Resonant tori and instabilities in Hamiltonian systems

Anna Litvak-Hinenzon* and Vered Rom-Kedar

The Faculty of Mathematics and Computer Sciences

The Weizmann Institute of Science, POB 26, Rehovot 76100, Israel

litvak@maths.warwick.ac.uk, vered@wisdom.weizmann.ac.il

Abstract

The existence of lower dimensional resonant tori of *parabolic*, hyperbolic and elliptic normal stability types is proved to be generic and persistent in a class of n degrees of freedom (d.o.f.) integrable Hamiltonian systems with $n \geq 3$. In particular, in such systems the existence of normally elliptic or hyperbolic $n-1$ dimensional torus of fixed points is persistent without the use of any external parameters, and the existence of an $n-1$ dimensional normally parabolic torus of fixed points is of co-dimension one. *Parabolic resonance* (respectively, hyperbolic or elliptic resonance) is created when a small Hamiltonian perturbation is added to an integrable Hamiltonian system possessing a resonant torus of the corresponding normal stability. It is numerically demonstrated that parabolic resonances cause intricate behavior and large instabilities. The place and role of lower dimensional parabolic resonant tori in the Arnold web, and the related structure of the unperturbed energy surfaces, are discussed and illustrated using models of near integrable Hamiltonian systems with three, four and five d.o.f.. Critical n values for which the system first persistently possesses mechanisms for large instabilities of a certain type are found. Initial numerical studies of the rate and time of development of the most significant instabilities are presented.

*Current address: Mathematics Institute, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, UK

1 Introduction

The main focus of this paper are the instabilities of near integrable Hamiltonian systems which posses resonant parabolic lower dimensional tori. Numerically, these appear to present stronger instabilities than the instabilities induced by transition chains of hyperbolic tori (e.g. [1, 41]) or even the instabilities induced by hyperbolic resonances (e.g. [7, 17] and references therein).

Consider autonomous near integrable n d.o.f. Hamiltonian systems of the form: $H_0(p,q) + \varepsilon H_1(p,q;\varepsilon), (p,q) \in M \subseteq \mathbb{R}^n \times \mathbb{R}^n$, M is a $2n$-dimensional, smooth, symplectic manifold, $H_0, H_1 \in C^\infty(M)$, and H_1 is analytic in ε. H_0 represents the completely integrable part of the Hamiltonian (the unperturbed system) and its structure is described below. H_1, the Hamiltonian perturbation, is assumed to be *uniformly bounded*. Generally, important instabilities of the perturbed system occur near unperturbed orbits which are structurally stable among integrable systems[1] (we assume $H_0(p,q)$ is generic in this sense), yet are structurally unstable under Hamiltonian perturbations. This observation is the basis for the systematic study of near-integrable Hamiltonian flows. For $n = 1$ this statement is null as the perturbed system is integrable. For one and a half[2] d.o.f. Hamiltonian systems, homoclinic orbits serve as the canonical example to this theme, leading to the appearance of chaotic behavior in the perturbed system (see, e.g. [16, 43]). The homoclinic tangle mechanism of instability exists for all $n \geq 1.5$. More generally, given any instability mechanism which appears for n_0 d.o.f. systems, it will clearly remain an instability mechanism for all $n \geq n_0$. Furthermore, as n_0 increases the number of different types of such instability mechanisms (orbits which are structurally stable for integrable systems, yet are structurally unstable for the near-integrable case) clearly increases. Finding such mechanisms is fundamental for understanding the dynamics in higher dimensional systems. The current study presents a new type of such instability mechanism, the parabolic resonance, which exists for all $n \geq 3$. Much of the analysis presented here deals with studying the special orbits of the integrable system (sections 2, 3 and 5) and proving that they are structurally stable among integrable Hamiltonians. Then, we demonstrate numerically that in the perturbed system they induce large instabilities (section 4).

[1] or special classes of such systems, e.g. symmetric integrable Hamiltonians

[2] A one d.o.f. Hamiltonian to which a Hamiltonian time periodic perturbation is applied. One and a half d.o.f. Hamiltonian systems are a subset of two d.o.f. Hamiltonian systems.

Consider an integrable n d.o.f. Hamiltonian system, $H_0(p,q)$; $(p,q) \in M \subseteq \mathbb{R}^n \times \mathbb{R}^n$, with n independent integrals of motion: $H_0 = F_1, F_2, ..., F_n \in C^\infty(M)$, pair wise in involution: $\{F_i, F_j\} = 0$; $i,j = 1, ...n$, where $M \subseteq \mathbb{R}^{2n}$ is a differentiable symplectic manifold, with the standard symplectic structure, and $\dim M = 2n$. Assume that $n \geq 3$ and that the Hamiltonian level sets $M_g = \{(p,q) \in M, \quad F_i = g_i; \quad i = 1, ..., n\}$, are closed. By the Liouville-Arnold theorem (see [2, 20] and references therein), the connected compact components of the Hamiltonian level sets, M_g, on which all of the dF_i are (point wise) linearly independent, are diffeomorphic to n - tori and hence a transformation to action-angle coordinates ($H_0 = H_0(I)$) near such level sets is nonsingular. These maximal tori are the regular fibers of the ramified torus bundle defined by the integrable Hamiltonian. By the Liouville-Arnold Theorem, generically, the motion on these connected components of the level sets is either quasi-periodic or periodic (see for example [2, 17, 20] and references therein). Generically, on each energy surface the non-resonant tori are of measure one, and the resonant tori[3] are of measure zero, yet both are dense. In particular, resonant tori of any dimension $2 \leq d \leq n$ are dense in each iso-energetically non-degenerate energy surface.

Now, consider a neighborhood of a possibly singular level set M_g. By the Liouville-Arnold theorem, on each such connected and closed Hamiltonian level set (energy surface) there is some neighborhood D, in which the Hamiltonian $H_0(p,q)$ may be transformed to the form:

$$H_0(x,y,I), \qquad (x,y,\theta,I) \in U \subseteq \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{T}^{n-k} \times \mathbb{R}^{n-k} \tag{1}$$

which does not depend on the angles of the tori, θ. The symplectic structure of the new integrable Hamiltonian (1) is $\sum_{j=1}^{k} dx_j \wedge dy_j + \sum_{i=1}^{n-k} d\theta_i \wedge dI_i$, where (θ, I) are the action-angle variables ($k = 0$ corresponds to the usual n-tori discussed above). The motion on the $(n-k)$-tori is described by the equations: $\dot{I}_i = 0$, $\dot{\theta}_i = \omega_i(x,y,I)$. The geometrical structure of the new Hamiltonian, $H_0(x,y,I)$, is such that an $(n-k)$-torus is attached to every point of the (x,y) plane (space, for $k > 1$). The (x,y) plane (space) is called the *normal plane* (*space*) [2, 36] of the $(n-k)$-tori, and defines their stability type. Invariant lower dimensional tori, of dimension $(n-k)$, generically exist for each $1 \leq k \leq n-1$, and are either families of tori

[3] A resonant torus is a torus with rationally dependent frequencies, i.e., the frequencies of a resonant torus satisfy the condition: $\langle \omega(I), k \rangle = 0$, for at least one nonzero vector of integers, k.

belonging to a higher dimensional resonant torus, or are isolated invariant tori. If an $(n-k)$-torus is attached to a fixed point of its k-dimensional normal space, then it is an invariant torus, on which each of the first k differentials of constants of motion, $dH_0 = dF_1, dF_2, ..., dF_k$, is linearly dependent on the remaining $n-k$ differentials of the integrals $I_1, ..., I_{n-k}$ (where all the remaining $n-k$ differentials, $dI_1, ..., dI_{n-k}$, are independent), and conversely. Such tori correspond to the singular leaves of the ramified n - tori bundle defined by the integrable Hamiltonian.

The invariant $(n-k)$-tori have an $(n-k)$-dimensional vector of inner frequencies, $\dot{\theta} = \omega(I)$ (the vector ω depends only on the I's when the torus is invariant). The *normal frequencies*[4], Ω, of the $(n-k)$-tori are defined as the imaginary parts of the purely imaginary characteristic eigenvalues (referred as Floquet multipliers for the corresponding Poincaré map) of the linearization of the system at the torus [2, 19, 36]. The latter define the normal stability type of the $(n-k)$-tori, which generically are either normally hyperbolic[5] or normally elliptic[6]. If the torus has at least one zero characteristic eigenvalue (and hence a pair of zero eigenvalues, as the system is Hamiltonian), then one of its normal frequencies vanishes, and it is said to be *normally parabolic*. For more details see [2, 10, 11, 12, 13, 17, 19, 15, 16, 20, 36] and references therein.

The survival of the confined and ordered motion along the invariant tori under small Hamiltonian perturbation is the subject of a large body of work. The Kolmogorov-Arnold-Moser (KAM) theory [2] states that in the non-degenerate case the sufficiently non-resonant[7] maximal ($k=0$) dimensional n-tori persists for sufficiently small perturbation. Following the KAM results, it has been proved (under certain Diophantine conditions which are specific for each one of the cases) that most (large Cantor sets of action values) non-resonant lower dimensional tori of hyperbolic [15], elliptic [10, 36] and parabolic [19] normal stability persist under small Hamiltonian perturbations, see [2, 17, 16, 43] for further references.

KAM tori are of co-dimension $n-1$ on each $(2n-1)$-dimensional en-

[4] In some references, these are called characteristic frequencies.

[5] If all the characteristic eigenvalues of a lower dimensional torus have a nonzero real part, it is said to be normally hyperbolic.

[6] If all the characteristic eigenvalues of a lower dimensional torus are purely imaginary (and do not vanish) it is said to be normally elliptic.

[7] Its frequencies, $\omega(I) = D_I(H_0(I)) = \dot{\theta}(I)$, are rationally independent, and satisfy certain Diophantine conditions.

ergy surface. It follows that in two d.o.f. systems, KAM tori divide iso-energetically non-degenerate energy surfaces into separated regions, hence their survival prevents large scale instabilities. For $n > 2$ this is no longer the case; on the contrary, for $n \geq 2.5$ families of resonant tori are connected, creating a resonance web (the so-called Arnold web) [2, 26].

Arnold described in [1] one possible universal scenario of instability which, by his description, leads to diffusion like motion *along* one resonance: For arbitrarily small perturbation, orbits follow the unstable manifold of an $n-1$ dimensional whiskered torus which intersects the stable manifold of a far away whiskered torus (this intersection is created by a transition chain of whiskers), leading to order one instability in the action variables. For generic analytic steep Hamiltonian systems of the form $H_0(I) + \varepsilon H_1(\theta, I; \varepsilon)$ Nekhoroshev's Theorem implies that if such orbits exist their phase-space measure and their velocities are at most exponentially small in ε, and that these orbits follow the transition chain of whiskers for exponentially long time. A proof for the existence of such orbits in systems of the form $H_0(I) + \varepsilon H_1(\theta, I; \varepsilon)$ has not been established yet.

It was later recognized that such kind of instability may not be observed in experiments or numerical simulations (see discussion in [17, 21]), and thus it cannot explain the observed instabilities in higher dimensional systems. It appears that instabilities which are the artifact of orbits starting in the vicinity of *junctions* of resonance surfaces of the integrable system are those which are frequently observed in applications. Such instabilities are often referred to as diffusion, or transport, *across* resonances. Heuristically, as in Arnold's work, one expects the behavior near a resonance junction to be similar to the behavior near a few (k) coupled pendulums which are coupled to a few $(n-k)$ oscillators (see more precise statement and derivation in [17] and references therein). Transport across resonances in such a priori unstable[8] systems may lead to observable fast instabilities in the action variables (e.g. [7, 17, 26, 30, 41, 42] and references therein). It has been suggested that variational methods may be of use in proving the existence of such instabilities, independent of the existence of a transition chain, see [3, 4, 5, 9, 44] and references therein.

The systems we study, of the form $H_0(x, y, I)$, for which lower dimensional hyperbolic tori exist for some range of action values, are a priori unstable in that range of actions. Hence, the mechanism for instabilities across res-

[8] An unperturbed system with some normally hyperbolic (or partially hyperbolic) tori.

onances is expected to play a role. For example, Treschev suggests in [41] that a priori unstable systems of the form $F(x,y)+H_0(I)+\varepsilon H_1(x,y,\theta,I;\varepsilon)$; $(x,y)\in R^2, I\in R^{n-1}$ (i.e., separable systems - systems in which the canonical variables, (x,y), may be separated from the action angle variables), which possess a figure eight like homoclinic loops to a hyperbolic saddle in their (x,y) plane for any value of the actions, exhibit instabilities in the action variables with averaged velocity of order $O(\frac{\varepsilon}{\log(\frac{1}{\varepsilon})})$.

While integrable separable systems possess generically either hyperbolic or elliptic lower dimensional tori, general *non separable* systems possess parabolic tori in a persistent way (see section 2). The lower dimensional tori undergo generic Hamiltonian bifurcations as the actions are varied: hyperbolic and elliptic tori coincide leading to the birth of a normally parabolic torus in a saddle-center bifurcation (or, in the symmetric case, a pitchfork bifurcation), see [19, 20, 31, 32, 33]. Furthermore, for sufficiently large n ($n\geq 3$), some of the action values for which the parabolic bifurcation is realized correspond to resonant tori. We call such tori parabolic resonant tori (more precise definitions are in section 2). It appears that under perturbations the unperturbed parabolic resonant tori give rise to larger instabilities than those induced by the co-existing homoclinic loops. For 2 d.o.f. systems it was proved that the appearance of such tori is a co-dimension one phenomena [37], hence that it is expected to appear in many applications, e.g. [35, 38, 39]. Here and in [28] we investigate the extension of the 2 d.o.f. results to higher dimensions, where unperturbed parabolic resonant tori appear persistently and induce various kinds of instabilities.

A main result here is that the *existence* of lower dimensional resonant parabolic tori (respectively, PRs) in a class of integrable (respectively, near integrable) $n\geq 3$ d.o.f. Hamiltonian systems is generic and persistent. In particular they are structurally stable among integrable systems. A certain set is called *generic* (or residual) if it contains a dense G_δ set[9]. A certain property of a dynamical system is said to be C^k- *generic*, for some $1\leq k\leq r\leq\infty$, if it occurs on a set of dynamical systems containing a C^r-dense C^k-G_δ (dense) set[10]. A property is said to be C^k-persistent if it occurs on a C^k-open set of a certain class, which means that this property persists under small C^r perturbations of this class on a C^r-open set of systems, for $1\leq k\leq r\leq\infty$. In this paper, when we say persistent (or generic),

[9] A set is called G_δ if it consists of a countable intersection of open sets.

[10] Note that C^k-open implies C^r-open, and C^r-dense implies C^k-dense; $1\leq k\leq r\leq\infty$.

we mean C^1-persistent (or generic). The basic tool for proving C^1-genericity is transversality. For an overview on these subjects with connection to dynamical systems we refer the reader to [16, 20] and references therein.

This paper is ordered as follows: In section 2 we prove persistence and genericity theorems for lower dimensional resonant tori. In section 3 we identify critical dimensions for which certain mechanisms for instabilities become persistent. In section 4 we illustrate these mechanisms of instability by introducing models of 4 and 5 d.o.f. near integrable Hamiltonian systems. In section 5 the structure of the energy surfaces of systems of the form (1), and of the corresponding resonance web, is studied. Finally, in section 6 conclusions and discussion are presented.

2 Genericity of lower dimensional resonant tori

Consider smooth n d.o.f. integrable Hamiltonian systems, with $n \geq 3$, which are already transformed to the form $H_0(x, y, I)$ (1) in some vicinity of one of their closed and connected level sets, M_g (as in the Liouville-Arnold Theorem). We define below the *class* $\mathcal{P}_k$ of such systems as systems in which there exists a lower dimensional torus which undergoes a change of stability as the actions are varied (namely truly coupled higher dimensional systems). Then, we prove that among such systems PR are generic.

Definition 1 *Let $H_0(x, y, I)$ be an n d.o.f. integrable Hamiltonian, which is obtained from some general smooth n d.o.f. integrable Hamiltonian by a non-singular transformation near some level set. The Hamiltonian is said to belong to the* class $\mathcal{P}_k$ *if $k \geq 1$ is the minimal integer for which the following conditions are satisfied:*

1. *$H_0(x, y, I)$ possesses at least one isolated equilibria in its $2k$-dimensional (x, y) normal space of the $(n-k)$ tori, parameterized by the actions I, where $x = (x_1, ..., x_k), y = (y_1, ..., y_k) \in \mathbb{R}^k$. Namely, at this equilibrium point p_f:*

$$\nabla_{(x,y)} H_0|_{p_f} = 0, \qquad p_f = (x_f, y_f, I_f). \tag{2}$$

2. *At the point p_f, $H_0(x, y, I)$ fulfills the following transversality condition, (3), for some $i = 1, ..., k$:*

$$\|\nabla_{(x_i,y_i)}\frac{\partial H_0}{\partial I_1}|_{p_f}\|+\|\nabla_{(x_i,y_i)}\frac{\partial H_0}{\partial I_2}|_{p_f}\|+\cdots+\|\nabla_{(x_i,y_i)}\frac{\partial H_0}{\partial I_{n-k}}|_{p_f}\|\neq 0, \quad (3)$$

where $\nabla_{(x_i,y_i)}$ denotes gradients with respect to the x_i and y_i variables only[11].

3. *There exists an $i = i_0$, $i_0 \in \{i = 1, ..., k\}$, for which the transversality condition (3) holds, and:*

$$\det\left(\frac{\partial^2 H_0}{\partial^2(x_{i_0}, y_{i_0})}\bigg|_{p_f}\right) = 0. \quad (4)$$

Indeed, systems from class $\mathcal{P}_k$ attain an invariant $(n-k)$-torus which is defined by a fixed point in its normal space (the (x, y) space), which by condition 3 undergoes a bifurcation. Generic integrable Hamiltonian systems which do not belong to this class, may possess isolated invariant tori of one stability type: Either elliptic or hyperbolic.

2.1 Maximal lower dimensional tori

The *maximal* lower dimensional tori are the $(n-1)$-tori of the integrable n d.o.f. Hamiltonian system. For this case $k = 1$, and the (x, y) normal space is a two dimensional plane.

Theorem 1 *Let $n, m \in \mathbb{N}$, where $n \geq 3$; $1 \leq m \leq n-2$, and $1 \leq r \leq \infty$. Smooth integrable n d.o.f. Hamiltonian systems which possess an $(n-2-m)$-dimensional family of normally parabolic, m-resonant, $(n-1)$-tori, form a C^1-open set in the space of smooth integrable n d.o.f. Hamiltonian systems, for each $n \geq 3$. Hence, the existence of such families is C^1-persistent in each such space.*
Each such normally parabolic $(n-1)$-torus generically consists of an m-dimensional family of nonresonant $(n-1-m)$-tori.

Proof: To prove Theorem 1 we first prove the following proposition:

[11] Equivalently, there exist some $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n-k\}$ such that $\nabla_{(x_i,y_i)}\frac{\partial H_0}{\partial I_j}|_{p_f} \neq 0$.

Proposition 1 *Let $n, m \in \mathbb{N}$, where $n \geq 3$; $1 \leq m \leq n-2$, and $1 \leq r \leq \infty$. Smooth integrable n d.o.f. Hamiltonian systems which possess an $(n-2-m)$-dimensional family of normally parabolic, m-resonant, $(n-1)$-tori, form a C^1 – open C^r – dense set in the class $\mathcal{P}_1$, hence are C^1-generic in this class.*

Remark 1 *By a 0-dimensional family we mean an isolated invariant torus, or a discrete set of isolated invariant tori.*

Proof of proposition: Let $H_0(x, y, I)$ be any integrable $n \geq 3$ d.o.f. Hamiltonian from class $\mathcal{P}_1$. Denote by N (here $N = \mathbb{R}^{n+1}$) the $(n+1)$-dimensional smooth manifold, on which the x, y, I variables of the Hamiltonian function $H_0(x, y, I)$ are defined. Each point in N corresponds to an $(n-1)$-torus in the full phase space. If an $(n-1)$-torus is attached to a fixed point of the (x, y) plane, it is an invariant $(n-1)$-torus. The conditions for existence of an equilibrium of the (x, y) system are:

$$\nabla_{(x,y)} H_0(x_f, y_f, I_f) = 0. \tag{5}$$

Since (5) defines two equations on the $n+1$ variables, (x, y, I), each of the two conditions in (5) is generically fulfilled on a smooth n-dimensional submanifold of N (this is a consequence of a corollary to the Sard Theorem; e.g. see [20] and references therein). By the Transversality Theorem [20], the n d.o.f. integrable Hamiltonian systems for which these two smooth submanifolds of N are transverse form a C^1 – *open* C^r – *dense* set in the space of n d.o.f. near integrable Hamiltonian systems (with $n \geq 2$). By the definition of the class $\mathcal{P}_1$ (definition 1), these two smooth n-dimensional submanifolds must intersect each other. Hence, they intersect each other transversely along an $(n-1)$-dimensional submanifold for a C^1 – *open* C^r – *dense* set of the class $\mathcal{P}_1$. If more than one fixed point exists in the (x, y) plane, generically each such equilibrium corresponds to one branch of solutions $\{(x_f, y_f, I_f)\}$, each defining a smooth $(n-1)$-dimensional submanifold of N. For a generic integrable Hamiltonian system, the set of such $\{(x_f, y_f, I_f)\}$ branches of isolated equilibria in the (x, y) plane is countable. Let us denote these smooth submanifolds by N_f^i, $i = 1, 2, ...$, or in short, N_f (when appropriate, with some abuse of notation, we will refer to N_f as the union of all the smooth submanifolds N_f^i) . Each such isolated equilibria in the (x, y) plane correspond to an invariant $(n-1)$-torus in the $2n$-dimensional phase space.

Since condition (5) and all conditions hereafter are imposed on the Hamiltonian function and its partial derivatives, and not on the jets of the vector field, the standard transversality theorems may be invoked to obtain genericity results in the class $\mathcal{P}_1$. In particular, in this formulation the symplectic structure plays no role.

The generic normal stability type of the invariant $(n-1)$-tori is either hyperbolic or elliptic, and for some action values such a torus may be parabolic. The condition for a fixed point from N_f to be parabolic is:

$$\det\left(\left.\frac{\partial^2 H_0}{\partial^2(x,y)}\right|_{(x_p,y_p,I_p)}\right)=0. \tag{6}$$

The variables which fulfill the (instantaneous) parabolicity condition, (6), generically span an n-dimensional submanifold $P\subset N$. For each $i=1,2,...$, by the Transversality Theorem, the n d.o.f. integrable Hamiltonian systems for which P is transverse to N_f in N form a $C^1-open\ C^r-dense$ set in the class $\mathcal{P}_1$, for each fixed $n\geq 2$. By the assumptions on the class $\mathcal{P}_1$, P must intersect at least one of the submanifolds N_f, at least in one point. However, if P and N_f intersect, they intersect transversely on a $C^1-open\ C^r-dense$ set of Hamiltonians from the class $\mathcal{P}_1$. The definition of a transverse intersection (e.g. see [20]) implies that if $P \pitchfork N_f$ and $P\cap N_f\neq\emptyset$ then P must intersect N_f transversely along an $(n-2)$-dimensional submanifold of N. Let us denote this $(n-2)$-dimensional submanifold by PN_f. Note that PN_f corresponds to an $(n-2)$-dimensional family of parabolic $(n-1)$-tori. It follows that n d.o.f. integrable Hamiltonian systems which possess an $(n-2)$-dimensional family of invariant normally parabolic $(n-1)$-tori form a $C^1-open\ C^r-dense$ set in the class $\mathcal{P}_1$, for each fixed $n\geq 2$. Moreover, taking intersections over all $2\leq n\in\mathbb{N}$, yields, by the Baire Theorem, that n d.o.f. integrable Hamiltonian systems possessing an $(n-2)$-dimensional family of normally parabolic $(n-1)$-tori form a $C^r-dense\ C^1-G_\delta$ (dense) set in the class $\mathcal{P}_1$. Hence, the existence of normally parabolic $(n-1)$-tori is C^1-generic in this class.

The inner frequencies, $\omega_i(x_f,y_f,I_f)$; $i=1,...,n-1$, of a resonant invariant $(n-1)$-torus, $(x_f,y_f,I_f)\in N_f$, are rationally dependent; there exists a vector of integers, $k^j\in\mathbb{Z}^{n-1}\backslash\{0\}$, such that:

$$\langle k^j,\omega(x_f,y_f,I_r)\rangle=0, \qquad (x_f,y_f,I_r)\in N_f\ . \tag{7}$$

Equation (7) is generically satisfied on an $(n-2)$-dimensional submanifold of N_f, the resonant submanifold $R^{k^j}N_f$. This follows from transversality and the fact that the rational numbers form a dense set in the space of real numbers. Consider the intersection of m such resonant submanifolds: Taking m independent vectors of integers, k^j, $j = 1,..,m$, the corresponding m resonant $(n-2)$-dimensional submanifolds of N_f must all intersect (as all resonant surfaces pass through the origin of the frequency space, and the inner frequencies depend smoothly on the actions in a neighborhood of the $(n-1)$-tori). Hence, by the Transversality Theorem, they intersect transversely on an $(n-1-m)$-dimensional submanifold of N_f for a C^1-*open* C^r-*dense* set of integrable Hamiltonians from the class $\mathcal{P}_1$. Let us denote this $(n-1-m)$-dimensional submanifold by $R^m N_f$. i.e.

$$R^m N_f = R^{k^1} N_f \cap \cdots \cap R^{k^m} N_f; 1 \leq m \leq n-2.$$

$R^m N_f$ consists of m-resonant $(n-1)$-tori, which are generically hyperbolic or elliptic. Note that if $m = n-1$, then $R^{n-1}N_f$ is a discrete set of points, each corresponding to an $(n-1)$-resonant $(n-1)$-torus of fixed points, which is generically hyperbolic or elliptic, and resides on a certain energy surface, which is determined by the point $(x_f, y_f, I_r) \in R^{n-1}N_f$.

By the Transversality Theorem, for each fixed $n \geq 3$, and for $1 \leq m \leq n-2$, those n d.o.f. integrable Hamiltonians for which the $(n-1-m)$-dimensional surface $R^m N_f$ of resonant tori is transverse to the $(n-2)$-dimensional surface PN_f of parabolic tori form a C^1- *open* C^r- *dense* set in the class $\mathcal{P}_1$. However, as PN_f corresponds to a smooth $(n-2)$-dimensional family of parabolic tori in N_f, it contains a dense set of m-resonant parabolic tori, hence it intersects $R^m N_f$ at least in one point. Therefore, by transversality, it must intersect $R^m N_f$ along an $(n-2-m)$-dimensional submanifold of N_f. Hence, the systems for which this intersection is transverse form a C^1-*open* C^r-*dense* set in the class $\mathcal{P}_1$. Let us denote this $(n-2-m)$-dimensional submanifold by PR^m, where $PR^m \subset R^m N_f \subset N_f$. Taking intersections over all $n \geq 3$, by the Baire Theorem, those n d.o.f. integrable Hamiltonian systems for which PR^m is transverse form a C^r-*dense* C^1-G_δ (dense) set in the class $\mathcal{P}_1$. Hence, they are C^1-*generic* in $\mathcal{P}_1$. $\square_{proposition}$

As transverse intersections is a C^1-open condition, and the condition of equation (3) is a transversality condition, it follows that systems from the class $\mathcal{P}_1$ for which the $(n-2-m)$-dimensional manifold, PR^m, of normally parabolic, m-resonant, $(n-1)$-tori, is transverse form a C^1-*open* set in the space of n d.o.f. integrable Hamiltonian systems for each $n \geq 3$, $1 \leq m \leq$

$n-2$. The assertion that each such normally parabolic m-resonant $(n-1)$-torus consists of an m-dimensional family of non-resonant, $(n-1-m)$-tori follows from Arnold [2]. Hence, we have established Theorem 1. □

Consider a small Hamiltonian perturbation of $H_0(x,y,I)$, $\varepsilon H_1(x,y,\theta,I;\varepsilon)$, with ε small. This perturbation may be considered as the higher order terms of the original Hamiltonian. The effect of such a perturbation clearly depends on the structure of the integrable flow, as will be discussed below. We define *parabolic m-resonance* in the following way (see also [37, 28, 27]):

Definition 2 *Parabolic m-resonance occurs when a small Hamiltonian perturbation $\varepsilon H_1(x,y,\theta,I;\varepsilon)$ is applied to an integrable n d.o.f. Hamiltonian system possessing an m-resonant lower dimensional normally parabolic torus.*

A parabolic m-resonance occurs for a set of initial conditions of the perturbed system in the vicinity of the previously existing normally parabolic resonant torus of the integrable system. A near integrable Hamiltonian system is said to belong to the class $\mathcal{P}_k$ if its integrable part is of this class. Hence, considering n d.o.f. near integrable Hamiltonian systems of the form: $H(x,y,\theta,I;\varepsilon) = H_0(x,y,I) + \varepsilon H_1(x,y,\theta,I;\varepsilon)$, we have the following corollary:

Corollary 1 *Let $n,m \in \mathbb{N}$, where $n \geq 3$; $1 \leq m \leq n-2$, and $1 \leq r \leq \infty$. Smooth near integrable n d.o.f. Hamiltonian systems of the form $H(x,y,\theta,I;\varepsilon) = H_0(x,y,I)+\varepsilon H_1(x,y,\theta,I;\varepsilon)$, exhibiting parabolic m-resonance, form a C^1-open C^r-dense set in the class $\mathcal{P}_1$, and a C^1-open set in the space of smooth near integrable n d.o.f. Hamiltonian systems, for each fixed $n \geq 3$. Hence, the existence of a parabolic m-resonance is C^1-generic in the class $\mathcal{P}_1$ and is C^1-persistent in the space of smooth near integrable n d.o.f. Hamiltonians, for each $n \geq 3$.*

Hereafter, by generic we mean C^1-generic, and by persistent we mean C^1-persistent.

In fact, we have established above the following Theorem:

Theorem 2 *Let $3 \leq n \in \mathbb{N}$ and $1 \leq r \leq \infty$.*

1. *Let $1 \leq m \leq n-2$. Smooth integrable n d.o.f. Hamiltonian systems which possess $(n-1-m)$-dimensional families of both normally hyperbolic and normally elliptic, m-resonant, $(n-1)$-tori, form a C^1-open*

C^r – dense set in the class $\mathcal{P}_1$, hence generic in this class.
Such systems are persistent in the space of smooth integrable n d.o.f. Hamiltonian systems for each $n \geq 3$.

2. *Let $m = n - 1$. Smooth integrable n d.o.f. Hamiltonian systems which possess a normally hyperbolic (respectively, normally elliptic), $(n-1)$-resonant, $(n-1)$-torus of fixed points, form a C^1 – open C^r – dense set in the class $\mathcal{P}_1$, hence generic in this class.*
   Such systems are persistent in the space of smooth integrable n d.o.f. Hamiltonian systems for each $n \geq 3$.

For $1 \leq m \leq n-2$, each such normally hyperbolic (respectively, elliptic) $(n-1)$-torus consists of an m-dimensional family of $(n-1-m)$-tori, where generically these are non-resonant.

We define *hyperbolic (elliptic) m-resonance* in the same manner as parabolic m-resonance (see [37], [18] for the 2 d.o.f. case):

Definition 3 *Hyperbolic (elliptic) m-resonance occurs when a small Hamiltonian perturbation of the form $\varepsilon H_1(x, y, \theta, I; \varepsilon)$ is applied to an integrable n d.o.f. Hamiltonian system possessing an m-resonant lower dimensional normally hyperbolic (elliptic) torus.*

Hence, considering smooth near integrable n d.o.f. Hamiltonian systems of the form: $H(x, y, \theta, I; \varepsilon) = H_0(x, y, I) + \varepsilon H_1(x, y, \theta, I; \varepsilon)$, where $\varepsilon H_1(x, y, \theta, I; \varepsilon)$ is a bounded Hamiltonian perturbation of the integrable Hamiltonian $H_0(x, y, I)$, we have the following corollary:

Corollary 2 *Let $3 \leq n \in \mathbb{N}$ and $1 \leq r \leq \infty$.*

1. *Let $1 \leq m \leq n-2$. Smooth near integrable n d.o.f. Hamiltonian systems of the form $H(x, y, \theta, I; \varepsilon) = H_0(x, y, I) + \varepsilon H_1(x, y, \theta, I; \varepsilon)$, exhibiting both hyperbolic and elliptic m-resonance, form a C^1 – open C^r – dense set in the class $\mathcal{P}_1$, hence generic in this class.*
   Such systems are persistent in the space of smooth near integrable n d.o.f. Hamiltonian systems for each $n \geq 3$.

2. *Let $m = n - 1$. Smooth near integrable n d.o.f. Hamiltonian systems exhibiting hyperbolic (respectively, elliptic) $(n-1)$-resonance, form a C^1 – open C^r – dense set in the class $\mathcal{P}_1$, hence generic in this class.*
   Such systems are persistent in the space of smooth near integrable n d.o.f. Hamiltonian systems for each $n \geq 3$.

The elliptic m-resonance occurs for a set of initial conditions of the perturbed system which are in the vicinity of the previously existing normally elliptic resonant torus. The hyperbolic m-resonance occurs for a set of initial conditions of the perturbed system which are in the vicinity of the previously existing normally hyperbolic resonant torus and its intersecting stable and unstable manifolds. The spatial extent of these manifolds in these higher dimensional systems is the subject of current research (see, for example, [17, 41] and references therein).

Theorem 1, Proposition 1, Corollary 1 and the first part of Theorem 2 and of Corollary 2 apply only for the case $n \geq 3$, while the second part of Theorem 2 and of Corollary 2 apply for the case $n = 2$ as well.

In Theorems 1, 2, Proposition 1 and Corollaries 1, 2, we prove that the existence of resonant tori of the above prescribed properties are generic and persistent - the actions of the unperturbed $(n-1)$-tori serve as internal parameters. Next we prove that the existence of $(n-1)$-resonant normally parabolic $(n-1)$-tori of fixed points is a co-dimension one phenomena, namely it requires one additional external parameter.

Consider one parameter families of integrable n d.o.f. Hamiltonian systems with Hamiltonians of the form: $H_0(x, y, I; \mu)$, where $\mu \in \mathbb{R}$ and all the assumptions stated in section 1, regarding smoothness, manifolds and constants of motion, remain the same. The class $\mathcal{P}_k^*$ is defined in this case (correspondingly to definition 1) as the class of one parameter families of integrable n d.o.f. Hamiltonians of the form $H_0(x, y, I; \mu)$, satisfying the conditions of definition 1 with $k = 1$, where the transversality condition in this case is:

$$\|\nabla_{(x,y)}\frac{\partial H_0}{\partial \mu}|_{p_f}\| + \|\nabla_{(x,y)}\frac{\partial H_0}{\partial I_1}|_{p_f}\| + \|\nabla_{(x,y)}\frac{\partial H_0}{\partial I_2}|_{p_f}\| + \cdots + \|\nabla_{(x,y)}\frac{\partial H_0}{\partial I_{n-1}}|_{p_f}\| \neq 0. \tag{8}$$

Denote by $N \subseteq \mathbb{R}^{n+2}$ the $(n+2)$-dimensional submanifold of M, spanned by the x, y, I, μ variables and parameter of the Hamiltonian. Then, N_f is an n-dimensional submanifold, P is an $(n+1)$-dimensional submanifold and PN_f is an $(n-1)$-dimensional submanifold. The resonance submanifolds, $R^{k^j}N_f$, are each of dimension $(n-1)$, and the submanifold $R^m N_f$, which corresponds to transverse intersections of m independent resonance conditions in N_f, is $(n-m)$-dimensional. The submanifold PR^{n-2}, which corresponds to normally parabolic $(n-2)$-resonant $(n-1)$-tori is 1-dimensional. Denote by k^{n-1} an $(n-1)$-dimensional vector of integers which is linearly independent of the

$n-2$ independent $(n-1)$-dimensional vectors of integers used to construct PR^{n-2}.

Remark 2 *By symplectic change of coordinates, the set of $n-1$ independent $(n-1)$-dimensional vectors of integers, $k^1, ..., k^{n-1}$, may be chosen as a set of $n-1$ unit vectors (i.e., $k^j = e_j^{n-1}$). Then each independent resonance condition of the form (7) corresponds to vanishing of the j^{th} frequency of the $(n-1)$-torus.*

The resonance condition (7) for k^{n-1} is generically met on an $(n-1)$-dimensional submanifold, $R^{k^{n-1}}N_f$, in N_f. Using the same reasoning as above, the two submanifolds of N_f, PR^{n-2} (a one dimensional curve) and $R^{k^{n-1}}N_f$ (a co-dimension one submanifold of N_f), intersect transversely in a C^1- *open* C^r- *dense* set of the class $\mathcal{P}_k^*$, and for a C^1- *open* set of one parameter families of integrable n d.o.f. Hamiltonian systems, for each $n \geq 3$. This transverse intersection occurs on a set of isolated points, each corresponding to a normally parabolic $(n-1)$-resonant $(n-1)$-torus of fixed points. Hence, we have established:

Theorem 3 *Let $3 \leq n \in \mathbb{N}$, $\mu \in \mathbb{R}$, and $1 \leq r \leq \infty$. The existence of one parameter families of smooth integrable n d.o.f. Hamiltonian systems which possess a normally parabolic $(n-1)$-resonant $(n-1)$-torus of fixed points is generic in the class $\mathcal{P}_k^*$, and persistent in the space of smooth integrable n d.o.f. Hamiltonian systems depending on one parameter, for each $n \geq 3$.*

Consider smooth near integrable systems of the form: $H(x, y, \theta, I; \varepsilon, \mu) = H_0(x, y, I; \mu) + \varepsilon H_1(x, y, \theta, I; \varepsilon, \mu)$, where such a system is said to be of class $\mathcal{P}_k^*$ if its integrable part is of this class. Then, we have the following corollary:

Corollary 3 *Let $3 \leq n \in \mathbb{N}$, $\mu \in \mathbb{R}$, and $1 \leq r \leq \infty$. The existence of smooth near integrable n d.o.f. Hamiltonian systems of the form $H(x, y, \theta, I; \varepsilon, \mu) = H_0(x, y, I; \mu) + \varepsilon H_1(x, y, \theta, I; \varepsilon, \mu)$, exhibiting parabolic $(n-1)$-resonance is generic in the class $\mathcal{P}_k^*$, and persistent in the space of n d.o.f. smooth near integrable Hamiltonian systems depending on one parameter, for each $n \geq 3$.*

Theorem 3 and Corollary 3 apply for the case $n = 2$ as well.

Remark 3 *In the theorems and corollaries above the actions (or constants of motion), I, of the integrable Hamiltonian serve as internal parameters,* *and govern the stability type of the lower dimensional invariant tori. Hence, (8) is a necessary condition for a given integrable Hamiltonian system of the form $H_0(x, y, I; \mu)$ to possess a normally parabolic resonant torus. Generally, systems which conserve angular momentum satisfy (8) yet separable Hamiltonian systems (e.g. systems of the form $H_0 = H_{0xy}(x, y) + \sum H_i(I_i; \mu)$) do not, see [37] for details (there only the 2-d.o.f. case is discussed, but the same ideas apply to the higher dimensional cases).*

2.2 The general case of lower dimensional tori: $k > 1$

When $2 \leq k \leq n-2$, an invariant $(n-k)$-torus of an integrable n d.o.f. system is still considered normally parabolic if one of its normal frequencies vanishes. When $k > 1$, an invariant $(n-k)$-torus, with one vanishing normal frequency has additional $k-1$ pairs of characteristic eigenvalues which may correspond to several possible stability types (elliptic, generically real hyperbolic, complex hyperbolic or parabolic). Analogous statements to theorems and corollaries 1-3, may be formulated, e.g.:

Theorem 4 *Let $n, k, m \in \mathbb{N}$, where $n \geq 4$, $2 \leq k \leq n-2$, $1 \leq m \leq n-k-1$, and $1 \leq r \leq \infty$. Smooth integrable n d.o.f. Hamiltonian systems which possess an $(n-k-m-1)$-dimensional family of normally parabolic, m-resonant, $(n-k)$-tori, form a C^1- open C^r- dense set in the class $\mathcal{P}_k$, and a C^1- open set in the space of smooth integrable n d.o.f. Hamiltonian systems, for each fixed $n \geq 3$. Hence, their existence is generic in the class $\mathcal{P}_k$, and persistent in the space of smooth integrable Hamiltonian systems, for each fixed $n \geq 3$. Each such normally parabolic $(n-k)$-torus consists of an m-dimensional family of $(n-k-m)$-tori.*

For example, it follows that $n = 4$ appears as a critical dimension for which n d.o.f. integrable Hamiltonian systems first possess a 1-resonant normally parabolic 2-torus persistently.

The behavior of perturbed orbits near a resonant $(n-k)$-torus is expected to be quite different and deserves a separate study. Hereafter we consider the case $k = 1$.

3 Underlying degeneracies and critical n values

The existence of families of iso-energetic lower dimensional resonant tori on one hand, and degenerate PR on the other, may both induce instabilities in the perturbed system, as demonstrated in section 4. Below we prove that the appearance of both is persistent.

3.1 Iso-energetic resonances

The appearance of a family of resonant tori on the same energy surface is one possible source of large instabilities in the perturbed flow, as postulated by Arnold. Families of lower dimensional tori appear on the boundary or singular folds of the energy surface, and are intersected by the resonance planes in the same fashion that the energy surface is intersected by them, with one additional co-dimension. Indeed, for a given n and $0 < m < n$ values, an l-dimensional family $(1 \leq l \leq n-m-2)$ of m-resonant tori of the same stability type generically intersects transversely the n-dimensional energy surface

$$H_0(x, y, I) - h = 0, \tag{9}$$

where h is some fixed energy value, along an $(l-1)$-dimensional manifold (by the same transversality arguments of the previous section). For $l \geq 2$ this leads to the non-trivial observation that families of tori of specific properties (e.g. m-resonant parabolic or hyperbolic) reside on the same energy surface. We have thus established that such a mechanism exists persistently for sufficiently large systems:

Corollary 4 *For $n \geq 4$, the existence of an $(n-m-2)$-dimensional family of normally hyperbolic or elliptic, m-resonant, $(n-1)$-tori on a given energy surface is persistent for $0 \leq m \leq n-3$. For $n = 3$ the existence of only isolated iso-energetic 1-resonant, 2-tori is persistent.*

Corollary 5 *For $n \geq 5$, the existence of an $(n-m-3)$-dimensional family of normally parabolic, m-resonant, $(n-1)$-tori on a given energy surface is persistent for $0 \leq m \leq n-4$. For $n = 3, 4$ the existence of only isolated iso-energetic parabolic 1-resonant, $(n-1)$-tori is persistent.*

Remark 4 *In the non-resonant case ($m = 0$), the statements in the above two corollaries apply to $n \geq 3$ and $n \geq 4$ respectively.*

The first corollary is a result from the realization that the critical dimension for having the resonance web on the energy surface boundaries or singular folds is larger by one from the critical dimension for having a resonance web on the energy surface itself. Similar realization regarding the boundary of the singular fold surface (a co-dimension 3 surface) implies the second corollary.

3.2 Tangential resonances

Consider the n-dimensional space (H_0, I) on which energy surfaces are presented as the hyper-planes $H_0 = h$. The family of lower dimensional invariant tori N_f is presented in this space as a co-dimension-one smooth surface. Notice that N_f is tangent to the energy surfaces containing lower dimensional resonant tori at these tori, $R^m N_f^h$ (namely at the $m-$resonant lower dimensional tori with energy h) along the m resonant directions: For example, let us assume that the co-ordinates (x, y, I) are chosen so that $\dot{\theta}_j = \left.\frac{\partial H_0}{\partial I_j}\right|_{p_{fr}} = 0; \quad j = 1, \ldots, m$ (see remark 2), where $p_{fr} \in R^m N_f^h$, and $PN_f = \{(x, y, I)| \, x = y = I_1 = 0\}$. Then, since on N_f : $\left.\frac{dH_0}{dI_j}\right|_{p_f} = \left.\frac{\partial H_0}{\partial I_j}\right|_{p_f}$ (by (2), and assuming that N_f may be expressed as a graph on the action variables), it follows that N_f is tangent to the energy surface in the directions $I_1, \ldots, I_m$ at p_{fr}. If this tangency occurs at the parabolic resonant torus $p_{pr} \in PR^{mh} \subset R^m N_f^h$, and if it is of *higher order* we expect stronger instability under perturbation. To leading order this occurs along the I_j direction when the next order derivative at p_{pr} vanishes:

$$\frac{d}{dI_j}\left(\frac{\partial H_0(x_f(I_{fr}), y_f(I_{fr}), I_{fr}; \mu_{fr})}{\partial I_j}\right)\Bigg|_{(x_{pr}, y_{pr}, I_{pr}; \mu_{pr})} = 0; \qquad \text{for some } j \in \{1, \ldots, m\}, \tag{10}$$

where $\{(x_f(I_{fr}), y_f(I_{fr}), I_{fr}; \mu_{fr})\} \subset R^m N_f$ denotes the resonant family of tori belonging to a smooth branch N_f^i of N_f which contains the point p_{pr} (corresponding to a parabolic m-resonant $(n-1)$-torus). In this case an infinitesimal family of m-resonant tori emanating from the parabolic torus belong to the same energy surface. We thus define:

Definition 4 *A tangential parabolic m-resonance (m-TPR), in the direction of the action I_j (for some $j = 1, ..., m$), occurs when a small Hamiltonian perturbation of the form $\varepsilon H_1(x, y, \theta, I; \mu, \varepsilon)$ is applied to an integrable Hamiltonian system, which possesses an energy surface on which an infinitesimal family of resonant tori emanating from a parabolic m-resonant torus exists in the direction of I_j.*

Condition (10) is set for C^∞ functions of the form $H_0(x, y, I; \mu)$. Furthermore, the arguments are smooth functions on each smooth branch of N_f, N_f^i. Hence, from the Sard Theorem it follows that condition (10) is satisfied for almost all systems in the $C^1 - open$ $C^r - dense$ set of n d.o.f. integrable Hamiltonian systems from the class $\mathcal{P}_k^*$, which possess an $(n-2-m+p)$-dimensional family of parabolic m-resonant $(n-1)$-tori (cf. Theorem 1), with $n \geq m+3-p$, $1 \leq m \leq n-1$ (i.e. on an $C^1 - open$ subset of systems of full Lebesgue measure). If $p = 0$ and $n = m+3$, each point from the discrete set of isolated points, $p_{tpr} = (x_{tpr}, y_{tpr}, I_{tpr}) \in PR^m$, which solves equation (10), defines an energy surface, $H_0(x, y, I) = h_{tpr}$, on which an infinitesimal family of resonant tori, containing a parabolic m-resonant torus, exists. Hence, by Theorem 1, we have established:

Theorem 5 *Let $m \geq 1$, $n = m+3$. Smooth integrable n d.o.f. Hamiltonian systems which possess, in the direction of one of the actions, an infinitesimal family of elliptic and/or hyperbolic resonant $(n-1)$-tori, containing a parabolic m-resonant $(n-1)$-torus, on some energy surface $H_0(x, y, I) = h_{tpr}$ (defined by the point p_{tpr}), are $C^1 - persistent$ in the space of smooth integrable $(m+3)$ d.o.f. Hamiltonian systems, for each $m \geq 1$.*

If $n + p > m + 3$, the TPR occurs on each energy surface for an open set of energy values; the condition $H_0(x, y, I; \mu) - h_{pr} = 0$ is generically satisfied on some $(n+p)$-dimensional energy surface, which generically intersects transversely the $(n-m-2+p)$-dimensional manifold PR^m on an $(n-m-3+p)$-dimensional manifold, HPR^m (by the same reasoning as given in the proofs above) for a $C^1 - open$ set of integrable n d.o.f. Hamiltonian systems. Then, condition (10) is satisfied on a subset of full measure of the $C^1 - open$ set of integrable n d.o.f. Hamiltonian systems, for which HPR^m is transverse. Hence, we have established:

Theorem 6 *Let $m \geq 1$, $n > m+3$. Integrable n d.o.f. Hamiltonian systems which possess in the direction of one of the actions an $(n-m-4)$-dimensional* *family of infinitesimal families of elliptic and/or hyperbolic resonant $(n-1)$-tori, containing a parabolic m-resonant $(n-1)$-torus, on each energy surface belonging to an open set of energy values are $C^1-persistent$ in the space of n d.o.f. smooth integrable Hamiltonian systems for each $n \geq m+4$.*

Theorem 7 *Let $3 \leq n \in \mathbb{N}$. The existence of two (respectively, one) parameter families of integrable n d.o.f. Hamiltonian systems which possess in the direction of one of the actions, an infinitesimal family of elliptic and/or hyperbolic resonant $(n-1)$-tori, containing a parabolic $(n-1)$-torus of fixed points (respectively, consisting of a family of invariant circles), on the energy surface $H_0(x,y,I;\mu) = H_0(x_{tpr}, y_{tpr}, I_{tpr}; \mu_{tpr}) = h_{tpr}$, is $C^1-persistent$ in the space of smooth integrable n d.o.f. Hamiltonian systems, for each $n \geq 3$.*

Now, consider a small Hamiltonian perturbation of the form: $\varepsilon H_1(x,y,I;\varepsilon,\mu)$:

Corollary 6 *Let $n \geq 4$, $1 \leq m \leq n-3$. The existence of a tangential parabolic m-resonance (m-TPR) is $C^1-persistent$ in the space of smooth near integrable n d.o.f. Hamiltonian systems, for each $n \geq 4$.*

Corollary 7 *Let $n \geq 3$, $m \in \{n-1, n-2\}$,*

$$p = \begin{cases} 1, & m = n-2 \\ 2, & m = n-1 \end{cases} .$$

The existence of a tangential parabolic m-resonance (m-TPR) is $C^1-persistent$ in p parameter families of smooth near integrable n d.o.f. Hamiltonian systems, for each $n \geq 3$.

It follows that $n = 4$ is the smallest number of d.o.f. for which the existence of 1-TPR is persistent without assuming dependence of the system on external parameters. For $n = 4+l$, $l \geq 1$, additional l conditions guarantying tangency of order $l+2$ may be imposed:

$$\begin{aligned} \frac{d^{k+1}}{dI_j^{k+1}} \left(\frac{\partial H_0(x_f(I_{fr}), y_f(I_{fr}), I_{fr}; \mu_{fr})}{\partial I_j} \right) \Bigg|_{(x_{pr}, y_{pr}, I_{pr}; \mu_{pr})} &= 0; \\ k &= 1, \ldots, l, \quad j \in \{1, \ldots, m\}. \end{aligned} \tag{11}$$

These conditions are satisfied on a C^1-open set in the space of integrable $(4+l)$ d.o.f. Hamiltonian systems, which persistently possess a parabolic 1-resonant $(3+l)$-torus. As l grows, higher order terms of $\dot{\theta}_j$ along some branch

N_f^i vanish, causing a "more flat" situation, which we denote by "$(l+1)-flat$". See section 4 for description of the corresponding numerical results. If (11) is satisfied for all positive k's ($l=\infty$), we say that the perturbed system has a *flat PR*.

Theorem 8 *Let $m=1$, $l\geq 1$, $n=4+l$. Smooth integrable n d.o.f. Hamiltonian systems which possess, in the direction of one of the actions, an $(l+1)$-flat family of elliptic and/or hyperbolic resonant $(n-1)$-tori, containing a parabolic m-resonant $(n-1)$-torus, on some energy surface, $H_0(x,y,I)=h_{fpr}$, are $C^1-persistent$ in the space of smooth integrable $(4+l)$ d.o.f. Hamiltonian systems, for each $l\geq 1$.*

Corollary 8 *Let $n\geq 5$, $m=1$, $l\geq 1$. The existence of an $(l+1)$-flat-TPR is $C^1-persistent$ in the space of smooth near integrable n d.o.f. Hamiltonian systems, with $n=4+l$, for each $l\geq 1$.*

Higher order tangencies at p_{pr} along all directions of $R^m N_f$ may be similarly formulated. However, such a situation is obviously of nonzero codimension for all n values.

4 Models

Here, the suggested mechanisms of instabilities are examined numerically. Using normal form techniques, $3,4$, and 5 d.o.f. integrable Hamiltonian systems are constructed, and typical parameter values are set. Then, the initial conditions for which tangential parabolic resonances, iso-energetic families of resonant hyperbolic tori and iso-energetic families of resonant parabolic tori appear are sought, and numerical solutions of nearby initial conditions of the perturbed system are recorded.

In [28], we studied the following phenomenological model for 3 d.o.f. near integrable Hamiltonian systems exhibiting PRs:

$$\begin{aligned}
& H(x,y,\theta_1,I_1,\theta_2,I_2;\alpha_1,\alpha_2,\alpha_3,\varepsilon,k)= \qquad (12)\\
= \; & H_0(x,y,I_1,I_2)+\varepsilon H_1(x,y,\theta_1,I_1,\theta_2,I_2)=\\
= \; & \frac{y^2}{2}-\frac{x^2}{2}I_1+\frac{x^4}{4}+(\alpha_1+\frac{1}{2})\frac{I_1^2}{2}+\frac{I_2^2}{2}+\alpha_2 I_2+\alpha_3 I_1 I_2+\\
& +\varepsilon_1(1-\frac{x^2}{2})\cos(k\theta_1)+\varepsilon_2\cos(k\theta_2)
\end{aligned}$$

The integrable case was studied analytically, by generalizing to 3 d.o.f. systems the method of constructing bifurcation diagrams in the space of constants of motion and analyzing unperturbed energy surfaces [38].

A partial study of the structure of the unperturbed energy surfaces of (12) in the three frequency space, and the structure of the resonance web on such energy surfaces, is presented in section 5.

4.1 Iso-energetic resonances in a 4 d.o.f. model

The simplest model of a 4 d.o.f. integrable Hamiltonian, which belongs to *class* $\mathcal{P}_1$ and is symmetric with respect to the x and y axis is:

$$\begin{aligned} & H_0(x,y,I_1,I_2,I_3;\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6) = \\ = \; & \frac{y^2}{2} - \frac{x^2}{2}I_1 + \frac{x^4}{4} + (\alpha_1 + \frac{1}{2})\frac{I_1^2}{2} + \frac{I_2^2}{2} + \frac{I_3^2}{2} + \\ & + \alpha_2 I_2 I_3 + \alpha_3 I_1 I_2 + \alpha_4 I_1 I_3 + \alpha_5 \frac{I_1^2}{2} I_3 + \alpha_6 I_3 + \alpha_7 \frac{I_1^3}{3}. \end{aligned} \tag{13}$$

The real parameters, $\alpha_i; i = 1, ...7$ are inserted for generality and compatibility with the 3 d.o.f. phenomenological model (12) which was studied in [27, 29, 28].

We show below that for any fixed non-zero value of the external parameters there exists an energy surface on which an infinitesimal family of 1-resonant elliptic tori, emanating from a 1-resonant parabolic torus in the direction of I_1, resides. The existence of this tangential 1-PR (1-TPR) instability mechanism for such generically constructed systems is in accordance with Theorem 5. The corresponding perturbed system exhibits 1-TPR, see figure 1.

All figures presented in this section are calculated for the fixed set of parameters: $\alpha_1 = 1.05, \alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 1, \alpha_5 = 1, \alpha_6 = \sqrt{2}$. This set of parameters is chosen to avoid other resonances, and/or additional non-generic degeneracies in the system. The role of the parameter α_7 is discussed below. Roughly, it serves to distinguish between 1-TPR ($\alpha_7 \neq 0$, see figure 1) and flat 1-PR ($\alpha_7 = 0$, hence all higher order terms of $\dot{\theta}_1$ in I_1 vanish, see figure 2 and discussion below).

The integrable Hamiltonian (13) is perturbed by:

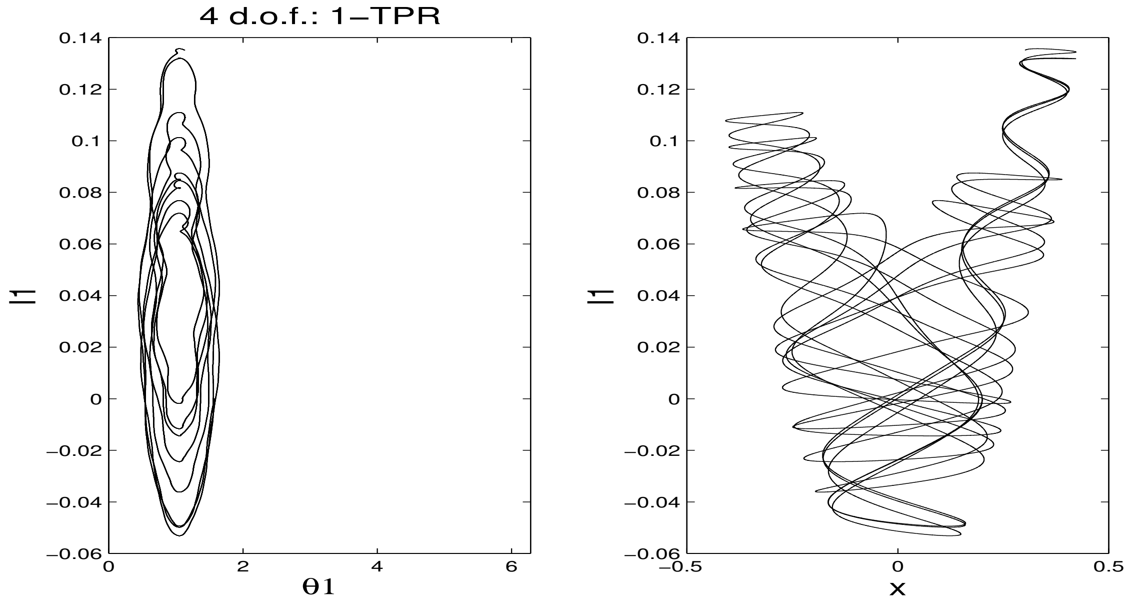

Figure 1: Tangential parabolic 1-resonance in a 4 d.o.f. system.
Initial conditions and parameters: $(x, y, I_1, I_2, I_3) = (0.2, 0., 0., 0.525, -1.05)$, $\theta_i = 1.57; i = 1, 2, 3$; $H_0 \approx -1.3467$; $\alpha_1 = 1.05, \alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 1, \alpha_5 = 1, \alpha_6 = \sqrt{2}, \alpha_7 = 1$; $k = 3, \varepsilon = \varepsilon_i = 1e - 3; i = 1, 2, 3$; $t = 1000$.

$$
\begin{aligned}
& \varepsilon H_1(x, y, \theta_1, I_1, \theta_2, I_2, \theta_3, I_3; \varepsilon, k) = \\
= \quad & \varepsilon_1 (1 - \frac{x^2}{2}) \cos(k\theta_1) + \varepsilon_2 \cos(k\theta_2) + \varepsilon_3 \cos(k\theta_3),
\end{aligned} \tag{14}
$$

Setting $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \varepsilon_3 \equiv 0$ in the integrable Hamiltonian (13) and in the perturbation (14), recovers the 3 d.o.f. Hamiltonian (12). In the figures we fix: $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 \equiv \varepsilon = 0.001$ and $k = 3$. The corresponding 4 d.o.f. Hamiltonian system is:

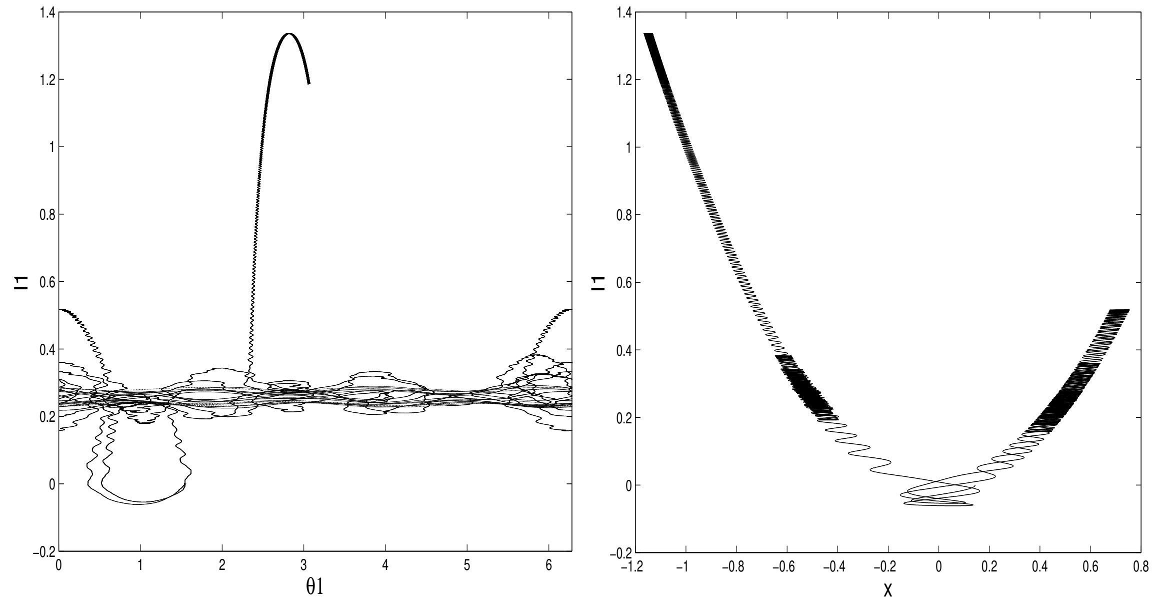

Figure 2: Flat parabolic 1-resonance in a 4 d.o.f. system.
Initial conditions and parameters: $(x, y, I_1, I_2, I_3) = (0.1454, 0., 0., 0.525, -1.05)$, $\theta_i = 1.57; i = 1, 2, 3$; $H_0 \approx -1.3467$; $\alpha_1 = 1.05, \alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 1, \alpha_5 = 1, \alpha_6 = \sqrt{2}, \alpha_7 = 0$; $k = 3, \varepsilon = \varepsilon_i = 1e-3; i = 1, 2, 3$; $t = 5000$.

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= xI_1 - x^3 + \varepsilon_1 x \cos(k\theta_1), \\
\dot{\theta}_1 &= -\frac{x^2}{2} + (\alpha_1 + \frac{1}{2})I_1 + \alpha_3 I_2 + \alpha_4 I_3 + \alpha_5 I_1 I_3 + \alpha_7 I_1^2, \\
\dot{I}_1 &= \varepsilon_1 k(1 - \frac{x^2}{2}) \sin(k\theta_1), \\
\dot{\theta}_2 &= I_2 + \alpha_2 I_3 + \alpha_3 I_1, \\
\dot{I}_2 &= \varepsilon_2 k \sin(k\theta_2), \\
\dot{\theta}_3 &= I_3 + \alpha_2 I_2 + \alpha_4 I_1 + \alpha_5 \frac{I_1^2}{2} + \alpha_6, \\
\dot{I}_3 &= \varepsilon_3 k \sin(k\theta_3).
\end{aligned}
\tag{15}
$$

The integrable Hamiltonian (13) fulfills the fixed point conditions (5) on $\{(x,y,I_1,I_2,I_3):(x,y)=(0,0),I_1,I_2,I_3\in\mathbb{R}\}\subset N_f$, and on $\{(x,y,I_1,I_2,I_3):(x,y)=(\pm\sqrt{I_1},0),I_1\geq 0,I_2,I_3\in\mathbb{R}\}\subset N_f$. The 3-torus corresponding to $(x,y)=(0,0)$ may be elliptic ($I_1<0$), hyperbolic ($I_1>0$) or parabolic ($I_1=0$). The family of 3-tori corresponding to $(x,y)=(\pm\sqrt{I_1},0)$ consists of two families of elliptic tori which meet at a parabolic 3-torus at $I_1=0$. Indeed, the parabolicity condition (6) for this system is:

$$0=\det\begin{pmatrix}0 & -I_1+3x^2\\ 1 & 0\end{pmatrix}|_{(0,0,I_{1p},I_{2p},I_{3p})}=I_{1p}, \tag{16}$$

hence it is fulfilled for $I_{1p}=0$, for any values of I_2 and I_3, i.e. it is fulfilled for the system (13) on the two dimensional manifold, $PN_f=\{(x,y,I_1,I_2,I_3)=(0,0,0,I_2,I_3);I_2,I_3\in\mathbb{R}\}$. Choosing the unit vector $k^1=(1,0,0)$, the resonance condition (7) is fulfilled for the system (13) on the submanifold $\{(x,y,I_1,I_2,I_3)=(\pm\sqrt{I_1},0,I_1\geq 0,I_2,I_3)\}$ of N_f iff:

$$\begin{aligned}\dot{\theta}_1\Big|_{(\pm\sqrt{I_{1r}},0,I_{1r},I_{2r},I_{3r})} &= \frac{\partial H_0}{\partial I_1}\Big|_{(\pm\sqrt{I_{1r}},0,I_{1r},I_{2r},I_{3r})} = \\ &= \alpha_1 I_{1r}+\alpha_3 I_{2r}+\alpha_4 I_{3r}+\alpha_5 I_{1r}I_{3r}+\alpha_7 I_{1r}^2=0\end{aligned} \tag{17}$$

Solutions to (17) define the resonance surface $R^{k^1}N_f$. Condition (17) is met on a parabolic 3-torus from PN_f iff $I_{1r}=I_{1p}=0$. Hence, PN_f intersects $R^1N_f=R^{k^1}N_f$ along $PR^1=\{(x,y,I_1,I_2,I_3)=(0,0,0,I_{2r},I_{3r})\}$, i.e. along the one dimensional manifold defined by:

$$I_{2pr}(I_3)=-\frac{\alpha_4 I_3}{\alpha_3}. \tag{18}$$

Each of the energy surfaces (a one dimensional family), containing a 1-resonant parabolic 3-torus, $p_{pr}\in PR^1$, is tangent to N_f at p_{pr} along I_1 (the resonant direction). One of these energy surfaces has a second order tangency to N_f along the I_1 direction, if at some point, $p_{tpr}=(x,y,I_1,I_2,I_3)=(0,0,0,I_{2tpr}(I_{3tpr}),I_{3tpr})$:

$$\begin{aligned}\frac{d}{dI_1}\left(\frac{\partial H_0(\pm\sqrt{I_1},0,I_1,I_2,I_3)}{\partial I_1}\right)\Big|_{p_{tpr}} &= (\alpha_1+\alpha_5 I_3+2\alpha_7 I_1)|_{p_{tpr}} = \\ &= \alpha_1+\alpha_5 I_{3tpr}=0.\end{aligned} \tag{19}$$

which implies:

$$I_{3tpr} = -\frac{\alpha_1}{\alpha_5}. \tag{20}$$

Substituting (20) into (18) yields:

$$p_{tpr} = (x, y, I_1, I_2, I_3) = (0, 0, 0, \frac{\alpha_1\alpha_4}{\alpha_3\alpha_5}, -\frac{\alpha_1}{\alpha_5}), \tag{21}$$

for any fixed pre-chosen set of nonzero parameters, $\alpha_i; i = 1, ...7$. The point (21) defines the energy level $H_0(x, y, I_1, I_2, I_3) = H_0(p_{tpr}) = h_{tpr}$ on which resides an infinitesimal family of elliptic 1-resonant tori emanating from the 1-resonant parabolic torus, p_{tpr}.

In figure 1 an example of a perturbed orbit exhibiting 1-TPR (starting near the point (21)) is shown. In the left plot we project the 8-dimensional orbit to the (θ_1, I_1) plane, and on the right plot to the (x, I_1) plane. The orbit performs fast travel through the successive elliptic resonance zones, and reaches the top of the perturbed energy surface (in the I_1direction) in quite a short time (see discussion in section 4.3), then goes back down to the parabolic resonance zone, reaches the bottom of the energy surface and starts to go up again. The displacements in the action I_1 are of size $\Delta I_1 \approx 0.2$, namely of two orders of magnitude larger than ε (see also the upper plot of figure 7 in subsection 4.3). We now show that $0.2 = O(\varepsilon^{1/3})$ is the maximal possible displacement of I_1 on this energy surface as long as I_2 and I_3 stay near their initial values, as is the case here.

The transversality condition which ensures that the existence of the second order tangency at (19) is persistent (while higher order tangencies are not) is:

$$\frac{d^2}{I_1^2}\left(\frac{\partial H_0(\pm\sqrt{I_1}, 0, I_1, I_2, I_3)}{\partial I_1}\right)\Bigg|_{p_{tpr}} = 2\alpha_7 \neq 0. \tag{22}$$

Namely, the term: $\alpha_7\frac{I_1^3}{3}$ in (13) ensures that condition (22) is fulfilled for nearby sets of nonzero parameters. More importantly here, the existence of higher order terms in $I_1$with positive coefficients insures that the energy surfaces are bounded in the positive I_1 direction; the bounds of the energy surfaces (if exist) in I_1 correspond to the elliptic equilibria of the (x, y) system (see [2], [28]), namely they may be calculated from the Hamiltonian equation by substituting $(x, y) = (x_{ell}, y_{ell})$ into the Hamiltonian function. For the system (15), $(x,y)=(0,0)$ is an elliptic fixed point of the (x,y) plane for $I_1<0$, for all ε values. Hence, the lower bound in I_1 of the perturbed energy surfaces, when exists, is obtained from solving the equation:

$$\begin{aligned}
& \alpha_7\frac{I_1^3}{3}+(\alpha_1+\frac{1}{2}+\alpha_5 I_3)\frac{I_1^2}{2}+(\alpha_3 I_2+\alpha_4 I_3)I_1+\frac{I_2^2}{2}+\frac{I_3^2}{2}+ \\
+ \quad & \alpha_2 I_2 I_3+\alpha_6 I_3+\varepsilon_1\cos(k\theta_1)+\varepsilon_2\cos(k\theta_2)+\varepsilon_3\cos(k\theta_3)-h=0,
\end{aligned} \tag{23}$$

with respect to I_1. The points $(x,y)=(\pm\sqrt{I_1+\varepsilon_1 cos(k\theta_1)},0)$ are elliptic relative equilibria of the (x,y) system for $I_1+\varepsilon_1 cos(k\theta_1)>0$. Hence, the upper bound in I_1 of the perturbed energy surfaces, when exists, is obtained by solving the equation:

$$\begin{aligned}
& \alpha_7\frac{I_1^3}{3}+(\alpha_1+\alpha_5 I_3)\frac{I_1^2}{2}+(\alpha_3 I_2+\alpha_4 I_3)I_1+\frac{I_2^2}{2}+\frac{I_3^2}{2}+\alpha_2 I_2 I_3+ \\
+ \quad & \alpha_6 I_3+\varepsilon_1(1-\frac{I_1}{2})\cos(k\theta_1)-O(\varepsilon_1^2)+\varepsilon_2\cos(k\theta_2)+\varepsilon_3\cos(k\theta_3)-h=0,
\end{aligned} \tag{24}$$

with respect to I_1. Consider first the unperturbed case $\varepsilon=\varepsilon_i=0$ $(i=1,2,3)$; at p_{tpr} (21), $\alpha_1+\alpha_5 I_3=0$; hence this branch of equilibrium manifold has a second order tangency to the energy surface with the energy value h_{tpr}, along the I_1 axis, corresponding to a tangent parabolic 1-resonance (1-TPR) in the perturbed system. Furthermore, near $\alpha_1+\alpha_5 I_3=0$, positive α_7 (or the coefficients of higher order terms in I_1) is a necessary condition for getting an upper finite bound on I_1. For $\alpha_7=0$ (as higher order terms in I_1 are not considered in (13)), this branch of equilibrium manifold is parallel to the I_1 axis to all orders, corresponding to a *flat parabolic resonance*. We claim that the instability in the I_1 direction near these parabolic resonances is maximal, namely the only barrier for it is the extent of the energy surface in the PR neighborhood. Indeed, solving equation (24) for I_1 with the parameters, initial actions and energy values as in figure 1, and with $\varepsilon=\varepsilon_i=0; i=1,2,3$, one obtains the upper bound in I_1 of the unperturbed energy surface as: $I_{1up}^{H_0}\approx 0.1$. The difference in the upper bounds in I_1 of the unperturbed and perturbed energy surfaces with $\varepsilon_1=\varepsilon_2=\varepsilon_3\equiv\varepsilon\neq 0$ is of order $O(\varepsilon^{\frac{1}{3}})$, i.e. $I_{1up}^{H}=I_{1up}^{H_0}+O(\varepsilon^{\frac{1}{3}})\approx 0.1+O(\varepsilon^{\frac{1}{3}})$. Hence, as long as the actions I_2 and I_3 do not vary much, as is the case here, the perturbed orbit in figure 1 indeed reaches the boundaries in I_1 of the perturbed energy surface.

Generally, $\alpha_7 = 0$ corresponds to a third order tangency in the I_1 direction (2-flat-TPR), a co-dimension one phenomenon for 4 d.o.f. integrable systems. Since we did not include higher order terms here, this situation corresponds to a flat PR. This flatness implies that the frequency $\dot{\theta}_1(\pm\sqrt{I_1}, 0, I_1, \frac{\alpha_1\alpha_4}{\alpha_3\alpha_5}, -\frac{\alpha_1}{\alpha_5})$ vanishes identically for all nonnegative values of I_1; the unperturbed energy surface which contains the point p_{tpr} (equation (21)) is unbounded in the positive direction of I_1 (see equation (24)). Perturbed orbits corresponding to this flat parabolic 1-resonance may exhibit order one instabilities, see figure 2. Compare this figure with figure 1: In both figures the orbit is seen to exhibit fast travels along the successive elliptic resonances, repeatedly passing through the parabolic resonance zone. The orbit in figure 1 exhibits displacements of size $\Delta I_1 \approx 0.2$ even after a much longer run time than shown ($t \geq 5000$). The orbit in figure 2 reaches displacements of size $\Delta I_1 \approx 0.5$ at $t \approx 1000$, occasionally gets trapped in a specific resonance zone, and then, after a longer time ($t = 5000$) it reaches I_1 values which are of order $O(1)$ (see also figures 7 and 6 in section 4.3, respectively). To control our simulations, we ensure that the value of the Hamiltonian is preserved at least up to the 12^{th} digit after the decimal point.

These simulations demonstrate the critical role of the extent of the energy surfaces in the action directions, and the connection between this simple geometrical phenomena and the dynamical phenomena of resonances - fast growing extent of the allowed range of actions in the energy surface corresponds to near-flatness in that direction, which corresponds to an infinitesimal family of bifurcating resonant tori.

In [37, 27, 29, 28] qualitatively similar behavior of perturbed orbits near a flat 1-PR was observed for 2 and 3 d.o.f. systems. We emphasize that in 2 d.o.f. 1-TPR is of co-dimension two, in 3 d.o.f. systems it is of co-dimension one, and only for systems with four or more d.o.f. it becomes of co-dimension zero (its existence is persistent). It is worth noting that in [37, 29, 28] simulations of flat (equivalent to looking at p_{tpr} with $\alpha_7 = 0$) and *near*-flat (equivalent to looking *near* p_{tpr} with $\alpha_7 = 0$) orbits were presented, and no TPR orbits were shown.

The example presented here illustrates the fact that $n = 4$ is the first value of n for which integrable n d.o.f. Hamiltonians persistently possess an infinitesimal family of resonant tori, containing a parabolic resonant torus on the same energy surface. It is the first demonstration that TPR is a strong enough mechanism for inducing instability.

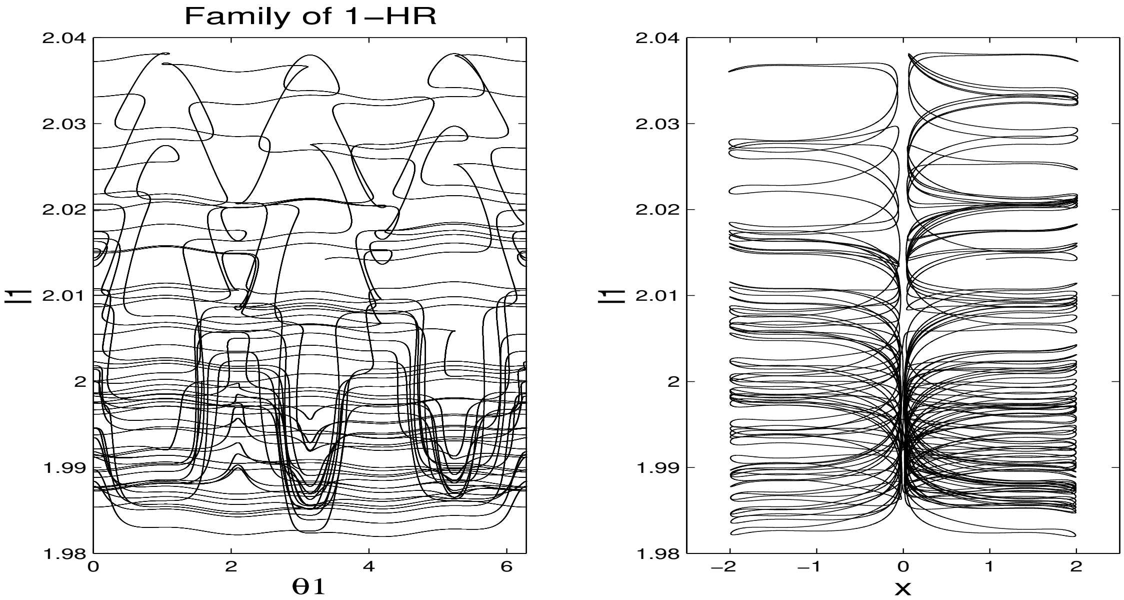

Figure 3: Iso energetic one dimensional family of hyperbolic 1-resonant tori in a 4 d.o.f. system.
Initial conditions and parameters: $(x, y, I_1, I_2, I_3) = (0.0015, 0., 2., -8.31829, 4.51219)$, $\theta_i = 1.57; i = 1, 2, 3$; $H_0 \approx 1.4999$; $\alpha_1 = 1.05, \alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 1, \alpha_5 = 1, \alpha_6 = \sqrt{2}, \alpha_7 = 0$; $k = 3, \varepsilon = \varepsilon_i = 1e-3; i = 1, 2, 3$; $t = 1000$.

In addition, $n = 4$ is also the first value of n for which the existence of a one dimensional family of hyperbolic (elliptic) 1-resonant tori on the same unperturbed energy surface becomes generic. This possible mechanism for instabilities occurs for the integrable Hamiltonian (13) when the three dimensional submanifold of N_f, $\{(x, y, I_1, I_2, I_3) : (x, y) = (0, 0), I_1 > 0, I_2, I_3 \in \mathbb{R}\}$, corresponding to normally hyperbolic 3-tori, intersects transversely the two dimensional submanifold, $R^1 N_f \subset N_f$, which corresponds to 1-resonant 3-tori, on a certain 4-dimensional energy surface, defined by the equation: $H_0(x, y, I_1, I_2, I_3) - h_{hr} = 0$, along a one dimensional curve of hyperbolic 1-resonant 3-tori. For $k^1 = (1, 0, 0)$, $R^1 N_f$ is defined by the equation:

$$
\begin{aligned}
\dot{\theta}_1\Big|_{(0,0,I_{1r},I_{2r},I_{3r})} &= \frac{\partial H_0}{\partial I_1}\Big|_{(0,0,I_{1r},I_{2r},I_{3r})} = \\
&= (\alpha_1+\frac{1}{2})I_{1r}+\alpha_3 I_{2r}+\alpha_4 I_{3r}+\alpha_5 I_{1r}I_{3r}+\alpha_7 I_{1r}^2 = 0,
\end{aligned}
\tag{25}
$$

and may be solved for I_2 as a function of the remaining actions:

$$
I_{2r}(I_1,I_3) = -\frac{(\alpha_1+\frac{1}{2})I_1+\alpha_4 I_3+\alpha_5 I_1 I_3+\alpha_7 I_1^2}{\alpha_3}, \tag{26}
$$

where for $I_1 > 0$, $I_{2r}(I_1,I_3)$ corresponds to a two dimensional family of hyperbolic resonant tori. The intersection of this two dimensional family with a certain energy surface, corresponding to some energy value $h_{hr} = H_0(0,0,I_{1hr},I_{2hr}(I_{1hr},I_{3hr}),I_{3hr})$, along a one dimensional curve is defined by the equation:

$$
H_0(0,0,I_1,I_{2r}(I_1,I_3),I_3) - h_{hr} = 0, \tag{27}
$$

which may be solved for $I_3 = I_{3hr}(I_1)$, for $I_1 > 0$. For example, to plot figure 3 we choose the energy value $H_0 = h_{hr} \approx 1.4999$, and $I_1 = 2$ as the initial value for I_1. In figure 3 projections of an orbit traveling through a family of hyperbolic 1-resonant tori (family of 1-HR) are presented. Notice that the structure of the orbit in these planes is very different from that of the 1-TPR and flat 1-PR cases (compare with figures 1, 2), and that the instabilities in I_1 seems to be smaller: In contrast to the 1-TPR case, this 1-HR orbit did not reach the boundaries of the perturbed energy surface during the time of integration ($t = 10000$), and the maximal displacement in I_1 is approximately of size $\Delta I_1 \approx 0.06$ (see also figure 7 in subsection 4.3).

4.2 Iso-energetic parabolic resonances in a 5 d.o.f. model

Here we present an example which illustrates the criticality of the dimension $n = 5$: The persistent appearance of a one dimensional family of 1-resonant parabolic 4-tori on the same energy surface, without dependence on external parameters. Consider:

$$H_0(x,y,I;\alpha) = \frac{y^2}{2} - \frac{x^2}{2}(I_1+\beta I_2) + \frac{x^4}{4} + (\alpha_1+\frac{1}{2})\frac{I_1^2}{2} + \frac{I_2^2}{2} + \frac{I_3^2}{2} + \qquad (28)$$
$$+\frac{I_4^2}{2} + \alpha_2 I_2 I_3 + \alpha_3 I_1 I_2 + \alpha_4 I_1 I_3 + \alpha_5 I_3 I_4 + \alpha_6 I_1 I_4 + \alpha_7 I_2 I_4 + \alpha_8 I_4,$$

The parameters, $\alpha_i; i = 1, ...8, \beta$, are real, and are fixed to generic values. The introduction of the new parameter β, which appears in the mixed term $\frac{x^2}{2}(I_1+\beta I_2)$ (namely was set to zero in (13)) is explained below. We use the perturbation:

$$\varepsilon H_1(x,y,\theta,I;\varepsilon,k) = \varepsilon[(1-\frac{x^2}{2})\cos(k\theta_1) + \cos(k\theta_2) + \cos(k\theta_3) + \cos(k\theta_4)]. \qquad (29)$$

The corresponding 5 d.o.f. Hamiltonian system of the Hamiltonian equations (28) and (29) is:

$$\begin{aligned}
\dot{x} &= y, \qquad (30)\\
\dot{y} &= x(I_1+\beta I_2) - x^3 + \varepsilon x\cos(k\theta_1),\\
\dot{\theta}_1 &= -\frac{x^2}{2} + (\alpha_1+\frac{1}{2})I_1 + \alpha_3 I_2 + \alpha_4 I_3 + \alpha_6 I_4,\\
\dot{I}_1 &= \varepsilon k(1-\frac{x^2}{2})\sin(k\theta_1),\\
\dot{\theta}_2 &= -\beta\frac{x^2}{2} + I_2 + \alpha_2 I_3 + \alpha_3 I_1 + \alpha_7 I_4,\\
\dot{I}_2 &= \varepsilon k\sin(k\theta_2),\\
\dot{\theta}_3 &= I_3 + \alpha_2 I_2 + \alpha_4 I_1 + \alpha_5 I_4,\\
\dot{I}_3 &= \varepsilon k\sin(k\theta_3),\\
\dot{\theta}_4 &= I_4 + \alpha_5 I_3 + \alpha_6 I_1 + \alpha_7 I_2 + \alpha_8,\\
\dot{I}_4 &= \varepsilon k\sin(k\theta_4).
\end{aligned}$$

By construction, the 5 d.o.f. integrable Hamiltonian (28) fulfills conditions (5) for equilibria in the (x,y) plane at $(x,y) = (0,0)$, for any values of the actions, I, and at $(x,y) = (\pm\sqrt{I_1+\beta I_2}, 0)$, for $I_1 \geq -\beta I_2$, and any $I_2, I_3, I_4 \in \mathbb{R}$. The system (28) fulfills the parabolicity condition (6) for $I_{1p}(I_2,I_3,I_4) = I_{1p}(I_2) = -\beta I_2$ (for any values of the actions: I_2, I_3, I_4). Hence for the system

(28), the three dimensional manifold corresponding to invariant parabolic 4-tori is: $PN_f = \{(x, y, I_1, I_2, I_3, I_4) : (x, y) = (0, 0), I_1 = -\beta I_2; \quad I_2, I_3, I_4 \in \mathbb{R}\}$. Choosing the unit vector $k^1 = (1, 0, 0, 0)$, the resonance condition (7) is fulfilled for the system (28) on PN_f iff:

$$\begin{aligned} \dot{\theta}_1\Big|_{(0,0,-\beta I_{2r}, I_{2r}, I_{3r}, I_{4r})} &= \frac{\partial H_0}{\partial I_1}\Big|_{(0,0,-\beta I_{2r}, I_{2r}, I_{3r}, I_{4r})} = \\ &= (-\beta(\alpha_1 + \frac{1}{2}) + \alpha_3) I_{2r} + \alpha_4 I_{3r} + \alpha_6 I_{4r} = 0, \end{aligned} \tag{31}$$

for any fixed set of parameters. Condition (31) is met on a two dimensional manifold, PR^1, defined by the equation:

$$I_{3pr}(I_2, I_4) = -\frac{(-\beta(\alpha_1 + \frac{1}{2}) + \alpha_3) I_2 + \alpha_6 I_4}{\alpha_4}. \tag{32}$$

Fixing an energy value, h_{pr}, imposes one extra condition on the remaining actions:

$$H_0(0, 0, I_{1p}(I_2), I_2, I_{3pr}(I_2, I_4), I_4) - h_{pr} = 0, \tag{33}$$

which may be solved for I_4, obtaining an iso-energetic one dimensional family of parabolic 1-resonant 4-tori on the curve:

$$HPR^1 = \{(x, y, I_1, I_2, I_3, I_4) = (0, 0, -\beta I_2, I_2, I_{3pr}(I_2, I_{4pr}(I_2)), I_{4pr}(I_2)); \quad I_2 \in \mathbb{R}\}. \tag{34}$$

Now the role of the parameter β becomes clear: for $\beta = 0$, this iso-energetic one dimensional family of resonant tori has no component in the I_1 direction, the direction in which the resonance occurs. It suggests that excursions associated with this resonance will lead to instabilities along this family only for non-zero β, which explains the choice of our mixed term. Indeed, when one sets $\beta = 0$, no new dynamical phenomena were numerically detected. This observation falls into a more general framework which distinguishes between different types of PR according to the relation between the actions which govern the bifurcating tori and the resonance directions. While the persistence theorems apply equally well to all cases, the corresponding perturbed orbits exhibit significantly different behaviors as demonstrated in [28] for the two possible types of 1-PR in 3 d.o.f. systems.

In figure 4 we present a perturbed orbit of the system (30), starting near a 1-resonant parabolic 4-torus of the unperturbed system, which is a part of

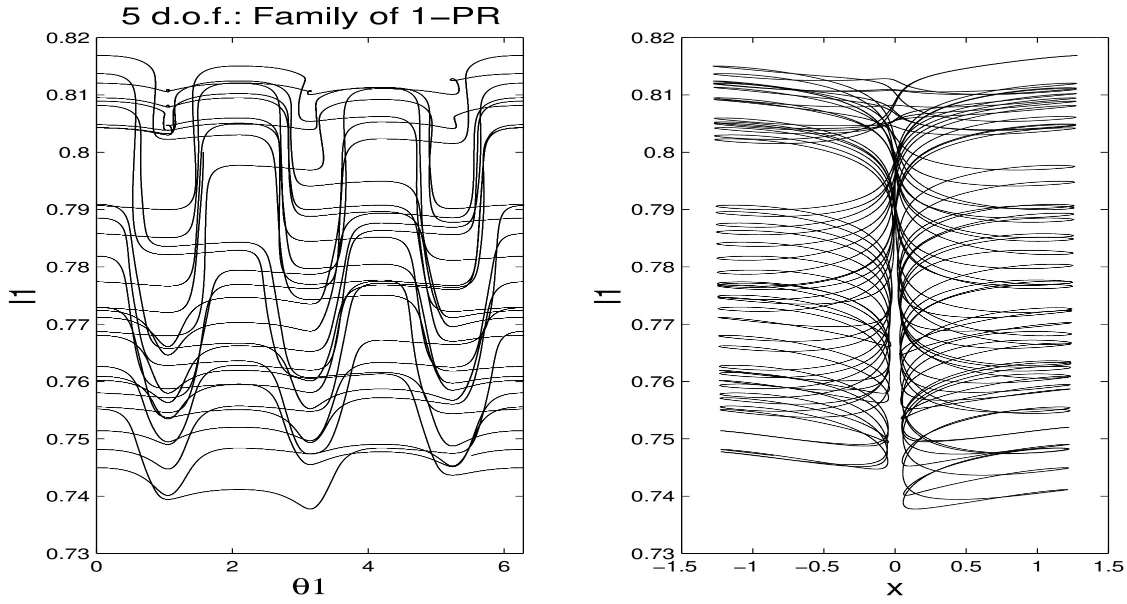

Figure 4: Iso energetic one dimensional family of parabolic 1-resonant tori in a 5 d.o.f. system.
Initial conditions and parameters: $(x, y, I_1, I_2, I_3, I_4) = (0.0002, 0., 0.8, -0.8, 0.4665, -0.999)$, $\theta_i = 1.57; i = 1, ..., 4$; $H_0 \approx -0.9626$; $\beta = 1, \alpha_1 = 1.05, \alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 3.8, \alpha_5 = \sqrt{3}, \alpha_6 = \sqrt{2}, \alpha_7 = 1.5, \alpha_8 = \sqrt{2}$; $k = 3, \varepsilon = 1e-3$; $t = 1000$.

a family of 1-resonant parabolic tori, all belonging to the same unperturbed energy surface, $h_{pr} = -0.9626$. In the left plot of this figure the projection of the 10-dimensional orbit on the (θ_1, I_1) plane is presented, and the projection of the orbit on the (x, I_1) plane is seen on the right. The remaining (nonresonant) actions stay near their initial values (performing infinitesimally small oscillations) for all integration times (up to $t = 10000$). The structure of the orbit in figure 4 somewhat resembles to the structure of the orbit of the 4 d.o.f. system (15) exhibiting a 1-family 1-HR, seen in figure 3. The perturbed orbit in this figure exhibits instabilities only in the direction of the resonant action, I_1, with displacements of size $\Delta I_1 \approx 0.08$, while the value of I_2 suffers a small drift with much weaker oscillation. Hence, the equation $I_1 + \beta I_2 = 0$, which corresponds to parabolic stability type of the tori, does not hold during the whole run time of the orbit. However, the orbit in

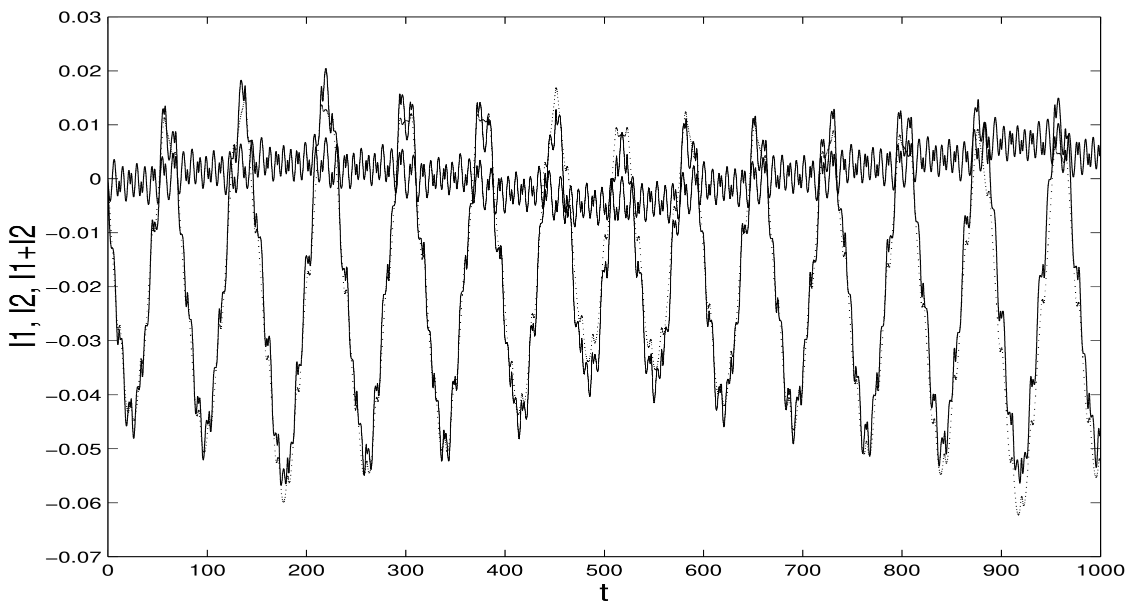

Figure 5: Iso energetic one dimensional family of parabolic 1-resonant tori in a 5 d.o.f. system.
The curves $\cdots$: $(I_1(t) - I_1(0))$, - - -: $(I_2(t) - I_2(0))$ and —: $(I_1(t) + I_2(t))$ are plotted as functions of the time, t. Initial conditions and parameters: as in figure 4.

figure 4 repeatedly passes through parabolic resonance zones, with a small drift along the parabolic family axis ($I_1 + \beta I_2 = 0$), and bifurcates to nearby elliptic and hyperbolic stability type behavior. This scenario is further illustrated in figure 5. In this figure we plot $I_1(t) - I_1(0)$ (dotted curve exhibiting large oscillations) $I_2(t) - I_2(0)$ (dashed curve exhibiting small oscillations) and $I_1(t) + \beta I_2(t)$ (solid curve) as functions of the time, t. In this figure, only the area near zero, which is covered by the curve $I_2(t) - I_2(0)$, corresponds to parabolic stability type. Below this region the actions correspond to elliptic stability type, and above it to hyperbolic stability type. See section 4.3 for further discussion, and comparison between all the mechanisms for instabilities proposed here.

When the initial values of I_1 and I_2 are zero, different behavior of nearby perturbed orbits is observed in simulations: Either one of the two orbit structures which are described for the case of isolated 1-PR in [27, 29, 28] is observed, depending on the initial values of the other actions.

4.3 Instabilities rate

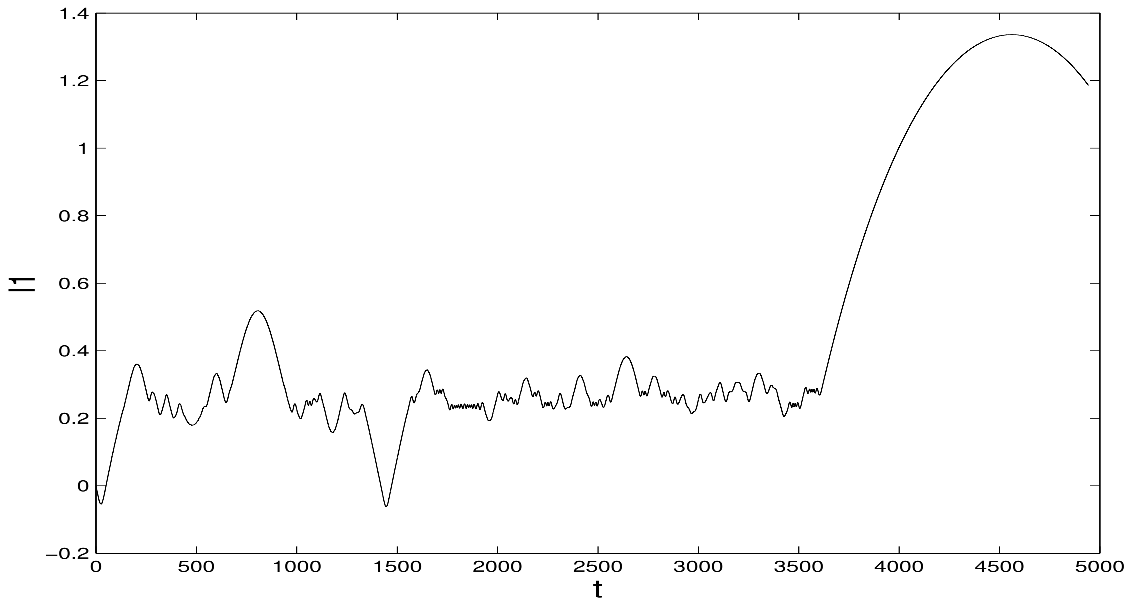

Figure 6: Flat 1-PR in a 4 d.o.f. system.
The curve $I_1(t) - I_1(0)$, is plotted as a function of the time, t. Initial conditions and parameters as in figure 2.

In subsections 4.1 and 4.2 we demonstrate that the different instability mechanisms result in qualitative different orbit structures (compare figures 1-4). Here we present some quantifiers for the resulting differences in the instability rates, which we define as the maximal observed displacement in the resonant action(s) over a certain time interval:

$$\Delta I(T, \varepsilon) = \max_{j;\ 0<t\leq T} |I_j(t) - I_j(0)|;$$

we compare these displacement - instability rates, for the different mechanisms described above, for a given fixed values of ε and T: $\varepsilon = 0.001$ and $T = 1000$ (or 5000). An exhaustive study of the dependence of these displacements and the averaged velocities by which they are reached as a function

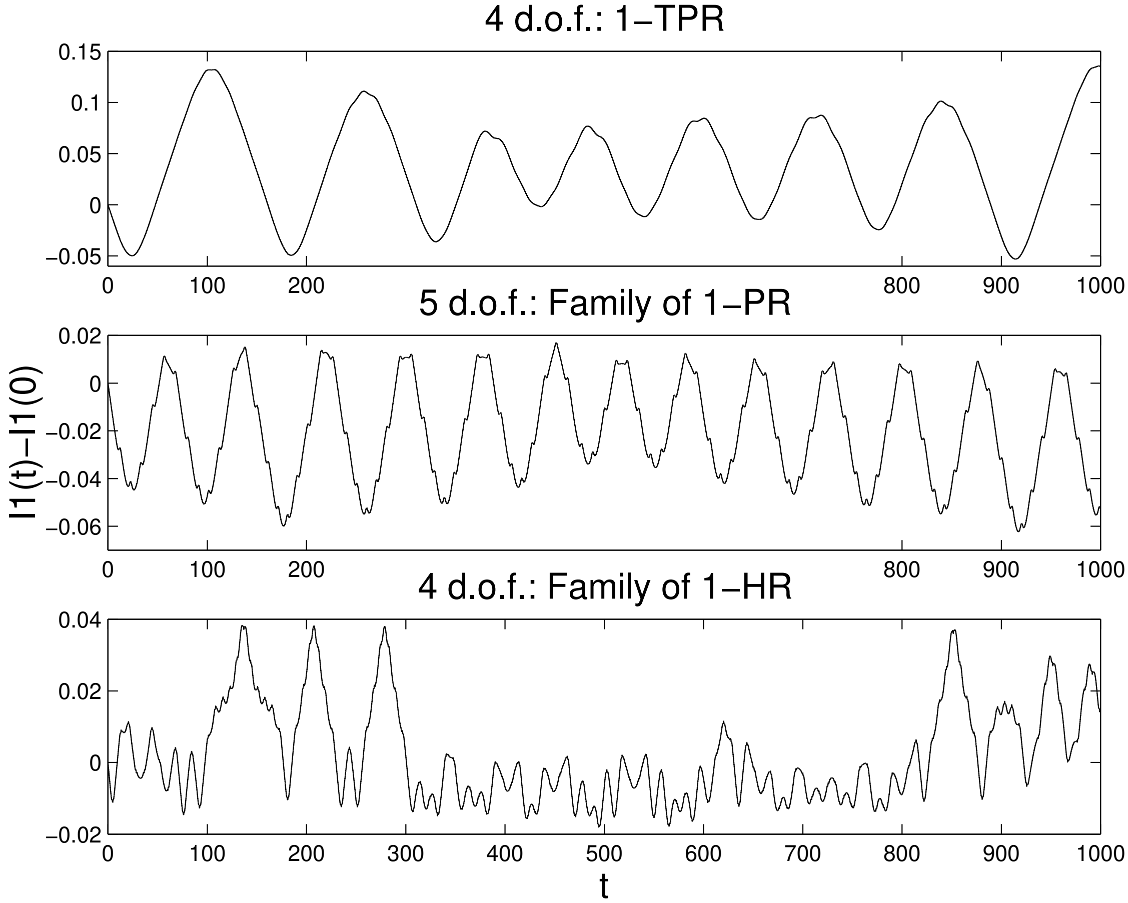

Figure 7: Upper plot: 1-TPR (see figure 1), Middle plot: 1-family of 1-PR (see figure 4), Lower plot: 1-family of 1-HR (see figure 3).

of ε and T for the different types of mechanism is certainly needed, yet it is beyond the scope of this work.

In figure 6 we present $I_1(t)$ for the degenerate case of flat 1-PR, where for fixed values of I_2 and I_3 (at which the flat 1-PR occurs) the energy surface is unbounded in the I_1 direction and large ($\Delta I(5000, 0.001) \approx 1.4$) fast ($t < 5000$, namely, probably not exponentially small in ε) instabilities occur. In all the other cases presented here (figure 7) for any fixed values of I_2 and I_3 the energy surfaces are bounded in I_1 and such large instabilities are prohibited as long as I_2 and I_3 remain close to their initial values.

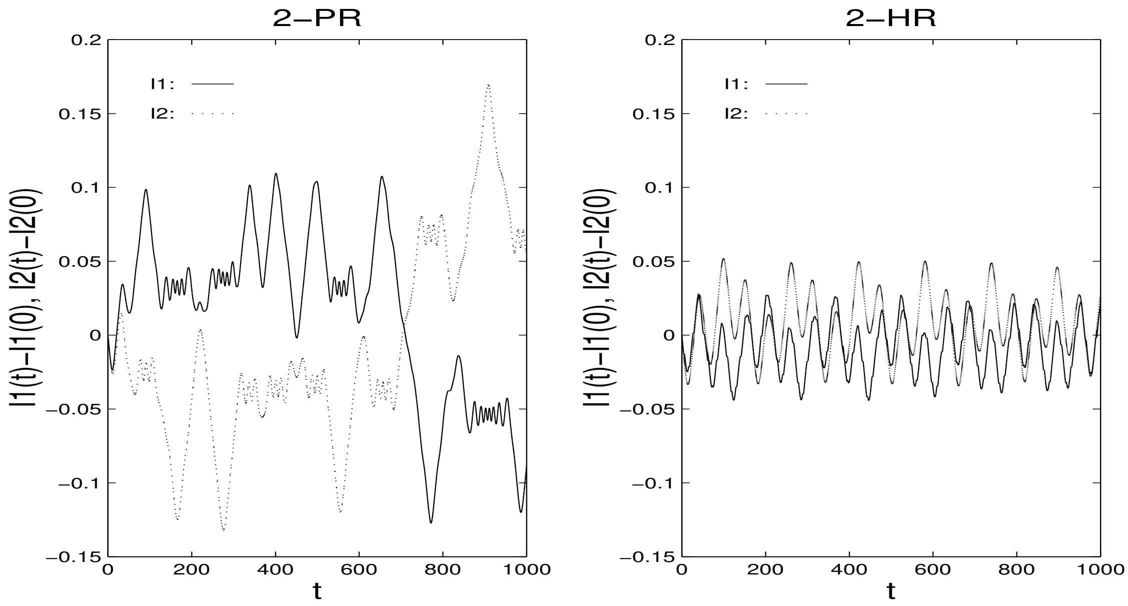

Figure 8: 2-PR versus 2-HR in the 3 d.o.f. system (12).
Initial conditions and parameters for the 2-PR: $(x, y, I_1, I_2) = (0.02, 0., 0., 0.)$; $(\alpha_1 = 1.05, \alpha_2 = 0., \alpha_3 = 2.)$; $\theta_1 = \theta_2 = 1.57$; $H_0 \approx 4e - 4$; $k = 3, \varepsilon_1 = \varepsilon_2 = 0.001$, (see figure 9).
Initial conditions and parameters for the 2-HR: $(x, y, I_1, I_2) = (0.02, 0., 1., -2.)$; $(\alpha_1 = 1.5, \alpha_2 = 1., \alpha_3 = 1.)$; $\theta_1 = \theta_2 = 1.57$; $H_0 \approx -1.0196$; $k = 3, \varepsilon_1 = \varepsilon_2 = 0.001$, (see figure 10).

In figure 7 we present the curves $I_1(t) - I_1(0)$ for time intervals $t \in [0, 1000]$ (in fact, the same instabilities are observed up to $t = 10,000$). The upper plot of this figure corresponds to the instabilities which are developed for the 4 d.o.f. system (15) as a result of a 1-TPR: $\Delta I(1000, 0.001) \approx 0.2$; in the middle plot the instabilities which develop as a result of an iso-energetic 1-family of 1-PR in the 5 d.o.f. system (30) are presented: $\Delta I(1000, 0.001) \approx 0.08$; in the lower plot the instabilities resulting from an iso-energetic 1-family of 1-HR in the 4 d.o.f. system (15) are shown: $\Delta I(1000, 0.001) \approx 0.06$. Though relatively ordered, the mechanism of TPR exhibits the strongest and the fastest instabilities. Moreover, in this case the displacements in I_1 are maximal; the orbit travels between the lowest and the uppermost values of I_1 on the energy surface at the given I_2 and I_3 values, on time intervals of order $O(100)$. In the other cases such displacements do not occur even on longer time intervals. We suggest that in the 1-TPR case when instabilities in I_1 develop and the orbit bifurcates to elliptic stability type it goes through successive elliptic resonance zones (as the first frequency in this case do not depend on the largest term in I_1), resulting in long runs on which $\dot{I}_1$ has one sign. In the other two cases the orbit visits near elliptic or hyperbolic tori (of the unperturbed system) which are only near resonance (as in these cases the first frequency does depend on all the terms in I_1) and $\dot{I}_1$ oscillates near zero. Moreover, the fact that the tangency point corresponds to a resonant parabolic torus is crucial, as a parabolic torus admits a bifurcation point near which resonant tori of other stability types reside. The numerical evidence presented here and in [37, 27, 29, 28] indicates that *the successive changes in the orbit through different stability types result in larger instabilities even if the zones it visits are only near resonant.*

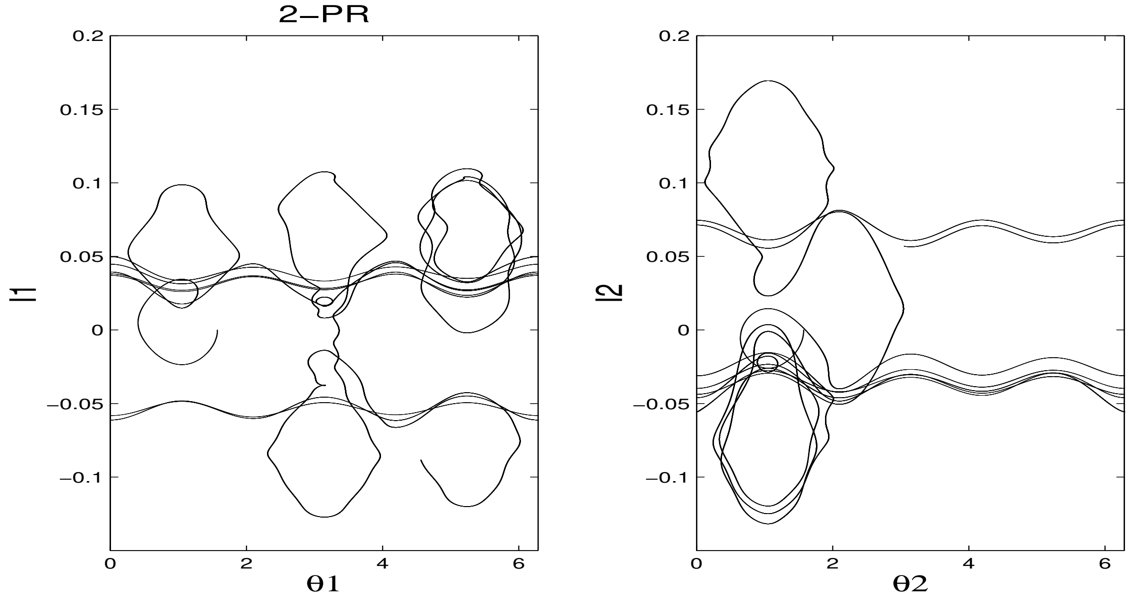

Figure 9: 2-PR in a 3 d.o.f. system.
Initial conditions and parameters: $(x, y, I_1, I_2) = (0.02, 0., 0., 0.)$; $(\alpha_1 = 1.05, \alpha_2 = 0., \alpha_3 = 2.)$; $\theta_1 = \theta_2 = 1.57$, $H_0 \approx 4e-4$; $k = 3, \varepsilon_1 = \varepsilon_2 = 0.001$, $t = 1000$.

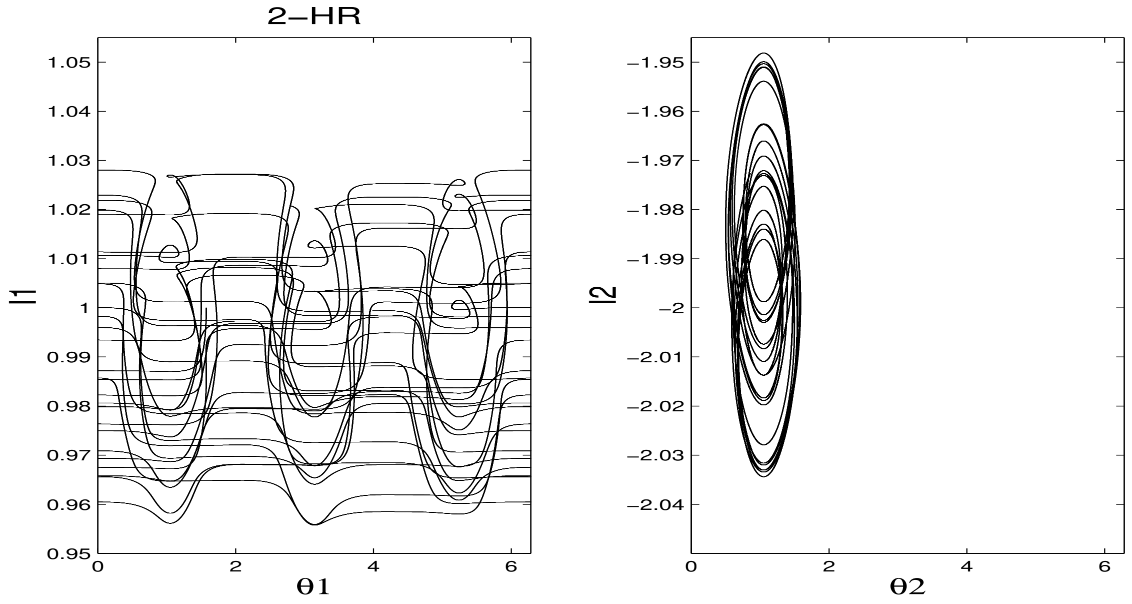

Figure 10: 2-HR in a 3 d.o.f. system.
Initial conditions and parameters: $(x, y, I_1, I_2) = (0.02, 0., 1., -2.)$; $(\alpha_1 = 1.5, \alpha_2 = 1., \alpha_3 = 1.)$; $\theta_1 = \theta_2 = 1.57$; $H_0 \approx -1.0196$; $k = 3, \varepsilon_1 = \varepsilon_2 = 0.001$, $t = 1000$.

The above statement is further supported when comparing 2-PR (parabolic torus of fixed points in the integrable system) versus 2-HR (hyperbolic torus of fixed points in the integrable system) in the 3 d.o.f. system (12). Figure 8 demonstrates that the instabilities developed in both actions are greater for the parabolic resonance than for the hyperbolic resonance; the maximal displacements observed for the resonant actions, I_1 and I_2, in the plotted time interval ($t = 1000$) are: $\Delta I_1 \approx 0.25$ and $\Delta I_2 \approx 0.3$ for 2-PR (figures 8-left and 9); $\Delta I_1 \approx 0.07$ and $\Delta I_2 \approx 0.08$ for 2-HR (figures 8-right and 10); Hence, $\Delta I(1000, 0.001) \approx 0.3$ for 2-PR, and $\Delta I(1000, 0.001) \approx 0.08$ for 2-HR. The different orbit structures for both cases are presented in figures 9 and 10, respectively. The orbit corresponding to 2-PR slides through bifurcations as the actions changes in time, while the orbit corresponding to 2-HR does not, and its behavior corresponds to a hyperbolic stability type behavior for the whole time of its run.

In 3 d.o.f. systems, 2-PR is a co-dimension one phenomenon; its existence becomes persistent (*without* dependence on external parameters) for $n \geq 4$ d.o.f. systems. In higher dimensional systems the orbit structure and the rate of instability of the resonant actions near double resonance appear to be similar. We present here the 3 d.o.f. orbits (figures 8, 9 and 10), as $n = 3$ is the first value of n for which such 2-resonances exist (2-HR persistently without dependence on external parameters, and 2-PR persistently as a co-dimension one phenomena).

5 Energy surfaces and the resonance web

For small perturbing Hamiltonians (namely if $|H_1(\cdot)| < |h|$ and $\varepsilon \ll 1$), a perturbed orbit with energy H resides, for its given realization[12] of the angle vector θ, on the energy surface $H_0(\cdot) = H - \varepsilon H_1(\cdot, \theta)$. Hence, the structure of the unperturbed energy surfaces and their resonance webs in an $O(\varepsilon h)$-interval of energies near H supplies global information on the allowed range of actions of the perturbed orbits. The classification of all the possible structures of the energy surfaces of integrable Hamiltonians is extremely challenging and has been completed for the 2 d.o.f. case only, see [6, 14, 23, 24, 25]. Some of these ideas have been extended to classify the integrable 3 d.o.f. dynamics of the Kovalevskaya top [8], the most complicated known example of an integrable 3 d.o.f. system. Below, we explore the mutual effects of the non-trivial energy surface structure and the nature of the resonance web for systems of the form (12): systems in which one d.o.f. undergoes a pitchfork bifurcation and all other d.o.f. correspond simply to motion along $n - 1$ dimensional tori. Related ideas appear in the work on "resonance streaming", where it is argued that in 2 d.o.f. integrable systems small angle between the intersecting resonance and energy curves (plotted in the frequency space) enhances the effect of added noise [26, 40]. The analysis of this mutual influence in the general case, the relation to Fomenko-Oshemkov graphs [14, 34] and to the energy-momentum bifurcation diagrams which explore the behavior of orbits in an *energy-neighborhood* of a given energy value (see [28, 38]), is the subject of current research.

To proceed, we first recall the trivial geometry of the resonance surfaces. The resonance surfaces are co-dimension one hyper-planes, passing through

[12] Due to resonances, in principle, there may exist ranges of θ values which are not realized by a perturbed orbit. It is clear that such ranges exist for finite time (finite segments of the orbit). It is still an open question if such exist asymptotically.

the origin of the frequency space, and as Arnold realized, this implies that they intersect densely on all energy surfaces of integrable Hamiltonian systems with 3 or more d.o.f.. The strength of a resonance is determined by $|k| = |k_1| + \cdots + |k_n|$, where k is the smallest integer vector for which a resonance condition of the form $\langle k, \omega \rangle = 0$ holds. The smaller $|k|$ is, the stronger the resonance is - the larger is its effect on the perturbed system. In particular, the strongest[13] resonances possible are realized when $k = e_n^j = (0, ..., 1, ...0)$ for some $j \in \{1, \ldots, n\}$ (where 1 is in the j^{th} place). Obviously, only finitely many ($\leq n$) strongest resonances may intersect a certain point of the unperturbed energy surface (see for example figures 11-16), and such an intersection does not necessarily occur on every energy surface (see for example figure 12). Detecting the *junctions* (intersections of several resonance surfaces) of the strongest resonances[14] and the energy surfaces which intersect them is pivotal for understanding the instabilities in the perturbed system. In figure 11 we present the standard picture of several strong resonance surfaces (with $|k| \leq 3$) in a three frequency space. On the right we include the vertical resonance surface, $\omega_0 \equiv 0$, whereas on the left we simply record its intersection with the other surfaces by a starred line, namely, starred lines denote double resonances, one of which is that of vanishing first frequency. Recall that in our model the first d.o.f. (the (x, y) plane) plays a special role - it is the only normal d.o.f. in which invariant tori of the unperturbed system change their stability.

An important observation is that *parabolic tori are a priori resonant*: their normal frequency vanishes. Indeed, let $\omega = (\Omega, \omega^{n-1}) \in \mathbb{R}^n$ denote the n dimensional vector of frequencies, including the normal frequency Ω, and the inner frequencies ($\omega^{n-1} \in \mathbb{R}^{n-1}$) of the $(n-1)$-torus. Parabolicity implies $\Omega = 0$, hence, $k^1 = e_n^1 = (1, 0, ..., 0)$ satisfies the resonance condi-

[13] In fact, defining "strong resonance" as above, in the context of the unperturbed system, is ill-defined; Formally, if the frequencies of a certain torus fulfill m independent resonance conditions, then, by using a symplectic change of variables these conditions may be replaced by m conditions of the form: $\langle e_n^j, \omega \rangle = 0$, $j = 1, \ldots, m$, converting any resonance into a strong resonance. Here, the form of the anticipated *perturbation* plays a major role - the minimal wave number multiplying the resonant terms in the perturbation determines the effective $|k|$ value, see [2] for exact definition and discussion. Conversely, the analysis determines which perturbation will cause the largest instability in the vicinity of a given resonance junction.

[14] For a study of resonance junctions, which correspond to one strong resonance intersecting with a weaker resonance (there are infinitely many such junctions), see e.g. [17] and references therein.

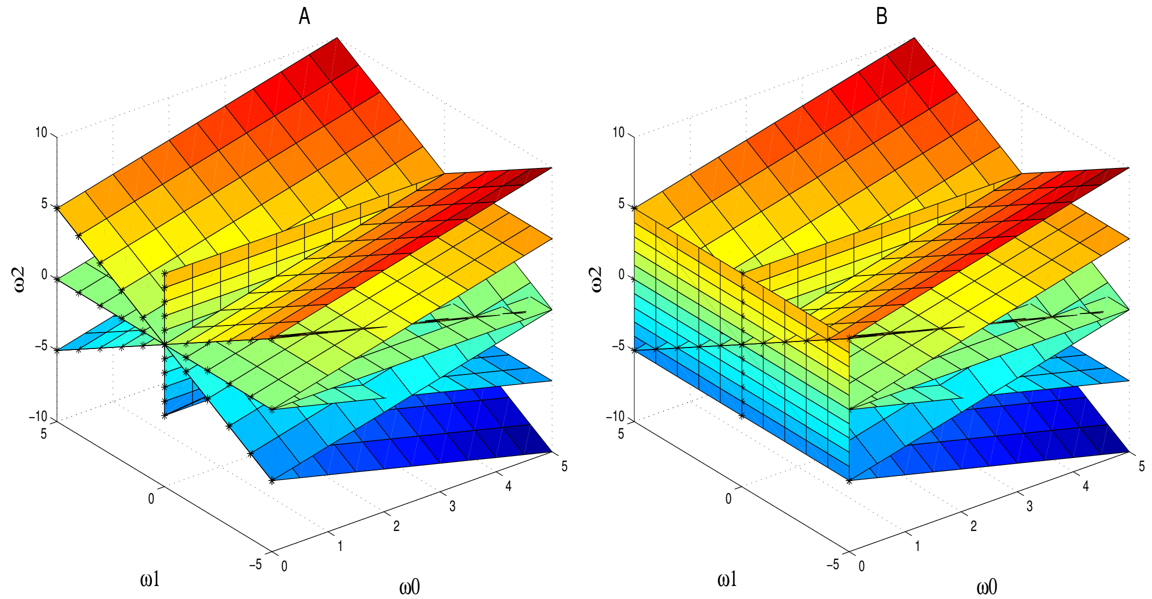

Figure 11: Intersection of few resonance surfaces, corresponding to strong resonances, in the three frequency space.
--*: Resonance lines, corresponding to vanishing of the first frequency.

tion $\langle k^1, \omega\rangle = 0$. Lower dimensional resonance implies that there exists at least one additional vector of integers, $k^2 = (0, l_{n-1}), l_{n-1} \in \mathbb{Z}^{n-1}$, such that $\langle k^2, \omega\rangle = 0$. Hence, *parabolic lower dimensional resonant tori correspond to junctions in the resonance web with at least one strongest resonance.* In particular, if the parabolic torus appears at the origin, where all resonances intersect, it corresponds to $(n-1)$-resonant $(n-1)$-torus, namely to a parabolic torus of fixed points. Theorem 3 implies that such a scenario is persistent in a one parameter family of integrable n d.o.f. Hamiltonian systems with $n \geq 3$.

In [26] an example of a 3 d.o.f. Hamiltonian of the form $H_0(I)$ is shown where the energy surfaces are spheres in the frequency space, centered at the origin. In this case, resonance junctions cover each energy surface densely and *uniformly*. Furthermore, the origin, where all resonance surfaces intersect, belongs to the spherical energy surface with diminishing radius. The nearby energy surfaces are again spheres of radius $O(\varepsilon h)$, hence the actions corresponding to frequencies near the origin remain near their initial values under small Hamiltonian perturbation.

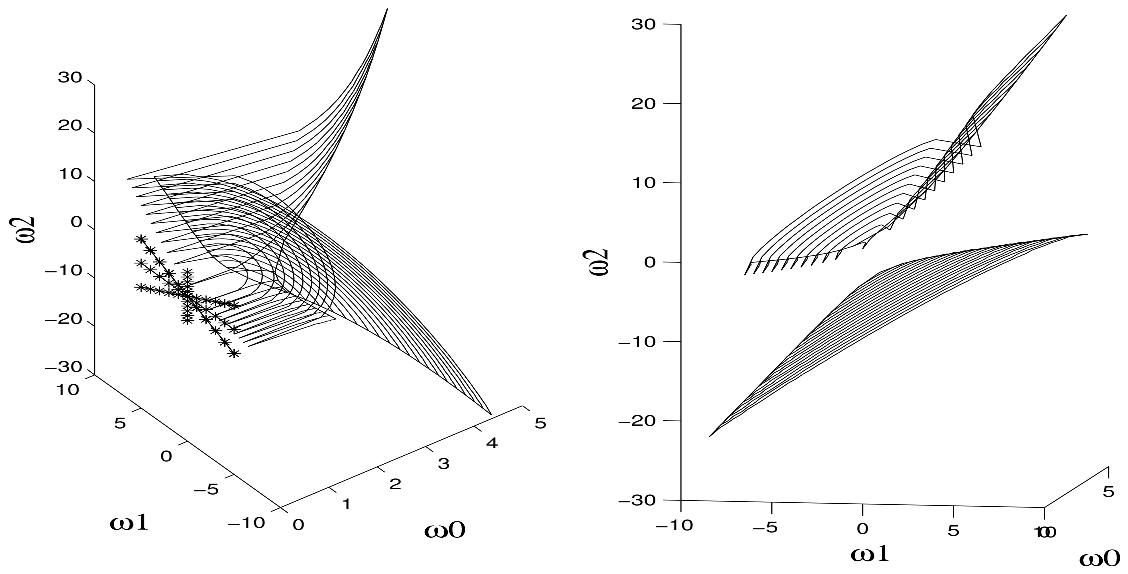

Figure 12: Energy surface with elliptic component and hyperbolic component. The $H_0(x, y, I) = -1.5$ surface and the strong resonances with vanishing first frequency are shown from two view points.
Parameters: $\alpha_1 = 1.05, \alpha_2 = 0., \alpha_3 = 2.$. *-*-*: Resonance lines.

In systems of the form $H_0(x, y, I)$ with a range of actions which includes hyperbolic lower dimensional tori, the energy surfaces have a completely different geometry as explained below. In figures 11-17 we plot the resonance web and the non spherical energy surfaces of the 3 d.o.f. integrable system (12) in the frequency space $\omega = (\omega_0, \omega_1, \omega_2)$, where ω_0 is the frequency of the motion in the (x, y) plane. Consider the energy surface $H_0(x, y, I) = H$. On it, almost all orbits reside on tori with a given frequency vector, hence the energy surface may be represented as a co-dimension one surface $\omega = \omega^H$ in the frequency space. The branches and components of ω^H are smooth as long as no singular points are encountered. Note that for the unperturbed Hamiltonian (12) ω_0 has one sign for the full phase space, hence only positive values of ω_0 are presented. Changes in the geometrical properties of the energy surfaces as a function of H correspond to bifurcation values of the energy[15].

Consider, for example, the structure of the energy surfaces of a system with a figure eight separatrix in the normal direction I_0, as (12). We want to explore the structure of its energy surfaces in the (x, y, I) co-ordinate system and its manifestation in the frequency space, $\omega = (\omega_0, \omega_1, \omega_2)$. We assume that the action angle coordinates (I, θ) are chosen, according to the applied perturbation, so that they are fixed. The simplest type of energy surface component contains only elliptic tori. For any fixed $(I_2, \ldots, I_{n-1})$ values on such a component, if the allowed motion in the I_1 direction is bounded for these values as in (12), the surface appears as a 2-sphere in the (x, y, I_1) space. The poles of this sphere are normally elliptic $n-1$ dimensional tori, and they correspond to the boundaries of this component of the energy surface. In the ω space, this component appears as a smooth co-dimension one surface with boundaries, see the lower component of the energy surface in figure 12. There, the $H_0 = -1.5$ energy surface is presented from two different view points; each line on the energy surface corresponds to a fixed value of I_2, and as before, the starred lines denote the strong double resonance lines. Transverse intersection of a smooth component of ω^H with one of the planes $\omega_j = 0$ corresponds to a strong resonance, as seen in this figure for $j = 1, 2$ (it cannot occur for $j = 0$ since $\omega_0 \geq 0$).

Now, examine the structure of a component of the energy surface which includes a hyperbolic torus with the figure eight separatrices in the normal direction I_j (in our case, in the $I_0 = x^2 + y^2$ direction). Consider a trajectory belonging to an n torus which is very close to the separatrix. The return time of this trajectory in the (x, y) plane approaches infinity, namely its frequency ω_0, is close to 0. It follows that the energy surface ω^H collides at the separatrix with the plane $\omega_0 = 0$, and then bounces back with the same sign of ω_0 (since the direction of motion does not change from the exterior to the interior tori). Furthermore, on one side of this singular manifold the energy surface splits to two parts, corresponding to the two families of periodic orbits which reside "inside" the separatrix (in our figures the two parts are

[15] Such bifurcations are better recorded with energy-momentum bifurcation diagrams [38, 28], where the ω_0 axis is replaced with an energy axis, H_0. Each energy surface corresponds to simple planes perpendicular to the H_0 axis, and the singular levels on each such surface are presented. Furthermore, following [8], we suggest that by adding the Fomenko tree associated with each region of the energy surfaces, a complete classification of an integrable system in an n dimensional space may be achieved. A full description along these lines is under preparation.

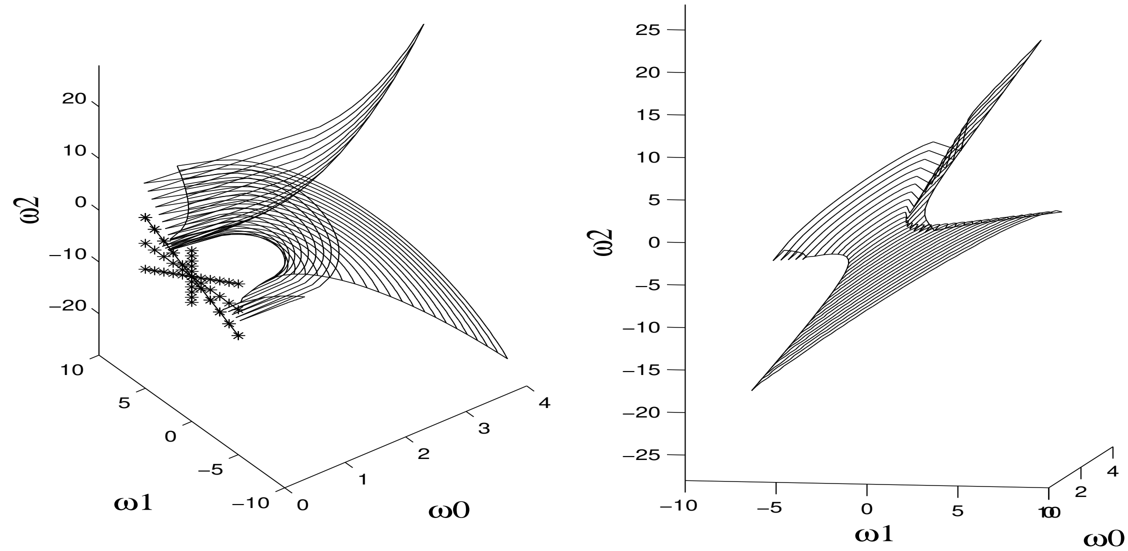

Figure 13: Intersection of the energy surface $H_0(x, y, I) = 1.5$ with strong resonances, corresponding to vanishing of the first frequency, from two view points.
Parameters: $\alpha_1 = 1.05, \alpha_2 = 0., \alpha_3 = 2..$ *-*-*: Resonance lines.

identified due to symmetry). The upper component of the energy surface of figure 12 ($H_0 = -1.5$) and the $H_0 = 1.5$ energy surface shown in figure 13 clearly demonstrate the emergence of these singular folds, corresponding to two families of hyperbolic 2-tori residing on these surfaces. Furthermore, in figure 13 one of the starred lines intersects one of the lines of singularities, at a hyperbolic 1-resonant 2-torus. Theorem 2 implies that the intersection of the singularity line with some resonant surface is unavoidable. Notice that even though the singular fold is contained in the $\omega_0 = 0$ plane, this intersection does not correspond to double resonance of the lower dimensional torus: $\omega_0 = 0$ at the homoclinic loop, whereas the normal frequency of the hyperbolic torus is imaginary and is non-zero. Figure 3 shows the motion of a perturbed orbit along the intersection of this singularity surface and a resonance plane (in a 4 d.o.f. system, where the intersection occurs along a one dimensional curve).

The colliding surface, at which ω^H is singular (non-smooth), is clearly of

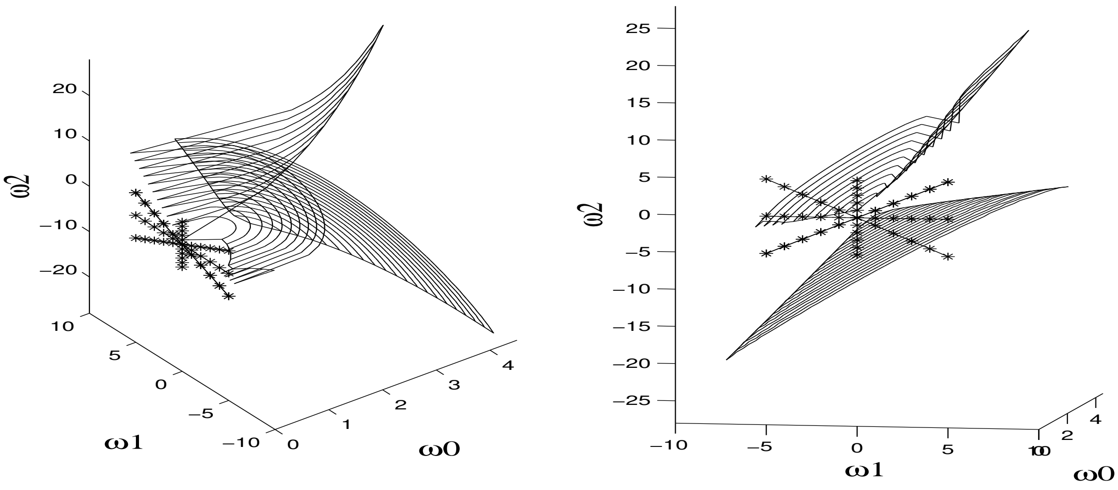

Figure 14: Intersection of the energy surface $H_0(x, y, I) = 0$, containing a 1-resonant parabolic 2-torus, with strong resonances, corresponding to vanishing of the first frequency, from two view points.
Parameters: $\alpha_1 = 1.05, \alpha_2 = 1., \alpha_3 = 2..$ *-*-*: Resonance lines.

co-dimension two, and it corresponds to the family of hyperbolic tori which live on the given energy surface, see corollary 4 and remark 4. The end points of this collision surface, where the projection singularity heals and the energy surface cease to contain hyperbolic tori, corresponds to parabolic tori (a co-dimension 3 surface, namely points in figures 12-15, see corollary 5 and remark 4. If such an end surface (in the figure, a point) intersects another resonance surface (here, for the strong resonances, one of the end points is contained in a starred line) a parabolic resonant torus is born. It is now clear that with additional d.o.f. such an intersection (of the boundary of the collision surface and the resonances on the $\omega_0 = 0$ plane) is generically transverse (Theorem 1 and Corollary 5), hence PR are expected to occur on surfaces corresponding to a range of energies. For the 3 d.o.f. case, since generically the end points (corresponding to the inner frequencies of the parabolic tori) change continuously with the energy values, there exists a set of dense values of energies for which these end points hit resonance surfaces and PR are created. In the figures, when the end points of the singularity curves touch starred lines (as in figure 14) we see that a strong PR occurs.

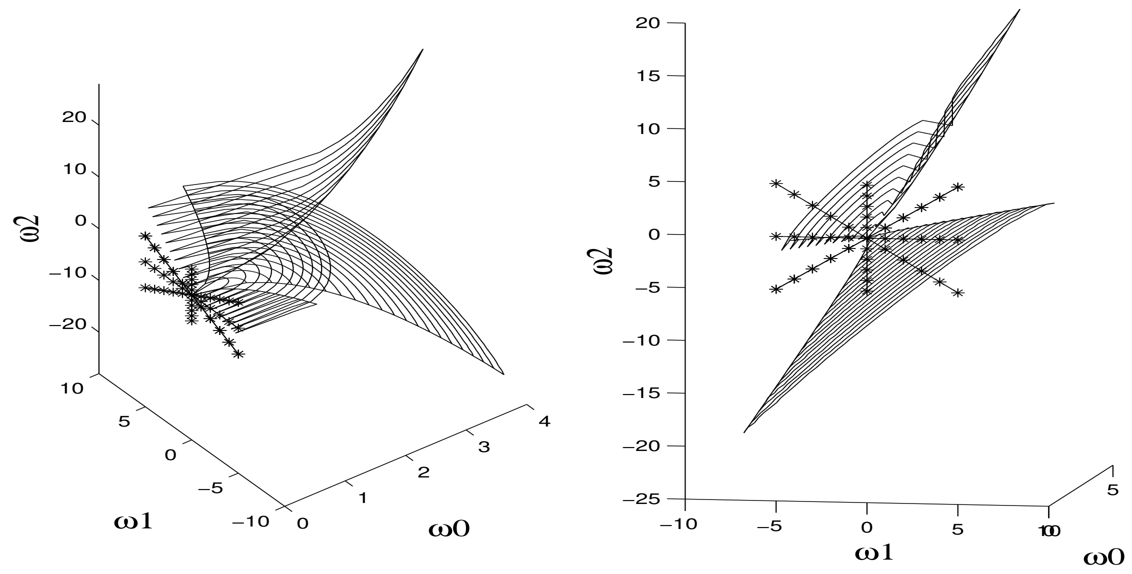

Figure 15: Intersection of the energy surface $H_0(x, y, I) = 0$, containing a parabolic 2-resonant 2-torus, with strong resonances, corresponding to vanishing of the first frequency, from two view points.
Parameters: $\alpha_1 = 1.05, \alpha_2 = 0., \alpha_3 = 2..$ *-*-*: Resonance lines.

In this case (figure 14) the starred point corresponds to a *double resonance*, as the normal frequency of the parabolic lower dimensional torus vanishes. The structure of perturbed orbits near 1-PR, which appear for a dense set of energy values, is similar to the one observed in the 2 d.o.f. case, see [28, 37].

In sections 3 and 4 we demonstrate that further degeneracies, which are persistent in higher dimensional systems without dependence on external parameters, result in strong instabilities. A full description of the manifestation of these scenarios in term of the structure of ω^H and the energy-momentum bifurcation diagram is under development.

One degeneracy we explore is the existence of a normally parabolic torus of fixed points, which is of co-dimension one (Theorem 3), and corresponds to *a local* violation of the KAM non-degeneracy condition. The induced strong instabilities of a perturbed orbit with initial values near this point is presented in figures 8 and 9. Graphically, such a scenario happens when the boundary of the singularity surface of the unperturbed energy surface (here

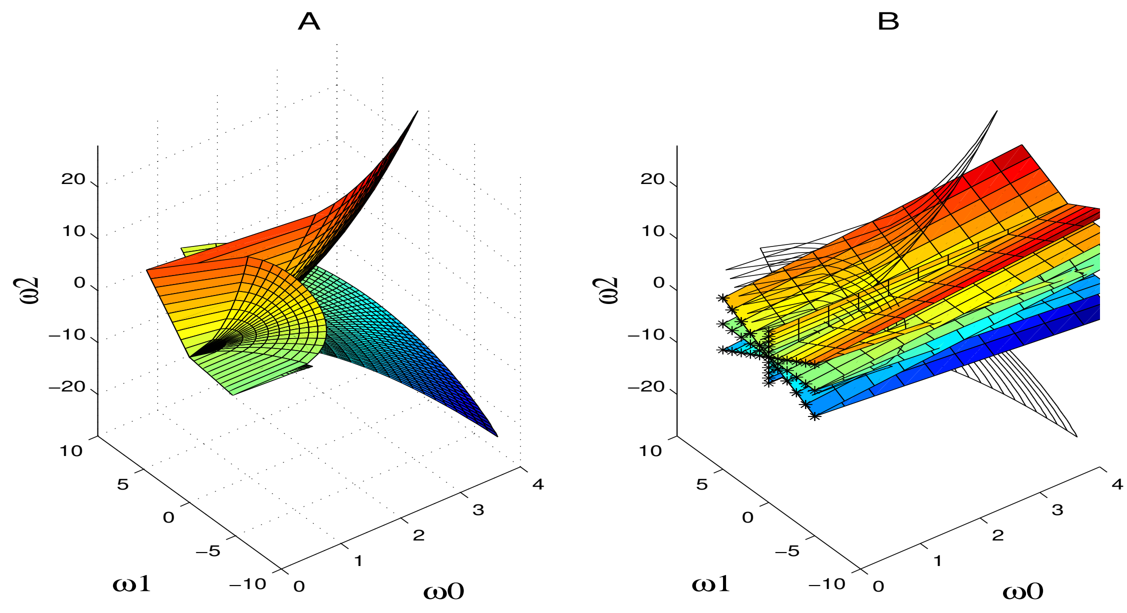

Figure 16: A. An energy surface, $H_0(x, y, I) = 0$, containing a parabolic 2-resonant 2-torus, in the three frequency space. B. Intersection of the energy surface with few resonance surfaces, corresponding to strong resonances. Parameters: $\alpha_1 = 1.05, \alpha_2 = 0., \alpha_3 = 2.$. *-*-*: Resonance lines, corresponding to vanishing of the first frequency.

the end points of the singularity lines) passes through the origin, where all the resonances (and starred lines) intersect, see figure 15. In 4 or more d.o.f. systems the existence of 2-PR is persistent without dependence of the system on external parameters (Theorem 1 and Corollary 1), and the local violation of the KAM iso-energetic non-degeneracy condition is avoided; yet, similar instabilities and orbit structure as shown in figures 8 and 9 are observed in numerical simulations.

To illustrate how the resonance surfaces intersect the unperturbed energy surfaces in the frequency space, namely to visualize the Arnold web for such systems, we present here figures 16 and 17. Figure 16A presents the energy surface using a surface plot, whereas figure 16B presents the intersection of the energy surface with the resonance surfaces of figure 11. Finally, in figure 17 we present the resulting resonance web (the lines of intersection between the $H_0 = 0$ energy surface and the strong resonance surfaces). Viewing all

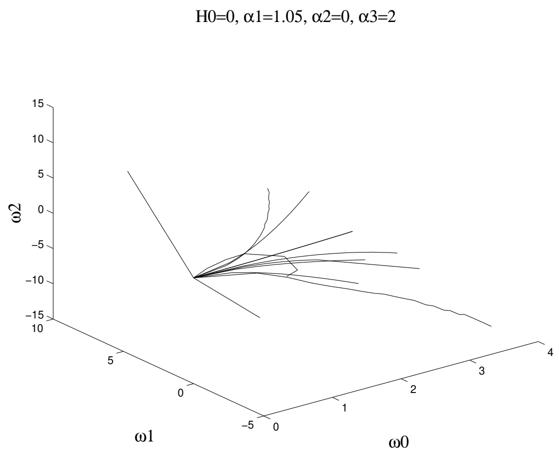

Figure 17: The intersection lines of the energy surface $H_0(x, y, I) = 0$, containing a parabolic 2-resonant 2-torus, with a few strong resonance surfaces.

the plots from figures 11, 15, 16 and 17 gives an idea on how the resonance surfaces intersect such a non spheric energy surface in the frequency space. Clearly, the structure of the resonance web for such systems is very far from being uniform, as it is for the spherical energy surfaces which are centered at the frequency origin. The many resonance lines which spring out from the origin are characteristic to the $(n-1)$-PR scenario by which the origin is contained in the energy surface.

Summarizing, we discovered that the projection of the energy surfaces of generic Hamiltonians of the form $H_0(x, y, I)$ (of class $\mathcal{P}_1$) to the frequency space has the following properties:

- The energy surface is singular along a co-dimension 2 surface belonging to the $\omega_0 = 0$ plane. This singularity surface corresponds to hyperbolic lower dimensional tori and their separatrices. The boundaries of the singularity surface (of co-dimension 3) correspond to parabolic tori.
- Parabolic resonant tori may be recognized as resonance junctions which belong to the boundary of the hyperbolic singular surface.
- Parabolic torus of fixed points appears when the boundary of the singular surface contains the origin.

- The zero energy surface which passes through the origin may have large extent in the actions directions.
- While the resonance surfaces still intersect the energy surfaces densely, the *uniformity seems to be lost.*

6 Conclusion and Discussion

Parabolic resonant tori always correspond to junctions of strong (low order) resonances in the unperturbed system. Hence, PRs always correspond to cross-resonance diffusion in action variables. Up till now special examples, exhibiting only hyperbolic behavior, were constructed to study cross-resonance diffusion, as people believed that hyperbolic resonances result in the strongest possible instabilities. Here and in [37, 27, 29, 28] we present numerical evidence that the existence of parabolic resonant tori in the unperturbed system not only does not interfere with creation of instabilities but may even induce stronger instabilities than the instabilities exhibited near hyperbolic resonant tori. Parabolic tori are a priori resonant and are a part of some bifurcation scenario, where near them tori of other stability types reside. We propose these reasons for the observation that PRs result in stronger instabilities than HRs. Moreover, we prove that the existence of parabolic resonances is generic in the class of near integrable systems with three or more d.o.f., whose integrable part possesses bifurcating tori, and is persistent in the space of near integrable n d.o.f. Hamiltonians for each $n \geq 3$. Hence, they are expected to appear in many applications, and their study is of major importance. In particular, PRs are expected to appear in systems with non-separable integrable skeleton. Such are typical, for example, to unperturbed systems which preserve angular momentum, yet are a-typical to systems which are composed of n slightly coupled oscillators. We suggest that since most higher-dimensional studies have concentrated on studying the latter models, the phenomena of PR and its higher order degeneracies has not received its proper exposure.

We propose here several mechanisms for instabilities, which are persistent without dependence on external parameters in systems with 4 or more d.o.f., hence $n = 4$ is a critical n value. In particular, we demonstrate that 1-TPR is a unique mechanism which results in fast strong instabilities, and it persistently exists for $n \geq 4$ d.o.f. systems. Orbits of near integrable 3, 4, and 5 d.o.f. systems exhibiting PRs are numerically observed to exhibit complicated behavior, which includes successive slides through bifurcations (similar to the ones observed in dynamical bifurcations, see [22]) with large variations in those actions which are involved in both the bifurcation and resonance.

It is numerically demonstrated that orbits exhibiting 2-PR exhibit relatively strong and fast instabilities in the direction of both resonant actions. It is proved that existence of 2-PR is persistent for near integrable Hamiltonian systems with 4 or more d.o.f. without dependence on external parameters. In 3 d.o.f. systems the existence of parabolic 2-resonant tori locally violates the KAM iso-energetically non-degeneracy condition. In $n \geq 4$ d.o.f. systems, parabolic 2-resonant tori exist on energy surfaces for which the KAM iso-energetically non-degeneracy condition does not fail on any of their points. In both these cases the corresponding near by perturbed orbits are observed to exhibit similar behavior, with similar size of displacements in the actions variables. This may indicate that the creation of such kind of cross resonance diffusion instabilities is independent of iso-energetic KAM degeneracies.

Finally, to some speculations. Arnold diffusion mechanism relies on the creation of whiskered nonresonant $(n-1)$-tori (whiskers) by the perturbation, which in turn create the transition chains. However, only sufficiently non-resonant whiskers are preserved under small Hamiltonian perturbations, and in generic systems gaps are created. What structures correspond to the boundaries of the gaps? Since on one side of the gap the tori are hyperbolic and on the other they are not (perhaps they disintegrate to smaller tori with different normal stabilities), it is quite possible that the boundary of the gaps is precisely given by families of parabolic tori, dense set of which are resonant. Furthermore, one expects that in the gaps the lower dimensional tori will undergo generic bifurcations. For example, since the existence of action values for which parabolic 1-resonant $(n-2)$-tori are created is generic in the class $\mathcal{P}_2$, and is persistent for smooth integrable $n \geq 4$ d.o.f. Hamiltonians (see section 2), we expect such parabolic resonant $(n-2)$-tori to exist in the gaps. The implications of such a scenario are vast. We have demonstrated numerically that orbits starting in the vicinity of parabolic resonant tori develop complicated behavior and relatively fast instabilities. It follows that if the various mechanisms of instability associated with parabolic resonance exist in the gaps and on their boundaries, they supply a much faster mechanism for transport than the instability through the whiskers. For example, recent works, such as [7, 17, 41] (see also references therein), showed that in a priori unstable systems "diffusion" in action variables exists in the gaps between the nonresonant whiskers (diffusion across resonances). In such systems, the relative instability rate in the gaps appears to be system dependent; in some examples it is faster than the instability rate along the nonresonant whiskers chain, while in other examples it is slower [7, 17]; in [41] a transition chain of both non-resonant and resonant hyperbolic tori was constructed, and it was suggested there that the development of instabilities (diffusion) in the action variables is independent of KAM tori. These findings are in accordance to our suggestion that parabolic resonant tori may play an important role in traversing the gaps.

References

[1] V. I. Arnol'd. Instability of dynamical systems with several degrees of freedom. *Dokl. Akad. Nauk SSSR (Russian)*, 156(1):9–12, 1964. English transl.: Sov. Math. Dokl., 5, 581-585; Russian Math. Surveys, 18, 85.

[2] V. I. Arnol'd. *Dynamical Systems III*, volume 3 of *Encyclopedia of Mathematical Sciences*. Springer-Verlag, second edition, 1993.

[3] V. I. Arnol'd. *Mathematical problems in classical physics*. Trends and Perspectives in Applied Mathematics. Springer-Verlag, NY, 1994.

[4] S. Bolotin and R. S. MacKay. Multibump orbits near the anti-integrable limit for lagrangian systems. *Nonlinearity*, 10:1015–1029, 1997.

[5] S. Bolotin and D. Treschev. Unbounded growth of energy in nonautonomous hamiltonian systems. *Nonlinearity*, 1999.

[6] A. V. Bolsinov and A. T. Fomenko. Trajectory classification of simple integrable hamiltonian systems on three-dimensional surfaces of constant energy. *Dokl. Akad. Nauk (Russian)*, 332(5):553–555, 1993. English trans. in Russian Acad. Sci. Dokl. Math., 48(2), pp 365-369, 1994.

[7] A. Delshams, R. de la Llave, and T. M. Seara. A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of $\mathbb{T}^2$. *Communications in Mathematical Physics*, 209(2):353–392, 2000.

[8] H. R. Dullin, M. Juhnke, and P. H. Richter. Action integrals and energy surfaces of the kovalevskaya top. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 4(6):1535–1562, 1994.

[9] R. W. Easton, J. D. Meiss, and G. Roberts. Drift by coupling to an anti-integrable limit. Preprint, August 2000.

[10] L. H. Eliasson. Perturbations of stable invariant tori for hamiltonian systems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. IV*, 15(1):115–147, 1988.

[11] N. Fenichel. Persistence and smoothness of invariant manifolds for flows. *Ind. Univ. Math. J.*, 21:193–225, 1971.

[12] N. Fenichel. Asymptotic stability with rate conditions. *Ind. Univ. Math. J.*, 23:1109–1137, 1974.

[13] N. Fenichel. Asymptotic stability with rate conditions, II. *Ind. Univ. Math. J.*, 26:81–93, 1977.

[14] A. T. Fomenko. Topological classification of all integrable hamiltonian differential equations of general type with two degrees of freedom. In *The geometry of Hamiltonian systems (Berkeley, CA, 1989)*, pages 131–339. Math. Sci. Res. Inst. Publ., 22, Springer, New York, 1991.

[15] S. M. Graff. On the conservation of hyperbolic invariant tori for Hamiltonian systems. *J. Diff. Eq.*, 15:1–69, 1974.

[16] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. New York: Springer, 1983.

[17] G. Haller. *Chaos Near Resonance*. Applied Mathematical Sciences; 138. Springer-Verlag, NY, 1999.

[18] G. Haller and S. Wiggins. N-pulse homoclinic orbits in perturbations of resonant Hamiltonian systems. *Arch. Rational Mech. Anl.*, 130(1):25–101, 1995. Communicated by P. Holmes.

[19] H. Handßmann. The quasi-periodic center-saddle bifurcation. *Journal of Differential Equations*, 142(2):305–370, January 1997.

[20] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1995.

[21] J. Laskar. Global dynamics and difussion. *Physica D*, 67:257–281, 1993.

[22] N. R. Lebovitz and A. I. Pesci. Dynamic bifurcation in Hamiltonian systems with one degree of freedom. *SIAM J. Appl. Math.*, 55(4):1117–1133, August 1995.

[23] L. M. Lerman and I. L. Umanskii. Classification of four-dimensional integrable hamiltonian systems and poisson actions of $\mathbb{R}^2$ in extended neighborhoods of simple singular points. I. *Mat. Sb. (Russian)*, 183(12), 1992. English trans. in Russian Acad. Sci. Sb. Math. 77(2), pp 511-542, 1994.

[24] L. M. Lerman and I. L. Umanskii. Classification of four-dimensional integrable hamiltonian systems and poisson actions of $\mathbb{R}^2$ in extended neighborhoods of simple singular points. II. *Mat. Sb. (Russian)*, 184(4):105–138, 1993. English trans. in Russian Acad. Sci. Sb. Math. 78(2), pp 479-506, 1994.

[25] L. M. Lerman and I. L. Umanskii. Classification of four-dimensional integrable hamiltonian systems and poisson actions of $\mathbb{R}^2$ in extended neighborhoods of simple singular points. III. realizations. *Mat. Sb. (Russian)*, 186(10):89–102, 1995. English trans. in Sb. Math. 186(10), pp 1477-1491, 1995.

[26] A. J. Lichtenberg and M. A. Lieberman. *Regular and Stochastic Motion*, volume 38 of *Applied Mathematical Sciences*. Springer-Verlag, 1983.

[27] A. Litvak-Hinenzon. *Parabolic resonances in Hamiltonian systems*. PhD thesis, The Weizmann Institute of Science, Rehovot, Israel, 2000.

[28] A. Litvak-Hinenzon and V. Rom-Kedar. Parabolic resonances in 3 d.o.f. near integrable Hamiltonian systems. Submitted.

[29] A. Litvak-Hinenzon and V. Rom-Kedar. Parabolic resonances in near integrable Hamiltonian systems. In D. S. Broomhead, E. A. Luchinskaya, P. V. E. McClintock, and T. Mullin, editors, *Stochaos: Stochastic and*

Chaotic Dynamics in the Lakes, pages 358–368. American Institute of Physics, Melville, NY, USA, 2000. Refereed.

[30] J. N. Mather. Graduate course in princeton, 95-96, and lectures at penn state, spring 96, paris, summer 96, austin, fall 96.

[31] K. R. Meyer. Generic bifurcations of periodic points. *Trans. AMS*, 149:95–107, May 1970.

[32] K. R. Meyer. Generic bifurcations in Hamiltonian systems. In A. Manning, editor, *Dynamical Systems - Warwick, 1974*, volume 468 of *Springer Lecture notes in Mathematics*, page 36. Springer, New York, 1975.

[33] K. R. Meyer and R. G. Hall. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, volume 90 of *Applied Mathematical Sciences*. Springer-Verlag, NY, 1991.

[34] A. A. Oshemkov. Description of isoenergetic surfaces of integrable hamiltonian systems with two degrees of freedom (russian). *Trudy Sem. Vektor. Tenzor. Anal.*, 23:122–132, 1988.

[35] N. Paldor and E. Boss. Chaotic trajectories of tidally perturbed inertial oscillations. *J. Atmos. Sci.*, 49:2306–2318, 1992.

[36] J. Pöschel. On elliptic lower dimensional tori in Hamiltonian systems. *Math. Z.*, 202:559–608, 1989.

[37] V. Rom-Kedar. Parabolic resonances and instabilities. *Chaos*, 7(1):148–158, 1997.

[38] V. Rom-Kedar, Y. Dvorkin, and N. Paldor. Chaotic Hamiltonian dynamics of particle's horizontal motion in the atmosphere. *Physica D*, 106:389–431, 1997.

[39] V. Rom-Kedar and N. Paldor. From the tropics to the poles in forty days. *Bulletin of the American Meteorological Society*, 78(12):2779–2784, December 1997.

[40] J. L. Tennyson, M. A. Lieberman, and A. J. Lichtenberg. Diffusion in near-integrable hamiltonian systems with three degrees of freedom. In *Nonlinear dynamics and the beam-beam interaction (Sympos.,*

Brookhaven Nat. Lab., New York, 1979), volume 57 of *AIP Conf. Proc.*, pages 272–301. Amer. Inst. Physics, New York, 1980.

[41] D. V. Treschev. Multidimensional simplectic separatrix maps. Preprint.

[42] D. V. Treschev. The mechanism of destruction of resonance tori of hamiltonian systems. *Math. USSR Sbornik*, 68:181–203, 1991.

[43] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Berlin: Springer, 1990.

[44] Z. Xia. Arnold diffusion: a variational construction. In *Proceedings of the International Congress of Mathematicians*, volume II of *Doc. Math., Extra Vol. ICM II*, pages 867–877, Berlin, 1998.