Band gap of the Schrödinger operator with a strong δ-interaction on a periodic curve

P. Exner[a,b] and K. Yoshitomi[c]

a) *Department of Theoretical Physics, Nuclear Physics Institute, Academy of Sciences, 25068 Řež, Czech Republic*
b) *Doppler Institute, Czech Technical University, Břehová 7, 11519 Prague, Czech Republic*
c) *Graduate School of Mathematics, Kyushu University, Hakozaki, Fukuoka 812-8581, Japan*
exner@ujf.cas.cz, yositomi@math.kyushu-u.ac.jp

In this paper we study the operator $H_\beta = -\Delta - \beta\delta(\cdot - \Gamma)$ in $L^2(\mathbb{R}^2)$, where Γ is a smooth periodic curve in $\mathbb{R}^2$. We obtain the asymptotic form of the band spectrum of H_β as β tends to infinity. Furthermore, we prove the existence of the band gap of $\sigma(H_\beta)$ for sufficiently large $\beta > 0$. Finally, we also derive the spectral behaviour for $\beta \to \infty$ in the case when Γ is non-periodic and asymptotically straight.

1 Introduction

In this paper we are going to discuss some geometrically induced spectral properties of singular Schrödinger operators which can be formally written as $H_\beta = -\Delta - \beta\delta(\cdot - \Gamma)$, where Γ is an infinite curve in the plane.

This problem stems from physical interest to quantum mechanics of electrons confined to narrow tubelike regions usually dubbed "quantum wires". Such systems are often modeled by means of Schrödinger operators on curves, or more generally, on graphs. This is an idealization, however, because in reality the electrons are confined in a potential well of a finite depth, and therefore one can find them also in the exterior of such a "wire", even if not too far since this a classically forbidden region. The generalized Schrödinger operators mentioned above provide us with a simple model which can take such tunneling effects into account.

Singular interactions have been studied by numerous authors – see the classical monograph [AGHH], and the recent volume [AK] for an up-to-date bibliography. While the general concepts are well known, the particular case of a δ-interaction supported by a curve attracted much less attention; we can mention [BT, BEKŠ] and a recent article [EI], where a nontrivial relation between spectral properties and the geometry of the curve Γ was found for the first time. It was followed by our previous paper [EY], where we posed the question about the strong coupling asymptotic behaviour, $\beta \to \infty$, of the eigenvalues of H_β in the case when Γ was a loop. We have shown there that the asymptotics is given by the spectrum of the Schrödinger operator on $L^2(\Gamma)$ with a curvature-induced potential.

Here we are going to discuss a similar problem in the situation when Γ is an infinite smooth curve without self-intersections. We pay most attention to the case of a periodic Γ where we find the asymptotic form of the spectral bands and prove existence of open band gaps for $\beta > 0$ large enough provided Γ is not a straight line. We also treat the case of a non-straight Γ which is straight asymptotically, and thus by [EI] it gives rise to a nonempty discrete spectrum; we find the behaviour of these eigenvalues for $\beta \to \infty$.

While the basic idea is the same as in [EY], namely combination of a bracketing argument with the use of suitable curvilinear coordinates in the vicinity of Γ, the periodic case requires several more tools. Let us review briefly the contents of the paper. In the following section we present a formulation of the problem and state the results. Section 3 is devoted to the proof of our main result, Theorem 2.1. We perform the Floquet-Bloch reduction and estimate the discrete spectrum of the fiber operator $H_{\beta,\theta}$ using a Dirichlet-Neumann bracketing and approximate operators with separated variables. As a corollary we obtain the existence of open gaps for β large enough. To get a more specific information on the last question, we derive in Section 4 a sufficient condition under which the nth gap is open for a given n. The final section deals with the case of an asymptotically straight Γ.

2 Main results

Let us first introduce the needed notation and formulate the problem. The main topic of this paper is the Schrödinger operator with a δ-interaction on a periodic curve. Let $\Gamma : \mathbb{R} \ni s \mapsto (\Gamma_1(s), \Gamma_2(s)) \in \mathbb{R}^2_{x,y}$ be a curve which is parametrized by its arc length. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be the signed curvature of Γ, i.e. $\gamma(s) := (\Gamma_1''\Gamma_2' - \Gamma_2''\Gamma_1')(s)$. We impose on it the following assumptions:

(A.1) $\gamma \in C^2(\mathbb{R})$.

(A.2) There exists $L > 0$ such that $\gamma(\cdot + L) = \gamma(\cdot)$ on $\mathbb{R}$.

(A.3) $\displaystyle\int_0^L \gamma(t)\, dt = 0$.

Given $\beta > 0$, we define

$$q_\beta(f, f) = \|\nabla f\|^2_{L^2(\mathbb{R}^2)} - \beta \int_\Gamma |f(x)|^2\, dS \quad \text{for} \quad f \in H^1(\mathbb{R}^2).$$

By H_β we denote the self-adjoint operator associated with the form q_β. The operator H_β can be formally written as $-\Delta - \beta\delta(\cdot - \Gamma)$. Our main purpose is to study the asymptotic behaviour of the band spectrum of H_β as β tends to infinity. Let $\alpha \in [0, 2\pi)$ be the angle between the vectors $\Gamma'(0)$ and $(1, 0)$: $\Gamma'(0) = (\cos\alpha, \sin\alpha)$. We define new coordinates (x', y') by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x - \Gamma_1(0) \\ y - \Gamma_2(0) \end{pmatrix}.$$

¿From now on, we work in the coordinates (x', y'), where the curve Γ assumes the form

$$\begin{aligned} \Gamma_1(s) &= \int_0^s \cos\left(-\int_0^t \gamma(u)\, du\right) dt, \\ \Gamma_2(s) &= \int_0^s \sin\left(-\int_0^t \gamma(u)\, du\right) dt. \end{aligned}$$

Combining these relations with $(A.3)$, we have

$$\Gamma(\cdot + L) - \Gamma(\cdot) = (K_1, K_2) \quad \text{on} \quad \mathbb{R}, \tag{2.1}$$

where

$$
\begin{aligned}
K_1 &= \int_0^L \cos\left(-\int_0^t \gamma(u)\,du\right)\,dt,\\
K_2 &= \int_0^L \sin\left(-\int_0^t \gamma(u)\,du\right)\,dt.
\end{aligned}
$$

In the vicinity of Γ one can introduce the natural locally orthogonal system of curvilinear coordinates. By Φ we denote the map

$$
\mathbb{R}^2 \ni (s,u) \mapsto (\Phi_1(s,u), \Phi_2(s,u)) = (\Gamma_1(s) - u\Gamma_2'(s), \Gamma_2(s) + u\Gamma_1'(s)) \in \mathbb{R}^2.
$$

We further impose the following assumptions on Γ:

(A.4) $K_1 > 0$.

(A.5) There exists $a_0 > 0$ such that the map $\Phi|_{[0,L)\times(-a,a)}$ is injective and $\Phi((0,L)\times(-a,a)) \subset (0,K_1)\times\mathbb{R}$ for all $a \in (0,a_0)$.

As in the proof of [Yo, Proposition 3.5], we notice that the assumptions $(A.4)$ and $(A.5)$ are satisfied, e.g., if $\max_{t\in[0,L]} |\int_0^t \gamma(s)\,ds| < \pi/2$; on the other hand, this condition is by no means necessary. Let us also remark that in general the choice of the initial point $s=0$ is important in checking the assumptions $(A.4)$ and $(A.5)$. We put

$$
\Lambda = (0,K_1)\times\mathbb{R}.
$$

For $\theta \in [0,2\pi)$, we define

$$
\begin{aligned}
Q_\theta &= \{u \in H^1(\Lambda);\quad u(K_1, K_2 + \cdot) = e^{i\theta}u(0,\cdot) \quad \text{on} \quad \mathbb{R}\},\\
q_{\beta,\theta}(f,f) &= \|\nabla f\|^2_{L^2(\Lambda)} - \beta \int_{\Gamma((0,L))} |f(x)|^2\,dS \quad \text{for} \quad f \in Q_\theta.
\end{aligned}
$$

By $H_{\beta,\theta}$ we denote the self-adjoint operator associated with the form $q_{\beta,\theta}$. We shall prove in Lemma 3.1 the unitary equivalence

$$
H_\beta \cong \int_0^{2\pi} \oplus H_{\beta,\theta}\,d\theta. \tag{2.2}
$$

By Lemma 3.3 this implies

$$
\sigma(H_\beta) = \bigcup_{\theta\in[0,2\pi)} \sigma(H_{\beta,\theta}). \tag{2.3}
$$

Since $\Gamma((0,L))$ is compact, we infer by Lemma 3.2 that

$$\sigma_{\mathrm{ess}}(H_{\beta,\theta}) = [0,\infty). \tag{2.4}$$

Next we need a comparison operator on the curve. For a fixed $\theta \in [0,2\pi)$ we define

$$S_\theta = -\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2 \quad \text{in} \quad L^2((0,L))$$

with the domain

$$P_\theta = \{u \in H^2((0,L)); \quad u(L) = e^{i\theta}u(0), \quad u'(L) = e^{i\theta}u'(0)\}.$$

For $j \in \mathbb{N}$, we denote by $\mu_j(\theta)$ the jth eigenvalue of the operator S_θ counted with multiplicity. This allows us to formulate our main result.

Theorem 2.1 *Let n be an arbitrary integer. There exists $\beta(n) > 0$ such that*

$$\sharp\sigma_{\mathrm{d}}(H_{\beta,\theta}) \geq n \quad \text{for} \quad \beta \geq \beta(n) \quad \text{and} \quad \theta \in [0,2\pi).$$

For $\beta \geq \beta(n)$ we denote by $\lambda_n(\beta,\theta)$ the nth eigenvalue of $H_{\beta,\theta}$ counted with multiplicity. Then $\lambda_n(\beta,\theta)$ admits an asymptotic expansion of the form

$$\lambda_n(\beta,\theta) = -\frac{1}{4}\beta^2 + \mu_n(\theta) + \mathcal{O}(\beta^{-1}\log\beta) \quad \text{as} \quad \beta \to \infty,$$

where the error term is uniform with respect to $\theta \in [0,2\pi)$.

Combining this result with Borg's theorem on the inverse problem for Hill's equation, we obtain the following corollary about the existence of the band gap of $\sigma(H_\beta)$.

Corollary 2.2 *Assume that $\gamma \neq 0$, i.e. that Γ is not a straight line. Then there exists $m \in \mathbb{N}$ and $G_m > 0$ such that*

$$\lim_{\beta\to\infty}\left(\min_{\theta\in[0,2\pi)}\lambda_{m+1}(\beta,\theta) - \max_{\theta\in[0,2\pi)}\lambda_m(\beta,\theta)\right) = G_m.$$

We would like to know, of course, which gaps in the spectrum open as $\beta \to \infty$. To this aim we prove a sufficient condition which guarantees this property for a fixed gap index n. Let $\{c_j\}_{j=1}^\infty$ and $\{d_j\}_{j=0}^\infty$ be the Fourier coefficients of $\frac{1}{4}\gamma(s)^2$:

$$\frac{1}{4}\gamma(s)^2 = \sum_{j=1}^\infty c_j \sin\frac{2\pi j}{L}s + \sum_{j=0}^\infty d_j \cos\frac{2\pi j}{L}s \quad \text{in} \quad L^2((0,L)). \tag{2.5}$$

Proposition 2.3 *Let $n \in \mathbb{N}$. Assume that* $0 < \sqrt{c_n^2 + d_n^2} < \frac{12\pi^2}{L^2} n^2$ *and*

$$\max_{s\in[0,L]} \left| \frac{1}{4}\gamma(s)^2 - d_0 - c_n \sin \frac{2n\pi}{L} s - d_n \cos \frac{2n\pi}{L} s \right| < \frac{1}{4}\sqrt{c_n^2 + d_n^2},$$

then we have

$$\lim_{\beta\to\infty} \left(\min_{\theta\in[0,2\pi)} \lambda_{n+1}(\beta,\theta) - \max_{\theta\in[0,2\pi)} \lambda_n(\beta,\theta) \right) > 0.$$

In particular, it is obvious that if the effective curvature-induced potential has a dominating Fourier component in the expansion (2.5), the band with the same index opens as $\beta \to \infty$. We also see that the second assumption of Proposition 2.3 is more difficult to satisfy as the index n increases.

3 Proof of Theorem 2.1

We first prove the unitary equivalence (2.2) by using the Floquet-Bloch reduction scheme – see, e.g., [RS, XIII.16]. For $u \in C_0^\infty(\mathbb{R}^2)$ and $\theta \in [0, 2\pi)$, we define

$$\mathcal{U}_0 u(x, y, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{im\theta} u(x - mK_1, y - mK_2), \quad (x, y) \in \Lambda.$$

Then $\mathcal{U}_0$ extends uniquely a unitary operator from $L^2(\mathbb{R}^2)$ to $\int_0^{2\pi} \oplus L^2(\Lambda)\, d\theta$, which we denote as $\mathcal{U}$. In addition, $\mathcal{U}$ is unitary also as an operator from $H^1(\mathbb{R}^2)$ to $\int_0^{2\pi} \oplus H^1(\Lambda) d\theta$. Let us check the following claim.

Lemma 3.1 *We have*

$$\mathcal{U} H_\beta \mathcal{U}^{-1} = \int_0^{2\pi} \oplus H_{\beta,\theta}\, d\theta. \tag{3.1}$$

Proof: We shall first show that

$$q_\beta(f, g) = \int_0^{2\pi} q_{\beta,\theta}((\mathcal{U}f)(\cdot,\cdot,\theta), (\mathcal{U}g)(\cdot,\cdot,\theta))\, d\theta \quad \text{for} \quad f, g \in H^1(\mathbb{R}^2). \tag{3.2}$$

Let $u, v \in C_0^\infty(\mathbb{R}^2)$. The quadratic form

$$q_\beta(u, v) = (\nabla u, \nabla v)_{L^2(\mathbb{R}^2)} - \beta \int_\Gamma u(x)\overline{v(x)}\, dS$$

can be in view of (2.1) written as

$$\sum_{m=-\infty}^{\infty}((\nabla u)(x-mK_1,x-mK_2),(\nabla v)(x-mK_1,y-mK_2))_{L^2(\Lambda)}$$
$$-\beta\sum_{m=-\infty}^{\infty}\int_{\Gamma((0,L))}u(x-mK_1,y-mK_2)\overline{v(x-mK_1,y-mK_2)}\,dS$$

and since $\{\frac{1}{\sqrt{2\pi}}e^{in\theta}\}_{n=-\infty}^{\infty}$ is a complete orthonormal system of $L^2((0,2\pi))$ we have

$$\begin{aligned}
&= \int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}e^{im\theta}(\nabla u)(x-mK_1,y-mK_2),\right.\\
&\qquad \left.\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}e^{in\theta}(\nabla v)(x-nK_1,y-nK_2)\right)_{L^2(\Lambda)} d\theta\\
&\qquad -\beta\int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}e^{im\theta}u(x-mK_1,y-mK_2),\right.\\
&\qquad \left.\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}e^{in\theta}v(x-nK_1,y-nK_2)\right)_{L^2(\Gamma((0,L)))} d\theta\\
&= \int_0^{2\pi}q_{\beta,\theta}((\mathcal{U}u)(\cdot,\cdot,\theta),(\mathcal{U}v)(\cdot,\cdot,\theta))\,d\theta. \qquad (3.3)
\end{aligned}$$

Let $f,g\in H^1(\mathbb{R}^2)$. Since $C_0^\infty(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, we can choose in it two sequences $\{u_j\}_{j=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ such that

$$u_j\to f \quad\text{in}\quad H^1(\mathbb{R}^2),\qquad v_j\to g\quad\text{in}\quad H^1(\mathbb{R}^2)\quad\text{as}\quad j\to\infty.$$

The form q_β is bounded in $H^1(\mathbb{R}^2)$, hence we get

$$\lim_{j\to\infty}q_\beta(u_j,v_j)=q_\beta(f,g). \qquad (3.4)$$

Notice that there exist a constant $C>0$ such that for any $\theta\in[0,2\pi)$ and $u,v\in Q_\theta$, we have

$$|q_{\beta,\theta}(u,v)|\le C\|u\|_{H^1(\Lambda)}\|v\|_{H^1(\Lambda)}. \qquad (3.5)$$

Since $\mathcal{U}$ is a unitary operator from $H^1(\mathbb{R}^2)$ to $\int_0^{2\pi} \oplus H^1(\Lambda)\, d\theta$, we have

$$\begin{aligned}
\mathcal{U}u_j \to \mathcal{U}f \qquad &\text{in} \qquad \int_0^{2\pi} \oplus H^1(\Lambda)\, d\theta, \\
\mathcal{U}v_j \to \mathcal{U}g \qquad &\text{in} \qquad \int_0^{2\pi} \oplus H^1(\Lambda)\, d\theta.
\end{aligned}$$

Combining these relations with (3.5), we have

$$\begin{aligned}
&\lim_{j\to\infty} \int_0^{2\pi} q_{\beta,\theta}((\mathcal{U}u_j)(\cdot,\cdot,\theta), (\mathcal{U}v_j)(\cdot,\cdot,\theta))\, d\theta \\
&\quad = \int_0^{2\pi} q_{\beta,\theta}((\mathcal{U}f)(\cdot,\cdot,\theta), (\mathcal{U}g)(\cdot,\cdot,\theta))\, d\theta. \qquad (3.6)
\end{aligned}$$

Putting (3.3), (3.4), and (3.6) together, we get (3.2).

Next we shall show that

$$\mathcal{U}^{-1}\left(\int_0^{2\pi} \oplus H_{\beta,\theta}\, d\theta\right)\mathcal{U} \subset H_\beta. \qquad (3.7)$$

Let $u \in L^2(\mathbb{R}^2)$ and $\mathcal{U}u \in \mathcal{D}(\int_0^{2\pi} \oplus H_{\beta,\theta}\, d\theta)$. By definition of the direct integral we have

$$\begin{aligned}
&(\mathcal{U}u)(\cdot,\cdot,\theta) \in \mathcal{D}(H_{\beta,\theta}) \quad \text{for} \quad \text{a.e.} \quad \theta \in [0, 2\pi), \\
&\int_0^{2\pi} \|H_{\beta,\theta}\mathcal{U}u(\cdot,\cdot,\theta)\|^2_{L^2(\Lambda)}\, d\theta < \infty. \qquad (3.8)
\end{aligned}$$

The first named property means in particular that $(\mathcal{U}u)(\cdot,\cdot,\theta) \in \mathcal{D}(H_{\beta,\theta})$ for a.e. $\theta \in [0, 2\pi)$, thus we have

$$q_{\beta,\theta}((\mathcal{U}u)(\cdot,\cdot,\theta), g) = (H_{\beta,\theta}\mathcal{U}u(\cdot,\cdot,\theta), g)_{L^2(\Lambda)} \quad \text{for} \quad \text{all} \quad g \in Q_\theta. \qquad (3.9)$$

Note that there exists a constant $b > 0$ such that for all $\theta \in [0, 2\pi)$ and $f \in Q_\theta$, we have

$$q_{\beta,\theta}(f, f) + b\|f\|^2_{L^2(\Lambda)} \geq \frac{1}{2}\|f\|^2_{H^1(\Lambda)}. \qquad (3.10)$$

It follows from (3.9) that

$$\begin{aligned}
|q_{\beta,\theta}(\mathcal{U}u(\cdot,\cdot,\theta), \mathcal{U}u(\cdot,\cdot,\theta))| &= |(H_{\beta,\theta}\mathcal{U}u(\cdot,\cdot,\theta), \mathcal{U}u(\cdot,\cdot,\theta))| \\
&\leq \frac{1}{2}\left(\|H_{\beta,\theta}\mathcal{U}u(\cdot,\cdot,\theta)\|^2_{L^2(\Lambda)} + \|\mathcal{U}u(\cdot,\cdot,\theta)\|^2_{L^2(\Lambda)}\right).
\end{aligned}$$

This together with (3.8) and (3.10) implies that $\mathcal{U}u \in \int_0^{2\pi} \oplus H^1(\Lambda)\, d\theta$, so we have $u \in H^1(\mathbb{R}^2)$. We pick any $v \in H^1(\mathbb{R}^2)$. Its image by $\mathcal{U}$ satisfies

$$(\mathcal{U}v)(\cdot,\cdot,\theta) \in Q_\theta \quad \text{for} \quad \text{a.e.} \quad \theta \in [0, 2\pi).$$

We put $w(\theta) = H_{\beta,\theta}\mathcal{U}u(\cdot,\cdot,\theta)$. From (3.2) we have

$$q_\beta(u,v) = \int_0^{2\pi} q_{\beta,\theta}((\mathcal{U}u)(\cdot,\cdot,\theta),(\mathcal{U}v)(\cdot,\cdot,\theta))\, d\theta$$

which can be using (3.9) rewritten as

$$q_\beta(u,v) = \int_0^{2\pi} (w(\theta),(\mathcal{U}v)(\cdot,\cdot,\theta))_{L^2(\Lambda)}\, d\theta = (\mathcal{U}^{-1}w, v)_{L^2(\mathbb{R}^2)}.$$

Using (3.8), we get

$$\mathcal{U}^{-1}w \in L^2(\mathbb{R}^2).$$

Thus we have $u \in \mathcal{D}(H_\beta)$ and

$$\mathcal{U}^{-1}\left(\int_0^{2\pi} \oplus H_{\beta,\theta}\, d\theta\right)\mathcal{U}u = H_\beta u,$$

which proves (3.7). Since the two operators in this inclusion are self-adjoint, we arrive at (3.1). ∎

Next we have to localize the essential spectrum of our operator.

Lemma 3.2 *We have*

$$\sigma_{\mathrm{ess}}(H_{\beta,\theta}) = [0,\infty).$$

Proof: We define

$$c_\theta(u,v) = \int_{\Gamma((0,L))} u(x)\overline{v(x)}\, dS, \quad u,v \in Q_\theta,$$

which allows us to write $q_{\beta,\theta} = q_{0,\theta} - \beta c_\theta$ on Q_θ. Let C_θ be the self-adjoint operator associated with the form $\overline{c_\theta}$. In view of the quadratic form version of Weyl's theorem (see [RS, XIII.4, Corollary 4]), it suffices to demonstrate that the operator $(H_{0,\theta}+1)^{-1}C_\theta(H_{0,\theta}+1)^{-1}$ is compact on $L^2(\Lambda)$. Let $\{u_n\}_{n=1}^\infty \subset L^2(\Lambda)$ be a sequence which converges to zero vector weakly in $L^2(\Lambda)$. We put $v_n = (H_{0,\theta}+1)^{-1}u_n$. Since $(H_{0,\theta}+1)^{-1}$ is a bounded operator from $L^2(\Lambda)$ to

$H^2(\Lambda)$ and the operator $H^2(\Lambda) \ni f \mapsto f|_{\Gamma((0,L))} \in L^2(\Gamma((0,L)))$ is compact, we have

$$\|C_\theta^{1/2}(H_{0,\theta}+1)^{-1}u_n\|^2_{L^2(\Lambda)} = c_\theta(v_n, v_n) = \|v_n\|^2_{L^2(\Gamma((0,L)))} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $C_\theta^{1/2}(H_{0,\theta}+1)^{-1}$ is a compact operator on $L^2(\Lambda)$, and consequently

$$(H_{0,\theta}+1)^{-1}C_\theta(H_{0,\theta}+1)^{-1} = [C_\theta^{1/2}(H_{0,\theta}+1)^{-1}]^*[C_\theta^{1/2}(H_{0,\theta}+1)^{-1}]$$

is a compact operator on $L^2(\Lambda)$. ∎

Lemma 3.3 *We have*

$$\sigma(H_\beta) = \bigcup_{\theta\in[0,2\pi)} \sigma(H_{\beta,\theta}).$$

Proof: We put

$$K_\beta = \int_0^{2\pi} \oplus H_{\beta,\theta}\, d\theta.$$

In view of Lemma 3.1, it suffices to prove that

$$\sigma(K_\beta) = \bigcup_{\theta\in[0,2\pi)} \sigma(H_{\beta,\theta}). \tag{3.11}$$

Combining Lemma 3.2 with [RS, Theorem XIII.85(d)], we have

$$\sigma(K_\beta) \cap [0,\infty) = \left(\bigcup_{\theta\in[0,2\pi)} \sigma(H_{\beta,\theta})\right) \cap [0,\infty) = [0,\infty). \tag{3.12}$$

Next we shall show that

$$\sigma(K_\beta) \cap (-\infty,0) = \left(\bigcup_{\theta\in[0,2\pi)} \sigma(H_{\beta,\theta})\right) \cap (-\infty,0). \tag{3.13}$$

For $n \in \mathbb{N}$, we put

$$\alpha_n(\beta,\theta) = \sup_{v_1,\cdots,v_{n-1}\in L^2(\Lambda)} \inf_{\phi\in\mathcal{P}(v_1,\cdots,v_{n-1})} q_{\beta,\theta}(\phi,\phi),$$

where $\mathcal{P}(v_1, \cdots, v_{n-1}) := \{\phi;\ \phi \in Q_\theta,\ \|\phi\|_{L^2(\Lambda)} = 1$, and $(\phi, v_j)_{L^2(\Lambda)} = 0$ for $1 \le j \le n-1\}$. In order to prove (3.13), we shall show that the functions $\alpha_n(\beta, \cdot)$ are continuous on $[0, 2\pi]$. Let $\theta, \theta_0 \in [0, 2\pi]$. We define

$$(V_{\theta,\theta_0} f)(x, y) = \exp\left\{ i \frac{\theta - \theta_0}{K_1} x \right\} f(x, y) \quad \text{for} \quad f \in L^2(\Lambda).$$

Then V_{θ,θ_0} is a unitary operator on $L^2(\Lambda)$ which maps Q_{θ_0} onto Q_θ bijectively. We have

$$\begin{aligned} & q_{\beta,\theta}(V_{\theta,\theta_0} g, V_{\theta,\theta_0} g) - q_{\beta,\theta_0}(g, g) \\ & \quad = \frac{(\theta - \theta_0)^2}{K_1^2} \|g\|^2_{L^2(\Lambda)} + 2\Re \left(i \frac{\theta - \theta_0}{K_1} V_{\theta,\theta_0} g,\ e^{i \frac{\theta - \theta_0}{K_1} x} \frac{\partial}{\partial x} g \right)_{L^2(\Lambda)} \end{aligned} \tag{3.14}$$

for $g \in Q_{\theta_0}$. Note that there exists $\alpha > 0$ such that

$$\left\| \frac{\partial}{\partial x} g \right\|^2_{L^2(\Lambda)} \le \frac{3}{2} q_{\beta,\theta_0}(g, g) + \alpha \|g\|^2_{L^2(\Lambda)} \quad \text{for} \quad g \in Q_{\theta_0}.$$

Combining this with (3.14), we obtain

$$\begin{aligned} & |q_{\beta,\theta}(V_{\theta,\theta_0} g, V_{\theta,\theta_0} g) - q_{\beta,\theta_0}(g, g)| \\ & \quad \le \frac{(\theta - \theta_0)^2}{K_1^2} \|g\|^2_{L^2(\Lambda)} + \frac{|\theta - \theta_0|}{K_1} \left((1 + \alpha) \|g\|^2_{L^2(\Lambda)} + \frac{3}{2} q_{\beta,\theta_0}(g, g) \right) \end{aligned}$$

for $g \in Q_{\theta_0}$. It proves the continuity of $\alpha_n(\beta, \cdot)$ on $[0, 2\pi]$. Combining this with the min-max principle and [RS, Theorem XIII.85(d)], we arrive at (3.13). The relations (3.12) and (3.13) together give (3.11) which completes the proof. ∎

The most important part of the proof is the analysis of the discrete spectrum of $H_{\beta,\theta}$. The tool we use is the Dirichlet-Neumann bracketing. Given $a > 0$, we put

$$\Sigma_a = \Phi((0, L) \times (-a, a)).$$

Note that Σ_a is a domain derived by transporting a segment of the length $2a$ perpendicular to Γ along the curve. Since $\Gamma'(0) = \Gamma'(L) = (1, 0)$, we have $\Phi_1(0, \cdot) = 0$ and $\Phi_1(L, \cdot) = K_1$ on $\mathbb{R}$. This together with $(A.5)$ implies, for

$|a| < a_0$, that $\Sigma_a \subset \Lambda$ and that $\Lambda \backslash \Sigma_a$ consists of two connected components, which we denote by Λ_a^1 and Λ_a^2. For $\theta \in [0, 2\pi)$, we define

$$\begin{aligned} R_{a,\theta}^+ &= \{u \in H^1(\Sigma_a); \quad u = 0 \quad \text{on} \quad \partial\Sigma_a \cap \Lambda, \\ &\qquad\qquad u(K_1, \cdot) = e^{i\theta} u(0, \cdot) \quad \text{on} \quad (-a, a)\}, \\ R_{a,\theta}^- &= \{u \in H^1(\Sigma_a); \quad u(K_1, \cdot) = e^{i\theta} u(0, \cdot) \quad \text{on} \quad (-a, a)\}, \\ q_{a,\beta,\theta}^+(f, f) &= \|\nabla f\|_{L^2(\Sigma_a)}^2 - \beta \int_{\Gamma((0,L))} |f(x)|^2 \, dS \quad \text{for} \quad f \in R_{a,\theta}^+, \\ q_{a,\beta,\theta}^-(f, f) &= \|\nabla f\|_{L^2(\Sigma_a)}^2 - \beta \int_{\Gamma((0,L))} |f(x)|^2 \, dS \quad \text{for} \quad f \in R_{a,\theta}^-. \end{aligned}$$

Let $L_{a,\beta,\theta}^+$ and $L_{a,\beta,\theta}^-$ be the self-adjoint operators associated with the forms $q_{a,\beta,\theta}^+$ and $q_{a,\beta,\theta}^-$, respectively. For $j = 1, 2$, we define

$$\begin{aligned} K_{a,j,\theta}^+ &= \{f \in H^1(\Lambda_a^j); \quad f(K_1, K_2 + u) = e^{i\theta} f(0, u) \quad \text{if} \quad (0, u) \in \partial\Lambda_a^j, \\ &\qquad\qquad f = 0 \quad \text{on} \quad \partial\Lambda_a^j \cap \Lambda\}, \\ K_{a,j,\theta}^- &= \{f \in H^1(\Lambda_a^j); \quad f(K_1, K_2 + u) = e^{i\theta} f(0, u) \quad \text{if} \quad (0, u) \in \partial\Lambda_a^j\}, \\ &\qquad\qquad e_{a,j,\theta}^\pm(f, f) = \|\nabla f\|_{L^2(\Lambda_a^j)}^2 \quad \text{for} \quad f \in K_{a,j,\theta}^\pm. \end{aligned}$$

Let $E_{a,j,\theta}^\pm$ be the self-adjoint operators associated with the forms $e_{a,j,\theta}^\pm$. By the bracketing bounds (see [RS, XIII.15, Proposition 4]) we obtain

$$E_{a,1,\theta}^- \oplus L_{a,\beta,\theta}^- \oplus E_{a,2,\theta}^- \leq H_{\beta,\theta} \leq E_{a,1,\theta}^+ \oplus L_{a,\beta,\theta}^+ \oplus E_{a,2,\theta}^+ \tag{3.15}$$

in $L^2(\Lambda_a^1) \oplus L^2(\Sigma_a) \oplus L^2(\Lambda_a^2)$. In order to estimate the negative eigenvalues of $H_{\beta,\theta}$, it is sufficient to estimate those of $L_{a,\beta,\theta}^+$ and $L_{a,\beta,\theta}^-$ because the other operators involved in (3.15) are non-negative.

To this aim we introduce two operators in $L^2((0, L) \times (-a, a))$ which are unitarily equivalent to $L_{a,\beta,\theta}^+$ and $L_{a,\beta,\theta}^-$, respectively. We define

$$\begin{aligned} Q_{a,\theta}^+ &= \{\varphi \in H^1((0, L) \times (-a, a)); \quad \varphi(K_1, \cdot) = e^{i\theta} \varphi(0, \cdot) \quad \text{on} \quad (-a, a), \\ &\qquad\qquad \varphi(\cdot, a) = \varphi(\cdot, -a) = 0 \quad \text{on} \quad (0, L)\}, \\ Q_{a,\theta}^- &= \{\varphi \in H^1((0, L) \times (-a, a)); \quad \varphi(K_1, \cdot) = e^{i\theta} \varphi(0, \cdot) \quad \text{on} \quad (-a, a)\}, \\ b_{a,\beta,\theta}^+(f, f) &= \int_0^L \int_{-a}^a (1 + u\gamma(s))^{-2} \left| \frac{\partial f}{\partial s} \right|^2 duds + \int_0^L \int_{-a}^a \left| \frac{\partial f}{\partial u} \right|^2 duds \\ &\quad + \int_0^L \int_{-a}^a V(s, u) |f|^2 \, dsdu - \beta \int_0^L |f(s, 0)|^2 \, ds \quad \text{for } f \in Q_{a,\theta}^+, \end{aligned}$$

$$
\begin{aligned}
b^-_{a,\beta,\theta}(f,f) = & \int_0^L \int_{-a}^{a} (1+u\gamma(s))^{-2} \left| \frac{\partial f}{\partial s} \right|^2 duds + \int_0^L \int_{-a}^{a} \left| \frac{\partial f}{\partial u} \right|^2 duds \\
& + \int_0^L \int_{-a}^{a} V(s,u)|f|^2 \, dsdu - \beta \int_0^L |f(s,0)|^2 \, ds \\
& - \frac{1}{2} \int_0^L \frac{\gamma(s)}{1+a\gamma(s)} |f(s,a)|^2 \, ds + \frac{1}{2} \int_0^L \frac{\gamma(s)}{1-a\gamma(s)} |f(s,-a)|^2 \, ds
\end{aligned}
$$

for $f \in Q^-_{a,\theta}$, where

$$
V(s,u) = \frac{1}{2}(1+u\gamma(s))^{-3} u\gamma''(s) - \frac{5}{4}(1+u\gamma(s))^{-4} u^2 \gamma'(s)^2 - \frac{1}{4}(1+u\gamma(s))^{-2}\gamma(s)^2.
$$

Let $B^+_{a,\beta,\theta}$ and $B^-_{a,\beta,\theta}$ be the self-adjoint operators associated with the forms $b^+_{a,\beta,\theta}$ and $b^-_{a,\beta,\theta}$, respectively. Acting as in the proof of Lemma 2.2 in [EY], we arrive at the following result.

Lemma 3.4 *The operators $B^+_{a,\beta,\theta}$ and $B^-_{a,\beta,\theta}$ are unitarily equivalent to $L^+_{a,\beta,\theta}$ and $L^-_{a,\beta,\theta}$, respectively.*

Next we estimate $B^+_{a,\beta,\theta}$ and $B^-_{a,\beta,\theta}$ by operators with separated variables. We put

$$
\gamma_+ = \max_{[0,L]} |\gamma(\cdot)|, \quad \gamma'_+ = \max_{[0,L]} |\gamma'(\cdot)|, \quad \gamma''_+ = \max_{[0,L]} |\gamma''(\cdot)|,
$$

and

$$
\begin{aligned}
V_+(s) &= \frac{1}{2}(1-a\gamma_+)^{-3} a\gamma''_+ - \frac{5}{4}(1+a\gamma_+)^{-4} a^2 (\gamma'_+)^2 - \frac{1}{4}(1+a\gamma_+)^{-2}\gamma(s)^2, \\
V_-(s) &= -\frac{1}{2}(1-a\gamma_+)^{-3} a\gamma''_+ - \frac{5}{4}(1-a\gamma_+)^{-4} a^2 (\gamma'_+)^2 - \frac{1}{4}(1-a\gamma_+)^{-2}\gamma(s)^2.
\end{aligned}
$$

If $0 < a < \frac{1}{2\gamma_+}$, we can define

$$
\begin{aligned}
\tilde{b}^+_{a,\beta,\theta}(f,f) &= (1-a\gamma_+)^{-2} \int_0^L \int_{-a}^{a} \left| \frac{\partial f}{\partial s} \right|^2 duds + \int_0^L \int_{-a}^{a} \left| \frac{\partial f}{\partial u} \right|^2 duds \\
& \quad + \int_0^L \int_{-a}^{a} V_+(s)|f|^2 \, duds - \beta \int_0^L |f(s,0)|^2 \, ds \quad \text{for } f \in Q^+_{a,\theta}, \\
\tilde{b}^-_{a,\beta,\theta}(f,f) &= (1+a\gamma_+)^{-2} \int_0^L \int_{-a}^{a} \left| \frac{\partial f}{\partial s} \right|^2 duds + \int_0^L \int_{-a}^{a} \left| \frac{\partial f}{\partial u} \right|^2 duds
\end{aligned}
$$

$$+\int_0^L\int_{-a}^a V_-(s)|f|^2\,duds-\beta\int_0^L|f(s,0)|^2\,ds$$
$$-\gamma_+\int_0^L(|f(s,a)|^2+|f(s,-a)|^2)\,ds\quad\text{for}\quad f\in Q^-_{a,\theta}.$$

Then we have

$$b^+_{a,\beta,\theta}(f,f)\le\tilde b^+_{a,\beta,\theta}(f,f)\quad\text{for}\quad f\in Q^+_{a,\theta},\tag{3.16}$$

$$\tilde b^-_{a,\beta,\theta}(f,f)\le b^-_{a,\beta,\theta}(f,f)\quad\text{for}\quad f\in Q^-_{a,\theta}.\tag{3.17}$$

Let $\tilde H^+_{a,\beta,\theta}$ and $\tilde H^-_{a,\beta,\theta}$ be the self-adjoint operators associated with the forms $\tilde b^+_{a,\beta,\theta}$ and $\tilde b^-_{a,\beta,\theta}$, respectively. Let $T^+_{a,\beta}$ be the self-adjoint operator associated with the form

$$t^+_{a,\beta}(f,f)=\int_{-a}^a|f'(u)|^2\,du-\beta|f(0)|^2,\quad f\in H^1_0((-a,a)).$$

Let finally $T^-_{a,\beta}$ be the self-adjoint operator associated with the form

$$t^-_{a,\beta}(f,f)=\int_{-a}^a|f'(u)|^2\,du-\beta|f(0)|^2-\gamma_+(|f(a)|^2+|f(-a)|^2)$$

for $f\in H^1((-a,a))$. We define

$$\begin{aligned}U^+_{a,\theta}&=-(1-a\gamma_+)^{-2}\frac{d^2}{ds^2}+V_+(s)\quad\text{in}\quad L^2((0,L))\quad\text{with the domain}\quad P_\theta,\\ U^-_{a,\theta}&=-(1+a\gamma_+)^{-2}\frac{d^2}{ds^2}+V_-(s)\quad\text{in}\quad L^2((0,L))\quad\text{with the domain}\quad P_\theta.\end{aligned}$$

Then we have

$$\begin{aligned}\tilde H^+_{a,\beta,\theta}&=U^+_{a,\theta}\otimes 1+1\otimes T^+_{a,\beta},\\ \tilde H^-_{a,\beta,\theta}&=U^-_{a,\theta}\otimes 1+1\otimes T^-_{a,\beta}.\end{aligned}\tag{3.18}$$

Next we consider the asymptotic behaviour for a fixed eigenvalue of $U^\pm_{a,\theta}$ as a tends to zero. Let $\mu^\pm_j(a,\theta)$ be the jth eigenvalue of $U^\pm_{a,\theta}$ counted with multiplicity. We recall the estimates contained in relations (2.25) and (2.26) of the paper [Yo].

Proposition 3.5 *For $j \in \mathbb{N}$ and $0 < a < \frac{1}{2\gamma_+}$, there exists $C_j > 0$ such that*

$$|\mu_j^+(a,\theta) - \mu_j(\theta)| \le C_j a$$

and

$$|\mu_j^-(a,\theta) - \mu_j(\theta)| \le C_j a,$$

where C_j is independent of a and θ.

We also need two-sided estimates for the first eigenvalue of the transverse operators $T_{a,\beta}^{\pm}$. They are obtained in the same way as in [EY]: we get

Proposition 3.6 *Assume that $\beta a > \frac{8}{3}$. Then $T_{a,\beta}^{+}$ has only one negative eigenvalue, which we denote by $\zeta_{a,\beta}^{+}$. It satisfies the inequalities*

$$-\frac{1}{4}\beta^2 < \zeta_{a,\beta}^{+} < -\frac{1}{4}\beta^2 + 2\beta^2 \exp\left(-\frac{1}{2}\beta a\right).$$

Proposition 3.7 *Let $\beta a > 8$ and $\beta > \frac{8}{3}\gamma_+$. Then $T_{a,\beta}^{-}$ has a unique negative eigenvalue $\zeta_{a,\beta}^{-}$, and moreover, we have*

$$-\frac{1}{4}\beta^2 - \frac{2205}{16}\beta^2 \exp\left(-\frac{1}{2}\beta a\right) < \zeta_{a,\beta}^{-} < -\frac{1}{4}\beta^2.$$

Now we are ready to prove our main result.

Proof of Theorem 2.1: We put $a(\beta) = 6\beta^{-1}\log\beta$. Let $\xi_{\beta,j}^{\pm}$ be the jth eigenvalue of $T_{a(\beta),\beta}^{\pm}$. From Propositions 3.6 and 3.7, we have

$$\xi_{\beta,1}^{\pm} = \zeta_{a(\beta),\beta}^{\pm} \quad \text{and} \quad \xi_{\beta,2}^{\pm} \ge 0.$$

¿From (3.18), we infer that $\{\xi_{\beta,j}^{\pm} + \mu_k^{\pm}(a(\beta),\theta)\}_{j,k\in\mathbb{N}}$, properly ordered, is the sequence of all eigenvalues of $\tilde{H}_{a(\beta),\beta,\theta}^{\pm}$ counted with multiplicity. Using Proposition 3.5, we find

$$\xi_{\beta,j}^{\pm} + \mu_k^{\pm}(a(\beta),\theta) \ge \mu_1^{\pm}(a(\beta),\theta) = \mu_1(\theta) + \mathcal{O}(\beta^{-1}\log\beta) \tag{3.21}$$

for $j \ge 2$ and $k \ge 1$, where the error term is uniform with respect to the quasimomentum $\theta \in [0, 2\pi)$. For $k \in \mathbb{N}$ and $\theta \in [0, 2\pi)$, we define

$$\tau_{\beta,k,\theta}^{\pm} = \zeta_{a(\beta),\beta}^{\pm} + \mu_k^{\pm}(a(\beta),\theta). \tag{3.22}$$

¿From Propositions 3.5–3.7 we get

$$\tau^{\pm}_{\beta,k,\theta} = -\frac{1}{4}\beta^2 + \mu_k(\theta) + \mathcal{O}(\beta^{-1}\log\beta) \quad \text{as} \quad \beta\to\infty, \tag{3.23}$$

where the error term is uniform with respect to $\theta\in[0,2\pi)$. Let $n\in\mathbb{N}$. Combining (3.21) with (3.23), we claim that there exists $\beta(n)>0$ such that

$$\tau^{+}_{\beta,n,\theta}<0, \quad \tau^{+}_{\beta,n,\theta}<\xi^{+}_{\beta,j}+\mu^{+}_{k}(a(\beta),\theta), \quad \text{and} \quad \tau^{-}_{\beta,n,\theta}<\xi^{-}_{\beta,j}+\mu^{-}_{k}(a(\beta),\theta)$$

for $\beta\ge\beta(n)$, $j\ge 2$, $k\ge 1$, and $\theta\in[0,2\pi)$. Hence the jth eigenvalue of $\tilde{H}^{\pm}_{a(\beta),\beta,\theta}$ counted with multiplicity is $\tau^{\pm}_{\beta,j,\theta}$ for $j\le n$, $\beta\ge\beta(n)$, and $\theta\in[0,2\pi)$. Let $\beta\ge\beta(n)$ and denote by $\kappa^{\pm}_j(\beta,\theta)$ the jth eigenvalue of $L^{\pm}_{a(\beta),\beta,\theta}$. From (3.16), (3.17), and the min-max principle, we obtain

$$\tau^{-}_{\beta,j,\theta}\le\kappa^{-}_j(\beta,\theta) \quad \text{and} \quad \kappa^{+}_j(\beta,\theta)\le\tau^{+}_{\beta,j,\theta} \quad \text{for} \quad 1\le j\le n, \tag{3.24}$$

so we have $\kappa^{+}_n(\beta,\theta)<0$. Hence the min-max principle and (3.15) imply that $H_{\beta,\theta}$ has at least n eigenvalues in $(-\infty,\kappa^{+}_n(\beta,\theta))$. For $1\le j\le n$, we denote by $\lambda_j(\beta,\theta)$ the jth eigenvalue of $H_{\beta,\theta}$. We have

$$\kappa^{-}_j(\beta,\theta)\le\lambda_j(\beta,\theta)\le\kappa^{+}_j(\beta,\theta) \quad \text{for} \quad 1\le j\le n.$$

This together with (3.23) and (3.24) implies that

$$\lambda_j(\beta,\theta) = -\frac{1}{4}\beta^2+\mu_j(\theta)+\mathcal{O}(\beta^{-1}\log\beta) \quad \text{as} \quad \beta\to\infty \quad \text{for} \quad 1\le j\le n,$$

where the error term is uniform with respect to $\theta\in[0,2\pi)$, and completes thus the proof of Theorem 2.1. ∎

Our next aim is to prove Corollary 2.2. As a preliminary, we denote by B_j and G_j, respectively, the length of the jth band and the jth gap of the spectrum of the operator $-\frac{d^2}{ds^2}-\frac{1}{4}\gamma(s)^2$ in $L^2(\mathbb{R})$ with the domain $H^2(\mathbb{R})$:

$$\begin{aligned} B_j &= \begin{cases} \mu_j(\pi)-\mu_j(0) & \text{for odd} \quad j, \\ \mu_j(0)-\mu_j(\pi) & \text{for even} \quad j, \end{cases} \\ G_j &= \begin{cases} \mu_{j+1}(\pi)-\mu_j(\pi) & \text{for odd} \quad j, \\ \mu_{j+1}(0)-\mu_j(0) & \text{for even} \quad j. \end{cases} \end{aligned}$$

Since $\mu_j(\cdot)$ is continuous on $[0,2\pi]$, we immediately obtain from Theorem 2.1 the following claim.

Lemma 3.8 *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} |\lambda_n(\beta, [0, 2\pi))| &= B_n + \mathcal{O}(\beta^{-1} \log \beta) \qquad \text{as} \qquad \beta \to \infty, \\ \min_{\theta \in [0,2\pi)} \lambda_{n+1}(\beta, \theta) - \max_{\theta \in [0,2\pi)} \lambda_n(\beta, \theta) &= G_n + \mathcal{O}(\beta^{-1} \log \beta) \qquad \text{as} \qquad \beta \to \infty. \end{aligned}$$

Now we recall Borg's theorem (see [Bo, Ho, Un]).

Theorem 3.9 (Borg) *Suppose that W is a real-valued, piecewise continuous function on $[0, L]$. Let $\alpha_j^{\pm}$ be the jth eigenvalue of the following operator counted with multiplicity:*

$$-\frac{d^2}{ds^2} + W(s) \quad \text{in} \quad L^2((0, L))$$

with the domain

$$\{v \in H^2((0, L)); \quad v(L) = \pm v(0), \quad v'(L) = \pm v'(0)\}.$$

Suppose that

$$\alpha_j^+ = \alpha_{j+1}^+ \quad \text{for all even} \quad j,$$

and

$$\alpha_j^- = \alpha_{j+1}^- \quad \text{for all odd} \quad j.$$

Then W is constant on $[0, L]$.

Proof of Corollary 2.2: Assume that γ is not identically zero. Then it follows from $(A.3)$ that γ is not constant on $[0, L]$. Combining this with Borg's theorem, we infer that there exists $m \in \mathbb{N}$ such that $G_m > 0$. From Lemma 3.8 we get

$$\lim_{\beta \to \infty} \left(\min_{\theta \in [0,2\pi)} \lambda_{m+1}(\beta, \theta) - \max_{\theta \in [0,2\pi)} \lambda_m(\beta, \theta) \right) = G_m > 0.$$

This completes the proof. ∎

4 The gaps of Hill's equation

It follows from Lemma 3.8 that if the mth gap of $-\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2$ in $L^2(\mathbb{R})$ is open, so is the mth gap of $H(\beta)$ for sufficiently large $\beta > 0$. It is thus useful to find a sufficient condition for which the mth gap of our comparison operator is open for a given $m \in \mathbb{N}$. Since a particular form of the effective potential is not essential, we will do that for gaps of the Hill operator with a general bounded potential.

Let $V \in L^\infty((-a/2, a/2))$ and denote by $\{a_j\}_{j=1}^\infty$ and $\{b_j\}_{j=0}^\infty$ the sequences of its Fourier coefficients:

$$V(x) = \sum_{j=1}^{\infty} a_j \sin \frac{2\pi j}{a} x + \sum_{j=0}^{\infty} b_j \cos \frac{2\pi j}{a} x \quad \text{in} \quad L^2((-a/2, a/2)),$$

where

$$\begin{aligned} a_j &= \frac{2}{a} \int_{-a/2}^{a/2} V(x) \sin \frac{2\pi j}{a} x \, dx, \\ b_j &= \frac{2}{a} \int_{-a/2}^{a/2} V(x) \cos \frac{2\pi j}{a} x \, dx. \end{aligned}$$

Let κ_j be the jth eigenvalue of the operator

$$-\frac{d^2}{ds^2} + V(x) \quad \text{in} \quad L^2((-a/2, a/2)) \quad \text{with periodic b.c.,} \tag{4.1}$$

and similarly, let ν_j be the jth eigenvalue of the operator

$$-\frac{d^2}{ds^2} + V(x) \quad \text{in} \quad L^2((-a/2, a/2)) \quad \text{with antiperiodic b.c..} \tag{4.2}$$

We are going to prove the following result.

Theorem 4.1 *Let $n \in \mathbb{N}$. Assume that*

$$0 < \sqrt{a_n^2 + b_n^2} < \frac{12\pi^2}{a^2} n^2$$

and

$$\left\| V(x) - b_0 - a_n \sin \frac{2\pi n}{a} x - b_n \cos \frac{2\pi n}{a} x \right\|_{L^\infty((-a/2, a/2))} < \frac{1}{4} \sqrt{a_n^2 + b_n^2}.$$

Then we have

$$\nu_{n+1} - \nu_n > 0 \quad \text{when} \quad n \quad \text{is odd,}$$

and

$$\kappa_{n+1} - \kappa_n > 0 \quad \text{when} \quad n \quad \text{is even.}$$

Proposition 2.3 immediately follows from Theorem 4.1. In order to prove the latter, we shall estimate the length of the first gap of the Mathieu operator. For $\alpha \in \mathbb{R}$, we define

$$M_\alpha = -\frac{d^2}{dx^2} + 2\alpha \cos\frac{2\pi}{a}x \quad \text{in} \quad L^2((-a/2, a/2))$$

with the domain

$$D = \{u \in H^2((-a/2, a/2)); \quad u(a/2) = -u(-a/2),\ u'(a/2) = -u'(-a/2)\}.$$

By $m_j(\alpha)$ we denote the jth eigenvalue of M_α counted with multiplicity. The sought estimate looks as follows:

Theorem 4.2 *We have*

$$m_2(\alpha) - m_1(\alpha) \geq |\alpha| \quad \text{provided that} \quad |\alpha| < \frac{6\pi^2}{a^2}.$$

Proof: We prove the assertion only for $\alpha < 0$ because that for $\alpha > 0$ is similar. We put

$$\begin{aligned} D^+ &= \{u \in H^2((0, a/2)); \quad u'(0) = u(a/2) = 0\}, \\ D^- &= \{u \in H^2((0, a/2)); \quad u(0) = u'(a/2) = 0\} \end{aligned}$$

and define

$$L^\pm_\alpha = -\frac{d^2}{dx^2} + 2\alpha\cos\frac{2\pi}{a}x \quad \text{in} \quad L^2((0, a/2)) \quad \text{with the domain} \quad D^\pm.$$

By $\mu^\pm_1(\alpha)$ we denote the first eigenvalue of $L^\pm_\alpha$. Since the function $\cos\frac{2\pi}{a}x$ is even, we infer that M_α is unitarily equivalent to the operator $L^+_\alpha \oplus L^-_\alpha$ in $L^2((0, a/2)) \oplus L^2((0, a/2))$. We put

$$\varphi_j(x) = \frac{2}{\sqrt{a}}\sin\frac{\pi}{a}(2j-1)x \quad \text{and} \quad \psi_j(x) = \frac{2}{\sqrt{a}}\cos\frac{\pi}{a}(2j-1)x.$$

It is clear that

$$\{\varphi_j\}_{j=1}^\infty \subset D^- \quad \text{and} \quad \{\psi_j\}_{j=1}^\infty \subset D^+,$$

and, in addition, $\{\varphi_j\}_{j=1}^\infty$ and $\{\psi_j\}_{j=1}^\infty$ are complete orthonormal systems of $L^2((0, a/2))$. We first estimate $\mu^+_1(\alpha)$ from above. By the min-max principle, we obtain

$$\mu^+_1(\alpha) \leq (L^+_\alpha\psi_1, \psi_1) = \left(\frac{\pi}{a}\right)^2 + \alpha. \tag{4.3}$$

Next we estimate $\mu_1^-(\alpha)$ from below. Let $\phi \in D^-$ and $\|\phi\|_{L^2((0,a/2))} = 1$. Since $\{\varphi_j\}_{j=1}^\infty$ is a complete orthonormal system of $L^2((0,a/2))$, we have

$$\phi(x) = \sum_{j=1}^\infty s_j \varphi_j, \quad \sum_{j=1}^\infty s_j^2 = 1,$$

where $s_j = (\phi, \varphi_j)_{L^2((0,a/2))}$ are the Fourier coefficients. We have

$$\begin{aligned}
&(L_\alpha^- \phi, \phi)_{L^2((0,a/2))} - \left(\frac{\pi}{a}\right)^2 \|\phi\|^2_{L^2((0,a/2))} \\
&= \sum_{j=2}^\infty s_j^2 \left(\frac{\pi}{a}\right)^2 4j(j-1) + \alpha \left(2 \sum_{j=1}^\infty s_j s_{j+1} - s_1^2\right) \\
&= \sum_{j=2}^\infty s_j^2 \left(\frac{\pi}{a}\right)^2 4j(j-1) + \alpha \left[2 \sum_{j=2}^\infty s_j s_{j+1} - (s_1 - s_2)^2 + s_2^2\right] \\
&\geq \sum_{j=2}^\infty s_j^2 \left(\frac{\pi}{a}\right)^2 4j(j-1) + \alpha \left[2 \sum_{j=2}^\infty s_j s_{j+1} + s_2^2\right] \\
&\geq \sum_{j=2}^\infty s_j^2 \left(\frac{\pi}{a}\right)^2 4j(j-1) + \alpha \left[\sum_{j=2}^\infty \left(\frac{1}{3} s_j^2 + 3 s_{j+1}^2\right) + s_2^2\right] \\
&= \left[8 \left(\frac{\pi}{a}\right)^2 + \frac{4}{3}\alpha\right] s_2^2 + \sum_{j=3}^\infty \left[\left(\frac{\pi}{a}\right)^2 4j(j-1) + \frac{10}{3}\alpha\right] s_j^2 \\
&\geq 0 \quad \text{for} \quad -\frac{6\pi^2}{a^2} < \alpha < 0.
\end{aligned}$$

This together with the min-max principle implies that

$$\mu_1^-(\alpha) \geq \left(\frac{\pi}{a}\right)^2 \quad \text{for} \quad -\frac{6\pi^2}{a^2} < \alpha < 0. \tag{4.4}$$

Combining (4.4) with (4.3), we obtain the assertion of the theorem. ∎

Now we are ready to prove the main result of this section.

Proof of Theorem 4.1: We prove the assertion for odd n only since the argument for even n is similar. We extend V to an a-periodic function which we denote by $\tilde{V}$. Let $\tau \in [0, 2\pi)$ be such that

$$\cos\tau = \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \quad \text{and} \quad \sin\tau = -\frac{a_n}{\sqrt{a_n^2 + b_n^2}}.$$

We have

$$a_n \sin\frac{2\pi n x}{a} + b_n \cos\frac{2\pi n x}{a} = \sqrt{a_n^2+b_n^2}\cos\frac{2n\pi}{a}\left(x+\frac{a}{2n\pi}\tau\right).$$

Let d_j be the jth eigenvalue of the operator with this potential,

$$-\frac{d^2}{dx^2}+\sqrt{a_n^2+b_n^2}\cos\frac{2n\pi}{a}\left(x+\frac{a}{2n\pi}\tau\right)\quad \text{in } L^2\left(\left(-\frac{a}{2}-\frac{a}{2n\pi}\tau, \frac{a}{2}-\frac{a}{2n\pi}\tau\right)\right)$$

with antiperiodic boundary condition. Since a coordinate shift amounts to a unitary transformation and does not change the spectrum, $d_{n+1}-d_n$ is equal to the difference of the first two eigenvalues of the operator

$$-\frac{d^2}{dx^2}+\sqrt{a_n^2+b_n^2}\cos\frac{2n\pi x}{a}\quad \text{in}\quad L^2\left(\left(-\frac{a}{2n}, \frac{a}{2n}\right)\right)$$

with antiperiodic boundary condition. Thus it follows from Theorem 4.2 that

$$d_{n+1}-d_n \geq \frac{1}{2}\sqrt{a_n^2+b_n^2}. \tag{4.5}$$

Let e_j be the jth eigenvalue of the operator

$$-\frac{d^2}{dx^2}+\tilde{V}(x)\quad \text{in}\quad L^2\left(\left(-\frac{a}{2}-\frac{a}{2n\pi}\tau, \frac{a}{2}-\frac{a}{2n\pi}\tau\right)\right)$$

with antiperiodic boundary condition. By the min-max principle, we get

$$|d_j-e_j| \leq \left\|\tilde{V}(x)-b_0-\sqrt{a_n^2+b_n^2}\cos\frac{2n\pi}{a}\left(x+\frac{a}{2n\pi}\tau\right)\right\|_{L^\infty((-\frac{a}{2}-\frac{a}{2\pi}\tau,\frac{a}{2}-\frac{a}{2\pi}\tau))}. \tag{4.6}$$

Notice that $\nu_j = e_j$ for all $j \in \mathbb{N}$. This together with (4.5) and (4.6) implies that $\nu_{n+1}-\nu_n > 0$, and completes therefore the proof of Theorem 4.1. ∎

5 Asymptotically straight curves

Finally, we are going to discuss briefly the case when Γ is non-periodic and asymptotically straight. We impose the following assumptions on γ:

(A.6) $\gamma \in C^2(\mathbb{R})$.

(A.7) The function γ is not identically zero.

(A.8) There exists $c \in (0,1)$ such that $|\Gamma(s) - \Gamma(t)| \geq c|t-s|$ for $s, t \in \mathbb{R}$.

(A.9) There exist $\tau > \frac{5}{4}$ and $K > 0$ such that $|\gamma(s)| \leq K|s|^{-\tau}$ for $s \in \mathbb{R}$.

¿From [EI, Proposition 5.1 and Theorem 5.2] we know that under these conditions

$$\sigma_{\mathrm{ess}}(H_\beta) = [-\frac{1}{4}\beta^2, \infty) \quad \text{and} \quad \sigma_{\mathrm{d}}(H_\beta) \neq \emptyset.$$

We define

$$S = -\frac{d^2}{ds^2} - \frac{1}{4}\gamma(s)^2 \quad \text{in} \quad L^2(\mathbb{R}) \quad \text{with the domain} \quad H^2(\mathbb{R}).$$

Since γ is not identically zero on $\mathbb{R}$, we have

$$\sigma_{\mathrm{d}}(S) \neq \emptyset$$

(see, e.g., [BGS] and [Si]). We put $n = \sharp\sigma_{\mathrm{d}}(S)$. For $1 \leq j \leq n$, we denote by μ_j the jth eigenvalue of S counted with multiplicity.

Theorem 5.1 *There exists $\beta_0 > 0$ such that $\sharp\sigma_{\mathrm{d}}(H_\beta) = n$ for $\beta \geq \beta_0$. For $\beta \geq \beta_0$ and $1 \leq j \leq n$, we denote by $\lambda_j(\beta)$ the jth eigenvalue of H_β counted with multiplicity. Then we have*

$$\lambda_j(\beta) = -\frac{1}{4}\beta^2 + \mu_j + \mathcal{O}(\beta^{-1}\log\beta) \quad \text{as} \quad \beta \to \infty \quad \text{for} \quad 1 \leq j \leq n.$$

We omit the proof, since it analogous to those of Theorem 2.1 and [EY, Theorem 1].

Acknowledments

K.Y. appreciates the hospitality in NPI, Academy of Sciences, in Řež where a part of this work was done. The research has been partially supported by GAAS and the Czech Ministry of Education within the projects A1048101 and ME170.

References

[AGHH] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden: *Solvable Models in Quantum Mechanics*, Springer, Heidelberg 1988.

[AK] S. Albeverio, P. Kurasov: *Singular Perturbations of Differential Operators*, London Mathematical Society Lecture Note Series 271, Cambridge Univ. Press 1999.

[BGS] R. Blanckenberger, M. Goldberger, B. Simon: The bound state of weakly coupled long-range one-dimensional quantum Hamiltonians, *Ann. Phys.* **108** (1977), 69-77.

[Bo] G. Borg: Eine Umkehrung der Sturm-Liouvillschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte, *Acta Math.* **78** (1946), 1-96.

[BEKŠ] J.F. Brasche, P. Exner, Yu.A. Kuperin, P. Šeba: Schrödinger operators with singular interactions, *J. Math. Anal. Appl.* **184** (1994), 112-139.

[BT] J.F. Brasche, A. Teta: Spectral analysis and scattering theory for Schrödinger operators with an interaction supported by a regular curve, in *Ideas and Methods in Quantum and Statistical Physics*, Cambridge Univ. Press 1992; pp. 197-211.

[EI] P. Exner, T. Ichinose: Geometrically induced spectrum in curved leaky wires, *J. Phys.* **A34** (2001), 1439-1450.

[EY] P. Exner, K. Yoshitomi: Asymptotics of eigenvalues of the Schrödinger operator with a strong δ-interaction on a loop, *math-ph/0103029 and mp-arc 01-108* (http://www.ma.utexas.edu/mp_arc/); submitted

[Ho] H. Hochstadt: On the determination of a Hill's equation from its spectrum, *Arch. Rat. Mech. Anal.* **19** (1965), 353-362.

[Ka] T. Kato: *Perturbation Theory for Linear Operators*, 2nd edition Springer, Heidelberg 1976.

[RS] M. Reed, B. Simon: *Methods of modern mathematical physics, IV. Analysis of operators*, Academic Press, New York 1978.

[Si] B. Simon: The bound state of weakly coupled Schrödinger operators in one and two dimensions, *Ann. Phys.* **97** (1976), 279-287.

[Un] P. Unger: Stable Hill equations, *Comm. Pure. Appl. Math.* **14** (1961), 707-710.

[Yo] K. Yoshitomi: Band gap of the spectrum in periodically curved quantum waveguides, *J. Diff. Eq.* **142** (1998), 123-166.