UNIFORM SPECTRAL PROPERTIES OF ONE-DIMENSIONAL QUASICRYSTALS, IV. QUASI-STURMIAN POTENTIALS

DAVID DAMANIK [1] AND DANIEL LENZ [2,3]*

[1] Department of Mathematics, University of California, Irvine, CA 92697, USA

[2] Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel
[3] Fachbereich Mathematik, Johann Wolfgang Goethe-Universität, 60054 Frankfurt, Germany

E-mail: damanik@math.uci.edu, dlenz@math.uni-frankfurt.de

1991 AMS Subject Classification: 81Q10, 47B80
Key words: Schrödinger operators, low-complexity potentials, singular continuous spectrum, zero-measure Cantor spectrum, Hausdorff dimensional spectral properties

Abstract. We consider discrete one-dimensional Schrödinger operators with quasi-Sturmian potentials. We present a new approach to the trace map dynamical system which is independent of the initial conditions and establish a characterization of the spectrum in terms of bounded trace map orbits. Using this, it is shown that the operators have purely singular continuous spectrum and their spectrum is a Cantor set of Lebesgue measure zero. We also exhibit a subclass having purely α-continuous spectrum. All these results hold uniformly on the hull generated by a given potential.

1. Introduction

This article studies spectral properties of discrete one-dimensional Schrödinger operators

$$(H\phi)(n) = \phi(n+1) + \phi(n-1) + V(n)\phi(n) \tag{1}$$

in $\ell^2(\mathbb{Z})$ with potential $V : \mathbb{Z} \to \mathbb{R}$ of low combinatorial complexity. Consider the case where V takes finitely many values. A particular example is given by the case where V is periodic. It is well known that periodicity of V implies that the spectrum of H is purely absolutely continuous. Among the aperiodic cases, several classes of potentials have been studied. These include, for example, Sturmian potentials [4, 13, 18, 19, 20], potentials generated by circle maps [23, 26], potentials generated by substitutions [3, 6, 14, 15, 26], and random (Bernoulli-type) potentials [7].

There are two important observations that can be made. In all the known results, the spectral type is pure, that is, we do not know of any example within the class of

* Research of D. L. was supported in part by THE ISRAEL SCIENCE FOUNDATION (grant no. 447/99) and the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).

operators with potential taking finitely many values that has mixed spectral type. On the other hand, any spectral type can occur: Periodic potentials always lead to purely absolutely continuous spectrum, Sturmian potentials always lead to purely singular continuous spectrum [18], and almost all random potentials lead to pure point spectrum [7]. Potentials generated by circle maps or substitutions seem to always lead to purely singular continuous spectrum; at least no counterexample is known yet.

A point of view that was raised in [16] and will be more comprehensively discussed in [2] is the following:

(2) The more complex the potential, the more singular the spectral type.

Here, complexity is to be understood in a combinatorial sense. By restricting V to finite intervals, we can speak of "finite subwords" of V. The (combinatorial) complexity function $p : \mathbb{N} \to \mathbb{N}$ then assigns to each $n \in \mathbb{N}$ the number $p(n)$ of finite subwords of V that have length n. As it is well known in the combinatorics on words community, periodicity of V is characterized by p being bounded and Sturmian V are (essentially) characterized by the fact that p has lowest possible growth among the unbounded complexity functions. On the other hand, random potentials have almost surely maximal complexity in the sense that if V takes values in $\mathcal{A} \subseteq \mathbb{R}$ and is generated randomly, then almost surely, $p(n) = |\mathcal{A}|^n$ for every $n \in \mathbb{N}$. Thus, reformulating (2) as

(3) The faster $p(n)$ grows, the more singular the spectral type.

we see that (3) holds true for the extreme cases, that is, maximal complexity growth, minimal complexity growth (no growth), and minimal complexity growth among the unbounded cases. Between Sturmian and random potentials, only few results are known. In particular it is not clear where the transition from singular continuous spectrum to pure point spectrum occurs in terms of complexity. Results in [22] show that one can construct potentials with very high complexity that lead to purely singular continuous spectrum.

A natural next step is to approximate the transition from above and below, that is, to extend [18] and [7]. Our purpose here is to do the former and we investigate the next natural complexity class: Quasi-Sturmian potentials. We will show that they, too, always lead to purely singular continuous spectrum.

It is an intriguing fact that the final result is of a very deterministing nature (every quasi-Sturmian operator has purely singular continuous spectrum), yet the proofs make crucial use of probabilistic methods by employing Kotani's theory for ergodic families of operators. It would be interesting to see whether one can do away with this and give a deterministic proof in the same generality.

Apart from Kotani theory, another crucial input is a detailed study of the orbits of a certain dynamical system, the quasi-Sturmian trace map, which is the heart of this paper. These results rely only on certain hierarchical structures and are to a large extent model-independent. They may thus be of of independent interest. Moreover, when specialized to the Sturmian case, they provide a different proof of a central result in [4].

Using our trace map results, we can show that the spectrum of Schrödinger operators with quasi-Sturmian potentials is always purely singular continuous; and in some cases it has even stronger continuity properties.

As we will see later, quasi-Sturmian potentials, while always finitely valued, can take arbitrarily many values, whereas Sturmian potentials take only two values. In this sense, the class of Sturmian potentials is much richer, in particular in terms of their local properties which can in fact be arbitrary.

This paper is a continuation of [18, 19, 20] (cf. [34] as well) and it further exploits the general theme of the series: Application of canonical partitions of potentials to spectral theory of the induced operators. Therefore, the three cornerstones of the theory are Kotani theory, trace map bounds, and partitions. Whenever one can establish these three pieces for a given class of models, one can obtain a good understanding of the behavior of generalized eigenfunctions of the operators and hence their spectral properties.

The article is organized as follows. In Section 2 we describe in detail the models we discuss and the results we obtain. The heart of the paper is Section 3 where we investigate the quasi-Sturmian trace map and characterize the energies from the spectrum in terms of trace map behavior. Sections 4 and 5 establish the singular continuous spectral type and Section 6 refines this from the point of view of Hausdorff dimensional properties. In the appendix we discuss some combinatorial properties of quasi-Sturmian sequences which we need in our proofs.

2. Models and Results

Let $\mathcal{A}$ be a finite set, called alphabet, and let $\mathcal{A}^*$ denote the set of finite words built from elements of $\mathcal{A}$. Given a word $w \in \mathcal{A}^*$, we define the length $|w|$ to be the number of symbols it is built from, that is, $|b_1 \dots b_n| = n$. The elements of $\mathcal{A}^{\mathbb{N}}, \mathcal{A}^{\mathbb{Z}}$ are called one-sided (resp., two-sided) infinite sequences over $\mathcal{A}$. Given a word or infinite sequence u over $\mathcal{A}$, any finite subword of it is called a factor. We denote by $F(u)$ the set of all factors of u.

Given a word or infinite sequence u, we define for $n \geq 1$,

$$p_u(n) = \#\{ \text{ factors of } u \text{ having length } n \ \}.$$

The function $p_u : \mathbb{N} \to \mathbb{N}$ is called factor complexity (or just complexity) of u. If u is a one-sided infinite sequence, then the following fundamental result holds [25]:

Proposition 2.1 (Hedlund-Morse)**.** *The following are equivalent:*

(i) *u is eventually periodic (i.e., there exist k, n_0 with $u(n+k) = u(n)$ for $n \geq n_0$).*
(ii) *p_u is bounded.*
(iii) *There exists n_0 with $p(n_0) \leq n_0$.*

This shows that the complexity function displays a dichotomy. It is either bounded (if u is eventually periodic) or it grows at least linearly (if u is aperiodic) with universal lower bound $p_u(n) \geq n + 1$ for every n.

In the following we will only consider sequences u that are recurrent, that is, every one of its subwords occurs infinitely often. We will not always make this assumption explicit but we will give a remark below which explains why these are the "interesting cases."

Recurrent sequences u with minimal complexity $p_u(n) = n + 1$ for every n exist and they are called Sturmian. More generally, a recurrent u is called quasi-Sturmian if there are k, n_0 with $p_u(n) = n + k$ for $n \geq n_0$.

From a combinatorial point of view, among the aperiodic sequences, Sturmian and quasi-Sturmian sequences are the closest to eventually periodic sequences. This has led mathematical physicists to consider their associated hulls (to be defined below) as standard models of one-dimensional quasicrystals (see [41] for the discovery of quasicrystals and [16] for a survey of their spectral theory in one dimension).

Given a one-sided infinite sequence u over $\mathcal{A}$, we define its associated hull (also called induced subshift) by

$$\Omega_u = \{\omega \in \mathcal{A}^{\mathbb{Z}} : \text{ every factor of } \omega \text{ is a factor of } u \}.$$

If we define the shift transformation $T : \mathcal{A}^{Z} \to \mathcal{A}^{\mathbb{Z}}$ by $(Tx)(n) = x(n+1)$ and endow $\mathcal{A}$ with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with product topology, then (Ω_u, T) is a topological dynamical system. If u is uniformly recurrent (i.e., every factor occurs infinitely often and with bounded gaps), (Ω_u, T) is minimal. If the frequencies of factors exist uniformly (see [38] for definitions and details), (Ω_u, T) is uniquely ergodic, that is, there exists a unique T-invariant measure μ which is necessarily ergodic. A subshift that is both minimal and uniquely ergodic is called strictly ergodic. It is known, and in fact can be shown using the methods we present in the appendix, that a quasi-Sturmian sequence u induces a strictly ergodic subshift Ω_u.

Given a subshift (Ω, T) and an injective function $f : \mathcal{A} \to \mathbb{R}$, we define, for $\omega \in \Omega$, potentials $V_\omega : \mathbb{Z} \to \mathbb{R}$ by $V_\omega(n) = f(\omega(n))$. This gives rise to a family $\{H_\omega\}_{\omega \in \Omega}$ of discrete one-dimensional Schrödinger operators in $\ell^2(\mathbb{Z})$ defined by

$$(H_\omega \phi)(n) = \phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n),$$

where $\omega \in \Omega$ and $\phi \in \ell^2(\mathbb{Z})$. If the subshift Ω is minimal, it follows by a strong approximation argument that the spectrum of H_ω is deterministic, that is, there exist a compact set $\Sigma \subseteq \mathbb{R}$ such that

$$\sigma(H_\omega) = \Sigma \text{ for every } \omega \in \Omega. \tag{4}$$

Our first main result is the following.

Theorem 1. *Suppose u is quasi-Sturmian and f is one-to-one. Then for every $\omega \in \Omega_u$, the operator H_ω has purely singular continuous spectrum and its spectrum is a Cantor set of Lebesgue measure zero.*

Remark. We do not really need that f is one-to-one. All results in this article hold under the weaker assumption that f is such that the resulting potentials V_ω are aperiodic. In fact, most results hold for arbitrary f. The only place where we need aperiodicity of the potentials is in Section 4 where we apply Kotani theory. Of course, for periodic potentials, the results in Sections 5 and 6 are well known.

This result has been known only in the Sturmian case and even there it was completed only very recently after a 1989 paper by Bellissard et al. had already established zero-measure spectrum and hence absence of absolutely continuous spectrum [4]. To establish uniform absence of eigenvalues [18], it has proved useful to employ combinatorial notions and methods; see [19] in particular.

The present paper pushes the approach of [18, 19, 20, 34] further and extends their result and that of [4] to all quasi-Sturmian cases. Not only is this a larger class, with still many claims to model quasicrystalline structures in one dimension; as we will see below, this class comprises models with arbitrary alphabet size, whereas

Sturmian models are always based on a two-letter alphabet (since $p(1) = 1+1 = 2$) which seems to be a physically ill-motivated restriction.

We will also be able to extend the other main result of [18] to the quasi-Sturmian case. Namely, the operators in question have always purely α-continuous spectrum for some strictly positive α which is uniform on the hull, provided that the underlying rotation number has bounded density; see below for a precise statement of the result and Section 6 for the proof. This implies in particular that all spectral measures associated with the operators H_ω give zero weight to sets with zero α-dimensional Hausdorff measure and one gets quantum dynamical implications by applying, for example, [32]; see [13, 18] for details.

Let us briefly comment on the case of non-recurrent quasi-Sturmian sequences. Using arguments similar to the ones used in the appendix one can relate this class to the class of non-recurrent Sturmian sequences. This class is in turn well understood [11] and contains only sequences which are eventually periodic and hence are not natural candidates for quasicrystal models.

Next we want to employ a result of Cassaigne (originally due to Coven and Paul) to establish for quasi-Sturmian sequences a weak analogue to the partitions of Sturmian sequences found in [19] (cf. [35] as well). Before we state Cassaigne's result, we recall the following. Let $\mathcal{A}, \mathcal{A}'$ be alphabets. A map $\mathcal{A} \to (\mathcal{A}')^*$ is called a substitution (or morphism). It can be morphically extended to $\mathcal{A}^*$ (resp., $\mathcal{A}^{\mathbb{N}}$) by $S(b_1 \dots b_n) = S(b_1) \dots S(b_n)$ (resp., $S(b_1 b_2 \dots) = S(b_1)S(b_2)\dots$). A substitution on a two-letter alphabet (say, $\mathcal{A} = \{a, b\}$) is called aperiodic if $S(ab) \neq S(ba)$. The following proposition, proved in [9], establishes a characterization of quasi-Sturmian sequences in terms of substitutive images of Sturmian sequences.

Proposition 2.2. *A one-sided sequence $u \in \mathcal{A}^{\mathbb{N}}$ is quasi-Sturmian if and only if there exist a word $w \in \mathcal{A}^*$, a Sturmian sequence $u_{\mathrm{St}} \in \{a,b\}^{\mathbb{N}}$, and an aperiodic substitution $S : \{a, b\} \to \mathcal{A}^*$ such that*

$$u = wS(u_{\mathrm{St}}). \tag{5}$$

Remark. While we base our presentation and discussion of Proposition 2.2 on [9], we would like to point out that the result has been known for a long time. It was shown in the papers [10] by Coven and [37] by Paul.

In order to appreciate the usefulness of the above proposition, let us recall that the local structure of Sturmian sequences is very well understood [5]: Given a Sturmian sequence u_{St}, there is a unique irrational number $\theta \in (0, 1)$ such that

$$F(u_{\mathrm{St}}) = F(c_\theta), \tag{6}$$

where c_θ is given by

- $s_{-1} = a$, $s_0 = b$, $s_1 = s_0^{a_1 - 1} s_{-1}$, and

$$s_n = s_{n-1}^{a_n} s_{n-2} \tag{7}$$

  for $n \geq 2$, where the a_n are the coefficients of the continued fraction expansion of θ,
- $c_\theta = \lim_{n\to\infty} s_n$ in the sense that s_n is a prefix of s_{n+1} for every $n \geq 1$ and $|s_n| \to \infty$.

For a given quasi-Sturmian sequence, it is therefore natural to associate with it the following set of words,

$$s'_n = S(s_n),\ n \geq -1. \tag{8}$$

By virtue of (5) and (6), it appears natural to carry over the partition approach to Sturmian sequences developed in [19] (cf. [35] as well), which decomposes a given sequence into blocks of type s_n and s_{n-1} (for every n), to the quasi-Sturmian case. In fact, we will use a weak analogue of the partition lemma from [19] to study quasi-Sturmian potentials.

We saw above that we can associate an irrational θ with a given quasi-Sturmian sequence u. Any such θ will be called a rotation number of u. We show in the appendix that there are in general multiple choices for θ, but the proof of Proposition 2.2, which will be sketched in the appendix, suggests a particular choice for θ, which will be called *canonical rotation number* and denoted by θ_c. Consider the continued fraction expansion of θ_c,

$$\theta_c = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

with uniquely determined $a_n \in \mathbb{N}$. The associated rational approximants $\frac{p_n}{q_n}$ are defined by

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2}, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2}. \end{aligned}$$

The number θ_c has bounded density if

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i < \infty.$$

The set of bounded density numbers is uncountable but has Lebesgue measure zero.

Now we are in position to state our second main result.

Theorem 2. *Suppose u is quasi-Sturmian with canonical rotation number θ_c having bounded density and f is one-to-one. Then there exists $\alpha > 0$ such that for every $\omega \in \Omega$, the operator H_ω has purely α-continuous spectrum.*

We finish this section with a discussion of certain symmetry properties of quasi-Sturmian models. These symmetry properties come from reflection symmetry of the underlying Sturmian dynamical systems. The results below will be used in Section 6. They extend the corresponding results of [20, 34] to our setting.

For a word $w = w_1 \dots w_n$ over $\mathcal{A} = \{a, b\} = \{0, 1\}$, we define the reflected word w^R by $w^R = w_n \dots w_1$. Similarly, for a two-sided infinite word ω over $\mathcal{A}$, we define ω^R by $\omega^R(n) = \omega(-n)$. It is well known that for arbitrary θ, the Sturmian dynamical system $\Omega(\theta)$ associated to c_θ is reflection invariant, that is, it satisfies $\Omega(\theta) = \Omega(\theta)^R = \{\omega^R : \omega \in \Omega(\theta)\}$.

A proof can be given as follows: By [5], for example, the s_n satisfy

$$s_{2k+1} = \pi_{2k+1}01,\ \ s_{2k} = \pi_{2k}10,\ k \in \mathbb{N}, \tag{9}$$

with suitable palindromes π_n (i.e., $\pi_n = \pi_n^R$). The discussion above shows that $\Omega(\theta)$ is determined by the set of s_n, $n \geq 2$, in the sense that a double-sided-infinite sequence ω over $\{0,1\}$ belongs to $\Omega(\theta)$ if and only if every factor of ω is a factor of a (suitable) s_n. By (9), we see that $\Omega(\theta)$ is determined in this sense by the set of π_n as well. As every π_n is a palindrome, we infer $\Omega(\theta) = \Omega(\theta)^R$.

Now, let $\Omega(\theta, S)$ be the quasi-Sturmian dynamical system associated to c_θ and the substitution S according to Proposition 2.2. Let S^R be the substitution on $\{0,1\}$ with $S^R(0) = S(0)^R$ and $S^R(1) = S(1)^R$.

Mimicking the reasoning leading to $\Omega(\theta) = \Omega(\theta)^R$, we directly infer the following proposition.

Proposition 2.3. $\Omega(\theta, S)^R = \Omega(\theta, S^R)$.

A direct calculation shows that $U_R H_\omega U_R^* = H_{\omega^R}$ holds for every double-sided-infinite ω over $\mathcal{A}$. Here, U_R is the unitary reflection operator in ℓ^2 given by $(U_R u)(n) = u(-n)$. Thus, the proposition immediately gives the following corollary.

Corollary 2.4. *Denote by $\Sigma(\Omega))$ the deterministic spectrum of random operators associated to Ω. Then, $\Sigma(\Omega(\theta, S)) = \Sigma(\Omega(\theta, S^R)$.*

3. The Quasi-Sturmian Trace Map

In this section we discuss the trace map associated with a quasi-Sturmian hull. We will show that the spectrum coincides with the set of energies for which the corresponding trace map orbit remains bounded and we give uniform bounds for these orbits. This result will be crucial to what follows and it allows us to pursue a strategy similar to [18, 19, 20, 34].

Before we actually begin our discussion, let us emphasize the following: While our results on trace maps are similar to the corresponding results in the Sturmian case treated in [4], our proofs are completely different and in fact both more general and more conceptual. This is necessary as the corresponding investigation of [4] cannot be carried over to quasi-Sturmian potentials (see below for details).

Recall that a quasi-Sturmian hull $\Omega = \Omega_u$ comes with a sequence of words s_n', $n \geq -1$, defined in (8). If s_n' has the form $s_n' = b_1 \ldots b_k$ with $b_i \in \mathcal{A}$ (leaving the dependence of k and the b_i's on n implicit) and $E \in \mathbb{C}$, we define the transfer matrix $M_E(n)$ by

$$M_E(n) = \begin{pmatrix} E - f(b_k) & -1 \\ 1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} E - f(b_1) & -1 \\ 1 & 0 \end{pmatrix}. \tag{10}$$

It follows from (7) and (8) that for every $E \in \mathbb{C}$ and every $n \geq 2$,

$$M_E(n) = M_E(n-2) M_E(n-1)^{a_n}. \tag{11}$$

In particular, the matrices $M_E(n)$ satisfy the same recursive relations as the transfer matrices in the Sturmian case; compare [4]. Thus, their traces

$$\tau_E(n) = \operatorname{tr}(M_E(n)) \tag{12}$$

satisfy the Sturmian recursive relations as well, however, with possibly different initial conditions. This presents a potential difficulty. The Sturmian trace map preserves a quantity known as the Fricke-Vogt invariant. In the Sturmian case, the initial conditions are always such that E drops out, that is, the invariant is E-independent. This may not be the case in our more general setting. However, since the invariant is given by a polynomial in E, and the spectrum is compact, it is uniformly bounded for energies from the spectrum. This fact will enable us to prove results similar to the Sturmian case. Another important remark is in order: The proof of the classification of trace map orbits in the Sturmian case [4] makes crucial use of a certain property of the initial conditions and hence does not extend to the quasi-Sturmian case where this property may not hold. We will therefore give a different proof of a similar classification result which works without any assumptions on the initial conditions.

When studying the traces $\tau_E(n)$ defined in (12), it is convenient to introduce the following variables:

(13)
$$(x_E(n), y_E(n), z_E(n)) = (\tfrac{1}{2}\mathrm{tr}(M_E(n-1)), \tfrac{1}{2}\mathrm{tr}(M_E(n)), \tfrac{1}{2}\mathrm{tr}(M_E(n)M_E(n-1))).$$

Using (11) and the Cayley-Hamilton theorem, it is possible to find for every $n \in \mathbb{N}$, a polynomial F_n from $\mathbb{R}^3$ to itself such that for every E, n, we have

$$F_n(x_E(n), y_E(n), z_E(n)) = (x_E(n+1), y_E(n+1), z_E(n+1)). \tag{14}$$

Namely, with the Chebyshev polynomials $U_m(x)$ (defined by $U_{-1} = 0$, $U_0 = 1$, $U_{m+1}(x) = 2xU_m(x) - U_{m-1}(x)$), we have

$$F_n(x, y, z) = (y, zU_{a_{n+1}-1}(y) - xU_{a_{n+1}-2}(y), zU_{a_{n+1}}(y) - xU_{a_{n+1}-1}(y)). \tag{15}$$

We want to study the orbits $(F_n \cdots F_1(x, y, z))_{n\in\mathbb{N}}$ for arbitrary initial vectors $(x, y, z) \in \mathbb{R}^3$. Our goal is to show that they are either bounded or super-exponentially diverging in every coordinate. Moreover, we want to show that the spectrum consists of exactly those energies E for which the corresponding orbit $((x_E(n), y_E(n), z_E(n)))_{n\in\mathbb{N}}$ is bounded. We will employ methods from combinatorial group theory which were introduced by Roberts in [40]; see [36] for background and [17] for extensions of [40] and further applications.

Let us first recall several known results. We refer the reader to [40] (and references therein). Each F_n preserves the Fricke-Vogt invariant, defined by

$$I(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1. \tag{16}$$

The set $\mathcal{M}$ of all such polynomial mappings with complex coefficients,

$$\mathcal{M} = \{F \in \mathbb{C}[x, y, z]^3 : I(F(x, y, z)) = I(x, y, z)\}, \tag{17}$$

is a group and can be written as a semidirect product of two groups,

$$\mathcal{M} = \Sigma \otimes_s \mathcal{G}, \tag{18}$$

where

$$\Sigma = \{\sigma_0 = Id, \sigma_1, \sigma_2, \sigma_3\} \tag{19}$$

is the normal subgroup. The involutions σ_i are the pairwise sign changes, for example, $\sigma_1(x,y,z)=(x,-y,-z)$. The group $\mathcal{G}$ can be generated by the involution p and the infinite-order mapping u, where

$$p(x,y,z)=(y,x,z),\ u(x,y,z)=(z,y,2yz-x). \tag{20}$$

For F_n from (15), we find

$$F_n=pu^{a_{n+1}}. \tag{21}$$

As a group, $\mathcal{G}$ is isomorphic to the projective linear group $\mathrm{PGl}(2,\mathbb{Z})$. The subgroup $\mathcal{G}_{\mathrm{OP}}\sim\mathrm{PSl}(2,\mathbb{Z})$ of orientation-preserving elements can be written as

$$\mathcal{G}_{\mathrm{OP}}=\langle v\rangle\times\langle q\rangle, \tag{22}$$

where

$$v(x,y,z)=(y,x,2xy-z),\ q(x,y,z)=(y,z,x). \tag{23}$$

Using

$$u=vq,\ pu^kp=(vq^{-1})^k, \tag{24}$$

we get the sequence G_n of orientation-preserving elements, defined by

$$G_n=F_{2n}F_{2n-1}=(vq^{-1})^{a_{2n+1}}(vq)^{a_{2n}}. \tag{25}$$

Let

$$U_n=G_n\cdots G_1, \tag{26}$$

so that

$$U_n(x_E(1),y_E(1),z_E(1))=(x_E(2n+1),y_E(2n+1),z_E(2n+1)). \tag{27}$$

Thus the U_n–orbit corresponds to trace map iterates with odd index. We get the trace map iterates with even index in a similar way: Let

$$H_n=F_{2n+1}F_{2n}=(vq^{-1})^{a_{2n+2}}(vq)^{a_{2n+1}}. \tag{28}$$

With

$$V_n=H_n\cdots H_1, \tag{29}$$

so that

$$V_n(x_E(2),y_E(2),z_E(2))=(x_E(2n+2),y_E(2n+2),z_E(2n+2)). \tag{30}$$

The following proposition is our central result on trace map orbits. It shows that unboundedness of orbits is equivalent to entry in the *escape set*

$$\mathcal{E} = \{(x, y, z) \in \mathbb{R}^3 : |y| > 1, |z| > 1, |yz| > |x|\}. \tag{31}$$

Moreover, once an orbit enters $\mathcal{E}$, it remains there and diverges super-exponentially in every coordinate. This exhibits a dichotomy in trace map orbit behavior. A trace map orbit is either bounded or superexponentially diverging, and one has in fact explicit bounds on the norms of iterates when the orbit is bounded.

Proposition 3.1. *For each energy E, the following are equivalent:*

(i) $(U_n(x_E(1), y_E(1), z_E(1)))_{n\in\mathbb{N}}$ *is unbounded.*
(ii) *There exists $N \in \mathbb{N}$ such that $U_N(x_E(1), y_E(1), z_E(1))) \in \mathcal{E}$.*
(iii) *There exists $N \in \mathbb{N}$ such that $U_n(x_E(1), y_E(1), z_E(1))) \in \mathcal{E}$ for each $n \geq N$.*
(iv) $(U_n(x_E(1), y_E(1), z_E(1))))_{n\in\mathbb{N}}$ *is super-exponentially diverging in every coordinate.*

and if (i)–(iv) *do not hold, then $(U_n(x_E(1), y_E(1), z_E(1)))_{n\in\mathbb{N}}$ is bounded by $C(E)$, where the constant $C(E)$ depends continuously on E.*

Proof. It follows from (25) and (26) that U_n can be written as a product of blocks of type $vq^{\pm 1}$. The strategy proposed by Roberts is to study the evolution of the initial vector $(x_E(1), y_E(1), z_E(1))$ under these elementary blocks, that is, to enlarge the orbit in question. This enlarged orbit will be denoted by $(\xi_m)_{m\in\mathbb{N}}$, where ξ_m is the vector obtained after applying the first m blocks of type $vq^{\pm 1}$ to $(x_E(1), y_E(1), z_E(1))$. It was shown in [40] that if $\xi_M \in \mathcal{E}$ for some $M \in \mathbb{N}$, then $\xi_m \in \mathcal{E}$ for every $m \geq M$ and ξ_m diverges super-exponentially in every coordinate. This readily gives $(ii) \Rightarrow (iii) \Rightarrow (iv)$. Of course, $(iv) \Rightarrow (i)$ is obvious.

To show that $(i) \Rightarrow (ii)$, we need some preparation. Let $b_i \in \{vq, vq^{-1}\}$, $i \in \mathbb{N}$, be chosen such that $\ldots G_3G_2G_1 = \ldots b_5b_4b_3b_2b_1$ and define $\xi_m = b_m \ldots b_1(x, y, z)$. Since q, q^{-1}, v preserve the invariant $I = I(x_E(1), y_E(1), z_E(1))$, we have $I(\xi_m) = I$ for every $m \in \mathbb{N}$. It is easily seen (cf. [40]) that, if

$$\|\xi_m\|_2^2 > 2 + (1 + \sqrt{I})^2, \tag{32}$$

ξ_m has at least two coordinates whose modulus is greater that one, where $\|\cdot\|_2$ denotes the norm $\|(x, y, z)\|_2^2 = |x|^2 + |y|^2 + |z|^2$. Thus every $\xi_m = (\xi_m^1, \xi_m^2, \xi_m^3)$ obeying (32) has to be of one of the following forms,

$$\begin{array}{llll} \text{(I)} & |\xi_m^1| > 1, & |\xi_m^2| > 1, & |\xi_m^3| > 1, \\ \text{(II)} & |\xi_m^1| \leq 1, & |\xi_m^2| > 1, & |\xi_m^3| > 1, \\ \text{(III)} & |\xi_m^1| > 1, & |\xi_m^2| \leq 1, & |\xi_m^3| > 1, \\ \text{(IV)} & |\xi_m^1| > 1, & |\xi_m^2| > 1, & |\xi_m^3| \leq 1. \end{array} \tag{33}$$

If a ξ_m satisfies (32), we say it is of type I, II, III, or IV if it satisfies the respective condition in (33).

Let us now describe a scenario sufficient for entry in $\mathcal{E}$ (cf. [17, 40] for similar reasoning). Suppose ξ_{m_0} obeys (32) and

$$\|\xi_{m_0+1}\| = \|vq^\varepsilon \xi_{m_0}\| > \|\xi_{m_0}\|, \tag{34}$$

where $\varepsilon \in \{1, -1\}$. We consider the case $\varepsilon = 1$, the other case is similar. If ξ_{m_0} is of type I, II, or IV, then ξ_{m_0+1} is in $\mathcal{E}$. This follows by simple direct calculations considering the cases $\xi_{m_0} \in \mathcal{E}$ and $\xi_{m_0} \notin \mathcal{E}$ (cf. Proposition 3.4 in [40] as well). If ξ_{m_0} is of type III, then we proceed as follows (cf. [17]). The iterate faces $vq^{-1}(vq)^j$ for some $j \geq 1$. If ξ_{m_0+j} obeys (32), then it must be one of the types I–IV. It is easy to see that $qv(\xi)$ fixes the second component of ξ. Thus, ξ_{m_0+j} must be of type III as well. Using this, it is straightforward to see that ξ_{m_0+j+1} belongs to $\mathcal{E}$. Thus, we infer that indeed the scenario

(S) $\|\xi_{m_0+1}\| = \|vq^\varepsilon \xi_{m_0}\| > \|\xi_{m_0}\|$ and $\|\xi_{m_0}\|, \|\xi_{m_0+j}\| \geq (2+(1+\sqrt{I})^2)^{1/2}$

forces entry in $\mathcal{E}$.

Our next step is to investigate how growing of U_n (or V_n) can force (S). We only consider U. Consider the situation

$$\|U_{n+1}(x_E(1), y_E(1), z_E(1))\| > \|U_n(x_E(1), y_E(1), z_E(1))\|. \tag{35}$$

Thus we have at least one situation of the form (34) between the two iterates and among these, we consider the one with largest m_0 for definiteness. To make sure that ξ_{m_0} obeys (32) it suffices to assume that $U_{n+1}(x_E(1), y_E(1), z_E(1)) = (x_E(2n+3), y_E(2n+3), z_E(2n+3))$ has norm sufficiently large. More precisely, let R be given with

$$\|vq(x)\|, \|vq^{-1}(x)\| > R$$

whenever $\|x\| \leq (2+(1+\sqrt{I})^2)^{1/2}$. Then (35) together with $\|U_{n+1}\| \geq R$ forces that ξ_{m_0} obeys (32). To make sure that ξ_{m_0+j} obeys (32), note that

$$\xi_{m_0+j} = p^{-1} F_{2n+1}(U_n(x_E(1), y_E(1), z_E(1))) = p^{-1} V_n(x_E(2), y_E(2), z_E(2))$$

if $\xi_{m_0+j} = (vq)^j \xi_{m_0}$ and

$$\xi_{m_0+j} = U_{n+1}((x_E(1), y_E(1), z_E(1))$$

if $\xi_{m_0+j} = (vq^{-1})^j \xi_{m_0}$ respectively. Thus, as p is an isometry, ξ_{m_0+j} obeys (32) if $V_n(x_E(2), y_E(2), z_E(2))$ and $U_{n+1}((x_E(1), y_E(1), z_E(1))$ have norm larger than $(2+(1+\sqrt{I})^2)^{1/2}$.

Let us summarize these considerations as follows: (35) together with sufficient largeness of $\|U_{n+1}(x_E(1), y_E(1), z_E(1))\|$ and $\|V_n(x_E(2), y_E(2), z_E(2))\|$ force (S) and thus entry in the ecape set. A completely analoguous arguments applies after interchanging V and U.

To conclude the proof, we show now that if (ii) fails, then (i) fails with explicit upper bound on the orbit. This will prove both $(i) \Longrightarrow (ii)$ and the last statement of the Proposition.

Thus, assume that (ii) fails for some energy E and write U_m for $U_m(x_E(1), y_E(1), z_E(1)) = (x_E(2m+1), y_E(2m+1), z_E(2m+1))$ and V_m for $V_m(x_E(2), y_E(2), z_E(2)) = (x_E(2m+2), y_E(2m+2), z_E(2m+2))$. By definition of U_n, V_n and the invariance of I, we have

$$\|U_{n+1}\| \text{ large implies } \|V_n\| \text{ or } \|V_{n+1}\| \text{ is larger than } R(2+(1+\sqrt{I})^2)^{1/2}. \tag{36}$$

(A similar argument is used in the proof of Corollary 3.6.) Now, we have for every n where (35) occurs with $\|U_{n+1}\|$ large enough for (36) to hold the following situation: If $\|V_n\|$ is large, then we have a scenario sufficient for entry in $\mathcal{E}$ as explained above, which is a contradiction to failure of (ii). If $\|V_{n+1}\|$ is large (but $\|V_n\|$ is not), then we consider the extended orbit between V_n and V_{n+1} and we are again in a scenario sufficient for entry in $\mathcal{E}$ because ξ_{m_0} can be chosen such that the role of the intermediate vector ξ_{m_0+j} is played by $p^{-1}U_{n+1}$ whose norm is large by assumption, again a contradiction. This shows that if (ii) fails, we get an explicit upper bound for $\|U_{n+1}\|$ in all situations where we have (35). In particular, (i) fails and we have an upper bound on the U_n–orbit in this case. Note that the upper bound we get is an explicit function of $I = I(x_E(1), y_E(1), z_E(1))$ and $\|(x_E(1), y_E(1), z_E(1))\|$ and hence is continuous in E. $\square$

Define the stable set

$$\mathcal{B} = \left\{ E \in \mathbb{R} : \sup_{n \in \mathbb{N}} \|(U_n(x_E(1), y_E(1), z_E(1))\| \le C(E) \right\}. \tag{37}$$

Proposition 3.2. *$\Sigma \subseteq \mathcal{B}$.*

Proof. Recall from (4) that Σ is equal to $\sigma(H_\omega)$ for every $\omega \in \Omega$. It follows from a standard strong approximation procedure that

$$\bigcup_{k \in \mathbb{N}} \mathrm{Int} \left(\bigcap_{m \ge k} \{E \in \mathbb{R} : |y_E(2m+1)| > 1\} \right) \subseteq \Sigma^c. \tag{38}$$

In fact, the set $\{E \in \mathbb{R} : |y_E(n)| > 1\}$ is just $\mathbb{R} \setminus \sigma(H_n)$ where H_n is a $|s'_n|$-periodic operator whose potential values over one period are given by $f(s'_n)$. Clearly, any operator H_ω is a strong limit of such operators, so we can apply Theorem VIII. 24 of [39] to get (38).

Using Proposition 3.1 and (38), we get the following chain of inclusions:

$$\begin{aligned}
\mathcal{B}^c &= \bigcup_{n \in \mathbb{N}} \{E : U_n(x_E(1), y_E(1), z_E(1)) \in \mathcal{E}\} \\
&= \bigcup_{n \in \mathbb{N}} \mathrm{Int}\left(\{E : U_n(x_E(1), y_E(1), z_E(1)) \in \mathcal{E}\}\right) \\
&= \bigcup_{n \in \mathbb{N}} \mathrm{Int} \left(\bigcap_{m \ge n} \{E : U_m(x_E(1), y_E(1), z_E(1)) \in \mathcal{E}\} \right) \\
&\subseteq \bigcup_{n \in \mathbb{N}} \mathrm{Int} \left(\bigcap_{m \ge n} \{E : |y_E(2m+1)| > 1\} \right) \\
&\subseteq \Sigma^c
\end{aligned}$$

$\square$

We now relate the stable set $\mathcal{B}$ to the set A of energies E for which the Lyapunov exponent $\gamma(E)$ vanishes. Let us briefly recall the definition of $\gamma(E)$. If $\Omega = \Omega_u$ is a quasi-Sturmian hull with potentials V_ω, $\omega \in \Omega$, then we define transfer matrices

$$M_{\omega,E}(n) = \begin{pmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} E - V_\omega(1) & -1 \\ 1 & 0 \end{pmatrix}.$$

If μ denotes the unique ergodic measure on Ω, it follows from the subadditive ergodic theorem that for every $E \in \mathbb{C}$, there exists a nonnegative number $\gamma(E)$ such that for almost every $\omega \in \Omega$ with respect to μ,

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \ln \|M_{\omega,E}(n)\|. \tag{39}$$

Let

$$A = \{E \in \mathbb{R} : \gamma(E) = 0\}.$$

The following proposition shows that a bounded trace map orbit implies vanishing Lyapunov exponent. The proof is modelled after [17] and is considerably shorter than the proof given in [4] for the Sturmian case (cf. [21] for related material).

Proposition 3.3. *$\mathcal{B} \subseteq A$.*

Proof. Assume there exists $E \in \mathcal{B}$ such that $\gamma(E) > 0$. Pick some $\omega \in \Omega$ for which the limit in (39) exists. By Osceledec's theorem [12] there exists a solution ϕ_+ of $H_\omega \phi = E\phi$ such that $\|\Phi_+(m)\|$ decays exponentially at $+\infty$ at the rate $\gamma(E)$, where $\Phi_+(m) = (\phi_+(m+1), \phi_+(m))^T$. Now, it follows from (5) and the fact that the words $s_n s_n$ occur infinitely often in u_{St} [5, 19] that the words $s'_n s'_n$, $n \in \mathbb{N}$, occur infinitely often in u and hence in ω. Since $E \in \mathcal{B}$ there is a constant $C \geq 1$ such that, for every $n \in \mathbb{N}$, we have

$$|\mathrm{tr}(M_E(n))| \leq C. \tag{40}$$

Pick m_0 such that, for every $m \geq m_0$ and every $k \in \mathbb{N}$, the solution ϕ_+ obeys

$$\|\Phi_+(m+k)\| \leq \exp(-\tfrac{1}{2}\gamma(E)k)\|\Phi_+(m)\|. \tag{41}$$

Choose n such that $\exp(-\frac{1}{2}\gamma(E)|s'_n|) < \frac{1}{2C}$. Look for an occurrence of $s'_n s'_n$ in ω, that is, $s'_n s'_n = \omega(l+1)\ldots\omega(l+2|s'_n|)$, such that $l \geq m_0$. It follows from the Cayley-Hamilton theorem that

$$\Phi_+(l+2|s'_n|) - \mathrm{tr}(M_E(n))\Phi_+(l+|s'_n|) + \Phi_+(l) = 0, \tag{42}$$

which in turn implies by (40)

$$\max(\|\Phi_+(l+|s'_n|)\|, \|\Phi_+(l+2|s'_n|)\|) \geq \frac{1}{2C}\|\Phi_+(l)\|, \tag{43}$$

contradicting (41). □

Proposition 3.4. *$A \subseteq \Sigma$.*

Proof. This is well known [8]. □

We collect the results of Propositions 3.2 through 3.4 in the following corollary which provides an extension of the main theorem of [4] to the quasi-Sturmian case.

Corollary 3.5. *$\Sigma = \mathcal{B} = A$.*

Since the spectrum is compact and the bounds on the trace map orbit for energies E in the spectrum depend continuously on E, we can find a global bound for these orbits. This observation will be important in Section 6, so we state the following corollary.

Corollary 3.6. *There is a constant $C = C(u, f) < \infty$ such that for every $E \in \Sigma$ and every $n \in \mathbb{N}$, we have*

$$\max\{\|\mathrm{tr}(M_E(n))\|, \|\mathrm{tr}(M_E(n)M_E(n-1))\|\} \le C.$$

Proof. As explained above, Corollary 3.5 and compactness of Σ provide uniform bounds on

$$\|U_n(x_E(1), y_E(1), z_E(1))\| = \|(x_E(2n+1), y_E(2n+1), z_E(2n+1))\|$$

for $E \in \Sigma$, $n \in \mathbb{N}$, that is, uniform bounds on

$$\|(x_E(n), y_E(n), z_E(n))\|$$

for every odd n and every $E \in \Sigma$. By $y_E(n) = x_E(n+1)$ this gives uniform bounds on $\|x_E(n)\|$ and $\|y_E(n)\|$ for *every* n and we can then use the invariant I to derive a uniform bound for $\|z_E(n)\|$, $E \in \Sigma$ and every $n \in \mathbb{N}$. □

4. Zero-Measure Spectrum and Absence of Absolutely Continuous Spectrum

Zero-measure spectrum and absence of absolutely continuous spectrum follows immediately from Corollary 3.5 and Kotani [31]; just as in the Sturmian case [4] (see also [43]). Namely, Kotani has shown the following: Given an ergodic family of discrete one-dimensional Schrödinger operators with aperiodic potentials taking finitely many values, we always have

$$|A| = 0, \tag{44}$$

where A is the set of energies for which the Lyapunov exponent vanishes and $|\cdot|$ denotes Lebesgue measure. These assumptions hold in our present context, so combining this equality with Corollary 3.5, we get that the spectrum has zero Lebesgue measure. Moreover, since a set of zero Lebesgue measure cannot support absolutely continuous spectrum, we also get absence of absolutely continuous spectrum for any quasi-Sturmian potential. More generally, uniform absence of absolutely continuous spectrum can also be proved in the following way: It follows from (44) that almost every operator in the ergodic family has empty absolutely continuous spectrum and since the hull is minimal in the quasi-Sturmian case this extends to *all* elements of the family by a result of Last and Simon [33]. We can therefore state the following:

Proposition 4.1. *Let u be quasi-Sturmian and let f be one-to-one. Then for every $\omega \in \Omega$, the operator H_ω has empty absolutely continuous spectrum and its spectrum as a set has zero Lebesgue measure.*

5. Absence of Point Spectrum

In this section we prove absence of eigenvalues for all operators with quasi-Sturmian potentials. Sections 4 and 5 thus imply Theorem 1. Proposition 2.2 provides us with an excellent tool to carry over the approach of [19] (cf. [35] as well) and extend the result of [18, 19] to the quasi-Sturmian case.

We will prove the following result which, together with Proposition 4.1, implies Theorem 1.

Proposition 5.1. *Let u be quasi-Sturmian and let f be one-to-one. Then for every $\omega \in \Omega$, the operator H_ω has empty point spectrum.*

To prove Proposition 5.1 we pursue a similar strategy as in [18, 19]. The criterion for excluding eigenvalues is Gordon's two-block method [16, 24] which can be phrased as follows [19].

Lemma 5.2. *Let V be a two-sided sequence such that for some $m \in \mathbb{Z}$, V has infinitely many squares $w_n w_n$ starting at m. If, for some energy $E \in \mathbb{R}$, the traces of the transfer matrices over the blocks w_n are bounded, E is not an eigenvalue of the discrete Schrödinger operator*

$$(H\phi)(n) = \phi(n+1) + \phi(n-1) + V(n)\phi(n).$$

We will show that sufficiently many squares can be found for every ω in a quasi-Sturmian family:

Lemma 5.3. *Let u be quasi-Sturmian. Then for every $\omega \in \Omega$, there is a site m such that the sequence ω has infinitely many squares conjugate to some s'_n or some $s'_n s'_{n-1}$ starting at m.*

Proof. Let $u \in \mathcal{A}^{\mathbb{N}}$ be a quasi-Sturmian sequence and let $\omega \in \Omega$. Since ω is necessarily recurrent, its restriction $\omega|_{\mathbb{N}}$ to $\mathbb{N}$ has the form (5) for some Sturmian $u_{\mathrm{St}} \in \{a,b\}^{\mathbb{N}}$, some aperiodic substitution $S : \{a,b\} \to \mathcal{A}$, and some finite word $w \in \mathcal{A}^*$; see the appendix and Proposition 7.1 in particular for details. Let $m = |w|$. It was shown in [18] that u_{St} starts with infinitely many squares conjugate to s_n or $s_n s_{n-1}$. Thus the lemma follows immediately from (5) and (8). □

Proof of Proposition 5.1. Using Corollary 3.6 and Lemma 5.3, we see that Lemma 5.2 applies to V_ω for every $\omega \in \Omega$ and every $E \in \Sigma = \sigma(H_\omega)$. Thus for every $\omega \in \Omega$, the operator H_ω has empty point spectrum. □

We end this section with a brief discussion of this result. For operators (1) with potential V taking finitely many values, the standard approaches to an "absence of eigenvalues"–type result employ either powers or palindromes. We have shown in this section that a Gordon criterion, which is based on powers (the occurrence of infinitely many squares), is applicable to quasi-Sturmian models and it gives uniform (i.e., for every $\omega \in \Omega$) absence of eigenvalues. In the Sturmian case, the palindrome method is also applicable, as shown by Hof et al. in [26], but is gives a weaker result: Absence of eigenvalues can be established for a dense G_δ set of ω's in Ω. For some quasi-Sturmian models, however, the palindrome method does not apply at all! Namely, if we take an arbitrary Sturmian sequence $u_{\mathrm{St}} \in \{a,b\}^{\mathbb{N}}$ and apply the aperiodic substitution $S(a) = 011001$, $S(b) = 001011$, we obtain a quasi-Sturmian sequence u which is not palindromic (see [1]) in the sense of [26], so the method of Hof et al. does not apply!

6. α-Continuity

In this section we prove Theorem 2. Corollary 3.6 provides us with the crucial tool, namely, uniformly bounded trace map orbits for energies from the spectrum. It has been demonstrated in [18] that α-continuity of a whole-line operator can be shown by proving power-law upper and lower bounds for local ℓ^2-norms of generalized eigenfunction on a half line.

The lower bound can be established by a technique similar to the one leading to absence of eigenvalues in Section 5. Thus, given boundedness of trace map orbits, it can be proved in the same way as in the Sturmian case. The proof of the upper bound extends the works [20, 27, 34]. This requires some extra care as the underlying shifts are not symmetric under reflection. To overcome this difficulty we use the results discussed at the end of Section 2. For further information and related material we refer the reader to [18, 20, 27, 34].

Let us first recall a criterion for α-continuity of a whole-line operator from [18]. It complements the half-line results of Jitomirskaya and Last which were established in [28]. Given an operator H as in (1), we consider the solutions of $H\phi = E\phi$, that is, solutions to the difference equation

$$\phi(n+1) + \phi(n-1) + V(n)\phi(n) = E\phi(n). \tag{45}$$

We define their local ℓ^2-norm by

$$\|\phi\|_L^2 = \sum_{n=0}^{\lfloor L \rfloor} \big|\phi(n)\big|^2 \; + \; (L - \lfloor L \rfloor)\big|\phi(\lfloor L \rfloor + 1)\big|^2. \tag{46}$$

We call a solution ϕ of (45) normalized if $|\phi(0)|^2 + |\phi(1)|^2 = 1$. The following result was proved in [18].

Proposition 6.1. *Let Σ be a bounded set. Suppose there are constants γ_1, γ_2 such that for each $E \in \Sigma$, every normalized solution of* (45) *obeys the estimate*

$$C_1(E)L^{\gamma_1} \le \|u\|_L \le C_2(E)L^{\gamma_2} \tag{47}$$

for $L > 0$ sufficiently large and suitable constants $C_1(E), C_2(E)$. Let $\alpha = 2\gamma_1/(\gamma_1 + \gamma_2)$. Then H has purely α-continuous spectrum on Σ, that is, for any $\phi \in \ell^2$, the spectral measure for the pair (H, ϕ) is purely α-continuous on Σ. Moreover, if the constants $C_1(E), C_2(E)$ can be chosen independently of $E \in \Sigma$, then for any $\phi \in \ell^2$ of compact support, the spectral measure for the pair (H, ϕ) is uniformly α-Hölder continuous on Σ.

We want to show that the bounds (47) can be established for every potential V_ω associated with some quasi-Sturmian sequence u provided that the rotation number has bounded density. In fact, the constants C_i, γ_i can be chosen uniformly in $\omega \in \Omega$ and $E \in \Sigma$, so we get that for any $\omega \in \Omega$ and any $\phi \in \ell^2$ of compact support, the spectral measure for the pair (H_ω, ϕ) is uniformly α-Hölder continuous.

We proceed similarly to the proof of Lemma 5.3. Namely, we consider an arbitrary element ω of the hull Ω and the associated operator H_ω. Since we are interested in the behavior of the solutions of

$$\phi(n+1) + \phi(n-1) + V_\omega(n)\phi(n) = E\phi(n) \tag{48}$$

on the right half-line, we consider the restriction of ω to the right half-line and its representation as in (5). Using this representation and properties of the associated Sturmian sequence (whose rotation number is θ_c and has bounded density; compare Proposition 7.1), along with the trace map bounds from Corollary 3.6, we can obtain the following two propositions.

Proposition 6.2. *Let θ_c be such that for some $B < \infty$, the denominators q_n of the associated rational approximants obey $q_n \le B^n$ for every $n \in \mathbb{N}$. Then for every injective f, there exist $0 < \gamma_1, C_1 < \infty$ such that for every $E \in \Sigma$ and every $\omega \in \Omega$, every normalized solution ϕ of* (48) *obeys*

$$\|\phi\|_L \ge C_1 L^{\gamma_1} \tag{49}$$

for L sufficiently large.

Remark. The set of θ_c's obeying the assumption of Proposition 6.2 has full Lebesgue measure [30] and clearly contains the set of bounded density numbers.

Proof. Using Corollary 3.6 and Proposition 7.1 this can be shown in the same way as in the Sturmian case. Details can be found in [18] (cf. [13] as well). □

Proposition 6.3. *Let θ_c be a bounded density number. Then for every injective f, there exist $0 < \gamma_2, C_2 < \infty$ such that for every $E \in \Sigma$ and every $\omega \in \Omega$, every normalized solution ϕ of* (48) *obeys*

$$\|\phi\|_L \le C_2 L^{\gamma_2} \tag{50}$$

for all L.

Proof. The proof is similar to the proof of the corresponding result in the Sturmian case (cf. [18, 20, 34]). However, it requires some additional effort as the quasi-Sturmian systems are not reflection invariant. To treat this case as well we will need the results (and the notation) of the discussion at the end of Section 2.

Recall that $\Omega = \Omega(\theta, S)$. Mimicking the argument of [27], which only relies on trace map bounds and recursions, one can easily infer that there exists C_1 and γ_1 with

$$\|M(S(y), E)\| \le C_1 |y|^{\gamma_1}$$

for every prefix y of c_θ and every $E \in \Sigma(\Omega(\theta, S))$. Now, every prefix of $S(c_\theta)$ can be written as $x = S(y)w$ with a prefix y of c_θ and a word w of length $|w| \le \max\{|S(0)|, |S(1)|\}$. Combining these estimates, we see that there exist C and γ with

$$\|M(x, E)\| \le C|x|^\gamma \quad \text{for every prefix } x \text{ of } S(c_\theta) \text{ and every } E \in \Sigma(\Omega(\theta, S)) \ . \tag{51}$$

By, Corollary 2.4, we have $\Sigma(\Omega(\theta, S)) = \Sigma(\Omega(\theta, S^R))$. Thus, replacing S by S^R but keeping the *same* set of energies, we can again apply the above reasoning and infer

$$\|M(x, E)\| \le C|x|^\gamma \quad \text{for every prefix } x \text{ of } S^R(c_\theta) \text{ and every } E \in \Sigma(\Omega(\theta, S)). \tag{52}$$

Now, the proof can be finished as follows. By results of [20, 34], every factor z of c_θ can be written as $z = xy$, where x is a suffix of a suitable s_n and y is a prefix of s_{n+1}. This implies that every factor z of a sequence in $\Omega(\theta, S)$ can be written as $z = xy$ with x a suffix of $S(s_n)$ and y a prefix of $S(s_{n+1})$ for a suitable n. Assume w.l.o.g. $n \ge 3$. (The case $n = 1, 2$ can be treated directly.) By equation (9), $s_n = \pi_n ab$ with $a, b \in \{0, 1\}$ and $a \ne b$.

We will assume that $|x| > |S(a)S(b)|$. The other case is similar (and, in fact, even simpler). Then x can be written as $x = \widetilde{x}S(a)S(b)$, where $\widetilde{x}$ is a suffix of $S(\pi_n)$. Now, we can estimate

$$\|M(z,E)\| \le \|M(S(a),E)\| \cdot \|M(S(b),E)\| \cdot \|M(\widetilde{x},E)\| \cdot \|M(y,E)\|.$$

We will provide suitable bounds for all these factors. The factor $\|M(S(a),E)\|\|M(S(b),E)\|$ is just a constant. The factor $\|M(y,E)\|$ can be estimated by equation (51) as y is a prefix of$S(s_{n+1})$. We find $\|M(y,E)\| \le C|y|^\gamma \le C|z|^\gamma$.

It remains to estimate the factor $\|M(\widetilde{x},E)\|$. As $\widetilde{x}$ is a suffix of $S(\pi_n)$, the word $\widetilde{x}^R$ is a prefix of $S(\pi_n)^R$. Since π_n is a palindrome, we have $S(\pi_n)^R = S^R(\pi_n)$. This means that $\widetilde{x}^R$ is a prefix of $S^R(c_\theta)$. By equation (52), we can thus estimate $\|M(\widetilde{x}^R,E)\|$. By general principles, however, we have $\|M(\widetilde{x},E)\| = \|M(\widetilde{x}^R,E)\|$ (cf. [20, 34]) and we see that $\|M(\widetilde{x},E)\|$ can be estimated by $\|M(\widetilde{x},E)\| \le C|\widetilde{x}|^\gamma \le C|z|^\gamma$.

Combining these estimates we arrive at the desired statement. □

Proof of Theorem 2. The claim follows from Propositions 6.2 and 6.3 along with Poposition 6.1. □

Acknowledgements. A substantial part of this work was done while one of the authors (D. L.) was visiting The Hebrew University, Jerusalem. He would like to thank Y. Last for hospitality and many useful discussions.

7. Appendix: Rotation numbers of quasi-Sturmian sequences

In this section we discuss Cassaigne's proof of Proposition 2.2 and its consequences for the rotation numbers associated with the elements in a quasi-Sturmian hull. We show in particular that given a quasi-Sturmian sequence u, every rotation number for u is a rotation number for every element of the associated subshift Ω and vice versa.

Let u be quasi-Sturmian, that is, u is recurrent and satisfies $p_u(n) = n + k$ for $n \ge n_0$ and let Ω be the induced subshift containing two-sided sequences whose factors are also factors of u. It is well known that u must be uniformly recurrent so that each factor of u is also a factor of every $\omega \in \Omega$ and hence all elements of Ω are factor-equivalent. We will show that the set of allowed rotation numbers is an invariant of the subshift.

To do so, let us briefly recall how the Sturmian sequence u_{St} in Proposition 2.2 is found. The Rauzy graphs $G(n)$, $n \in \mathbb{N}$ associated with u are defined as follows: For every $n \in \mathbb{N}$, the graph $G(n)$ contains $p_u(n)$ vertices corresponding to the factors of u having length n and it has $p_u(n+1)$ edges corresponding to the factors of u having length $n+1$. There is an edge from w_1 to w_2 if and only if $w_1 = ax$, $w_2 = xb$, $|a| = |b| = 1$ ($|x| = n-1$), and axb is a factor of u. Since u is quasi-Sturmian, for large enough n, the topology of $G(n)$ is the same as the topology of a Rauzy graph of a Sturmian sequence. Namely, $p_u(n+1) - p_u(n) = 1$ and there is a unique right-special factor (a vertex with out-degree 2) and a unique left-special factor (a vertex with in-degree 2). Similarly, one may define Rauzy graphs for every $\omega \in \Omega$. Observe that the family of Rauzy graphs is the same for u and every $\omega \in \Omega$. For certain n, it can be shown that the right-special factor and the left-special factor coincide (i.e., there exists a bispecial factor). Fix such an n and consider the graph

$G(n)$. It has a vertex with in-degree 2 and out-degree 2 and exactly two paths starting and ending at this vertex. Each vertex in these two paths (other than the vertex B corresponding to the bispecial factor) has in-degree 1 and out-degree 1 and one finds the Sturmian sequence u_{St} by running through u (or one of the $\omega \in \Omega$) and recording the sequence of choices as to which of the two paths is taken after every passage through the vertex B. The prefix w in Proposition 2.2 is just the word we have to read to get from the starting site to the first occurrence of the bispecial factor.

From the above remarks, we can deduce the following. Since u and all $\omega \in \Omega$ are factor-equivalent, the Sturmian sequences u_{St} that can be obtained must be factor-equivalent as well. This shows that the rotation numbers we can get this way must be valid for all of these sequences. Moreover, the set of rotation numbers is naturally labelled by the set of integers n for which there exists a bispecial factor. For definiteness, we may choose the rotation number corresponding to the shortest bispecial factor as the *canonical rotation number* associated with Ω.

We summarize our observations in the following proposition.

Proposition 7.1. *Let u be a quasi-Sturmian sequence. Then there is a canonical rotation number θ_c which is a rotation number for all the elements of the subshift Ω. By restricting the elements $\omega \in \Omega$ to the right half-line, we obtain one-sided infinite words that all have a representation of the form (5). In this representation, one can accomplish the following: The substitution S is independent of ω, while the Sturmian sequence $u_{\mathrm{St}} = u_{\mathrm{St}}(\omega)$ does depend on ω, however, its rotation number is θ_c and hence independent of ω. The prefixes $w = w(\omega)$ depend on ω and they are suffixes of some s'_n.*

References

[1] J.-P. Allouche, M. Baake, J. Cassaigne, and D. Damanik, Palindrome complexity, to appear in *Theoret. Comput. Sci.*

[2] J.-P. Allouche and D. Damanik, Applications of combinatorics on words to physics, in preparation

[3] J. Bellissard, A. Bovier, and J.-M. Ghez, Spectral properties of a tight binding Hamiltonian with period doubling potential, *Commun. Math. Phys.* **135** (1991), 379–399

[4] J. Bellissard, B. Iochum, E. Scoppola, and D. Testard, Spectral properties of one-dimensional quasi-crystals, *Commun. Math. Phys.* **125** (1989), 527–543

[5] J. Berstel, Recent results in Sturmian words, in *Developments in Language Theory*, Eds. J. Dassow and A. Salomaa, World Scientific, Singapore (1996), 13–24

[6] A. Bovier and J.-M. Ghez, Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions, *Commun. Math. Phys.* **158** (1993), 45–66; Erratum: *Commun. Math. Phys.* **166** (1994), 431–432

[7] R. Carmona, A. Klein, and F. Martinelli, Anderson localization for Bernoulli and other singular potentials, *Commun. Math. Phys.* **108** (1987), 41–66

[8] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Boston (1990)

[9] J. Cassaigne, Sequences with grouped factors, in *Developments in Language Theory III*, Aristotle University of Thessaloniki (1998), 211–222

[10] E. M. Coven, Sequences with minimal block growth, II, *Math. Systems Theory* **8** (1975), 376–382

[11] E. M. Coven and G. A. Hedlund, Sequences with minimal block growth, *Math. Systems Theory* **7** (1973), 138–153

[12] H. L. Cycon, R. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators, with Application to Quantum Mechanics and Global Geometry*, Springer, Berlin-Heidelberg-New York (1987)

[13] D. Damanik, α-continuity properties of one-dimensional quasicrystals, *Commun. Math. Phys.* **192** (1998), 169–182

[14] D. Damanik, Singular continuous spectrum for the period doubling Hamiltonian on a set of full measure, *Commun. Math. Phys.* **196** (1998), 477–483

[15] D. Damanik, Singular continuous spectrum for a class of substitution Hamiltonians, *Lett. Math. Phys.* **46** (1998), 303–311

[16] D. Damanik, Gordon-type arguments in the spectral theory of one-dimensional quasicrystals, in *Directions in Mathematical Quasicrystals*, Eds. M. Baake and R. V. Moody, CRM Monograph Series **13**, AMS, Providence, RI (2000), 277–305

[17] D. Damanik, Substitution Hamiltonians with bounded trace map orbits, *J. Math. Anal. Appl.* **249** (2000), 393–411

[18] D. Damanik, R. Killip, and D. Lenz, Uniform spectral properties of one-dimensional quasicrystals, III. α-continuity, *Commun. Math. Phys.* **212** (2000), 191–204

[19] D. Damanik and D. Lenz, Uniform spectral properties of one-dimensional quasicrystals, I. Absence of eigenvalues, *Commun. Math. Phys.* **207** (1999), 687–696

[20] D. Damanik and D. Lenz, Uniform spectral properties of one-dimensional quasicrystals, II. The Lyapunov exponent, *Lett. Math. Phys.* **50** (1999), 245–257

[21] D. Damanik and D. Lenz, Half-line eigenfunction estimates and singular continuous spectrum of Lebesgue measure zero, preprint

[22] D. Damanik and B. Solomyak, Some high-complexity Hamiltonians with purely singular continuous spectrum, preprint

[23] F. Delyon and D. Petritis, Absence of localization in a class of Schrödinger operators with quasiperiodic potential, *Commun. Math. Phys.* **103** (1986), 441–444

[24] A. Gordon, On the point spectrum of the one-dimensional Schrödinger operator, *Usp. Math. Nauk* **31** (1976), 257–258

[25] G. A. Hedlund and M. Morse, Symbolic dynamics, *Amer. J. Math.* **60** (1938), 815–866

[26] A. Hof, O. Knill and B. Simon, Singular continuous spectrum for palindromic Schrödinger operators, *Commun. Math. Phys.* **174** (1995), 149–159

[27] B. Iochum, L. Raymond, and D. Testard, Resistance of one-dimensional quasicrystals, *Physica A* **187** (1992), 353–368

[28] S. Jitomirskaya and Y. Last, Power-law subordinacy and singular spectra. I. Half-line operators, *Acta Math.* **183** (1999), 171–189

[29] M. Kaminaga, Absence of point spectrum for a class of discrete Schrödinger operators with quasiperiodic potential, *Forum Math.* **8** (1996), 63–69

[30] A. Ya. Khinchin, *Continued Fractions*, Dover Publications, Mineola (1997)

[31] S. Kotani, Jacobi matrices with random potentials taking finitely many values, *Rev. Math. Phys.* **1** (1989), 129–133

[32] Y. Last, Quantum dynamics and decompositions of singular continuous spectra, *J. Funct. Anal.* **142** (1996), 406–445

[33] Y. Last and B. Simon, Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators, *Invent. Math.* **135** (1999), 329–367

[34] D. Lenz, Aperiodische Ordnung und gleichmässige spektrale Eigenschaften von Quasikristallen, Dissertation, Frankfurt am Main, Logos (2000)

[35] D. Lenz, Hierarchical structures in Sturmian dynamical systems, preprint

[36] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial Group Theory*, Dover, New York (1976)

[37] M. E. Paul, Minimal symbolic flows having minimal block growth, *Math. Systems Theory* **8** (1975), 309–315

[38] M. Queffélec, *Substitution Dynamical Systems - Spectral Analysis*, Lecture Notes in Mathematics, Vol. 1284, Springer, Berlin, Heidelberg, New York (1987)

[39] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis*, Academic Press, New York (1972)

[40] J. A. G. Roberts, Escaping orbits in trace maps, *Physica A* **228** (1996), 295–325

[41] D. Shechtman, I. Blech, D. Gratias, and J. V. Cahn, Metallic phase with long-range orientational order and no translational symmetry, *Phys. Rev. Lett.* **53** (1984), 1951–1953

[42] A. Sütő, The spectrum of a quasiperiodic Schrödinger operator, *Commun. Math. Phys.* **111** (1987), 409–415

[43] A. Sütő, Singular continuous spectrum on a Cantor set of zero Lebesgue measure for the Fibonacci Hamiltonian, *J. Stat. Phys.* **56** (1989), 525–531