Ground state and functional integral representations of the CCR algebra with free evolution

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Abstract

The ground state representations on the CCR algebra with free evolution are classified and shown to be either non regular or indefinite. In both cases one meets mathematical structures which appear as prototypes of phenomena typical of gauge quantum field theory. The functional integral representation in the positive non egular case is discussed in terms of a generalized stochastic process satisfying the Markov property. In the indefinite case the ground state is faithful and its GNS representation is characterized in terms of a KMS operator. In the corresponding euclidean formulation, one has a generalization of the Osterwalder-Schrader reconstruction and the indefinite Nelson space, defined by the Schwinger functions, has a unique Krein structure allowing for the construction of Nelson projections, which satisfy the Markov property. The Schwinger functions can be represented in terms of a functional measure and complex variables.

1 Introduction

Most of the wisdom on quantum field theory and more generally on quantum systems with infinite degrees of freedom relies on the existence of a vacuum or ground state, so that the formulation and the control of the model is obtained in terms of the ground state correlation functions. In particular the analytic continuation to imaginary time and the functional integral approach exploits this basic property. [1, 2]

For a large class of models involving infrared singular canonical variables or fields the existence of a ground state is linked to non regular representations of such variables [3, 4, 5, 6] if positivity is required, whereas regular, i.e. weakly continuous, representations are available if one allows indefinite metric [7, 8, 9].

In order to clarify the mathematical structures emerging in these cases, including the euclidean functional integral formulation, we investigate the simple case of the ground state representations of the Heisenberg algebra with free time evolution.

Even if the model may look trivial, it reproduces the basic properties of the temporal gauge in quantum electrodynamics [10], where the longitudinal variables $div\mathbf{A}$, $div\mathbf{E}$ correspond to a collection of free quantum mechanical canonical variables [11].

The analysis of the model sheds light on the occurrence of unusual features, like the interplay between the time translation invariance, the energy spectral condition, the positivity and the regularity of the vacuum state etc. [11].

The model also provides non trivial information on the functional integral formulation of the temporal gauge and in general of models with infrared singular variables.

In Sect.2 we show that the ground state representations of the Heisenberg algebra, which correspond to the the standard field quantization of the temporal gauge, violate positivity and the energy spectral condition and share the basic features of the KMS representations in statistical mechanics [12, 13]. The ground state turns out to be faithful on the Heisenberg algebra and the corresponding GNS representation, with cyclic ground state vector Ψ_0, is given by the tensor product of a Fock and anti-Fock representations [14, 15] of the canonical variables $Q_\pm \equiv (q \pm p')/\sqrt{2}$, $P_\pm \equiv (\pm p + q')/\sqrt{2}$, with $q' \equiv iSqS$, $p' \equiv -iSpS$ and S the antiunitary KMS operator defined by $SA\Psi_0 = A^*\Psi_0$.

In Sect.3 we show that the ground state positive representations of the Weyl algebra are non regular and satisfy the energy spectral condition. The corresponding functional integral formulation and the relation with stochastic processes is discussed in Sect.4, where we show that one has Osterwalder-Schrader positivity and that the Markov property holds [2, 16].

Actually, the euclidean correlation functions can be expressed as expectations of the euclidean fields $U(\alpha,\tau) = e^{i\alpha x(\tau)}$ with a functional measure which is the product of the conditional Wiener measure $dW_{0,x}(x(\tau))$ on trajectories $x(\tau)$ starting at x at time $\tau = 0$, times the ergodic mean $d\nu(x)$, which defines a measure on the Gelfand spectrum of the Bohr algebra [5] generated by the $U(\alpha,\tau)$. The Osterwalder-Schrader reconstruction theorem provides the (non regular) quantum mechanical representation, with a ground state, which is cyclic with respect to the euclidean algebra at time zero.

In Sect. 5 we analyze the euclidean correlation functions of the indefinite representation of the Heisenberg algebra. The Ostervalder-Schrader positivity does not hold but an Osterwalder-Schrader reconstruction can be done, yielding the indefinite quantum mechanical representation.

Nelson positivity fails and we have an indefinite inner product space with a Krein structure very similar to that of the massless scalar field in two space time dimensions [7]. A generalization of Nelson strategy [16] can be performed and projection operators $E_{\pm}, E_0$ can be defined, yielding the same operator formulation of the Markov property as in the positive case.

The euclidean (indefinite) correlation functions have a functional integral representation in terms of the conditional Wiener measure $dW^{x,v} \equiv dW_{0,0}(x(\tau) - x - v|\tau|)$ and the quantum mechanical expectation on the variables $x = x(0)$, and $v = \lim_{\tau\to+\infty} x(\tau)/\tau$.

The correlation functions $< x(\tau_1)...x(\tau_n) >$ can also be obtained as a functional integral of complex variables $z_\alpha(\tau_1)...z_\alpha(\tau_n)$,

$$z_\alpha(\tau) \equiv (1-\alpha^2|\tau|)(x-\alpha^{-2}v)/2 + i(1+\alpha^2\,|\tau|)(x+\alpha^{-2}\,v)/2 + x(\tau) - x - v|\tau|,$$

with a measure which is the product of the conditional Wiener measure $dW_{0,0}(x(\tau) - x - v|\tau|)$ times the gaussian measure $d\gamma_{\alpha^{-1}}(x)\, d\gamma_\alpha(v)$, where $d\gamma_\alpha(.)$ denotes the gaussian measure with variance $\alpha^2/2$.

In view of the strong analogies between the simple model discussed in this note and the massless scalar field in two space time dimensions, the above analysis can be extended to the quantum field theory model to provide a functional integral representation of its euclidean correlation functions.

2 Ground state representations of the Heisenberg algebra with free evolution

At the algebraic level the quantum free particle is described by the Heisenberg *-algebra $\mathcal{A}_H$ generated by q, p (for simplicity we consider the one dimensional case), satisfying the canonical commutation relations (CCR)

$$[\, q,\, p\,] = i \tag{2.1}$$

and the invariance under the *-operation (selfadjointness): $q = q^*$, $p = p^*$. The time evolution is defined by the following one parameter group of *-automorphisms α_t, $t \in \mathbf{R}$

$$\alpha_t(q) = q + p\, t \equiv q(t)\ , \qquad \alpha_t(p) = p\ . \tag{2.2}$$

It is natural to ask whether there are time translationally invariant linear functionals ω on $\mathcal{A}_H$, $\omega(\alpha_t(A)) = \omega(A)$, and to investigate the generalized GNS representations defined by them.

Proposition 2.1 *A state ω on the Heisenberg algebra invariant under time translations, i.e. such that $\forall A \in \mathcal{A}_H$, $\omega(\alpha_t(A)) = \omega(A)$, has the following properties:*
i) it is not positive;
ii) the Fourier transform of $\omega(A\alpha_t(B))$, $A, B \in \mathcal{A}_H$ is not a measure, in general;
iii) its two point function has the following form

$$\omega(q^2) = c, \qquad \omega(p^2) = 0,\ , \qquad \omega(qp) = -\omega(pq) = i/2\ , \tag{2.3}$$

where c is a constant;
iv) furthermore, if ω is Gaussian, i.e. its truncated correlation functions vanish, the GNS representation $(\mathcal{H}_\omega, \pi_\omega)$ defined by it has a non trivial commutant $\pi_\omega(\mathcal{A}_H)'$.
v) if ω is Gaussian and $c = 0$, then ω is invariant under scale transformations

$$q \to \lambda\, q, \quad p \to \lambda^{-1}\, p, \quad \lambda \in \mathbf{R},$$

and such transformations are implemented by isometric operators in $\mathcal{H}_\omega$.

Proof Property i) follows from

$$\omega(p^2) = \omega(d\alpha_t(q)/dt\ p) = d/dt\ \omega(\alpha_t(qp)) = 0.$$

Property iii) follows from the condition of time translation invariance of the two point function $\omega(q(t)\, q(s))$ and the CCR. The Fourier transform of the two point function

$$W(t) \equiv \omega(q\, \alpha_t(q)) = c + i\, t/2$$

is given by

$$\tilde{W}(\omega) = \sqrt{2\pi}\, (c\, \delta(\omega) - \delta'(\omega)/2)$$

has support at the origin and it is not a measure.

If ω is Gaussian, the matrix elements of the operator $U(t)$ which implements the time translations on $D \equiv \pi_\omega(\mathcal{A}_H)\Psi_0$ can be explicitly computed and the (weak) derivative $dU(t)/dt$ exists and defines a hermitean operator H on D. Without loss of generality, we can redefine H so that $H\Psi_0 = 0$. Now, the CCR imply

$$H = p^2/2 + h\ , \qquad h \in \pi_\omega(\mathcal{A}_H)' \tag{2.4}$$

and, if $\pi_\omega(\mathcal{A}_H)' = \{\lambda 1\}\ , \lambda \in \mathbf{C}$, the equations $(\Psi_0, H\Psi_0) = 0$, $\omega(p^2) = 0$ imply $h = 0$. This leads to a contradiction

$$0 = \omega(q^2 H) = \omega(qp)^2 = -1/4.$$

The existence of a non-trivial commutant is reminiscent of the structure which characterizes the theory of KMS states and more generally the Tomita-Takesaki theory. Actually, one can show

Proposition 2.2 *The Gaussian states ω of Proposition 2.1 are faithful, i.e.,*

$$\omega(BA) = 0, \quad \forall B \in \mathcal{A}_H \quad \Rightarrow \quad A = 0. \tag{2.5}$$

Proof If, for some polynomial $P(q,p)$, $\omega(BP(q,p)) = 0$, $\forall B \in \mathcal{A}_H$, then, by time translation invariance of ω, the same holds for $\alpha_t(P) = P(q+pt, p)$, for all t. This implies that the coefficient of the highest power of t vanishes, i.e., $P(p,p)\Psi_0 = 0$. Then, $\omega(q^k P(p,p)) = 0$, $\forall k$, which implies $P = 0$.

By exploiting the property of ω being faithful, one has

Proposition 2.3 *If ω is a state on the Heisenberg algebra $\mathcal{A}_H$, with the properties i-iv) of the above Proposition with $c = 0$, then $\pi_\omega(\mathcal{A}_H)'$ is the (concrete) algebra $\mathcal{A}'_H$ generated by pseudo-canonical variables q', p' satisfying*

$$[\, q', \, p' \,] = -i \tag{2.6}$$

and the GNS representation defined by ω is the direct product of two Fock representations for the pairs of canonical variables

$$Q_\pm \equiv (q \pm p')/\sqrt{2} \ , \quad P_\pm \equiv (\pm p + q')/\sqrt{2} \tag{2.7}$$

satisfying

$$[Q_\pm, P_\pm] = \pm i \ , \quad [Q_\mp, P_\pm] = 0 \ , \tag{2.8}$$

namely, by introducing the destruction operators operators

$$A \equiv (Q_+ + iP_+)/\sqrt{2} \ , \quad B \equiv (Q_- + iP_-)/\sqrt{2} \ , \tag{2.9}$$

which satisfy

$$[A, A^*] = 1 \ , \quad [B, B^*] = -1 \ , \quad [A, B] = [A, B^*] = 0 \ , \tag{2.10}$$

one has that the vector Ψ_0 which represents the state ω satisfies the Fock conditions

$$A\Psi_0 = 0 \ , \quad B\Psi_0 = 0 \ . \tag{2.11}$$

Proof Since ω is faithful, in $\mathcal{H}_\omega$ one can introduce the analogue of the Tomita-Takesaki antilinear operator S

$$SA\Psi_0 \equiv A^*\Psi_0 \qquad \forall A \in \pi_\omega(\mathcal{A}_H). \tag{2.12}$$

As in the standard case, quite generally one has

$$S\Psi_0 = \Psi_0 \ , \quad S^2 = 1 \ , \quad SAS \in \pi_\omega(\mathcal{A}_H)' \ , \quad \forall A \in \pi_\omega(\mathcal{A}_H) \ . \tag{2.13}$$

Similarly, the adjoint S^* satisfies

$$S^*\Psi_0 = \Psi_0 \ , \quad S^{*\,2} = 1 \ , \quad S^*A'S^* \in \pi_\omega(\mathcal{A}_H) \ , \quad \forall A' \in \pi_\omega(\mathcal{A}_H)' \ . \tag{2.14}$$

In fact, e.g., one has

$$(A\Psi_0, S^*\Psi_0) = \overline{(SA\Psi_0, \Psi_0)} = (A\Psi_0, \Psi_0).$$

By using the properties i) - iv) of the state ω, one easily checks that

$$(Sq^jS)^* = (-1)^j Sq^jS \ ; \qquad (Sp^jS)^* = (-1)^j Sp^jS \ . \tag{2.15}$$

We may then introduce the following hermitean operators

$$q' \equiv iSqS \ , \qquad p' \equiv -iSpS \ . \tag{2.16}$$

They satisfy the following pseudo-canonical commutation relations

$$[q', p'] = S[q, p]S = -i.$$

Furthermore, by the definition of the antilinear operator S, we have

$$(q + iq')\Psi_0 = (q - SqS)\Psi_0 = 0 \ , \qquad (p - ip')\Psi_0 = (p - SpS)\Psi_0 = 0.$$

The above equations imply the Fock conditions for the destruction operators A, B defined above. They also imply that the operator

$$H \equiv (p^2 + p'^2)/2 = (p + ip')(p - ip')/2 \tag{2.17}$$

annihilates Ψ_0 and can be taken as the Hamiltonian, since it has the right commutation relations with q, p. By the same kind of argument, Ψ_0 is annihilated also by $q^2 + (q')^2$.

Finally, if an operator $C : D \mapsto D$ commutes with $\pi_\omega(\mathcal{A}_H)$, then C is identified by its action on Ψ_0. Now, $\forall \psi \in D$, there exists a polynomial $P(q', p')$ such that $P(q', p')\Psi_0 = \psi$, since

$$D = SD = S\pi_\omega(\mathcal{A}_H)\Psi_0 = S\pi_\omega(\mathcal{A}_H)S\Psi_0 = \mathcal{A}'_H\Psi_0$$

and therefore $\pi_\omega(\mathcal{A}_H)' = \mathcal{A}'_H$.

The above characterized representation of $\mathcal{A}_H$ in an indefinite inner product space can be given a Krein structure.

Proposition 2.4 *The indefinite inner product representation space $D = \pi_\omega(\mathcal{A}_H))\Psi_0$ can be given a Krein structure by introducing a metric operator η with $\eta\Psi_0 = \Psi_0$ and*

$$\eta A\eta = A \ , \qquad \eta B\eta = -B \ , \tag{2.18}$$

equivalently,

$$\eta q\eta = p' \ , \qquad \eta p\eta = q' \ . \tag{2.19}$$

Proof It follows easily from the positivity of the two point function $(B\Psi_0, \eta B\Psi_0)$.

The general case $c \neq 0$ can be reduced to the one discussed above by putting

$$q = c_+ Q_+ + c_- Q_-, \quad p = d_+ P_+ - d_- P_-, \qquad c_\pm,\ d_\pm \in \mathbf{R} \tag{2.20}$$

with

$$c_+ d_+ + c_- d_- = 1 \qquad c_+^2 - c_-^2 = 2c\ . \tag{2.21}$$

A possible choice is

$$d_+ = d_- = (c_+ + c_-)/2 = 1/\sqrt{2}\ , \qquad c_+ - c_- = 2c \tag{2.22}$$

The so obtained structure provides an example of the Tomita-Takesaki theory in indefinite inner product spaces; surprisingly, the commutant is described by pseudo-canonical variables, as it happens for the time component of the electromagnetic potential in the Gupta-Bleuler formulation of the free electromagnetic field. The above representation of B, B^* is the same as the anti-Fock representation of the CCR discussed in [14, 15]

Because of the indefinite inner product, the analog of the modular operator $\triangle \equiv S^* S$ is not positive and it is fact given by

$$S^* S p^k q^j \Psi_0 = S^* q^j p^k S^* \Psi_0 = (-1)^{j+k} p^k q^j \Psi_0 .$$

One may then introduce a square root of $\triangle$

$$\triangle^{1/2} p^k q^j \Psi_0 \equiv i^k (-i)^j p^k q^j \Psi_0 .$$

It is not difficult to check that the so defined $\triangle^{1/2}$ is hermitean

$$(\triangle^{1/2} p^l q^m \Psi_0,\, p^k q^j \Psi_0) = (-i)^l i^m\, \omega(q^m p^l p^k q^j) =$$

$$i^k (-i)^j\, \omega(q^m p^l p^k q^j) = (\, p^l q^m \Psi_0,\, \triangle^{1/2} p^k q^j \Psi_0)\ ,$$

since the above correlation functions vanish unless $k + l = m + j$.

In analogy with the standard case, one may then introduce the analog of the modular conjugation J defined by

$$J \equiv S \triangle^{-1/2} \tag{2.23}$$

Since also $\triangle^{-1/2}$ (defined by changing i into $-i$ in the definition of $\triangle^{1/2}$) is hermitean, one has

$$JJ^* = J^*J = 1 \ , \tag{2.24}$$

i.e. J is an anti-unitary operator (with respect to the indefinite inner product of D). As in the standard case one has $J\triangle^{1/2}J = \triangle^{-1/2}$.

The above representation of the Heisenberg algebra in a Krein space allows for the construction of the Weyl algebra (equivalently of the Heisenberg group) as the algebra generated by the (pseudo) unitary operators $U(\alpha) \equiv \exp i\alpha q$, $V(\beta) \equiv \exp i\beta p$; therefore, in this way one gets a ground state (regular) representation of the Weyl algebra for the free particle.

Proposition 2.5 *The formal series corresponding to the exponentials $U(\alpha)$, $V(\beta)$, $\alpha, \beta \in \mathbf{R}$ converge strongly in the Krein topology defined in Proposition 2.4 and define (pseudo) unitary operators in the corresponding Krein space. The so obtained Weyl operators $U(\alpha)$, $V(\beta)$ generate a Weyl algebra $\mathcal{A}_W$ and in this way one gets a regular representation in a Krein space, defined by a state ω invariant under the free time evolution and characterized by the following expectations*

$$\omega(e^{i\alpha q}\, e^{i\beta p}) = e^{-i\alpha\beta/2} \qquad \forall \alpha, \beta \in \mathbf{R} \tag{2.25}$$

Proof By expressing q and p in terms of the destruction and creation operators A, B defined in Proposition 2.3, one gets (by Fock space methods) the analog of the standard Fock estimate

$$||q^n \Psi_0||_K \leq 2^n \sqrt{n!},$$

where $||\ ||_K$ denotes the Krein-Hilbert norm, and a similar estimate for $p^n \Psi_0$. As in the standard case, this yields the strong convergence of the series and the existence of the (pseudo) unitary operators $U(\alpha)$, $V(\beta)$. The above expectations follow from the correlation functions

$$\omega(q^n p^m) = \delta_{n,m} (i/2)^n \, n! \tag{2.26}$$

It is worthwhile to mention that the operators $U(\alpha)$, $V(\beta)$ do not commute with the metric operator η, and therefore they are not unitary with respect to the positive (Hilbert) scalar product $(\cdot, \eta\cdot)$ defined by η. Since ω is time translation invariant, the time evolution automorphisms α_t, $t \in \mathbf{R}$ are implemented by a one parameter group $\mathcal{U}(t)$ of (pseudo) unitary operators, and information about their spectral properties is given by the Fourier transforms of the correlation functions $\omega(A\, \alpha_t(B))$.

Proposition 2.6 *In the representation of the Weyl algebra in the Krein space discussed above, the Fourier transform of the correlation functions* $\omega(A\,\alpha_t(B))$ *are tempered distributions with the following properties:*
i) for $A, B \in \mathcal{A}_H$ *they have support at the origin,*
i) for $A, B \in \mathcal{A}_W$ *their support is the entire real line.*

Proof In fact, by the definition of the time evolution and the invariance of ω one has

$$\omega(p^k q^j\, \alpha_t(p^l q^m)) = P_m(t),$$

with $P_m(t)$ a polynomial of degree m. On the other hand

$$\omega(U(\alpha)\, \alpha_t(U(\beta))) = e^{-i\alpha\beta t/2}.$$

In the latter case, the support of the Fourier transform is contained in the positive real axis for the diagonal expectations, $\alpha = \beta$, but not in general.

3 Ground state positive representations of the Weyl algebra

The lack of positivity of the time translationally invariant state can be avoided by looking for representations of the Weyl algebra which do not yield representations of the Heisenberg algebra. The analog of eq.(2.2) is now

$$\alpha_t(W(\alpha, \beta)) = W(\alpha, \beta + \alpha t). \tag{3.1}$$

In this case one can also achieve the positivity of the energy spectrum, in the sense that the Fourier transforms of the correlation functions $\Omega(A\alpha_t(B))$ are tempered distributions with support contained in $\mathbf{R}^+$

Proposition 3.1 *A time translationally invariant positive state* Ω *on the Weyl algebra* $\mathcal{A}_W$ *satisfying the positivity of the energy spectrum is identified by having the following expectations of* $W(\alpha, \beta) \equiv U(\alpha)V(\beta)\exp(i\alpha\beta/2)$

$$\Omega(W(\alpha, \beta)) = 0, \qquad if \quad \alpha \neq 0\ ; \qquad \Omega(W(0, \beta)) = 1\ . \tag{3.2}$$

Proof Time translation invariance implies that the above expectation is independent of β if $\alpha \neq 0$. On the other hand,

$$\Omega(W(\alpha, 0)\, \alpha_t(W(\gamma, 0)) = \Omega(W(\alpha + \gamma, \gamma t))\, e^{-i\alpha\gamma t/2}, \tag{3.3}$$

so that, for $\alpha = \gamma$,

$$\Omega(W(\alpha+\gamma,\gamma t)) = \Omega(W(\alpha+\gamma,0))$$

and positivity of the energy requires that it vanishes. For $\alpha = -\gamma$, the Fourier transform of eq.(3.3) can be supported in $\mathbf{R}^+$, $\forall\alpha \in \mathbf{R}$, only if the Fourier transform of $\Omega(W(0,\alpha t))$ is supported at the origin. Positivity of the state Ω then implies that $\Omega(W(0,\alpha t)) = 1$. Thus, eqs.(3.2) hold. Conversely, eqs.(3.2) define a positive state(as limit of ground states of harmonic oscillators [3]). Moreover, eqs.(3.2) imply

$$\Omega(W(\alpha,\beta)\,\alpha_t(W(\gamma,\delta))) = \delta_{-\alpha,\gamma}\, e^{-i\alpha(\delta+\beta)/2}\, e^{i\alpha^2 t/2}$$

and therefore positivity of the energy follows.

In conclusion, by the GNS construction, one has a nonregular representation of the Weyl algebra in a Hilbert space $\mathcal{H}$, with cyclic vector Ψ_Ω. Since eqs.(3.2) imply

$$V(\beta)\,\Psi_\Omega = \Psi_\Omega, \tag{3.4}$$

Ψ_Ω is also cyclic with respect to the algebra generated by the $U(\alpha)$'s. The occurrence of non regular representations should not be regarded as too bizarre, since they can be related to reasonable physical descriptions. In fact, if we consider a free particle in a bounded volume V, it is reasonable to consider the algebra of canonical variables $\mathcal{A}_V$ generated by $\exp(i\beta p)$, $\beta \in \mathbf{R}$ and by the (continuous) functions of q, $f_V(q)$, with support contained in V.

There is a natural embedding of $\mathcal{A}_V$ into the Weyl algebra $\mathcal{A}_W$, i.e., if

$$f_V(x) = \sum c_n\, e^{ik_n x}\ , \qquad x \in V,$$

then its periodic extension

$$f(q) = \sum c_n\, e^{ik_n q}\ , \qquad q \in \mathbf{R},$$

defines a corresponding element of $\mathcal{A}_W$. Then, if $\Psi_V(x) = const$ denotes the ground state in the volume V (with periodic or Neumann boundary conditions for the Hamiltonian), one has

$$(\Psi_V, f_V(x)\, e^{i\,\beta\,p}\,\Psi_V) = \Omega(f(q)\, e^{i\,\beta\,p}),$$

where Ω is the nonregular state characterized in the above Proposition. The nonregular representation provides therefore a volume independent mathematical description of the above concrete situation.

4 Euclidean formulation and stochastic processes. Positive case

In order to discuss the stochastic processes associated to the quantum free particle, it is convenient to derive the corresponding euclidean formulation.

For the representation characterized in the previous section, we have for the two point (Wightman) function for the Weyl operators $U(\alpha)$

$$\Omega(U(-\alpha)\, e^{iH\,t}\, U(\alpha')) = \delta_{\alpha,-\alpha'}\, e^{i\alpha^2\, t/2} \tag{4.1}$$

with Fourier transform

$$\tilde{W}(\omega) = \sqrt{2\pi}\, \delta(\omega - \alpha^2/2)\, \delta_{\alpha,-\alpha'}.$$

The n-point (Wightman)functions of the Weyl operators $U(\alpha)$ are obtained by induction from

$$\alpha_t(U(\alpha)\, V(\beta)) = U(\alpha)\, V(\beta + \alpha t)\, e^{i\alpha^2 t/2}$$

using eq.(3.4), and are given by

$$\begin{aligned}\Omega(\alpha_{t_1}(U(\alpha_1))...\alpha_{t_n}(U(\alpha_n))) &= \Omega(\alpha_{t_1}(U(\alpha_1)\, \alpha_{t_2-t_1}(U(\alpha_2)...\alpha_{t_n-t_{n-1}}(U(\alpha_n))...) \\ &= \delta_{\sum \alpha_i,\,0}\, e^{i\sum_{i=2}^{n}(\tau_i-\tau_{i-1})(\sum_{k=i}^{n}\alpha_k)^2/2}. \end{aligned} \tag{4.2}$$

The corresponding n-point Schwinger functions are immediately obtained by analytic continuation to ordered imaginary times $\tau_1 \leq \tau_2... \leq \tau_n$:

$$\mathcal{S}(\alpha_1\tau_1, ...\alpha_n\tau_n) = \delta_{\sum \alpha_i,\,0}\, e^{-\sum_{i=2}^{n}(t_i-t_{i-1})(\sum_{k=i}^{n}\alpha_k)^2/2}. \tag{4.3}$$

and extended by symmetry to all euclidean times $\tau_1, \tau_2...\tau_n \in \mathbf{R}$.

¿From the existence and positivity of the Hamiltonian in the representation defined by eq.(3.2), it follows that the above Schwinger functions can also be written as

$$\mathcal{S}(\alpha_1\tau_1, ...\alpha_n\tau_n) = (\Psi_\Omega, U(\alpha_1)e^{-(\tau_2-\tau_1)H}U(\alpha_2)...e^{-(\tau_n-\tau_{n-1})H}U(\alpha_n)\Psi_\Omega) \tag{4.4}$$

By standard arguments, one can introduce the corresponding Borchers algebra and euclidean fields $U(\alpha, \tau)$ so that the Schwinger functions define a linear functional E on the euclidean fields

$$\mathcal{S}(\alpha_1\tau_1, ...\alpha_n\tau_n) = E(U(\alpha_1, \tau_1)...U(\alpha_n, \tau_n)).$$

Equation (4.3) implies the Osterwalder-Schrader (OS) positivity of the Schwinger functions, with the OS reflection operator θ defined by

$$\theta U(\alpha, \tau) = U(\alpha, -\tau),$$

i.e. one has

$$E(\overline{\theta B}\, B) \geq 0,$$

$\forall B$ belonging to the algebra generated by $U(\alpha, \tau)$, $\alpha \in \mathbf{R}$, $\tau \geq 0$, where $\overline{U(\alpha, \tau)} \equiv U(-\alpha, \tau)$.

It is a non trivial fact that also Nelson positivity holds. As a matter of fact, the above Schwinger functions can be expressed as expectations of fields $U(\alpha, \tau) \equiv e^{i\alpha x(\tau)}$ with functional measure

$$d\mu(x(\tau)) = dW_{0,x}(x(\tau))\, d\nu(x),$$

where $dW_{s,y}$ denotes the Wiener measure for trajectories starting at the point y at time $t = s$ and $d\nu$ denotes the ergodic mean; in fact $d\nu$ defines a measure on the Gelfand spectrum Σ of the Bohr algebra generated by $e^{i\alpha x}$ [5] and

$$\int d\nu(x)\, dW_{0,x}(x(\tau))\, e^{i\alpha_1 x(\tau_1)} ... e^{i\alpha_n x(\tau_n)}$$

$$= \int d\nu(x)\, e^{i\sum_i \alpha_i x} \int dW_{0,x}(x(\tau))\, e^{i\sum_i \alpha_i (x(\tau_i) - x)}$$

$$= \delta_{\sum_i \alpha_i, 0} \int dW_{0,0}(y(\tau))\, e^{i\sum_i \alpha_i y(\tau_i)} = \mathcal{S}(\alpha_1\tau_1, ...\alpha_n\tau_n), \quad y(\tau_i) \equiv x(\tau_i) - x.$$

The measure $d\mu$ is invariant under time translations,

$$dW_{0,y}(x(\tau))d\nu(y) = dW_{s,y}(x(\tau))d\nu(y)$$

because $d\nu(x)$ is invariant under translations $x \to x + a$ and therefore stationary for the Brownian motion.

The measure $d\mu$ is invariant under translations $x(\tau) \to x(\tau) + a$; they can be given the meaning of gauge trasformations and can be regarded as the analog of the gauge group of traslations of step 2π for a particle on a circle [5]. In conclusion, the euclidean correlation functions can be obtained as the stochastic process with expectations defined by $d\mu$.

Since the ground state Ψ_Ω is cyclic with respect to the euclidean algebra at time zero in the space $\mathcal{H}$, obtained by the OS recostruction theorem, such a space can be identified with $L^2(\Sigma, d\nu)$.

One can explicitly check that the Markov property holds; if $\tau_1 \leq 0 \leq \tau_2$ one has

$$\int d\mu\, e^{i\alpha x(\tau_1)}\, e^{i\beta x(\tau_2)} = \int d\nu(x) e^{i(\alpha+\beta)x} \int dW_{0,x}(x(\tau)) e^{i\alpha(x(\tau_1)-x)} e^{i\beta(x(\tau_2)-x)}$$

$$= \int d\nu(x) e^{i(\alpha+\beta)x)} dW^-_{0,0}(y(\tau))\, e^{i\alpha y(\tau_1)}\, dW^+_{0,0}(y(\tau))\, e^{i\beta y(\tau_2)}.$$

5 Euclidean formulation and stochastic processes. Indefinite case

a. Schwinger functions and Osterwalder-Schrader reconstruction

We start by deriving the euclidean formulation by analytic continuation. The Laplace transforms of the two point function $W(t)$, derived in Sect.2, gives an analytic function $\mathcal{W}(z_1, z_2) = W(\zeta)$, $\zeta \equiv z_2 - z_1 = \xi + i\tau$, $\xi \in \mathbf{R}$, $\tau > 0$, and its restrictions at the euclidean points $\xi = 0, \tau > 0$ define the Schwinger two point function

$$\mathcal{S}(\tau_1, \tau_2) = S(\tau) = W(i\tau) = c - \tau/2, \;\; \tau \equiv \tau_2 - \tau_1 \; > 0.$$

The extension to $\tau \leq 0$ is given by the symmetry condition $S(-\tau) = S(\tau)$, so that in conclusion

$$\mathcal{S}(\tau_1, \tau_2) = S(\tau) = c - |\tau|/2.$$

The n-point Schwinger functions factorize as usual for quasi free states and define a functional Φ_S on the euclidean Borchers'algebra, fully determined by the two point function

$$\Phi_S(\overline{f} \times g) \equiv < f,\, g > \equiv \int d\tau_1\, d\tau_2\, \overline{f}(\tau_1)\, \mathcal{S}(\tau_1, \tau_2)\, g(\tau_2),$$

$f, g \in \mathcal{S}(\mathbf{R}) \equiv \mathcal{S}$.

Euclidean fields $x(f)$, $f \in \mathcal{S}$, f real, can be introduced as usual, with two point function

$$< x(f)\, x(g) > = < f,\, g >$$

and n-point functions given by the "gaussian"functional (f real)

$$< e^{ix(f)} > \equiv e^{-<f,f>/2} = e^{\int f(\tau) f(s)\, |\tau - s|\, d\tau\, ds/4}. \tag{5.1}$$

The Osterwalder-Schrader positivity is not satisfied , since $\mathcal{S}(-\tau_1, \tau_2) = c - |\tau_1 + \tau_2|/2$ is not a positive kernel and therefore

$$\Phi_S(\overline{\theta f} \times f), \quad (\theta f)(\tau) \equiv f(-\tau),$$

is not positive.

However, the Osterwalder-Schrader (O-S) reconstruction of the real time indefinite vector space, in terms of the Schwinger functions can be done by the standard extension of the recostruction without positivity [17]. The O-S scalar product is defined by factorization starting from

$$< f,\, g >_{OS} \equiv < \theta f, g >= i/2\, (\overline{\tilde{f}(0)}\, \tilde{g}'(0) - \overline{\tilde{f}'(0)}\, \tilde{g}(0)) + c\overline{\tilde{f}(0)}\, \tilde{g}(0))$$

for $f, g \in \mathcal{S}(\mathbf{R}^+) \equiv \mathcal{S}^+$.

The null space of the O-S scalar product,

$$N_{OS} \equiv \{f \in \mathcal{S}(\mathbf{R}^+) : < f, g >_{OS} = 0 \quad \forall g \in \mathcal{S}(\mathbf{R}^+)\} \quad ,$$

has codimension two, so that the space $K \equiv \mathcal{S}(\mathbf{R}^+)/N_{OS}$ is two–dimensional.

b. Nelson space

The two point Schwinger function does not satisfy positivity, i.e. $\mathcal{S}(\tau_1, \tau_2)$ is not a positive kernel; in more detail one has

Proposition 5.1 *The vector space $\mathcal{S}(\mathbf{R})$ with inner product*

$$< f,\, g >= \int d\tau_1\, d\tau_2\, \overline{f}(\tau_1)\, \mathcal{S}(\tau_1, \tau_2)\, g(\tau_2) \tag{5.2}$$

is an indefinite inner product space with one negative dimension (pre-Pontriagin space): $<,>$ is positive on $\mathcal{S}_0(\mathbf{R}) \equiv \{f \in \mathcal{S}(\mathbf{R}), \int f d\tau = 0\}$, and there is a function χ with $\int \chi d\tau \neq 0$ such that $< \chi,\, \chi >< 0$.

Proof In fact,for $f, g \in \mathcal{S}(\mathbf{R}^+)$, by $(\theta f(\tau) \equiv f(-\tau))$

$$< f,\, g >_{OS} \equiv < \theta f, g >= i/2\, (\overline{\tilde{f}(0)}\, \tilde{g}'(0) - \overline{\tilde{f}'(0)}\, \tilde{g}(0)) + c\overline{\tilde{f}(0)}\, \tilde{g}(0)).$$

the Fourier transform of S is

$$\tilde{S}(\omega) = 2\pi c\, \delta(\omega) - (d/d\omega) P(1/\omega),$$

where P denotes the principal value. Then, if $\tilde{f}(0) = 0$

$$\int d\omega \, \overline{\tilde{f}}(\omega) \, \tilde{S}(\omega) \tilde{f}(\omega) > 0.$$

It is easy to see that

$$\chi(\tau) = [e^{-(\tau+T)^2/L} + e^{-(\tau-T)^2/L}](2\sqrt{\pi}\, L)^{-1}$$

has the properties of the Proposition, for L small enough and T large enough. **Remark**. The above indefinite inner product space has a functional structure very similar to the indefinite inner product space defined by the (real time) two point function of the massless scalar field in two space time dimensions [7]. For simplicity, in the following we consider the case $c = 0$.

Proposition 5.2 *The above indefinite inner product space can be given a Krein structure by introducing e.g. the following Krein inner product*

$$[f, g] =< f, g > +(< f, \chi > -\overline{\tilde{f}}(0))(< \chi, g > -\tilde{g}(0)), \qquad (5.3)$$

where χ has been chosen such that $\tilde{\chi}(0) = 1, \ < \chi, \chi >= 0$

Proof The proof is the same as for the massless scalar field [7].

Even if the Krein topology that can be associated to the above inner product space is unique, as in the case of the massless field in two dimensions [7], there are many possible choices for the positive scalar product which turns $\mathcal{S}(\mathbf{R})$ into a (pre-)Krein space. In the following we shall exploit this freedom in order to obtain a scalar product with a more direct interpretation, in particular in terms stochastic processes.

For this purpose, we first remark that the sequences f_n, with $f_n(\tau)$ smooth approximations of the Dirac delta function $\delta(\tau)$, converge weakly in the topology of the inner product (5.1); they actually define strongly compatible sequences in the sense of [15], i.e. $\lim_{n,m} < f_n, \, f_m >$ exists and $\lim_n < f_n, \, g >$ exists $\forall g \in \mathcal{S}(\mathbf{R})$. Therefore, one can consider the weak extension of $\mathcal{S}(\mathbf{R})$ defined by them. Such an extension is contained in the above Krein space as a consequence of its uniqueness.

In particular such extension contains Dirac delta functions $\delta_\sigma \equiv \delta(\tau - \sigma)$ and one has

$$< \delta_\sigma, \, g >= -\tfrac{1}{2} \int g(s)|s - \sigma| ds, \quad < \delta_\sigma, \delta_\rho >= -\tfrac{1}{2} \, |\sigma - \rho|. \qquad (5.4)$$

A further weak extension is defined by the set of strongly compatible sequences $f_n(\tau) = f(\tau - n)/n$, $f \in \mathcal{S}(\mathbf{R})$, $\tilde{f}(0) = 1$, since one has

$$\lim_{n\to\pm\infty} < f_n,\, g >= \mp\tilde{g}(0)/2,$$

$$\lim_{(n,m)\to\infty} < f_n,\, f_m >= 0, \quad \lim_{n\to\infty} < f_n,\, \delta_\sigma >= -1/2.$$

Thus, by introducing

$$w \equiv weak - \lim_{n\to\infty} f_n(\tau), \tag{5.5}$$

we have

$$< w,\, g >= -\tilde{g}(0)/2, \;\; < w,\, w >= 0, \;\; < \delta_0,\, w >= -1/2. \tag{5.6}$$

As a consequence of eq.(5.5), a convenient class of Krein scalar products can be parametrized in the following way

$$[f,\, g]_\alpha \equiv < f,\, g > +2 < f,\, \alpha\delta_0 + \alpha^{-1}w >< \alpha\delta_0 + \alpha^{-1}w,\, g >, \tag{5.7}$$

α playing the rôle of a scale parameter. Thus, even if the correlation functions are scale invariant, the Krein product is not. Summarizing, we have

Proposition 5.3 *The Krein product (5.7) defines a Krein topology equivalent to that defined by eq.(5.2). The resulting Krein space K_α has the following decomposition*

$$K_\alpha = \overline{(\frac{d}{d\tau})^2\mathcal{S}^+} \oplus \overline{(\frac{d}{d\tau})^2\mathcal{S}^-} \oplus \{a\delta_0 + bw,\; a, b \in \mathbf{C}\}, \tag{5.8}$$

where $\mathcal{S}^\pm \equiv \mathcal{S}(\mathbf{R}^\pm)$; the decomposition is both orthogonal with respect to the indefinite inner product $< .\,, . >$ as well as with respect to the positive product $[.\,, .]$. The metric operator η_α satisfies $\eta_\alpha^2 = \mathbf{1}$ and has a unique negative eigenvector $\alpha\,\delta_0 + \alpha^{-1}\,w$.

In particular, eqs.(5.7), (5.8) imply that $\forall f, g \in (d/d\tau)^2\mathcal{S}^\pm$, i.e. $f = F''$, $g = G''$, $F, G \in \mathcal{S}^\pm$ one has

$$[f,\, g] = < f,\, g >= (F',\, G')_{L^2}.$$

As usual one can construct the Fock space $\Gamma(K)$ over K. Euclidean fields $x(f)$, f real, act on $\Gamma(K)$ and by standard arguments they can be extended from $\mathcal{S}$ to K; the existence of δ_0 and w are equivalent to the existence of the

fields $x(\tau)$ at fixed times, $x \equiv x(0)$, and of the (positive time) ergodic mean of the velocity, namely $v \equiv \lim_{\tau\to+\infty} x(\tau)/\tau$.

Eq. (5.7) defines the following positive bilinear functional on the fields at fixed time,

$$[x(\tau)\, x(s)]_\alpha = -(1/2)\,(|\tau - s| - |\tau| - |s| - \alpha^{-2} - \alpha^2\,|\tau|\,|s|). \tag{5.9}$$

The OS scalar product extends by continuity to the Krein closure of $\mathcal{S}^+$ and K_{OS} can be identified with the subspace of the Krein closure $\overline{\mathcal{S}}(\mathbf{R}^+)$ generated by $x \equiv x(0) = x(\delta_0)$ and $v \equiv x(w)$. In fact, by eqs. (5.4), (5.6),

$$< f,\, g >_{OS} = < \tilde{f}(0)\, x - i\tilde{f}'(0)\, v\ ,\ \tilde{g}(0)\, x - i\tilde{g}'(0)\, v > \quad .$$

By comparison with eqs. (2.3), one has the correspondence $x \sim q\Psi_0,\ v \sim ip\Psi_0$. Clearly, the one particle space determines the whole (indefinite) OS reconstruction space, which coincides with the real time space of Prop. 2.1, with the correspondence

$$P(x)\, Q(-iv) \sim P(q)\, Q(p)\, \Psi_0.$$

c. Markov property

The structure displayed by the model is actually an example of a general structure which substantially generalizes Nelson strategy to cases in which positivity fails and in particular allows for a generalization of the formulation of the Markov property in terms of projections.

For simplicity, we discuss the problem at the level of the two point function, which is assumed to define a non degenerate inner product $< .\,, . >$ on $\mathcal{S}$, satisfying the following properties:

i) $< \theta f,\, \theta g >=< f,\, g >,\ \ (\theta f)(\tau) \equiv f(-\tau)$,

ii) there exists an operator D on $\mathcal{S}$, such that, $\forall f, g \in \mathcal{S}$

$$< f,\, Dg >= (f, g)_{L^2},\ \ D\mathcal{S}^\pm \subseteq \mathcal{S}^\pm,$$

iii) there is a Krein structure $[.\,, .]$, namely a Hilbert majorant of $< .\,, . >$, such that the corresponding closure of $\mathcal{S}$ decomposes as

$$\overline{\mathcal{S}} = \overline{\mathcal{S}^-} < + > \overline{D\mathcal{S}^+}, \tag{5.10}$$

with $< + >$ denoting a $<, >$ orthogonal sum.

Then, by i) one also has

$$\overline{\mathcal{S}} = \overline{\mathcal{S}^+} < + > \overline{D\mathcal{S}^-}. \tag{5.11}$$

Furthermore, since $\overline{D\mathcal{S}^\pm} \subseteq \overline{\mathcal{S}^\pm}$, $\forall f \in \mathcal{S}$, by eq.(5.10) one has

$$f = f_- + f_{0+}, \quad f_- \in \overline{\mathcal{S}^-}, \quad f_{0+} \in \overline{D\mathcal{S}^+}$$

and by eq.(5.11) $f_- = (f_-)_+ + (f_-)_{0-}$, $(f_-)_+ \in \overline{\mathcal{S}^+}$, $(f_-)_{0-} \in \overline{D\mathcal{S}^-}$. Hence,

$$f = f_{0+} + (f_-)_{0-} + (f_-)_+, \quad (f_-)_+ \in \overline{\mathcal{S}^+} \cap \overline{\mathcal{S}^-}.$$

Moreover, if $f \in \overline{\mathcal{S}^+} \cap \overline{\mathcal{S}^-}$, then $f_{0+} = 0 = (f_-)_{0-}$, i.e. $f = (f_-)_+$. In conclusion, one has

$$\overline{\mathcal{S}} = \overline{D\mathcal{S}^-} < + > \overline{D\mathcal{S}^+} < + > \mathcal{V}, \qquad \mathcal{V} = \overline{\mathcal{S}^-} \cap \overline{\mathcal{S}^-}. \tag{5.12}$$

Since the inner product space $\mathcal{S}$ defined by the two point function can always be taken non degenerate, also the Krein closure $\bar{\mathcal{S}}$ is non degenerate so that the decompositions (5.10-5.12) are unique; in fact, non uniqueness is possible only if the spaces occurring in them have a non zero intersection, but this would imply degeneracy. Thus, any vector $f \in \bar{\mathcal{S}}$ has unique decompositions according to eqs.(5.10-5.12), which means that correspondingly there are everywhere defined idempotent operators, which are hermitean with respect to $<,>$. In particular, there are $E_\pm$, E_0 defined by $E_\pm \overline{\mathcal{S}} = \overline{\mathcal{S}^\pm}$, and $E_0 \bar{\mathcal{S}} = \mathcal{V}$ and eqs.(5.10-5.12) give

$$E_+ E_- = E_- E_+ = E_0. \tag{5.13}$$

Thus, we get the same operator formulation of the Markov property as in the positive case.

For quasi free states, the above construction extends in the usual way to the n-point Schwinger functions, in terms of the sum of the symmetric tensor products of the above indefinite spaces. Similarly, the euclidean $x(f)$ has an extension to $f \in \bar{\mathcal{S}}$.

In the example discussed in this note, $D = d^2/d\tau^2$ with the Krein structure given by eq.(5.7). The decomposition (5.12) reduces to (5.8) and it is also orthogonal in the positive scalar product. The lesson from the model is that the space $\mathcal{V}$ can be larger than the standard time zero space; in the model it contains a time translationally invariant variable, actually the ergodic limit of the velocity.

d. Functional integral representation

In order to represent euclidean correlation functions, eq. (5.1), in terms of a measure we start by noting that, for $f(\tau) = dF(\tau)/d\tau$, $F \in \mathcal{S}$, the functional (5.1) coincides with the functional defined by the Wiener measure. For functions which are not derivatives, positivity is lost.

Nevertheless, a functional integral representation of the functional (5.1) can be given in terms of complex variables in the gaussian measure spaces defined by the Krein scalar product, eq. (5.7).

A gaussian measure $d\mu_\alpha$ is in fact defined on $\mathcal{S}'$ by the expectation (f real)

$$\int d\mu_\alpha \, e^{ix(f)} = [e^{ix(f)}]_\alpha \equiv e^{-[f,f]_\alpha/2}. \tag{5.14}$$

The corresponding process can be explicitly analyzed as follows: let $y(\tau)$ denote the Wiener process, x and v independent gaussian variables, with variances

$$[x^2] = \alpha^{-2}/2 \quad , \quad [v^2] = \alpha^2/2. \tag{5.15}$$

It is immediately seen that the process

$$x + v|\tau| + y(\tau), \tag{5.16}$$

defined as the product measure space of x, v and $y(\tau)$ has the variance given by eq. (5.9), and can therefore be identified with the process defined by eq. (5.15), i.e.

$$d\mu_\alpha(x(\tau)) \;=\; d\gamma_{\alpha^{-1}}(x) \;\; d\gamma_\alpha(v) \;\; dW_{0,0}(y(\tau)) \tag{5.17}$$

with $x(\tau) = x + v|\tau| + y(\tau)$, and $d\gamma_\alpha(x)$ the gaussian measure of variance $\alpha^2/2$.

An explicit analysis of the gaussian measure defined by the Krein scalar product (generating kernel, Markov properties etc.) immediately follows from the representation of eq.(5.17). In particular, we have the representation (written for simplicity for $0 < \tau_1 < \ldots < \tau_n$)

$$[f_1(x(\tau_1))...f_n(x(\tau_n))]_\alpha = \int d\mu_\alpha(x(\tau)) \;\; f_1(x(\tau_1)) \;\ldots\; f_n(x(\tau_n)) =$$

$$= \int d\gamma_{\alpha^{-1}}(x) \;\, d\gamma_\alpha(v) \;\, K(y_n - y_{n-1}, \tau_n - \tau_{n-1}) \;\ldots\; K(y_2 - y_1, \tau_2 - \tau_1)$$

$$K(y_1 - x, \tau_1) \;\, f_n(y_n + v\tau_n)) \;\ldots\; f_1(y_1 + v\tau_1)) \, dy_1 ...dy_n,$$

with $K(x, \tau)$ the kernel defining Brownian motion.

Thus, the Markov property for $d\mu_\alpha$ holds in the variables $\xi(\tau) \equiv x + y(\tau)$, $v(\tau)$, with $v(\tau)$ constant in τ. The measure given by eq.(5.17) clearly defines a measure on trajectories $x(\tau)$, $v(\tau)$, concentrated on $v(\tau) = v$ and satisfies the Markov property in the two variables $x(\tau)$, $v(\tau)$.

The decomposition of eq.(5.16) for $x(f)$, f real, corresponds to the decomposition of f according to eq.(5.12)

$$f = \delta_0 \int f(\tau) d\tau + w \int f(\tau) |\tau| d\tau + h.$$

Clearly, the process defined by $d\mu_\alpha$ corresponds to Brownian motion, with gaussian distribution of variance $\alpha^{-2}/2$ for the position at $\tau = 0$, with an additional gaussian variable, with variance $\alpha^2/2$, describing non-zero mean velocities, with opposite and constant values for $\tau > 0$ and $\tau < 0$. The limit $\alpha \to 0$ exists in the sense of positive functionals on the Bohr algebra and coincides with the measure on its spectrum discussed in Sect.4.

The gaussian functional defined by eq.(5.1) can be represented in the measure space $(x(\tau),\, d\mu_\alpha)$, eqs. (5.16), (5.17), by introducing complex variables $z(\tau)$ in order to have

$$< e^{ix(f)} >= \int d\mu_\alpha(x(\tau))\, e^{\, i \int z_\alpha(\tau) f(\tau)\, d\tau}. \tag{5.18}$$

The simplest choice is

$$z_\alpha(f) = x(J_\alpha f), \;\; J_\alpha(\alpha\, \delta_0 + \alpha^{-1}\, w) = i(\alpha\, \delta_0 + \alpha^{-1}\, w),$$

$$J_\alpha f = f, \;\; \forall f \in (\alpha\, \delta_0 + \alpha^{-1}\, w)^\perp.$$

J_α satisfies

$$[J_\alpha,\, \eta_\alpha] = 0, \;\; J_\alpha^2 = \eta_\alpha, \;\; < J_\alpha f,\, J_\alpha g >=< f,\, g > .$$

Then, by using the extension of eq.(5.14) to complex f, since $\overline{J_\alpha g} = \eta_\alpha\, J_\alpha \overline{g}$, we have ($g$ real)

$$\int d\mu\, e^{ix(J_\alpha g)} = e^{-[\overline{J_\alpha g},\, J_\alpha g]/2} = e^{-<g,\, g>/2}.$$

In conclusion, the euclidean correlation functions of the model have the following functional integral representation

$$< x(\tau_1), ...x(\tau_n) >= \int d\gamma_\alpha(x, v)\, dW_{0,0}(x(\tau) - x - v|\tau|)\, z_\alpha(\tau_1)...z_\alpha(\tau_n) =$$

$$< \int dW_{0,0}(x(\tau) - x - v|\tau|)\, x(\tau_1)...x(\tau_n) >,$$

where

$$z_\alpha(\tau) \equiv x(\tau) - x - v|\tau| + i(1+\alpha^2|\tau|)\,(x+\alpha^{-2}v)/2 + (1-\alpha^{-2}|\tau|)\,(x-\alpha^{-2}v)/2,$$

and $< \, . \, >$ denotes the (quantum mechanical) expectation, eq.(5.1), on the variables x, v; thus, the expectations at all times are uniquely given by the conditional Wiener measure

$$dW^{x,v}(x(\tau)) \equiv dW_{0,0}(x(\tau) - x - v|\tau|)$$

and by the (quantum mechanical) expectations of x, v.

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