Potential approximations to δ': an inverse Klauder phenomenon with norm-resolvent convergence

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Abstract: We show that there is a family Schrödinger operators with scaled potentials which approximates the δ'-interaction Hamiltonian in the norm-resolvent sense. This approximation, based on a formal scheme proposed by Cheon and Shigehara, has nontrivial convergence properties which are in several respects opposite to those of the Klauder phenomenon.

1 Introduction

Point interactions are often used for constructing solvable models of quantum mechanical systems [AGHH]. To judge a quality of such models one has to be able, of course, to decide how well does a point interaction approximate the "actual" interaction. In the simplest case of a one-dimensional point interaction introduced originally by Kronig and Penney [KP] the answer is easy: the appropriate Hamiltonian is a norm-resolvent limit of a family of Schrödinger operators with squeezed potentials, which physically means that a slow particle with a widely smeared wave packet "sees" only the mean value of a localized potential. The problem is more complicated in dimension two and three where squeezed potentials can also be used, however, with a renormalization such that the limiting coupling is "infinitely weak". The idea belongs to Friedman [Fr]; a detailed discussion for a general shape of the approximating potential can be found in [AGHH] together with description of other, nonlocal, approximations and the corresponding bibliography. More generally, we have here an important particular case of the question what is the "right Hamiltonian" for a strongly singular perturbation of the Laplace operator – see [NZ1, NZ2] and references therein.

A peculiarity of the one-dimensional situation is that not all point interactions are of the δ type. This follows from the standard construction of a point interaction [BF, AGHH] which relies on restriction of the free Hamiltonian to functions which vanish in the vicinity of the interaction support, and a consecutive construction of self-adjoint extensions of the obtained symmetric operator. For a single center in dimension one the latter has deficiency indices $(2, 2)$ leading thus to a four-parameter family of extensions. A subset of them usually called δ' interactions was introduced in [GHM], the whole family was later studied in [GK, Še1, GH] and subsequent papers by other authors. In distinction to the usual δ interactions the other extensions were constructed as mathematical objects and the question about their physical meaning arose naturally.

Šeba [Še2] was the first who addressed the question of approximation of δ' Hamiltonians by those with "regular" interactions. He showed, in particular, that the name is misleading because such Hamiltonians *cannot* be obtained using families of scaled zero-mean potentials. At the same time he demonstrated that the δ' interaction can be approximated in a nonlocal way using a suitable family of rank-one operators with a nontrivial coupling-constant renormalization. Later local approximations were constructed [Ca, CH] but they were not of potential type since they involved first-derivative terms.

The question about the δ' interaction meaning became more appealing when interesting physical properties of this coupling were discovered. Specifically, it was shown that Wannier-Stark systems with an array of δ' interactions have no absolutely continuous spectrum [AEL, Ex, MS] and even that the spectrum is pure point for most values of the parameters [ADE]. The origin of this effect is the high-energy behaviour of the δ' scattering, with the transmission amplitude vanishing as $k \to \infty$. Such a behaviour can be approximated, up to a phase factor, within a fixed finite interval of energies by small complicated graph scatterers [AEL], and the qualitatively same scattering picture, up to a series of resonances, was found for a sphere with two halflines attached [Ki].

Until recently it was believed, however, that no potential-type approximation to the δ' interaction existed. It came thus as a surprise, when two years ago Cheon and Shigehara (CS) constructed an approximation by means of a triple of δ interactions with the coupling constants scaled in a nonlinear way as their distances tend to zero [CS]. In distinction to the situations mentioned above this renormalization leads to an "infinitely strong" coupling in the limit. The said authors computed formally the limiting wave function and showed that it obeyed the δ' boundary conditions [AGHH]; they also presented an alternative argument based on convergence of the corresponding transfer matrices [SMMC].

It is natural to ask in which sense does the limit exist and whether one can construct a similar approximation using regular potentials. We shall answer the second question affirmatively and show that the approximating families converge in a rather strong topology, namely norm resolvent. A nontrivial character of the approximation will be seen from the fact that we do not recover the sought limit when the involved operators are expressed through the respective quadratic forms, in particular, because the form domain of the limiting operator is *larger* than those of the approximating ones.

Such a disparity between the form domains reminds of the *Klauder phenomenon* [Kl, Si] where a singular perturbation is switched off in the strong resolvent sense yielding an operator different from the free one obtained as the formal limit by putting the coupling constant equal to zero. Here the situation is in several respects opposite. First of all, the coupling here is not switched off as in [Kl, Si] but rather becomes *infinitely strong*, so it is not straightforward to identify the formal limit. On the other hand, the larger form domain corresponds to the *true* norm-resolvent limit. In addition, the CS-approximation requires a subtle interplay of the coupling constants. If we change this choice, we arrive at an operator the form domain of which is *smaller* than those of the approximants, namely to the Laplacian with Dirichlet decoupling at the δ' interaction position.

Let us review briefly the contents of the paper. In the next section we will formulate the approximation by triple δ interaction and examine it using the explicit form of the operators involved. Then we combine this result with the known squeezed-potential approximation of the δ interaction [AGHH, Thm. I.3.2.3] to show that a δ' can be approximated by a family of potentials consisting of three $a(\epsilon)$-spaced parts of a "size" ϵ which approach each other as $\epsilon \to 0+$ and at the same time undergo a CS-type scaling. Furthermore, we determine a squeezing rate which yields a convergent approximation: it is sufficient that $\epsilon\, a(\epsilon)^{-12}$ tends to zero. In Section 4 we illustrate the mentioned non-stability of the approximation: if we disbalance only slightly the ϵ dependence of the coupling constants we get a family which converges in the norm-resolvent sense to the Dirichlet decoupled Laplace operator on the line. To keep things simple we do not strive for a maximum generality. We restrict ourselves to the δ' case, because an extension to the general four-parameter point interaction is easy to obtain by adapting the scheme of [SMMC]. We also do not ask about the optimal rate between ϵ and $a(\epsilon)$ needed for the convergence.

2 Resolvent approach to the CS approximation

In the following we use the notations and definitions of [AGHH]. Let $H_0 = -\Delta$ be free one-dimensional Schrödinger operator in the Hilbert space $L^2(\mathbb{R})$. Its resolvent is an integral operator with the kernel

$$G_k(x-x') \equiv (-\Delta - k^2)^{-1}(x,x') := \frac{i}{2k}\, e^{ik|x-x'|} \tag{2.1}$$

for any $\Im\mathrm{m}\, k > 0$ and $x, x' \in \mathbb{R}$. The related function

$$\tilde{G}_k(x-x') := \mathrm{sgn}(x-x')\, \frac{i}{2k}\, e^{ik|x-x'|} \tag{2.2}$$

allows us to express the resolvent for the δ'-perturbation of H_0 centered at the point y and having the "strength" β, denoted by $\Xi_{\beta,y}$, in the form [AGHH, Sec. I.4]:

$$(\Xi_{\beta,y} - k^2)^{-1}(x,x') = G_k(x-x') - \frac{2\beta k^2}{2 - i\beta k}\, \tilde{G}_k(x-y)\tilde{G}_k(x'-y)\,. \tag{2.3}$$

Recall that $\Xi_{\beta,y}$ acts as H_0 away of y and its domain consists of those $f \in W^{2,2}(\mathbb{R} \setminus \{y\})$ which satisfy the boundary conditions

$$\psi'(y+) = \psi'(y-) =: \psi'(y)\,, \quad \psi(y+) - \psi(y-) = \beta\psi'(y)\,. \tag{2.4}$$

Our first aim is to approximate the resolvent (2.3) of $\Xi_{\beta,y}$ by a family of those corresponding to the triple δ-perturbation of H_0 with the couplings $\mathcal{A}_a = \{\alpha_j\}_{j=-1,0,1} = \{2\beta^{-1}-a^{-1}, \beta a^{-2}, 2\beta^{-1}-a^{-1}\}$ localized at $Y_a = \{y_j\}_{j=-1,0,1} = \{y-a, y, y+a\}$ for $a \geq 0$ letting $a \to 0$. Denote this perturbed operator by $-\Delta_{\mathcal{A}_a,Y_a}$. Then by [AGHH, Sec. II.2] the corresponding resolvent has the kernel

$$(-\Delta_{\mathcal{A}_a,Y_a} - k^2)^{-1}(x,x') = G_k(x-x') - \sum_{j,j'=-1,0,1} [\Gamma_a(k)]^{-1}_{jj'}\, G_k(x-y_j)\, G_k(x'-y_{j'})\,, \tag{2.5}$$

where $[\Gamma_a(k)]_{jj'} := \left[\alpha_j^{-1}\delta_{jj'} + G_k(y_j - y_{j'})\right]_{jj'}$ and $j,j' = -1,0,1$. In particular, for a purely imaginary $k = i\kappa$, $\kappa > 0$, we get

$$\Gamma_a(i\kappa) = \frac{1}{2\kappa}\begin{pmatrix} 1+u & w & w^2 \\ w & 1+v & w \\ w^2 & w & 1+u \end{pmatrix}. \tag{2.6}$$

where

$$u := 2\beta\kappa a/(2a-\beta), \quad v := 2\kappa a^2/\beta, \quad w := e^{-\kappa a}. \tag{2.7}$$

Let us look how the spectrum of the operators $\{-\Delta_{\mathcal{A}_a,Y_a}\}_{a\geq 0}$ behaves as $a \to 0$ for a fixed β. Since the perturbation in (2.5) is a rank three operator, $\sigma_{ess}(H_0) = \sigma_{ac}(H_0) = [0,\infty)$ is not affected by the perturbation and the point spectrum consists of at most three negative eigenvalues, with the multiplicity taken into account [We, Sec. 8.3]. Here we have:

Proposition 2.1 *For small enough spacing a the operator $-\Delta_{\mathcal{A}_a,Y_a}$ has at most one eigenvalue. This happens if and only if $\beta < 0$, and in that case*

$$\inf\sigma(-\Delta_{\mathcal{A}_a,Y_a}) = -\frac{4}{\beta^2} + \mathcal{O}(a)\,. \tag{2.8}$$

Proof: Since the negative part of $\sigma(-\Delta_{\mathcal{A}_a,Y_a})$ is the point spectrum determined by zeros of $\det\Gamma_a(i\kappa)$ by [AGHH, Sec. II.2] we arrive at the equation

$$(1+u-w^2)\left[(1+u)(1+v) - w^2(1-v)\right] = 0\,, \tag{2.9}$$

or

$$e^{-2\kappa a} = 1 + \frac{2\beta\kappa a}{2a-\beta} \tag{2.10}$$

and

$$e^{-2\kappa a} = \left(1 + \frac{2\beta\kappa a}{2a-\beta}\right)\frac{1+2\kappa a^2\beta^{-1}}{1-2\kappa a^2\beta^{-1}}\,. \tag{2.11}$$

Expanding the left- and right-hand sides of the last two equations around $a = 0$, one finds that only (2.10) has a solution for a sufficiently small $a > 0$ and that it equals

$$\kappa(a) = -\frac{2}{\beta} + \mathcal{O}(a).$$

Since $k = i\kappa$ corresponds to an isolated eigenvalue if and only if $\Im\mathrm{m}\, k > 0$, the assertion follows readily. $\square$

Proposition 2.1 also shows that if $\kappa > -2/\beta$, $\beta \neq 0$, is fixed, then there is $a_0(\kappa) > 0$ such that $-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2 > 0$ and the resolvent $(-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2)^{-1}$ exists for all $a \in (0, a_0(\kappa))$. We further note that the operator $-\Delta_{\mathcal{A}_a,Y_a}$ admits a definition in the sense of quadratic forms. Denoting by $\mathcal{Q}_{\mathcal{A}_a,Y_a}[\cdot,\cdot]$ this quadratic form one has

$$\begin{aligned} \mathcal{Q}_{\mathcal{A}_a,Y_a}[u,v] = (u',v') + \frac{\beta}{a^2}\, u(y)\overline{v(y)} \\ + \left(\frac{2}{\beta} - \frac{1}{a}\right)\left\{u(y+a)\overline{v(y+a)} + u(y-a)\overline{v(y-a)}\right\} \end{aligned} \tag{2.12}$$

for $u, v \in \mathrm{dom}(\mathcal{Q}_{\mathcal{A}_a,Y_a}) = W^{1,2}(\mathbb{R})$. When equipped with the scalar product

$$(u,v)_{\mathcal{Q}_{\mathcal{A}_a,Y_a}} := \left(\sqrt{-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2}\, u, \sqrt{-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2}\, v\right), \tag{2.13}$$

where $\kappa > -2/\beta$ and $a \in (0, a_0(\kappa))$, the domain $\mathrm{dom}(\mathcal{Q}_{\mathcal{A}_a,Y_a})$ becomes a Hilbert space. It is important to note that the norm $\|\cdot\|_{\mathcal{Q}_{\mathcal{A}_a,Y_a}}$ arising from this scalar product is equivalent to the norm of the Hilbert space $W^{1,2}(\mathbb{R})$.

Proposition 2.1 shows that up to an $\mathcal{O}(a)$ error the spectral properties of $-\Delta_{\mathcal{A}_a,Y_a}$ coincide with those of $\Xi_{\beta,y}$. Next we compare the corresponding resolvents.

Theorem 2.2 *Let $\kappa \neq -2/\beta$ and $\beta \neq 0$ be fixed. Then the relation*

$$\lim_{a\to 0+} \left(-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2\right)^{-1}(x,x') = \left(\Xi_{\beta,y} + \kappa^2\right)^{-1}(x,x') \tag{2.14}$$

holds for any $x, x' \in \mathbb{R}$. Consequently, $-\Delta_{\mathcal{A}_a,Y_a} \to \Xi_{\beta,y}$ as $a \to 0+$ in the norm-resolvent sense.

Proof: By virtue of (2.3), to check (2.14) it is sufficient to compute the pointwise limit of the second term at the right-hand side of (2.5). Using the notations introduced in the preceding proof, we obtain an explicit expression for the inverse matrix in (2.5):

$$\begin{aligned} [\Gamma_a(i\kappa)]^{-1} = \frac{2\kappa}{(w^2-1-u)[(1+u)(1+v) - w^2(1-v)]} \\ \times \begin{pmatrix} w^2-(1+u)(1+v) & -w(w^2-1-u) & w^2 v \\ -w(w^2-1-u) & (w^2+1+u)(w^2-1-u) & -w(w^2-1-u) \\ w^2 v & -w(w^2-1-u) & w^2-(1+u)(1+v) \end{pmatrix}. \end{aligned} \tag{2.15}$$

Without loss of generality we may assume $y = 0$. Suppose, for instance, that $x, x' > a$, then the resolvent difference kernel is obtained by sandwiching the above matrix between the vectors $G(x)$, $G(x')$, where

$$G(x) := \begin{pmatrix} G_{i\kappa}(x+a) \\ G_{i\kappa}(x) \\ G_{i\kappa}(x-a) \end{pmatrix} = \frac{1}{2\kappa}\, e^{-\kappa x} \begin{pmatrix} w \\ 1 \\ w^{-1} \end{pmatrix}, \tag{2.16}$$

which yields the expression

$$\sum_{j,j'=-1,0,+1} [\Gamma_a(i\kappa)]^{-1}_{jj'} G(x-y_j) G(x-y_{j'}) = \frac{1}{4\kappa^2}\, e^{-\kappa x}\, e^{-\kappa x'}\, \frac{N}{D} \tag{2.17}$$

with

$$D = \frac{(w^2-1-u)[(1+u)(1+v) - w^2(1-v)]}{2\kappa} \tag{2.18}$$

and

$$N = (w^2 + w^{-2})[w^2 - (1+u)(1+v)] + 2w^2 v + (w^2-1-u)(u-1-w^2)\,. \tag{2.19}$$

It is straightforward if tedious to compute the Taylor expansions of the denominator and numerator: we get

$$D = -2\kappa^2 a^4 \left(\kappa + 2\beta^{-1}\right) + \mathcal{O}(a^5)\,, \tag{2.20}$$

while in the other expression all the terms cancel up to the third order giving

$$N = 4\kappa^4 a^4 + \mathcal{O}(a^5)\,. \tag{2.21}$$

The sought kernel is thus

$$\sum_{j,j'=-1,0,1} [\Gamma_a(k)]^{-1}_{jj'}\, G_k(x-y_j) G_k(x'-y_{j'}) = -\frac{\beta}{2(2+\beta\kappa)}\, e^{-\kappa x} e^{-\kappa x'}\, (1+\mathcal{O}(a)) \tag{2.22}$$

as expected. In the same way one can treat the other situations with x, x' belonging to $(-\infty, a)$, $(-a, 0)$, $(0, a)$, and (a, ∞). In the coefficient this corresponds to different combinations of $(w, 1, w^{-1})$ and $(w^{-1}, 1, w)$ in (2.16). Due to the symmetry of $[\Gamma_a(i\kappa)]^{-1}$, however, there are just two different expressions, the other one having the numerator replaced by

$$N = (w^4 + 1)v + 2[w^2 - (1+u)(1+v)] + (w^2-1-u)(u-1-w^2) \tag{2.23}$$

leading to

$$N = -4\kappa^4 a^4 + \mathcal{O}(a^5) \tag{2.24}$$

and the correct kernel again; recall the sign factor in (2.2). This yields the relation (2.14).

For a fixed $\kappa > 0$ we see from the relation (2.16) that its left-hand side can be majorized by a function from $L^2(\mathbb{R}^2)$ which is independent of a. The same is, of course, true for the last term in (2.3). Then by (2.3), (2.5), (2.17), and dominated convergence the resolvent converges in the Hilbert-Schmidt norm,

$$\lim_{a\to 0} \left\| (-\Delta_{\mathcal{A}_{a(\epsilon)},Y_{a(\epsilon)}} + \kappa^2)^{-1} - (\Xi_{\beta,y} + \kappa^2)^{-1} \right\|_2 = 0\,, \tag{2.25}$$

and thus, *a fortiori*, $\{-\Delta_{\mathcal{A}_a,Y_a}\}_{a\geq 0}$ approximates $\Xi_{\beta,y}$ in the norm-resolvent topology. □

Remark 2.3 The result remains valid if the coupling constants $\mathcal{A}_a$ are replaced by

$$\alpha_{\pm 1}(a) = \frac{2}{\beta} - \frac{1}{a} + \varphi_1(a)\,, \quad \alpha_0(a) = \frac{\beta}{a^2}\,(1+\varphi_0(a))\;, \tag{2.26}$$

where φ_j are smooth functions behaving as $\mathcal{O}(a)$ for $a \to 0+$.

3 Approximation of δ' by regular potentials

It is easy to use the above result to prove the existence of an approximation of δ' by local potentials. After a suitable translation we can put $y=0$ and we seek in the form

$$W^a_{\epsilon,0}(x) = \frac{\beta}{\epsilon a(\epsilon)^2}\, V_0\left(\frac{x}{\epsilon}\right) + \left(\frac{2}{\beta} - \frac{1}{a(\epsilon)}\right)\left\{\frac{1}{\epsilon}\, V_{-1}\left(\frac{x+a(\epsilon)}{\epsilon}\right) + \frac{1}{\epsilon}\, V_1\left(\frac{x-a(\epsilon)}{\epsilon}\right)\right\}; \tag{3.1}$$

the general potential approximation $W^a_{\epsilon,y}(x)$ is obtained by replacing x by $x-y$ at the right-hand side. In this expression $\beta \in \mathbb{R}\setminus\{0\}$, and the involved potentials are supposed to satisfy $V_j \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} V_j(x)\, dx = 1 \tag{3.2}$$

for $j=-1,0,1$. The function $a:\mathbb{R}_+ \to \mathbb{R}_+$, to be specified later, is supposed to be continuous at $\epsilon = 0$ with $a(0)=0$. The family of one-dimensional Schrödinger operator used to approximate $\Xi_{\beta,y}$ will be of the form

$$H^a_{\epsilon,y} := -\Delta + W^a_{\epsilon,y}\,. \tag{3.3}$$

If $V_j \in L^1(\mathbb{R})$ the r.h.s. is defined in the sense of the corresponding quadratic forms. If we add the requirement $V_j \in L^2(\mathbb{R})$, then $W^a_{\epsilon,y}(x)$ is an infinitely small perturbation of the Laplacian and (3.3) as a self-adjoint operator is defined on $\mathrm{dom}(H^a_{\epsilon,0}) = \mathrm{dom}(-\Delta) = W^{2,2}(\mathbb{R})$ as an operator sum. We will make this assumption everywhere in the following, except for Theorem 3.1 where we refer directly to a result in [AGHH].

To compare the resolvents, we choose $k=i\kappa$ which belongs to the resolvent sets of both $H^a_{\epsilon,y}$ and the operator $\Xi_{\beta,y}$ introduced above; this can be achieved if k^2 is nonreal or with $\kappa>0$ large enough. Then we may employ the elementary estimate

$$\begin{aligned} &\left\|(H^a_{\epsilon,y}+\kappa^2)^{-1} - (\Xi_{\beta,y}+\kappa^2)^{-1}\right\| \\ &\quad\leq \left\|(H^a_{\epsilon,y}+\kappa^2)^{-1} - (-\Delta_{\mathcal{A}_{a(\epsilon)},Y_{a(\epsilon)}}+\kappa^2)^{-1}\right\| + \left\|(-\Delta_{\mathcal{A}_{a(\epsilon)},Y_{a(\epsilon)}}+\kappa^2)^{-1} - (\Xi_{\beta,y}+\kappa^2)^{-1}\right\| \end{aligned} \tag{3.4}$$

to prove the following claim:

Theorem 3.1 *Let $V_j \in L^1(\mathbb{R})$, $j=-1,0,1$. For any sequence $\{a_n\} \subset (0,\infty)$ with $a_n \to 0$ there is a sequence $\{\epsilon_n\}$ of positive numbers with $\epsilon_n \to 0$ such that*

$$\lim_{n\to\infty} \left\|(H^{a_n}_{\epsilon_n,y}+\kappa^2)^{-1} - (\Xi_{\beta,y}+\kappa^2)^{-1}\right\| = 0 \tag{3.5}$$

holds for any $\kappa > 2|\beta|^{-1}$.

Proof: Without loss of generality we may put $y=0$. In view of Theorem 2.2 it is sufficient to deal with the first term at the right-hand side of (3.4). By [AGHH, Thm. II.2.2.2] for each $a_n>0$, $n=1,2,\ldots$, there exists a sequence of $\{\epsilon_{nm}\}_{m=1}^{\infty}$ with $\lim_{m\to\infty}\epsilon_{nm}=0$ such that

$$\lim_{m\to 0} \left\|(H^{a_n}_{\epsilon_{nm},0}+\kappa^2)^{-1} - (-\Delta_{\mathcal{A}_{a_n},Y_{a_n}}+\kappa^2)^{-1}\right\| = 0\,, \tag{3.6}$$

where $Y_{a_n}=\{y_j\}_{j=-1,0,1}=\{-a_n,0,a_n\}$, $A_{a_n}=\{\alpha^{(n)}_j\}_{j=-1,0,1}=(2\beta^{-1}-a_n^{-1},\beta a_n^{-2},2\beta^{-1}-a_n^{-1})$ and $\{H^{a_n}_{\epsilon_{nm},0}\}_{n\ge1}$ are defined by local potentials

$$\begin{aligned} W^{a_n}_{\epsilon_{nm},0}(x) = {} & \frac{\beta}{\epsilon_{nm}\,a_n^2} V_0\left(\frac{x}{\epsilon_{nm}}\right) \\ & + \left(\frac{2}{\beta}-\frac{1}{a_n}\right)\left\{\frac{1}{\epsilon_{nm}}V_{-1}\left(\frac{x+a_n}{\epsilon_{nm}}\right)+\frac{1}{\epsilon_{nm}}V_1\left(\frac{x-a_n}{\epsilon_{nm}}\right)\right\}. \end{aligned} \tag{3.7}$$

Indeed, in view of (3.2) Theorem II.2.2.2 of [AGHH] applies if we choose the real analytic function $\lambda_j(\cdot)$, which enters into Theorem II.2.2.2, of the form $\lambda_j(\epsilon_{nm}) := \epsilon_{nm}\alpha^{(n)}_j$. If $\Im\mathrm{m}\, k^2 \neq 0$ the norms at the right-hand side of (3.6) are uniformly bounded and the claim is valid for the diagonal sequence, $\epsilon_n := \epsilon_{nn}$ – cf. [RS, Sec. I.3]. By the first resolvent identity its validity extends to any point outside the spectrum of $\Xi_{\beta,0}$. □

The diagonal trick used in the above proof introduces a relation between the parameters a and ϵ. Since to a given a we choose ϵ small enough to meet the requirements, the procedure works if $a(\epsilon)$ tends to zero sufficiently slowly as $\epsilon\to 0+$. Put like that the claim is, of course, very vague. Even without computing the resolvents, e.g., we can conjecture that the family (3.3) will *not* yield the sought approximation if $a(\epsilon)\sim\epsilon^{\nu}$ with $\nu>1$ since then the three potentials will overlap substantially for small values of ϵ and eventually the (divergent) overall mean value will prevail.

The question about a rate between a and ϵ which is sufficient to yield a convergent approximation is subtle, and the rest of the section is devoted to it. As above we put $y=0$ in the following argument restoring a general y only in the final result. First we introduce the sesquilinear forms $t^{(0)}_{a,\epsilon}[\cdot,\cdot]$,

$$t^{(0)}_{a,\epsilon}[u,v] := \frac{\beta}{a^2}\left\{u(0)\overline{v(0)} - \frac{1}{\epsilon}\int_{-\infty}^{+\infty} dx\; V_0(x/\epsilon)u(x)\overline{v(x)}\right\},$$

and $t^{(j)}_{a,\epsilon}[\cdot,\cdot]$,

$$t^{(j)}_{a,\epsilon}[u,v] := \left(\frac{2}{\beta}-\frac{1}{a}\right)\left\{u(ja)\overline{v(ja)} - \frac{1}{\epsilon}\int_{-\infty}^{+\infty} dx\; V_j(x-ja/\epsilon)u(x)\overline{v(x)}\right\},$$

where $j = \pm 1$ and $\mathrm{dom}(t^{(0)}_{a,\epsilon}) = \mathrm{dom}(t^{(j)}_{a,\epsilon}) = W^{1,2}(\mathbb{R})$. We set

$$t_{a,\epsilon}[\cdot,\cdot] := t^{(0)}_{a,\epsilon}[\cdot,\cdot] + t^{(-1)}_{a,\epsilon}[\cdot,\cdot] + t^{(+1)}_{a,\epsilon}[\cdot,\cdot]$$

with $\mathrm{dom}(t_{a,\epsilon}) = W^{1,2}(\mathbb{R})$. To proceed further we need stronger hypotheses about the potentials, namely the conditions (3.8) and (3.11) below. It can be shown that in combination with $V_j \in L^2(\mathbb{R})$ they imply $V_j \in L^1(\mathbb{R})$.

Lemma 3.2 *Let $V_0 \in L^2(\mathbb{R})$. If the conditions (3.2) and*

$$\int_{-\infty}^{+\infty} dx\ |x|^{1/2}\ |V_0(x)| < +\infty\,, \tag{3.8}$$

are valid, then $|t^{(0)}_{a,\epsilon}[u,v]| \le \sqrt{2}\sqrt{\epsilon}|\beta|a^{-2}\int_{-\infty}^{+\infty} dx\ |x|^{1/2}\ |V_0(x)|\ \|u\|_{W^{1,2}}\|v\|_{W^{1,2}}$ holds for $u, v \in W^{1,2}(\mathbb{R})$.

Proof: Changing the integration variable $x \to \epsilon x$ in the definition of $t^{(0)}_{a,\epsilon}[u,v]$ we get

$$t^{(0)}_{a,\epsilon}[u,v] = \frac{\beta}{a^2}\int_{-\infty}^{+\infty} dx\ V_0(x)\left\{u(0)\overline{v(0)} - u(\epsilon x)\overline{v(\epsilon x)}\right\}\,,$$

which yields

$$t^{(0)}_{a,\epsilon}[u,v] = -\frac{\beta}{a^2}\int_{-\infty}^{+\infty} dx\ V_0(x)\left\{(u(0)-u(\epsilon x))\overline{v(0)} + u(\epsilon x)\overline{(v(0)-v(\epsilon x))}\right\}.$$

Since

$$|f(x)| \le \frac{1}{\sqrt{2}}\|f\|_{W^{1,2}}, \quad f \in W^{1,2}(\mathbb{R}), \tag{3.9}$$

and

$$|f(x) - f(y)| \le \sqrt{|x-y|}\,\|f\|_{W^{1,2}}, \quad f \in W^{1,2}(\mathbb{R}), \tag{3.10}$$

as it follows from $f(x) - f(y) = -\int_x^y f'(t)\,dt$ and the Hölder inequality, we find

$$|t^{(0)}_{a,\epsilon}[u,v]| \le 2\sqrt{\frac{\epsilon}{2}}\frac{|\beta|}{a^2}\int_{-\infty}^{+\infty} dx\ \sqrt{|x|}\,|V_0(x)|\ \|u\|_{W^{1,2}}\|v\|_{W^{1,2}}$$

for $u, v \in W^{1,2}(\mathbb{R})$ which proves the lemma. □

Lemma 3.3 *Let $V_j \in L^2(\mathbb{R})$, $j = \pm 1$, and $\beta \ne 0$. If the conditions (3.2) and*

$$\int_{-\infty}^{+\infty} dx\ |x|^{1/2}\ |V_j(x)| < +\infty\,, \quad j = \pm 1\,, \tag{3.11}$$

are satisfied, then

$$\left|t^{(j)}_{a,\epsilon}[u,v]\right| \le \sqrt{2}\sqrt{\epsilon}\left|\frac{2}{\beta} - \frac{1}{a}\right|\int_{-\infty}^{+\infty} dx\ |x|^{1/2}\ |V_j(x)|\ \|u\|_{W^{1,2}}\|v\|_{W^{1,2}} \tag{3.12}$$

holds for any $u, v \in W^{1,2}(\mathbb{R})$ and $j = \pm 1$.

Proof: Let $j = -1$. Changing the integration variable to $\epsilon x - a$ in the definition of $t_{a,\epsilon}^{(-1)}[u, v]$ we get

$$t_{a,\epsilon}^{(-1)}[u,v] = \left(\frac{2}{\beta} - \frac{1}{a}\right) \int_{-\infty}^{+\infty} dx\, V_{-1}(x) \left\{ u(-a)\overline{v(-a)} - u(\epsilon x - a)\overline{v(\epsilon x - a)} \right\}.$$

¿From here we infer

$$t_{a,\epsilon}^{(-1)}[u,v] = \left(\frac{2}{\beta} - \frac{1}{a}\right) \times \tag{3.13}$$
$$\int_{-\infty}^{+\infty} dx\, V_{-1}(x) \left\{ (u(-a) - u(\epsilon x - a))\overline{v(-a)} + u(\epsilon x - a)\overline{(v(-a) - v(\epsilon x - a))} \right\}.$$

Using again (3.9) and (3.10) we complete the proof. □

Corollary 3.4 *Let $V_j \in L^2(\mathbb{R})$, $j = -1, 0, +1$, and $\beta \neq 0$. If the potentials V_j satisfy the conditions (3.2), (3.8), (3.11), then the estimate*

$$|t_{a,\epsilon}[u,v]| \leq \sqrt{\epsilon}\, C(a)\ \|u\|_{W^{1,2}} \|v\|_{W^{1,2}}$$

is valid for $u, v \in \mathrm{dom}(t_{a,\epsilon}) = W^{1,2}(\mathbb{R})$, where the constant $C(a)$ is given by

$$C(a) := \sqrt{2} \left\{ \frac{|\beta|}{a^2} \int_{-\infty}^{+\infty} dx\ |x|^{1/2}\ |V_0(x)| + \left| \frac{2}{\beta} - \frac{1}{a} \right| \int_{-\infty}^{+\infty} dx\ |x|^{1/2} \{ |V_{-1}(x)| + |V_{+1}(x)| \} \right\}. \tag{3.14}$$

Let us next introduce the operator $G(a) : L^2(\mathbb{R}) \to \mathbb{C}^3$,

$$G(a)f := \begin{pmatrix} \int_{-\infty}^{+\infty} dx\ G_{i\kappa}(x+a) f(x) \\ \int_{-\infty}^{+\infty} dx\ G_{i\kappa}(x) f(x) \\ \int_{-\infty}^{+\infty} dx\ G_{i\kappa}(x-a) f(x) \end{pmatrix}$$

for $f \in \mathrm{dom}(G(a)) = L^2(\mathbb{R})$. Obviously, the action of the adjoint operator $G(a)^* : \mathbb{C}^3 \to L^2(\mathbb{R})$ is given by

$$(G(a)^* \xi)(x) = G_{i\kappa}(x+a)\xi_{-1} + G_{i\kappa}(x)\xi_0 + G_{i\kappa}(x-a)\xi_{+1},$$

where

$$\xi := \begin{pmatrix} \xi_{-1} \\ \xi_0 \\ \xi_{+1} \end{pmatrix} \in \mathbb{C}^3.$$

With these definitions the r.h.s. of (2.5) can rewritten as

$$(-\Delta_{\mathcal{A}_a,Y_a}+\kappa^2)^{-1}f=(H_0+\kappa^2)^{-1}f+G(a)^*\Gamma_a(i\kappa)^{-1}G(a)f\,, \tag{3.15}$$

where $Y_a=\{y_j\}_{j=-1,0,+1}$ with $y_j=ja$ and the matrix $\Gamma_a(i\kappa)$ is given by (2.6). Furthermore, we introduce the operator $\hat{G}(a)$:

$$\hat{G}(a)f:=\begin{pmatrix}(H_0+\kappa^2)^{-1/2}f\\ G(a)f\end{pmatrix}:L^2(\mathbb{R})\longrightarrow\begin{array}{c}L^2(\mathbb{R})\\ \oplus\\ \mathbb{C}^3\end{array} \tag{3.16}$$

and the operator $\hat{\Gamma}_a(i\kappa)$:

$$\hat{\Gamma}_a(i\kappa):=\begin{pmatrix}I&0\\0&\Gamma_a(i\kappa)\end{pmatrix}:\begin{array}{c}L^2(\mathbb{R})\\ \oplus\\ \mathbb{C}^3\end{array}\longrightarrow\begin{array}{c}L^2(\mathbb{R})\\ \oplus\\ \mathbb{C}^3\end{array} \tag{3.17}$$

Using the definitions (3.16) and (3.17) we can rewrite (3.15) as

$$(-\Delta_{\mathcal{A}_a,Y_a}+\kappa^2)^{-1}f=\hat{G}(a)^*\hat{\Gamma}_a(i\kappa)^{-1}\hat{G}(a)f\,.$$

Since $G_{i\kappa}(x-ja)\in W^{1,2}(\mathbb{R})$ for $j=-1,0,+1$, one gets that $\mathrm{ran}(G(a)^*)\subseteq W^{1,2}(\mathbb{R})$, and consequently, $\mathrm{ran}(\hat{G}(a)^*)\subseteq W^{1,2}(\mathbb{R})$. Thus it makes sense to define the following sesquilinear form

$$d_{a,\epsilon}[\hat{\xi},\hat{\eta}]:=t_{a,\epsilon}[\hat{G}(a)^*\hat{\xi},\hat{G}(a)^*\hat{\eta}],\quad \hat{\xi},\hat{\eta}\in\mathrm{dom}(d_{a,\epsilon})=\hat{\mathcal{H}}:=\begin{array}{c}L^2(\mathbb{R})\\ \oplus\\ \mathbb{C}^3\end{array}$$

where

$$\hat{\xi}:=\begin{pmatrix}f\\ \xi\end{pmatrix}\quad\text{and}\quad\hat{y}:=\begin{pmatrix}g\\ \eta\end{pmatrix}$$

with $f,g\in L^2(\mathbb{R})$ and $\xi,\eta\in\mathbb{C}^3$. By construction, the form $d_{a,\epsilon}[\cdot,\cdot]$ defines a bounded operator $D_{a,\epsilon}:\hat{\mathcal{H}}\to\hat{\mathcal{H}}$.

Lemma 3.5 *Let $V_j\in L^2(\mathbb{R})$, $j=-1,0,+1$, and $\beta\neq 0$. If the potentials V_j satisfy the conditions (3.2), (3.8) and (3.11) and $\kappa\geq 1$, then one has*

$$\|D_{a,\epsilon}\|_{\mathcal{B}(\hat{\mathcal{H}},\hat{\mathcal{H}})}\leq 4\sqrt{\epsilon}\,C(a) \tag{3.18}$$

for $a>0$.

Proof: Using Corollary 3.4 we find

$$|d_{a,\epsilon}[\hat{\xi},\hat{\eta}]|=|t_{a,\epsilon}[\hat{G}(a)^*\hat{\xi},\hat{G}(a)^*\hat{\eta}]|\leq\sqrt{\epsilon}\,C(a)\|\hat{G}(a)^*\hat{\xi}\|_{W^{1,2}}\,\|\hat{G}(a)^*\hat{\eta}]\|_{W^{1,2}}\,.$$

Since

$$(\hat{G}(a)^*\hat{\xi})(x) = (H_0+\kappa^2)^{-1/2} f + G_{i\kappa}(x+a)\xi_{-1} + G_{i\kappa}(x)\xi_0 + G_{i\kappa}(x-a)\xi_{+1}\,,$$

we have

$$\begin{aligned}\|\hat{G}(a)^*\hat{\xi}\|^2_{W^{1,2}} \le 4\Big(&\|(H_0+\kappa^2)^{-1/2} f\|^2_{W^{1,2}} + \|G_{i\kappa}(\cdot+a)\xi_{-1}\|^2_{W^{1,2}} \\ &+ \|G_{i\kappa}(\cdot)\xi_0\|^2_{W^{1,2}} + \|G_{i\kappa}(\cdot-a)\xi_{+1}\|^2_{W^{1,2}}\Big)\end{aligned}$$

The assumption $\kappa \ge 1$ yields

$$\|(H_0+\kappa^2)^{-1/2} f\|^2_{W^{1,2}} \le \|f\|^2, \quad f \in L^2(\mathbb{R})\,,$$

and

$$\|G_{i\kappa}(\cdot - ja)\xi_j\|^2_{W^{1,2}} = \frac{1}{4}\left(\kappa^{-1}+\kappa^{-3}\right)|\xi_j|^2 \le |\xi_j|^2, \quad j=-1,0,+1\,,$$

for $a \ge 0$. In this way we get the estimate

$$\|\hat{G}(a)^*\hat{\xi}\|^2_{W^{1,2}} \le 4\left(\|f\|^2 + \|\xi\|^2_{\mathbb{C}^3}\right) \le 4\|\hat{\xi}\|^2_{\hat{\mathcal{H}}}\,.$$

or for $a \ge 0$. This leads to the estimate

$$|d_{a,\epsilon}[\hat{\xi},\hat{\eta}]| = |t_{a,\epsilon}[\hat{G}(a)^*\hat{\xi},\hat{G}(a)^*\hat{\eta}]| \le 4\sqrt{\epsilon}\,C(a)\|\hat{\xi}\|_{\hat{\mathcal{H}}}\|\hat{\eta}\|_{\hat{\mathcal{H}}}$$

from which (3.18) follows readily. □

Let us further introduce the Neumann iterations $R^{(n)}_{a,\epsilon}(i\kappa)$ defined by

$$R^{(n)}_{a,\epsilon}(i\kappa) := \hat{G}^*(a)\hat{\Gamma}_a(i\kappa)^{-1}\left(D_{a,\epsilon}\hat{\Gamma}_a(i\kappa)^{-1}\right)^n \hat{G}(a), \quad n=0,1,2,\dots\,.$$

for $k > \max(-2/\beta, 1)$ and $a \in (0, a_0(\kappa))$. The meaning of these expressions will become clear below; we note that

$$R^{(0)}_{a,\epsilon}(i\kappa) = (-\Delta_{\mathcal{A}_a,Y_a}+\kappa^2)^{-1}. \tag{3.19}$$

We also need to know how the norm of $\Gamma_a(i\kappa)^{-1}$ behaves as $a \to 0$. The Taylor expansion for all the expressions contained in (2.15) yields

$$\begin{aligned}[\Gamma_a(i\kappa)]^{-1} &= \frac{2\beta a^{-2}}{2+\beta\kappa} \\ &\times \begin{pmatrix} 2\kappa(\kappa+\beta^{-1}) & -2\kappa(\kappa+2\beta^{-1}) & 2\kappa\beta^{-1} \\ -2\kappa(\kappa+2\beta^{-1}) & 4\kappa(\kappa+2\beta^{-1}) & -2\kappa(\kappa+2\beta^{-1}) \\ 2\kappa\beta^{-1} & -2\kappa(\kappa+2\beta^{-1}) & 2\kappa(\kappa+\beta^{-1}) \end{pmatrix}(1+\mathcal{O}(a))\,.\end{aligned}$$

Consequently, for $\kappa > \max(-2/\beta, 1)$ there is a constant $C_\Gamma(\kappa) > 0$ such that

$$\left\|\Gamma_a(i\kappa)^{-1}\right\|_{\mathcal{B}(\mathbb{C}^3,\mathbb{C}^3)} \le C_\Gamma(\kappa)\, a^{-2} \tag{3.20}$$

holds for any $a \in (0, a_0(\kappa))$.

Lemma 3.6 *Let $V_j \in L^2(\mathbb{R})$, $j = -1, 0, +1$, and $\kappa > \max(-2/\beta, 1)$, $\beta \neq 0$. If the potentials V_j satisfy the conditions (3.2), (3.8) and (3.11), then the Neumann iterations obey the estimate*

$$\left\| R^{(n)}_{a,\epsilon}(i\kappa) \right\| \leq \frac{2C_\Gamma(\kappa)}{a^2} \left(\frac{4\sqrt{\epsilon}\, C_\Gamma(\kappa) C(a)}{a^2} \right)^n \tag{3.21}$$

for $a \in (0, a_0(\kappa))$ and $n = 1, 2, \ldots$, where $C(a)$ is given by (3.14).

Proof: Since $\kappa > 1$, we have

$$\|\hat{G}(a)\|_{\mathcal{B}(\mathcal{H},\hat{\mathcal{H}})} = \|\hat{G}^*(a)\|_{\mathcal{B}(\hat{\mathcal{H}},\mathcal{H})} \leq \sqrt{2}\,.$$

An elementary estimate, $\left\| R^{(n)}_{a,\epsilon}(i\kappa) \right\| \leq 2 \, \|\Gamma_a(i\kappa)^{-1}\|^{n+1}_{\mathcal{B}(\mathbb{C}^3,\mathbb{C}^3)} \; \|D_{a,\epsilon}\|^n_{\mathcal{B}(\hat{\mathcal{H}},\hat{\mathcal{H}})}$, gives

$$\left\| R^{(n)}_{a,\epsilon}(i\kappa) \right\| \leq 2 \cdot 4^n \epsilon^{n/2} C_\Gamma(\kappa)^{n+1} C(a)^n a^{-(2n+2)}$$

so (3.21) follows readily. □

If $\kappa > \max\{-2/\beta, 1\}$ and the condition

$$\tau(\epsilon, a, \kappa) := \frac{4\sqrt{\epsilon}\, C_\Gamma(\kappa) C(a)}{a^2} < 1 \tag{3.22}$$

is satisfied for some $a \in (0, a_0(\kappa))$, then the operator $R_{a,\epsilon}(i\kappa)$,

$$R_{a,\epsilon}(i\kappa) := \sum_{n=0}^{\infty} R^{(n)}_{a,\epsilon}(i\kappa)\,,$$

is well defined. We denote the closed quadratic form which is associated with the self-adjoint operator $H^a_{\epsilon,0}$ by $h^a_{\epsilon,0}[\cdot,\cdot]$. Obviously, its domain is $\mathrm{dom}(h^a_{\epsilon,0}) = W^{1,2}(\mathbb{R})$; we note that the natural norm $\|\cdot\|_{h^a_{\epsilon,0}}$ on $\mathrm{dom}(h^a_{\epsilon,0})$ is equivalent to the norm of the Hilbert space $W^{1,2}(\mathbb{R})$.

Lemma 3.7 *Let $V_j \in L^2(\mathbb{R})$, $j = -1, 0, +1$, and $\kappa > \max(-2/\beta, 1)$, $\beta \neq 0$. If the potentials V_j satisfy the conditions (3.2), (3.8) and (3.11) and $\tau(\epsilon, a, \kappa) < 1$ is valid for some $a \in (0, a_0(\kappa))$, then $-\kappa^2$ belongs to the resolvent set of the operator $H^a_{\epsilon,0}$ given by (3.3), and moreover, one has*

$$(H^a_{\epsilon,0} + \kappa^2)^{-1} = R_{a,\epsilon}(i\kappa)\,. \tag{3.23}$$

Proof: Combining the above definitions of the quadratic forms, we get

$$\left(\sqrt{-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2}\; u, \sqrt{-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2}\; v \right) = h^a_{\epsilon,0}[u,v] + \kappa^2(u,v) + t_{a,\epsilon}[u,v] \tag{3.24}$$

for $u, v \in W^{1,2}(\mathbb{R})$. We use this relation for $u = R_{a,\epsilon}(i\kappa)f$ and $v = \left(-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2\right)^{-1} g$ with $f, g \in L^2(\mathbb{R})$. Since

$$R_{a,\epsilon}(i\kappa) = \hat{G}(a)^* \hat{\Gamma}_a(i\kappa)^{-1} \sum_{n=0}^{\infty} \left(D_{a,\epsilon} \hat{\Gamma}_a(i\kappa)^{-1} \right)^n \hat{G}(a) \tag{3.25}$$

and $\operatorname{ran}(\hat{G}(a)^*) \subseteq W^{1,2}(\mathbb{R})$ we get $u \in W^{1,2}(\mathbb{R})$. Since $v = \left(-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2\right)^{-1} g \in W^{1,2}(\mathbb{R})$, we can insert u and v into (3.24). This yields

$$\begin{aligned}(R_{a,\epsilon}(i\kappa)f, g) &= h^a_{\epsilon,0}[R_{a,\epsilon}(i\kappa)f, (-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2)^{-1} g] \\ &\quad + \kappa^2 (R_{a,\epsilon}(i\kappa)f, (-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2)^{-1} g) + t_{a,\epsilon}[R_{a,\epsilon}(i\kappa)f, (-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2)^{-1} g] .\end{aligned}$$

Using (3.19) and (3.25) we find

$$\begin{aligned}& t_{a,\epsilon}[R_{a,\epsilon}(i\kappa)f, (-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2)^{-1} g] \\ &= t_{a,\epsilon}[\hat{G}(a)^* \hat{\Gamma}_a(i\kappa)^{-1} \sum_{n=0}^{\infty} \left(D_{a,\epsilon} \hat{\Gamma}_a(i\kappa)^{-1} \right)^n \hat{G}(a) f, \hat{G}(a)^* \hat{\Gamma}(i\kappa)^{-1} \hat{G}(a) g] \\ &= \left(D_{a,\epsilon} \hat{\Gamma}_a(i\kappa)^{-1} \sum_{n=0}^{\infty} \left(D_{a,\epsilon} \hat{\Gamma}_a(i\kappa)^{-1} \right)^n \hat{G}(a) f, \hat{\Gamma}(i\kappa)^{-1} \hat{G}(a) g \right) \\ &= \left(\sum_{n=1}^{\infty} R^{(n)}_{a,\epsilon}(i\kappa) f, g \right) .\end{aligned}$$

Furthermore, from (3.19) we infer that

$$\begin{aligned}&\left((-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2)^{-1} f, g \right) \\ &\quad = h^a_{\epsilon,0}[R_{a,\epsilon}(i\kappa)f, (-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2)^{-1} g] + \kappa^2 (R_{a,\epsilon}(i\kappa)f, (-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2)^{-1} g) .\end{aligned}$$

Setting now $h := (-\Delta_{\mathcal{A}_a,Y_a} + \kappa^2)^{-1} g$ we find

$$(f, h) = h^a_{\epsilon,0}[R_{a,\epsilon}(i\kappa)f, h] + \kappa^2 (R_{a,\epsilon}(i\kappa)f, h) \tag{3.26}$$

for $h \in \operatorname{dom}(-\Delta_{\mathcal{A}_a,Y_a})$. Since $\operatorname{dom}(-\Delta_{\mathcal{A}_a,Y_a})$ is a core for the quadratic form $h^a_{\epsilon,0}[\cdot,\cdot]$ one concludes that the equality (3.26) extends to each $h \in \operatorname{dom}(h^a_{\epsilon,0})$. In particular, if $h \in \operatorname{dom}(H^a_{\epsilon,0})$ we have

$$(f, h) = (R_{a,\epsilon}(i\kappa)f, (H^a_{\epsilon,0} + \kappa^2)h) .$$

In this way we find $R_{a,\epsilon}(i\kappa)f \in \operatorname{dom}(H^a_{\epsilon,0}))$ and

$$(H^a_{\epsilon,0} + \kappa^2) R_{a,\epsilon}(i\kappa) f = f, \quad f \in \mathcal{H} ,$$

and

$$R_{a,\epsilon}(i\kappa)(H^a_{\epsilon,0} + \kappa^2) h = h, \quad h \in \operatorname{dom}(H^a_{\epsilon,0}) .$$

Hence $\ker(H^a_{\epsilon,0} + \kappa^2) = \{0\}$ and $\operatorname{ran}(H^a_{\epsilon,0} + \kappa^2) = \mathcal{H}$, so the operator $H^a_{\epsilon,0} + \kappa^2$ is boundedly invertible and $(H^a_{\epsilon,0} + \kappa^2)^{-1} = R_{a,\epsilon}(i\kappa)$. □

With the help of Lemma 3.7 one can prove the following estimate.

Lemma 3.8 *Let $V_j \in L^2(\mathbb{R})$, $j = -1, 0, +1$, and $\kappa > \max(-2/\beta, 1)$, $\beta \neq 0$. If the potentials V_j satisfy the conditions (3.2), (3.8) and (3.11) and $\tau(\epsilon, a, \kappa) < 1$ is valid for some $a \in (0, a_0(\kappa))$, then*

$$\left\|(H^a_{\epsilon,0} + \kappa^2)^{-1} - (-\Delta_{\mathcal{A}_a, Y_a} + \kappa^2)^{-1}\right\| \leq 2C_\Gamma(\kappa)\,\frac{\tau(\epsilon, a, \kappa)}{a^2}\,(1 - \tau(\epsilon, a, \kappa))^{-1}. \tag{3.27}$$

Proof: Taking into account (3.23) and (3.19) we find

$$(H^a_{\epsilon,0} + \kappa^2)^{-1} - (-\Delta_{\mathcal{A}_a, Y_a} + \kappa^2)^{-1} = \sum_{n=1}^{\infty} R^{(n)}_{a,\epsilon}(i\kappa)\,.$$

Using the notation (3.22) and taking into account the estimate (3.21) one gets

$$\left\|(H^a_{\epsilon,0} + \kappa^2)^{-1} - (-\Delta_{\mathcal{A}_a, Y_a} + \kappa^2)^{-1}\right\| \leq \frac{2C_\Gamma(\kappa)}{a^2}\sum_{n=1}^{\infty} \tau(\epsilon, a, \kappa)^n.$$

If $\tau(\epsilon, a, \kappa) < 1$ is satisfied, we obtain (3.27) easily. □

Now we are ready to say something about the rate of the potential approximation in terms of the relation between a and ϵ. Consider a function $a : (0, \infty) \to (0, \infty)$.

Theorem 3.9 *Let $V_j \in L^2(\mathbb{R})$, $j = -1, 0 + 1$, and $\kappa > \max(-2/\beta, 1)$, $\beta \neq 0$. Moreover, suppose that $a(\epsilon) \to 0$ as $\epsilon \to 0+$. If the potentials V_j satisfy the conditions (3.2), (3.8) and (3.11) for $j = -1, 0, +1$, and*

$$\lim_{\epsilon\to 0} \frac{\epsilon}{a(\epsilon)^{12}} = 0\,, \tag{3.28}$$

then

$$\lim_{\epsilon\to 0}\left\|(H^a_{\epsilon,y} + \kappa^2)^{-1} - (-\Delta_{\mathcal{A}_{a(\epsilon)}, Y_{a(\epsilon)}} + \kappa^2)^{-1}\right\| = 0 \tag{3.29}$$

and

$$\lim_{\epsilon\to 0}\left\|(H^a_{\epsilon,y} + \kappa^2)^{-1} - (\Xi_{\beta,y} + \kappa^2)^{-1}\right\| = 0\,. \tag{3.30}$$

Proof: Since $H^a_{\epsilon,y}$ is unitarily equivalent to $H^a_{\epsilon,0}$ by translation and the same is true for the other involved operators, we can again put $y = 0$ without loss of generality. By assumption, $a(\epsilon) \in (0, a_0(\kappa))$ for ϵ sufficiently small. Further, we note that there is a constant $C = C(V_j, \beta)$ such that $C(a) \leq Ca^{-2}$ for $a > 0$. Using that we can estimate

$$\tau(\epsilon, a(\epsilon), \kappa) \leq \frac{4\sqrt{\epsilon}\,C_\Gamma(\kappa)C}{a(\epsilon)^4} = 4C_\Gamma(\kappa)Ca(\epsilon)^2\frac{\sqrt{\epsilon}}{a(\epsilon)^6}\,,$$

so $\lim_{\epsilon\to 0+} \tau(\epsilon, a(\epsilon), \kappa) = 0$ by (3.28) and $\lim_{\epsilon\to 0} a(\epsilon) = 0$. Hence, $\tau(\epsilon, a(\epsilon), \kappa) < 1$ holds for ϵ sufficiently small. Applying Lemma 3.8 we get

$$\left\|(H^a_{\epsilon,0} + \kappa^2)^{-1} - (-\Delta_{\mathcal{A}_{a(\epsilon)}, Y_{a(\epsilon)}} + \kappa^2)^{-1}\right\| \leq 8C_\Gamma(\kappa)^2C\frac{\sqrt{\epsilon}}{a(\epsilon)^6}(1 - \tau(\epsilon, a(\epsilon), \kappa))^{-1}.$$

Taking into account once again the assumption (3.28) we prove (3.29). Moreover, using Theorem 2.2 together with the estimate (3.4) we arrive at (3.30). □

4 Exceptional character of the CS approximation

In conclusion we want to show that it is sufficient to disbalance the limiting procedure slightly, say by changing the normalization (3.2), and the result will be completely different than that in Theorem 3.9. For simplicity we will consider the case $y=0$ only. Denote by $-\Delta_{D,0}$ the Laplace operator with Dirichlet boundary conditions at the origin, i.e

$$\mathrm{dom}(-\Delta_{D,0}) = \left\{ f \in W^{2,2}(\mathbb{R}_-) \oplus W^{2,2}(\mathbb{R}_+) : f(0-) = f(0+) = 0 \right\}$$

and

$$(-\Delta_{D,0} f)(x) = -\frac{d^2}{dx^2} f(x)\,, \quad f \in \mathrm{dom}(-\Delta_{D,0})\,.$$

With respect to $L^2(\mathbb{R}) = L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+)$ the operator $-\Delta_{D,0}$ decomposes into

$$-\Delta_{D,0} = -\Delta^-_{D,0} \oplus -\Delta^+_{D,0}$$

with $\mathrm{dom}(-\Delta^\pm_{D,0}) = \{f \in W^{2,2}(\mathbb{R}_\pm) : f(0\pm) = 0\}$. We note that $\sigma(-\Delta^\pm_{D,0}) = [0, +\infty)$. The resolvents $(-\Delta^\pm_{D,0} + \kappa^2)^{-1}$ are integral operators with the kernels

$$D^\pm_{i\kappa}(x,x') := \begin{cases} \pm\frac{1}{\kappa}\, e^{\mp\kappa x}\, \sinh(\kappa x') & \ldots \quad \pm x' \in [0, \pm x) \\ \pm\frac{1}{\kappa}\, \sinh(\kappa x)\, e^{\mp\kappa x'} & \ldots \quad \pm x' \in [\pm x, +\infty) \end{cases}$$

A straightforward computation shows that

$$G_{i\kappa}(x,x') = D^-_{i\kappa}(x,x') \oplus D^+_{i\kappa}(x,x') + \frac{1}{2\kappa} e^{-\kappa|x|} e^{-\kappa|x'|}. \tag{4.1}$$

The indicated modification corresponds to the changed $-\Delta_{\mathcal{A}_a,Y_a}$ with $\mathcal{A}_a$ replaced by $\alpha\mathcal{A}_a$,

$$\alpha\mathcal{A}_a := \left\{ \alpha(2\beta^{-1} - a^{-1}), \alpha\beta a^{-2}, \alpha(2\beta^{-1} - a^{-1}) \right\},$$

where α, $\beta \in \mathbb{R} \setminus \{0\}$. The form $\mathcal{Q}_{\alpha\mathcal{A}_a,Y_a}[\cdot,\cdot]$ associated with the operator $-\Delta_{\alpha\mathcal{A}_a,Y_a}$ is given by

$$\mathcal{Q}_{\alpha\mathcal{A}_a,Y_a}[u,v] = (u',v') + \alpha\frac{\beta}{a^2}\, u(0)\overline{v(0)} + \alpha\left(\frac{2}{\beta} - \frac{1}{a}\right)\{u(+a)\overline{v(+a)} + u(-a)\overline{v(-a)}\}\,,$$

where $u, v \in \mathrm{dom}(\mathcal{Q}_{\alpha\mathcal{A}_a,Y_{a,\alpha}}) = W^{1,2}(\mathbb{R})$, which means that $\alpha \neq 1$ amounts to a simultaneous change of all the δ coupling parameters. The resolvent $(-\Delta_{\alpha\mathcal{A}_a,Y_a} + \kappa^2)^{-1}$ is again given by Krein's formula

$$(-\Delta_{\alpha\mathcal{A}_a,Y_a} + \kappa^2)^{-1}(x,x') = G_{i\kappa}(x,x') - \sum_{j,j'=-1,0,+1} [\Gamma_{a,\alpha}(i\kappa)]^{-1}_{jj'} G_{i\kappa}(x-y_j) G_{i\kappa}(x-y_{j'})\,, \tag{4.2}$$

where

$$\Gamma_{a,\alpha}(i\kappa) = \frac{1}{2\kappa}\begin{pmatrix} 1+\alpha u & w & w^2 \\ w & 1+\alpha v & w \\ w^2 & w & 1+\alpha u \end{pmatrix},$$

i.e., in comparison with (2.6) we have $u \to \alpha u$, $v \to \alpha v$, while w is preserved.

Lemma 4.1 *Let $\kappa > 0$. The resolvent $(-\Delta_{\alpha\mathcal{A}_a,Y_a} + \kappa^2)^{-1}$ exists for sufficiently small $a > 0$ if $\alpha \neq 1$.*

Proof: It is sufficient that $-\kappa^2$ is not an eigenvalue. As in Proposition 2.1 this would be true for $-\Delta_{\alpha\mathcal{A}_a,Y_a}$ if κ satisfies one of the equations analogous to (2.10) and (2.11), with κ replaced by $\alpha\kappa$ at the r.h.s. The Taylor expansion around $a = 0$ shows that this cannot happen unless $\alpha = 1$. $\square$

In the following we fix $\kappa > 0$, $\alpha \notin \{0, 1\}$, and $\beta \neq 0$. Then there is $a_0(\kappa) > 0$ such that for all $a \in (0, a_0(\kappa))$ the resolvent $(-\Delta_{\alpha\mathcal{A}_a,Y_a} + \kappa^2)^{-1}$ exists.

Theorem 4.2 *Let $\kappa > 0$, $\alpha \neq 0, 1$, and $\beta \neq 0$ be fixed. Then the relation*

$$\lim_{a\to 0+} \left(-\Delta_{\alpha\mathcal{A}_a,Y_a} + \kappa^2\right)^{-1}(x, x') = D^-_{i\kappa}(x, x') \oplus D^+_{i\kappa}(x, x') \tag{4.3}$$

holds for any $x, x' \in \mathbb{R}$. Consequently, $-\Delta_{\alpha\mathcal{A}_a,Y_a} \to -\Delta_{D,0}$ as $a \to 0+$ in the norm-resolvent sense.

Proof: Considering the case $x, x' \geq a$ and following the line of reasoning from (2.15) to (2.20) we obtain

$$\sum_{jj'=-1,0+1} [\Gamma_{a,\alpha}(i\kappa)]^{-1}_{jj'} G(x-y_j)G(x-y_{j'}) = \frac{1}{4\kappa^2} e^{-\kappa x} e^{-\kappa x'} \frac{N_\alpha}{D_\alpha} \tag{4.4}$$

with

$$D_\alpha := \frac{(w^2 - 1 - \alpha u)[(1+\alpha u)(1+\alpha v) - w^2(1-\alpha v)]}{2\kappa}$$

and

$$N_\alpha := (w^2 + w^{-2})[w^2 - (1+\alpha u)(1+\alpha v)] + 2\alpha w^2 v + (w^2 - 1 - \alpha u)(\alpha u - 1 - w^2)\,.$$

If $\alpha \neq 1$, one gets

$$D_\alpha = -2\kappa a^2(1-\alpha) + \mathcal{O}(a^3)$$

and

$$N_\alpha = -4\kappa^2 a^2(1-\alpha) + \mathcal{O}(a^3)\,, \tag{4.5}$$

so the r.h.s. of (4.4) equals $2\kappa e^{-\kappa x} e^{-\kappa x'}(1 + \mathcal{O}(a))$. Inserting (4.5) into (4.2) and using (4.1), we find

$$\lim_{a\to +0} (-\Delta_{\alpha\mathcal{A}_a,Y_a} + \kappa^2)^{-1}(x, x') = D^+_{i\kappa}(x, x') \tag{4.6}$$

for $x, x' \in [a, +\infty)$. In the same way one can treat the other combinations with x, x' belonging to $(-\infty, a]$, $(-a, 0)$, $(0, a)$ and $[a, +\infty)$; doing so we check (4.3) for $x, x' \in \mathbb{R}$.

Taking into account (4.1) and (4.2) one easily verifies that $\left(-\Delta_{\alpha\mathcal{A}_a,Y_a} + \kappa^2\right)^{-1}(x, x') - D^-_{i\kappa}(x, x') \oplus D^+_{i\kappa}(x, x')$ can be majorized by a function from $L^2(\mathbb{R}^2)$ which is independent of a. By (4.3) and the Lebesgue convergence theorem the difference $\left(-\Delta_{\alpha\mathcal{A}_a,Y_a} + \kappa^2\right)^{-1} -$

$\left(-\Delta_{D,0}+\kappa^2\right)^{-1}$ converges to zero in the Hilbert-Schmidt norm, so $-\Delta_{\alpha\mathcal{A}_a,Y_a} \to -\Delta_{D,0}$ as $a \to 0+$ in the norm-resolvent sense. $\square$

Let us introduce the Schrödinger operator $H^a_{\epsilon,0,\alpha}$ defined by

$$H^a_{\epsilon,0,\alpha} := -\Delta + \alpha W^a_{\epsilon,0}$$

for $\alpha \in \mathbb{R}\setminus\{0\}$ as in the previous section. It corresponds to rescaling of the original approximation potential: we have $H^a_{\epsilon,0,\alpha} = H^a_{\epsilon,0}$ if $\alpha = 1$. The Neumann iterations are now defined by

$$R^{(n)}_{a,\epsilon,\alpha}(i\kappa) := \hat{G}^*(a)\hat{\Gamma}_{a,\alpha}(i\kappa)^{-1}\left(D_{a,\epsilon}\hat{\Gamma}_{a,\alpha}(i\kappa)^{-1}\right)^n \hat{G}(a), \quad n = 0,1,2,\dots.$$

for $k > 1$ and $a \in (0, a_0(\kappa))$ where the definition of $\hat{\Gamma}_{a,\alpha}(i\kappa)$ is obvious. We note that for for $\kappa > 1$ and $\alpha \neq 1$ there is a constant $C_{\Gamma_\alpha}(\kappa) > 0$ such that instead of (3.20) one has the estimate

$$\|\Gamma_{a,\alpha}(i\kappa)^{-1}\|_{\mathcal{B}(\mathbb{C}^3,\mathbb{C}^3)} \le C_{\Gamma_\alpha}(\kappa)a^{-1}$$

for $a \in (0, a_0(\kappa))$. Lemma 3.6 reads now as follows.

Lemma 4.3 *Let $V_j \in L^2(\mathbb{R})$, $j - 1, 0, +1$, and $\kappa > 1$, $\beta \neq 0$. If the potentials V_j satisfy the conditions (3.2), (3.8) and (3.11), then the Neumann iterations obey the estimate*

$$\left\|R^{(n)}_{a,\epsilon,\alpha}(i\kappa)\right\| \le \frac{2C_{\Gamma_\alpha}(\kappa)}{a}\left(\frac{4\sqrt{\epsilon}\,C_{\Gamma_\alpha}(\kappa)C(a)}{a}\right)^n$$

for $a \in (0, a_0(\kappa))$ and $n = 1, 2, \dots,$ where $C(a)$ is given by (3.14).

The proof is similar to that of Lemma 3.6. In view of Lemma 4.3 one has to modify the parameter $\tau(\epsilon, a, \kappa)$ to

$$\tau_\alpha(\epsilon, a, \kappa) := \frac{4\sqrt{\epsilon}\,C_{\Gamma_\alpha}(\kappa)C(a)}{a}.$$

If $\alpha\tau_\alpha(\epsilon, a, \kappa) < 1$ is satisfied, then the operator $R_{a,\epsilon,\alpha}(i\kappa) := \sum_{n=0}^\infty \alpha^n R^{(n)}_{a,\epsilon,\alpha}(i\kappa)$ is well defined. With obvious modifications Lemma 3.7 takes the following form.

Lemma 4.4 *Let $V_j \in L^2(\mathbb{R})$, $j = -1, 0, +1$, and let $\kappa > 1$, $\alpha \notin \{0,1\}$, and $\beta \neq 0$. If the the potentials V_j satisfy the conditions (3.2), (3.8) and (3.11) and $\alpha\tau_\alpha(\epsilon, a, \kappa) < 1$ is valid for some $a \in (0, a_0(\kappa))$, then $-\kappa^2$ belongs to the resolvent set of the operator $H^a_{\epsilon,0,\alpha}$, and, moreover, one has*

$$(H^a_{\epsilon,0,\alpha} + \kappa^2)^{-1} = R_{a,\epsilon,\alpha}(i\kappa)\,.$$

Lemma 3.8 modifies similarly but we get a slightly stronger result because the matrix $\Gamma_{a,\alpha}(i\kappa)^{-1}$ is now less singular for any $\kappa > 0$ as $a \to 0$.

Lemma 4.5 *Under the assumptions of the preceding lemma,*

$$\left\|(H^a_{\epsilon,0,\alpha} + \kappa^2)^{-1} - (-\Delta_{\alpha\mathcal{A}_a,Y_a} + \kappa^2)^{-1}\right\| \le 2\alpha C_{\Gamma_\alpha}(\kappa)\frac{\tau_\alpha(\epsilon, a, \kappa)}{a}\left(1 - \alpha\tau_\alpha(\epsilon, a, \kappa)\right)^{-1}.$$

Taking into account Theorem 4.2 and Lemmata 4.4, 4.5 we thus prove the following theorem.

Theorem 4.6 *Let $V_j \in L^2(\mathbb{R})$, $j = -1, 0, 1$, and let $\kappa > 1$, $\alpha \notin \{0, 1\}$, and $\beta \neq 0$. Furthermore, let $\lim_{\epsilon\to 0} a(\epsilon) = 0$. If the potentials V_j satisfy the conditions (3.2), (3.8) and (3.11) and*

$$\lim_{\epsilon\to 0} \frac{\epsilon}{a(\epsilon)^8} = 0\,,$$

then

$$\lim_{\epsilon\to 0} \left\| \left(H^a_{\epsilon,0,\alpha} + \kappa^2\right)^{-1} - \left(-\Delta_{\alpha\mathcal{A}_{a(\epsilon)},Y_{a(\epsilon)}} + \kappa^2\right)^{-1} \right\| = 0\,.$$

and

$$\lim_{\epsilon\to 0} \left\| \left(H^a_{\epsilon,0,\alpha} + \kappa^2\right)^{-1} - \left(-\Delta_{D,0} + \kappa^2\right)^{-1} \right\| = 0.$$

Using a translation, the analogous conclusion can be made for the family $\{H^a_{\epsilon,y,\alpha}\}$ with the potential center shifted to a point y, which naturally converges for $\alpha \notin \{0, 1\}$ to the Laplacian with the Dirichlet decoupling at y.

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