+1)$-th up-stair and let $a_\kappa<c<b$ be the biggest number such that $\pi(c)<\pi(a_\kappa)$. Such number exists, since any valley between a_κ and b must lie below $\pi(a_\kappa)$, otherwise there would also be a peak somewhere strictly between a_κ and b higher than $\pi(a_\kappa)$ which contradicts to the choice of b. Now we simply choose c to be the bottom of the $(\kappa+1)$-th up-stair. The proof of the decreasing statement is similar.

Now $f(\alpha,1)=\alpha+1$ is proven as follows. Suppose that there are two peaks in decreasing position; $a_1<a_2$ and $\pi(a_1)>\pi(a_2)$, then it gives rise to a decreasing 2-staircase by the observation above. Hence if we have $\alpha+1$ peaks, and there is no decreasing 2-staircase, then the peaks must be monotonically increasing, which gives rise to an increasing $(\alpha+1)$-staircase inductively applying the observation above. The relation $f(1,\beta)=\beta+1$ is analogous.

The proof of (B.1) is done by induction on $\alpha+\beta$. Consider a permutation that has at least $f(\alpha,\beta+1)$ peaks. We have to show that it either has an increasing $(\alpha+1)$-staircase or a decreasing $(\beta+2)$-staircase.

The permutation up to the first $f(\alpha-1,\beta+1)$ peaks either has a decreasing $(\beta+2)$-staircase or it has an increasing α-staircase. In the latter case let $a_1<a_2<\ldots<a_\alpha$ be the location of the tips. Now we look at the next $f(\alpha,\beta)$ peaks. If they have an increasing $(\alpha+1)$-staircase (within themselves), we are done, so can assume that it has a decreasing $(\beta+1)$-staircase, let $b_1<b_2<\ldots<b_{\beta+1}$ be the locations of the tips and certainly $a_\alpha<b_1$.

Now if $\pi(a_\alpha) < \pi(b_1)$, then we found a new peak after and above $\pi(a_\alpha)$, hence by the observation on increasing peaks there is an increasing $(\alpha+1)$-staircase.

If $\pi(a_\alpha) > \pi(b_1)$, then we found a higher peak before $\pi(b_1)$, hence by the observation on the decreasing peaks, there is a decreasing $(\beta+2)$-staircase. This shows that if the permutation has at least $f(\alpha-1,\beta+1)+f(\alpha,\beta)$ peaks, then it either has a $(\beta+2)$-staircase or it has an increasing α-chain. $\square$

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