Localized breathing solutions for Bose-Einstein condensates in periodic traps.

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We consider Bose-Einstein condensates confined by a periodic potential with widely separated wells. Through a coupled soliton ansatz we reduce the full dynamics to a Toda lattice with on-site potentials. This lattice differential equation possesses a variety of oscillatory breathing solutions. In particular we construct localized breathers, which correspond to homoclinic orbits of an appropriate recurrence map. We also explore the possibility of extended breathers and chaotic oscillations.

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The recent experimental realization of dilute Bose-Einstein condensates (BECs) has triggered a vast interest in the physics community. Novel techniques of generating condensates are permitting the investigation of a wide range of phenomenology and applications. Typically, condensates are grown in optical/magnetic traps produced from standing light waves. Much attention has focused on condensates extending over several periods of the trapping potential. Typical filling factors for these lattices are of the order of one atom per ten lattice sites [1]. Here we are interested in densely populated sites where a whole condensate is grown at each potential minimum. This will obviously require lower temperatures and adjustments to the optical trap techniques.

The macroscopic mean field of dilute BECs is governed by the Gross-Pitaevskii equation (GPE) [2]. In 3D, the GPE takes the form of a cubic, time dependent, nonlinear Schrödinger equation (NLS) with an external potential due to the trap. We limit our study to one-dimensional models of BECs. The GPE has been shown to reduce to the 1D NLS in the quasi-1D regime when the longitudinal direction is much larger than the transverse dimension (of the order of the healing length) [3]. Another possible scenario in which the GPE can be reduced to the 1D NLS arises for a stack of 'flat pancakes' of condensates formed from two superimposed optical lattices [4].

We assume that the dynamics of the BEC trapped in a periodic potential V is governed by the periodic NLS which in dimensionless units takes the form

$$\mathrm{i}u_t + \frac{1}{2}u_{xx} \pm |u|^2 u = V(x)u. \tag{1}$$

Here $u = u(x,t)$ is the mean field value for the density of the condensate and the $\pm$ sign correspond to attractive $(+)$ and repulsive $(-)$ atoms. In the present work we address the case of attractive atoms, however the same approach carries over to the repulsive case. We take the external potential V to be given by

$$V(x) = V_0\,\mathrm{sn}^2(x;\kappa), \tag{2}$$

where $\mathrm{sn}(x;\kappa)$ is the Jacobi elliptic sine, V_0 is the potential strength and κ is a tuning parameter. By varying κ from 0 to 1 we deform the potential from $\sin^2(x)$ to $\tanh^2(x)$. We consider the case $\kappa \to 1^-$ in which the potential can be well approximated by a sum of equispaced potentials $V_j = V_0\,\tanh^2(x-\xi_{0;j})$:

$$V(x) \simeq -2N\,V_0 + \sum_{j=-N}^{N} V_j(x). \tag{3}$$

The centers $\xi_{0;j}$ of the potentials are given by $\xi_{0;j} = j\,R$ where $R = 2\,K(\kappa)$ is the inter-site spacing and $K(\kappa)$ is the complete elliptic integral of the first kind that prescribes the period of the potential. For $\kappa = 1$, when the potential V reduces to $V(x) = V_0\,\tanh^2(x)$, the condensate

$$\phi(x,t) = \sqrt{1-V_0}\;\mathrm{sech}(x)\,e^{\mathrm{i}\omega t}, \tag{4}$$

where $\omega = 1/2 - V_0$, is an exact solution of (1) [5], corresponding to a rescaled NLS soliton solution. We wish to consider the dynamics of the whole train of condensates,

$$u(x,t) = \sum_{j=-\infty}^{\infty} u_{0;j}(x,t), \tag{5}$$

where $u_{0;j}(x,t) = \phi(x-\xi_{0;j},t)$. We approximate the overall dynamics, focusing on two major contributions: a) the internal dynamics due to interactions between $u_{0;j}$ and V_j and b) the tail-tail interactions between consecutive solitons. We address each of these terms individually and use the linear superposition afforded by soliton perturbation theory [6].

For a single soliton $u_{0;j}$ inside its on-site potential V_j the dynamics for position perturbations can be treated using variational techniques, perturbation techniques or energy methods [7]. The details and improvements on these methods will be covered elsewhere [8]. The effect of the on-site potential to a non-centered BEC soliton is an oscillatory pattern approximated by

$$u_{0;j}(x,t) = H_j\,\mathrm{sech}\left[c_j\,(x-\xi_j)\right]\,e^{\mathrm{i}\,W_j\,t} \tag{6}$$

where $H_j,\ c_j,\ W_j,\ \xi_j$ are the time dependent height, width, phase and position of the soliton. It is well known

[7] that the oscillations of the jth soliton position $\xi_j(t)$ can be approximated by a particle inside an effective potential

$$\ddot{\xi}_j = -V'_{\text{eff}}(\xi - \xi_{0;j}). \tag{7}$$

In the limit of small amplitude oscillations we may neglect, to leading order, the small variations in the height, width and position and consider $H \simeq \sqrt{1 - V_0}$, $c \simeq 1$ and $W(t) \simeq \omega$ [8]. We also mention that the rate of decay due to radiation is small compared to survival times of condensates in experiments [8]. In this regime it can be shown that the effective potential admits an expansion $V_{\text{eff}}(x)/V_0 = \frac{4}{15}x^2 - \frac{4}{63}x^4 + o(x^6)$; and we restrict our attention to effective potentials of the form

$$V_{\text{eff}}(x) = \frac{\alpha}{2}\,x^2 + \frac{\beta}{4}\,x^4 + o(x^6). \tag{8}$$

It remains to consider the interaction of the tail of the jth condensate with the other condensates and potentials. More specifically, the jth condensate interacts with:

(A) the $(j-1)$th and $(j+1)$th condensates,
(B) the ith condensate for $|i - j| > 1$,
(C) the ith potential V_i for $i \neq j$.

Interactions of type (B) and (C) are neglected since they are small compared to (A). Interactions of type (A) are approximated using perturbation techniques based on inverse scattering techniques [6]. Tail-tail interactions of type (A) are exponentially small and can give rise to a complex version of the Toda lattice involving position and phases of the solitons [9]. It is possible to further reduce the dynamics of the interacting condensates under the assumption that the relative phases are constant. This is a reasonable approximation in our regime where the solitons are kept well apart and at an almost constant spacing $R = 2\,K(\kappa)$. With a relative phase of π for consecutive condensates the tail-tail interactions reduce to a *real* Toda lattice (TL) [10] on the positions: $\ddot{\xi}_j = 8\,(e^{-(\xi_j - \xi_{j-1})} - e^{-(\xi_{j+1} - \xi_j)})$. In [11] it is shown that the only stable stationary solution is a $\text{cn}(x;\kappa)$-type solution —Jacobi elliptic cosine or cnoidal wave— which corresponds (for the $\kappa \to 1^-$ limit) to a concatenation of sech solitons with a π phase shift. In practice, the π phase shift can be implementing by phase design of the initial configuration of the condensate [12].

Exploiting the linearity of soliton perturbation theory we combine the soliton on-site potential interaction (7) with the tail-tail interactions to find a lattice differential equation (LDE) on the condensates positions

$$\ddot{\xi}_j = 8\left(e^{-(\xi_j - \xi_{j-1})} - e^{-(\xi_{j+1} - \xi_j)}\right) - V'_{\text{eff}}(\xi_j - \xi_{0,j}). \tag{9}$$

Our goal is to find oscillatory solutions to (9). A standard approach is to propose an oscillating ansatz for solutions of the LDE [13]. Namely, write the position of the jth condensate as a combination of oscillatory modes $\xi_j(t) = \sum_{k=-\infty}^{+\infty} A_j(k)\,\exp(\mathrm{i}k\Omega t)$. We take $A_j(k) = A_j^*(-k)$ to allow our solution to be real and $A_j(0) = \xi_{0,j}$ since the jth soliton is centered at $\xi_{0,j}$. After substituting this oscillatory ansatz into (9) it is possible to find a recurrence relation between the $A_j(k)$. However, our goal is to find *breathers*, that is, structures which oscillate with a single, locked frequency Ω. To this end we consider the *one-mode* ansatz

$$\xi_j(t) = \xi_{0,j} + A_j\,\cos(\Omega t). \tag{10}$$

We substitute (10) into (9), expand and match terms to find the recurrence relation between the oscillation amplitudes A_j

$$A_{j+1} = (a + b\,A_j^2)\,A_j - A_{j-1}, \tag{11}$$

with $a = 2 - \Omega^2 + \alpha\,e^R/4$ and $b = 3\,\beta\,e^R/16$, where R is the inter-site spacing between potentials ($R = |\xi_{0,j+1} - \xi_{0,j}| = 2\,K(\kappa)$) and ($\alpha$,$\beta$) are the harmonic and nonharmonic parts of the expansion for the effective potential (8). We introduce $y_n = A_n$ and $x_n = A_{n-1}$ and recast the second order recurrence relation (11) as a 2D map

$$\begin{aligned}(x_{n+1}, y_{n+1}) &= M(x_n, y_n) \\ &= (y_n, (a + b\,y_n^2)\,y_n - x_n).\end{aligned} \tag{12}$$

Up to the approximations made to obtain equation (11), every solution of (12) represents an approximate breathing solution of the LDE (9). It should be noticed that the frequency Ω of the breather is incorporated into the 2D map equation through the parameter a of (11). Thus one expects families of breathers parameterized by their frequency Ω. It is worth mentioning that the iteration index n for the 2D map (12) corresponds to the site index j for the amplitude of the oscillation (10).

We now construct *localized breathers* of the LDE (9). These are nontrivial solutions of the 2D recurrence map (12) which are homoclinic to the origin, corresponding to solutions of (11) which satisfy $\lim_{n\to\pm\infty} A_n = 0$. A necessary condition for the existence of an orbit homoclinic to the rest state is that the origin be a hyperbolic fixed point of the 2D map. This hyperbolicity condition is satisfied for all $|a| > 2$ or equivalently $\left|2 - \Omega^2 + \alpha\,e^R/4\right| > 2$, which provides constraints on the inter-site spacing R and breather frequency Ω. We find numerically, for reasonable values of the inter-site spacing R, that the stable and unstable manifolds of the origin intersect in a homoclinic tangle (see left panel in Fig. 1). The localized breather is then given by $A_n = \mathcal{P}(P_n)$ where $\mathcal{P}$ is the projection to the y-axis and $\{..., P_{-1}, P_0, P_1, ...\}$ is a homoclinic orbit of M (see right panel in Fig. 1).

A proof of the existence of breathers can be obtained from the so called anticontinuum or anti-integrable limit [14]. In our model, this limit corresponds to the inter-site coupling in (9) being driven to zero either by increasing the inter-site spacing R or increasing the relative strength of the local potential V_{eff}. For general nonlinear Hamiltonian lattices of weakly interacting oscillators, the existence of localized breathers has been established in [15].

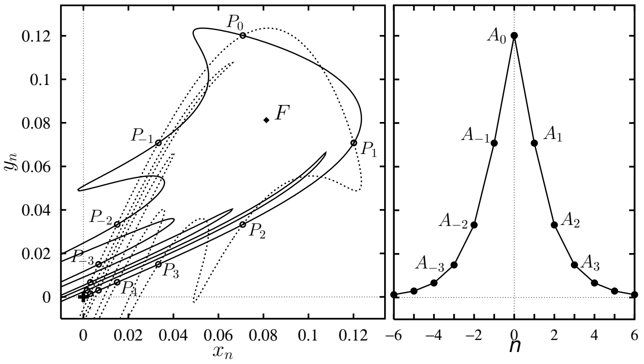

FIG. 1. Left: Homoclinic tangle for the two-dimensional map (12) emanating from the origin. The homoclinic orbit $\{..., P_{-1}, P_0, P_1, ...\}$ belongs to the stable (solid line) and unstable (dash line) manifolds. The physical parameters correspond to $R = 2\,K(0.99963620547) \simeq 10$, $\Omega = 17.671$ and $V_0 = 0.1$. Right: The corresponding configuration for the oscillation amplitudes.

The homoclinic orbit of the 2D map induces a breather in the LDE through the relation (10) and in the periodic NLS through the ansatz (6). An example of such a localized breathing solution in the periodic NLS is depicted in Fig. 2, in which a central condensate oscillates which a maximal amplitude A_0 and the condensates on either side oscillate with amplitudes $A_{\pm n}$ which decrease exponentially in increasing n (see right panel in Fig. 1). This localized oscillation corresponds to the exponential rate of approach of the homoclinic orbit to the origin along the stable and unstable manifold as $n \to \pm\infty$. Indeed the exponential decay rate is proportional to $\lambda_-^{|n|}$ as $n \to -\infty$ and $\lambda_+^{-|n|}$ as $n \to +\infty$ where $\lambda_\pm = (a \pm \sqrt{a^2-4})/2$ are respectively the stable and unstable eigenvalues of the map M linearized at the origin ($\lambda_- = \lambda_+^{-1} \simeq 0.4448$ in Fig. 1). Note that, because (12) has $x \leftrightarrow y$ symmetry, the eigenvalues are related through $\lambda_- = \lambda_+^{-1}$ and the breather is symmetric with respect to the central condensate.

The structural stability of the homoclinic tangle implies persistence of the homoclinic orbit of the 2D map under external perturbations. Indeed, this structural stability ensures that a localized breather for the reduced dynamics (12) corresponds to a nearby localized breather for the full system. However it is important to point out that while the localized breathers are a robust phenomena, they are also non-generic in the sense that initial data for (1) comprised of a sum of pulses with arbitrary oscillation amplitudes A_n which tend to zero as $n \to \pm\infty$, will typically degenerate into a non-localized solution whose energy radiates away to infinity. The localized breathers represent an important structure in the dynamics of (1) which enable energy to be pinned down on the lattice.

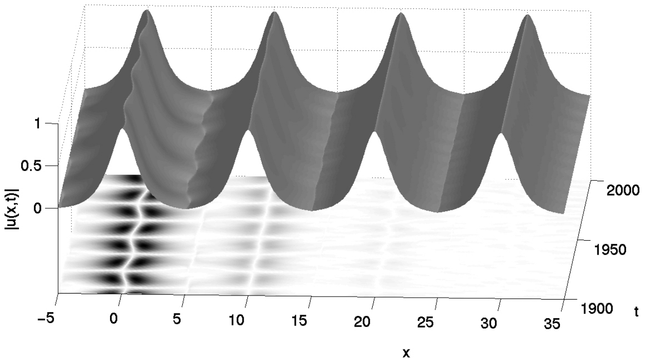

FIG. 2. Typical localized breather solution of (1) showing the dynamics of coupled Bose-Einstein condensates in a periodic potential. The system parameter values are taken as in Fig. 1. Only condensates with index 0, 1, 2 and 3 are shown (the oscillations are symmetric with respect to the origin). The amplitude of oscillation of condensates with index larger than 2 is imperceptible at this scale. In this plot, the bottom plane depicts $d|u(x,t)|/dt$ —darker areas corresponds to regions where $u(x,t)$ undergoes greater variation.

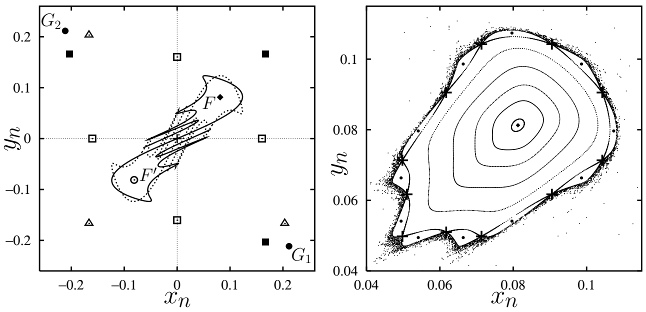

FIG. 3. Left: some periodic orbits and the homoclinic tangle of the 2D map (12). F and F' correspond to fixed points. $\{G_1, G_2\}$ corresponds to a period 2 orbit. Also depicted are some higher order periodic orbits: period 3 (triangles and filled squares) and period 4 (open squares). Right: behaviour near the fixed point F (diamond). Close to F the 2D map (12) displays quasiperiodic orbits which disappear at the separatrix originating from the manifolds of a period 11 orbit (crosses). Outside this separatrix a single chaotic orbit is depicted (small dots).

There are other breathing phenomena of (1) which are described by orbits of the 2D map (12). Any nontrivial periodic points of (12) gives rise to *global* oscillations [5]. The map M has three fixed points: the origin, $F = (x^*, y^*)$ and $F' = (-x^*, -y^*)$ (see Fig. 3), where $x^* = y^* = \sqrt{(2-a)/b}$. The fixed point at the origin gives rise to the trivial stationary solution $\{A_n = 0\}_{-\infty}^{\infty}$ of (11). The points F and F' give rise to a solution of the LDE (9) in which all condensates oscillate in phase with same amplitude $\{A_n = x^*\}_{-\infty}^{\infty}$. The corresponding breathing solution for the periodic NLS is depicted in Fig. 4.a. Other possible dynamics could arise from higher order periodic points. For example, a possible period two orbit of (12) is $G_1 = M(G_2) = M^2(G_1)$ (see Fig. 3) where $G_1 = (\hat{x}, -\hat{x})$ and $G_2 = (-\hat{x}, \hat{x})$ with $\hat{x} = \sqrt{-(2+a)/b}$. This period two orbit gives rise to an amplitude configuration $\{\ldots, \hat{x}, -\hat{x}, \hat{x}, -\hat{x}, \hat{x}, \ldots\}$. The resulting anti-phase global oscillation for the periodic NLS is depicted in Fig. 4.b. It is in principle possible to construct more complex patterns for the global oscillations by using higher order periodic orbits of the 2D map (cf. Fig. 3).

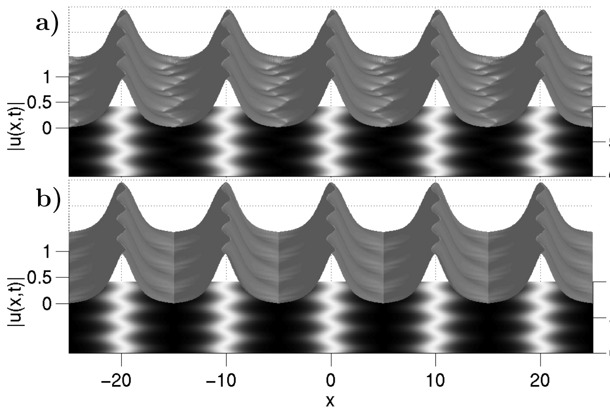

FIG. 4. Global oscillations of the condensates. a) in-phase oscillations corresponding to the nontrivial fixed point F (see Fig. 1). b) anti-phase oscillations corresponding to a period two cycle of (12) with alternating amplitude sign.

The 2D map also predicts global oscillations which are quasi-periodic in site index n. These orbits exist near the nontrivial fixed point F (see right panel in Fig. 3). The corresponding breather for the full periodic NLS (1) has an in-phase global oscillation with a small modulation of the amplitudes (given by the rotation number of the quasi periodic orbit around F).

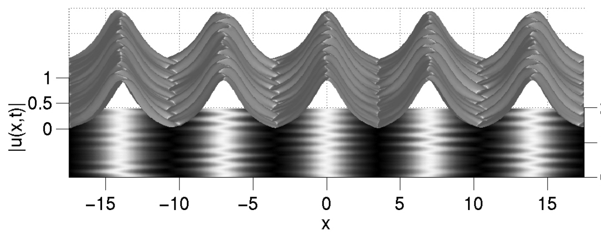

FIG. 5. Chaotic interaction of the condensates.

Another interesting possibility for the dynamics of the coupled condensates is the prospect of chaotic evolution. In fact, close to the non-trivial fixed point F there seems to be a region displaying irregular motion (see right panel in Fig. 3). This corresponds to chaotic oscillations for the condensates (see Fig. 5). It should be possible, in principle, to find the onset of chaos for the condensates by analyzing in more detail the reduced dynamics on (12).

In summary, we have shown [8] that the periodic NLS (1) supports a wide variety of breathing solutions described by the ansatz (10) and (6) and parameterized by their breathing frequency Ω. In particular we have identified localized breathers which correspond to homoclinic orbits of the reduced map (12). These are important structures in the phase space of (1) since they permit localization of energy on the optical lattice. The stability of these breathers under the dynamics of the full periodic NLS is a subject of further study.

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