Pauli operator and Aharonov Casher theorem for measure valued magnetic fields

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Abstract

We define the two dimensional Pauli operator and identify its core for magnetic fields that are regular Borel measures. The magnetic field is generated by a scalar potential hence we bypass the usual $\mathbf{A} \in L^2_{loc}$ condition on the vector potential which does not allow to consider such singular fields. We extend Aharonov-Casher theorem for magnetic fields that are measures with finite total variation and we present a counterexample in case of infinite total variation. One of the key technical tools is a weighted L^2 estimate on a singular integral operator.

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1 Introduction

We consider the usual Pauli operator in $d = 2$ dimensions with a magnetic field B

$$H = [\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A})]^2 = (-i\nabla + \mathbf{A})^2 + \sigma_3 B \qquad \text{on} \quad L^2(\mathbf{R}^2, \mathbf{C}^2),$$

$B := \text{curl}(\mathbf{A}) = \nabla^\perp \cdot \mathbf{A}$ with $\nabla^\perp := (-\partial_2, \partial_1)$. Here $\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A})$ is the two dimensional Dirac operator on the trivial spinorbundle over $\mathbf{R}^2$ with vector potential $\mathbf{A}$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Precise conditions on $\mathbf{A}$ and B will be specified later.

The Aharonov-Casher Theorem [A-C] states that the dimension of the kernel of H is given

$$\dim \text{Ker}(H) = \lfloor |\Phi| \rfloor, \tag{1}$$

where

$$\Phi := \frac{1}{2\pi} \int_{\mathbf{R}^2} B(x) \mathrm{d}x$$

(possibly $\pm\infty$) is the flux (divided by 2π) and $\lfloor \quad \rfloor$ denotes the lower integer part ($\lfloor n \rfloor = n - 1$ for $n \geq 1$ integer and $\lfloor 0 \rfloor = 0$). Moreover, $\sigma_3 \psi = -s\psi$ for any $\psi \in \text{Ker}(H)$, where $s = \text{sign}(\Phi)$.

On a $Spin^c$-bundle over S^2 with a smooth magnetic field the analogous theorem is equivalent to the index theorem (for a short direct proof see [E-S]). From topological reasons the analogue of Φ, the total curvature of a connection, is an integer (the Chern number of the determinant line bundle), and the number of zero modes of the corresponding Dirac operator is $|\Phi|$.

In the present paper we investigate two related questions:

(i) What is the most general class of magnetic fields for which the Pauli operator can be properly defined on $\mathbf{R}^2$?

(ii) What is the most general class of magnetic fields for the Aharonov-Casher theorem to hold on $\mathbf{R}^2$?

Pauli operators are usually defined either via the magnetic Schrödinger operator, $(-i\nabla + \mathbf{A})^2$, by adding the magnetic field $\sigma_3 B$ as an external potential, or directly by the quadratic form of the Dirac operator $\boldsymbol{\sigma} \cdot (-i\nabla + \mathbf{A})$ (see Section 2.1). In both ways, the standard condition $\mathbf{A} \in L^2_{loc}$ is necessary.

On the other hand, the statement of the Aharonov-Casher theorem uses only that $B \in L^1$, and in fact B can even be a measure. It is therefore a natural question to extend the Pauli operator for such magnetic fields and investigate the validity of the Aharonov-Casher theorem. However, even if $B \in L^1$, it might not be generated by an $\mathbf{A} \in L^2_{loc}$. For example, any gauge $\mathbf{A}$ generating the radial field $B(x) = |x|^{-2}|\log|x|\ |^{-3/2}\mathbf{1}(|x| \leq \frac{1}{2}) \in L^1$ satisfies $\int_{|x|\leq 1/2} |\mathbf{A}(x)|^2 \mathrm{d}x \geq \int_0^{1/2}(r|\log r|)^{-1}\mathrm{d}r = \infty$ (here $\mathbf{1}$ is the characteristic function). Hence the Pauli operator cannot be defined in the usual way on C_0^∞ as its core. In case of a point singularity at $p \in \mathbf{R}^2$ one can study the extensions from $C_0^\infty(\mathbf{R}^2 \setminus \{p\})$, but such approach may not be possible for B with a more complicated singular set.

In this paper we present an alternative method which enables us to define the Pauli operator for any magnetic field that is a regular Borel measure (Theorem 2.7).

The basic idea is to define the Pauli operator via a real generating potential function h, satisfying

$$\Delta h = B \tag{2}$$

instead of the usual vector potential $\mathbf{A}$. This potential function appears in the original proof of the Aharonov-Casher theorem. The key identity is the following

$$\int \Big|\boldsymbol{\sigma}\cdot(-i\nabla + \mathbf{A})\psi\Big|^2 = 4\int \Big|\partial_{\bar{z}}(e^{-h}\psi_+)\Big|^2 e^{2h} + \Big|\partial_z(e^{h}\psi_-)\Big|^2 e^{-2h} \tag{3}$$

for regular data, with $\mathbf{A} := \nabla^\perp h$ (integrals without specified domains are understood on $\mathbf{R}^2$ with respect to the Lebesgue measure). We will *define* the Pauli quadratic form by the right hand side even for less regular data. It turns out that any magnetic field that is a regular Borel measure can be handled by an h-potential.

The main technical tool is that for an appropriate choice of h, the weight function $e^{\pm 2h}$ (locally) belongs to the Muckenhoupt A_2 class ([G-R], [St]). Therefore the maximal operator and certain singular integral operators are bounded on the weighted L^2 spaces. This will be essential to identify the core of the operators.

We point out that this approach does *not* apply to the magnetic Schrödinger operator $(-i\nabla + \mathbf{A})^2$.

The Aharonov-Casher theorem has been rigorously proven only for a restricted class of magnetic fields on $\mathbf{R}^2$. The conditions involve some control on the decay at infinity and on local singularities. In fact, to our knowledge, the optimal conditions have never been investigated. The original paper [A-C] does not focus on conditions. The exposition [CFKS] assumes compactly supported bounded magnetic field $B(x)$. The Ph.D. thesis by K. Miller [Mi] assumes boundedness, and assumes that $\int |B(x)| \log |x| \mathrm{d}x < \infty$. The boundedness condition is clearly too strong, and it can be easily replaced with the assumption that $B \in \mathcal{K}(\mathbf{R}^2)$ Kato class.

Miller also observes that in case of integer $\Phi \neq 0$ there could be either $|\Phi|$ or $|\Phi| - 1$ zero states, but if the field is compactly supported then the number of states is always $|\Phi| - 1$ [CFKS].

The idea behind each proof is to construct a potential function h satisfying (2). *Locally*, $H\psi = 0$ is equivalent to $\psi = (e^h g_+, e^{-h} g_-)$ with $\partial_{\bar{z}} g_+ = 0$, $\partial_z g_- = 0$, where we identify $\mathbf{R}^2$ with $\mathbf{C}$ and use the notations $x = (x_1, x_2) \in \mathbf{R}^2$ and $z = x_1 + ix_2 \in \mathbf{C}$ simultaneously. The condition $\psi \in L^2(\mathbf{R}^2, \mathbf{C}^2)$ together with the explicit growth (or decay) rate of h at infinity determines the *global* solution space by identifying the space of (anti)holomorphic functions $g_\pm$ with a controlled growth rate at infinity.

For bounded magnetic fields decaying fast enough at infinity, a solution to (2) is given by

$$h(x) = \frac{1}{2\pi} \int_{\mathbf{R}^2} \log |x - y| B(y) \mathrm{d}y \tag{4}$$

and $h(x)$ behaves as $\approx \Phi \log |x|$ for large x. If $\Phi \geq 0$, then $e^h g_+$ is never in L^2, and $e^{-h} g_- \in L^2$ if $|g_-|$ grows at most as the $\big(\lfloor \Phi \rfloor - 1\big)$-th power of $|x|$. If $\Phi < \infty$, then g_- must be a polynomial of degree at most $\lfloor \Phi \rfloor - 1$. If $\Phi = \infty$, then the integral in (4) is not absolutely convergent. If the radial behavior of B is regular enough, then h may still be defined via (4) as a conditionally convergent integral and we then have a solution space of infinite dimension.

Conditions on local regularity and decay at infinity are used to establish bounds on the auxiliary function h given by (4), but they are not a priori needed for the Aharonov-Casher Theorem (1). We show that local regularity conditions are irrelevant by proving the Aharonov-Casher theorem for any measure valued magnetic fields with finite total variation (Theorem 3.1). Many fields with infinite total variation can also be covered; some regular behavior at infinity is sufficient (Corollary 3.2). However, some control is needed in general, as we present a counterexample to the Aharonov-Casher theorem for a magnetic field with infinite total variation.

Counterexample 1.1 *There exists a continuous bounded magnetic field B such that $\int_{\mathbf{R}^2} |B| = \infty$ and*

$$\Phi := \lim_{r\to\infty} \Phi(r) = \lim_{r\to\infty} \frac{1}{2\pi} \int_{|x|\le r} B(x)\mathrm{d}x \tag{5}$$

exists and $\Phi > 1$, but dim $KerH = 0$.

Finally, we recall a conjecture from [Mi]:

Conjecture 1.2 *Let $B(x) \ge 0$ with $\Phi := \frac{1}{2\pi}\int B$, which may be infinite. Then the dimension of $Ker(H)$ is at least $\lfloor \Phi \rfloor$.*

The proof in [Mi] failed because it would have relied on the conjecture that for any continuous function $B \ge 0$ there exists a positive solution h to (2). This is false. A counterexample (even with finite Φ) was given by C. Fefferman and B. Simon and it was presented in [Mi]. However, the same magnetic field does *not* yield a counterexample to Conjecture 1.2.

Theorem 3.1 settles this conjecture for $\Phi < \infty$, but the case $\Phi = \infty$ remains open. The magnetic field in our counterexample does not have a definite sign, in fact Φ is defined only as an improper integral.

2 Definition of the Pauli operator

2.1 Standard definition for $\mathbf{A} \in L^2_{loc}$

The standard definition of the *magnetic Schrödinger operator*, $(-i\nabla + \mathbf{A})^2$, or the *Pauli operator*, $[\boldsymbol{\sigma}\cdot(-i\nabla + \mathbf{A})]^2$, as a quadratic form, requires $\mathbf{A} \in L^2_{loc}$ (see e.g. [L-S, L-L] and for the Pauli operator [So]). We define $\Pi_k := -i\partial_k + A_k$, $Q_\pm := \Pi_1 \pm i\Pi_2$, or with complex notation $Q_+ = -2i\partial_{\bar z} + a$, $Q_- = -2i\partial_z + \bar a$ with $a := A_1 + iA_2$. These are closable operators, originally defined on $C_0^\infty(\mathbf{R}^2)$. Their closures are denoted by the same letter on the minimal domains $\mathcal{D}(\Pi_j)$ and $\mathcal{D}(Q_\pm)$.

Let

$$s_{\mathbf{A}}(u,u) := \|\Pi_1 u\|^2 + \|\Pi_2 u\|^2 = \int |(-i\nabla + \mathbf{A})u|^2 , \qquad u \in C_0^\infty(\mathbf{R}^2)$$

be the closable quadratic form associated with the magnetic Schrödinger operator on the minimal form domain $\mathcal{D}(s_{\mathbf{A}})$. It is known [Si] that the minimal domain coincides with the maximal domain which consists of all $u \in L^2$ such that $s_{\mathbf{A}}(u,u) < \infty$.

The closable quadratic form associated with the *Pauli operator* is

$$p_{\mathbf{A}}(\psi,\psi) := \|Q_+\psi_+\|^2 + \|Q_-\psi_-\|^2 = \int |\boldsymbol{\sigma}\cdot(-i\nabla + \mathbf{A})\psi|^2 \,, \qquad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \in C_0^\infty(\mathbf{R}^2, \mathbf{C}^2) \,,$$

again on the minimal form domain $\mathcal{D}(p_{\mathbf{A}}) = \mathcal{D}(Q_+) \otimes \mathcal{D}(Q_-)$. The condition $\mathbf{A} \in L^2_{loc}$ is obviously necessary. In both cases there exists a unique self-adjoint operator, $S_{\mathbf{A}}$ and $P_{\mathbf{A}}$, associated with these quadratic forms.

A simple calculation shows that $\mathcal{D}(s_{\mathbf{A}}) \otimes \mathcal{D}(s_{\mathbf{A}}) \subset \mathcal{D}(p_{\mathbf{A}})$. This inclusion already indicates that the Pauli operator may be defined in a more general setup. In case of $B = \nabla^\perp \cdot \mathbf{A} \in L^\infty$, these two domains are equal and $P_{\mathbf{A}} = S_{\mathbf{A}} \otimes I_2 + \sigma_3 B$. If $B \in L^\infty_{loc}$ only, then the form domains coincide locally. For more details on these statements, see Section 2 of [So].

2.2 Measures and integer point fluxes

Let $\mathcal{M}$ be the set of signed real Borel measures $\mu(\mathrm{d}x)$ on $\mathbf{R}^2$ with finite total variation, $|\mu|(\mathbf{R}^2) = \int_{\mathbf{R}^2} |\mu|(\mathrm{d}x) < \infty$. Let $\overline{\mathcal{M}}$ be the set of signed real regular Borel measures μ on $\mathbf{R}^2$, in particular they have σ-finite total variation. If $\mu(\mathrm{d}x) = B(x)\mathrm{d}x$ is absolutely continuous, then $\mu \in \mathcal{M}$ is equivalent to $B \in L^1$. Let $\overline{\mathcal{M}^*}$ be the set of all measures $\mu \in \overline{\mathcal{M}}$ such that $\mu(\{x\}) \in (-2\pi, 2\pi)$ for any point $x \in \mathbf{R}^2$, and $\mathcal{M}^* := \mathcal{M} \cap \overline{\mathcal{M}^*}$.

Definition 2.1 *Two measures $\mu, \mu' \in \overline{\mathcal{M}}$ are said to be* **equivalent** *if $\mu - \mu' = 2\pi \sum_j n_j \delta_{z_j}$, where $n_j \in \mathbf{Z}$, $z_j \in \mathbf{R}^2$. The equivalence class of any measure $\mu \in \overline{\mathcal{M}}$ contains a unique measure, denoted by μ^*, such that $\mu^*(\{x\}) \in [-\pi, \pi)$ for any $x \in \mathbf{R}^2$. In particular, $\mu^* \in \overline{\mathcal{M}^*}$.*

The Pauli operator associated with $\mu \in \overline{\mathcal{M}}$ will depend only on the equivalence class of μ up to a gauge transformation, so we can work with $\mu \in \overline{\mathcal{M}^*}$. This just reflects the physical expectation that any magnetic point flux $2\pi n \delta_z$, with integer n, is removable by the gauge transformation $\psi(x) \to e^{in\varphi}\psi(x)$, where $\varphi = \arg(x - z)$. In case of several point fluxes, $2\pi \sum_j n_j \delta_{z_j}$, the phase factor should be $\exp\Big(i \sum_j n_j \arg(x - z_j)\Big)$, but it may not converge for an infinite set of points $\{z_j\}$. However, any $\mu \in \overline{\mathcal{M}}$ can be uniquely written as $\mu = \mu^* + 2\pi \sum_j n_j \delta_{z_j}$ with a set of points z_j which do not accumulate in $\mathbf{R}^2 \equiv \mathbf{C}$. Hence there exist analytic functions $F_\mu(x)$ (recall $x = x_1 + ix_2$) that has zeros exactly at z_j with multiplicity n_j for $n_j > 0$ and $G_\mu(x)$ having zeros at z_j with multiplicity $-n_j$ for $n_j < 0$ by Weierstrass theorem. Let $L_\mu(x) := F_\mu(x)\overline{G}_\mu(x)$. Then the integer point fluxes can be removed by the unitary gauge transformation

$$U_\mu : \psi(x) \to \frac{L_\mu(x)}{|L_\mu(x)|}\,\psi(x) \tag{6}$$

and clearly $L_\mu(x)/|L_\mu(x)| = \exp\Big(in_j\mathrm{arg}(x - z_j) + iH_j(x)\Big)$, where H_j is a real harmonic function for x near z_j. In particular $U_\mu^*(-i\nabla)U_\mu\psi = \Big(-i\nabla + n_j\mathbf{A}_j + \nabla H_j\Big)\psi$ for ψ whose support contains only z_j among the flux points, and here $\nabla^\perp \cdot \mathbf{A}_j = 2\pi\delta_{z_j}$.

2.3 Potential function

The Pauli quadratic form for magnetic fields $\mu \in \overline{\mathcal{M}^*}$ will be defined via the right hand side of (3), where h is a solution to $\Delta h = \mu$. The following theorem shows that for $\mu \in \mathcal{M}^*$ one can always choose a good potential function h. Later we will extend it for $\mu \in \overline{\mathcal{M}^*}$.

Theorem 2.2 *Let $\mu \in \mathcal{M}^*$ and $\Phi := \frac{1}{2\pi}\int \mu(\mathrm{d}x)$ be the total flux (divided by 2π). There is $0 < \varepsilon(\mu) \le 1$ such that for any $0 < \varepsilon < \varepsilon(\mu)$ there exists a real valued function $h = h_\varepsilon \in \cap_{p<2} W^{1,p}_{loc}$ with $\Delta h = \mu$ (in distributional sense), such that*

(i) For any compact set $K \subset \mathbf{R}^2$ and any square $Q \subset K$

$$\Big(\frac{1}{|Q|}\int_Q e^{2h}\Big)\Big(\frac{1}{|Q|}\int_Q e^{-2h}\Big) \le C_1(K, \varepsilon, \mu) \tag{7}$$

(ii) $e^{\pm 2h} \in L^{1+\varepsilon}_{loc}$.

(iii) h can be split as $h = h_1 + h_2$ with the following estimates:

$$\left|\frac{h_1(x)}{\log|x|} - \Phi\right| \le \varepsilon \qquad for \quad |x| \ge R(\varepsilon, \mu) \tag{8}$$

and

$$\int_{Q(u)} e^{\pm 2h_2} \le C_2(\varepsilon)\langle u\rangle^{2\varepsilon} \tag{9}$$

with some constants $C_1(K, \varepsilon, \mu), C_2(\varepsilon)$ and $R(\varepsilon, \mu)$. Here $Q(u) = [u - \frac{1}{2}, u + \frac{1}{2}]^2$ denotes the unit square about $u \in \mathbf{R}^2$ and $\langle u\rangle = (u^2 + 1)^{1/2}$.

Remark. The property (i) means that e^{2h} satisfies a certain reversed Hölder inequality locally; in fact the constant in (7) is independent of K if the scale of Q is big enough. If (7) were true for any square, then e^{2h} would be in the weight-class A_2 used in harmonic analysis (see [G-R, St]). Nevertheless, this property will allow us to use weighted L^2-bounds on a certain singular integral operator locally (Lemma 2.9).

We also remark that property (ii) follows from the local analog of the well-known fact that $\omega \in A_2 \Longrightarrow \omega \in A_p$ for some $p < 2$.

Corollary 2.3 *If $h \in L^1_{loc}$ satisfies $\Delta h \in \overline{\mathcal{M}^*}$, then $h \in W^{1,p}_{loc}$ for all $p < 2$ and $e^{\pm 2h} \in L^1_{loc}$. If, in addition, $\Delta h \in \mathcal{M}^*$, then $e^{\pm 2h} \in L^{1+\varepsilon}_{loc}$ with some $\varepsilon > 0$.*

Proof. Suppose first that $\mu := \Delta h \in \mathcal{M}^*$. Choose $\varepsilon < \varepsilon(\mu)$ and consider $h_\varepsilon \in \bigcap_{p<2} W^{1,p}_{loc}$ constructed in Theorem 2.2 with $e^{\pm 2h_\varepsilon} \in L^{1+\varepsilon}_{loc}$. Since $\Delta(h - h_\varepsilon) = 0$, we have $h = h_\varepsilon + \varphi$ with a smooth function φ so the statements follow for h as well.

If $\mu = \Delta h$ has infinite total variation, then Theorem 2.2 cannot be applied directly. But for any compact set K one can find another compact set K^* with $K \subset \mathrm{int}(K^*)$ and then the measure $\Delta h \in \overline{\mathcal{M}^*}$ restricted to K^* has finite total variation. Therefore one can find a function $h^* \in \bigcap_{p<2} W^{1,p}_{loc}$, $e^{\pm h^*} \in L^2_{loc}$ with $\Delta h^* = \mu$ on K^*, i.e., $h - h^*$ is harmonic on K^*, hence it is smooth and bounded on K. So $h \in \bigcap_{p<2} W^{1,p}(K)$ and $e^{\pm h} \in L^2(K)$ follows from the same properties of h^*. $\square$

Proof of Theorem 2.2. Step 1. First we write $\mu = \mu_d + \mu_c$, where $\mu_d := 2\pi \sum_j C_j \delta_{z_j}$ ($C_j \in (-1, 1)$, $z_j \in \mathbf{C} \equiv \mathbf{R}^2$) is the discrete part of the measure μ, and μ_c is continuous, i.e., $\mu_c(\{x\}) = 0$ for any point $x \in \mathbf{R}^2$. The summation can be infinite, finite or empty, but $\sum_j |C_j| < \infty$. We also assume that z_j's are distinct. Let

$$\varepsilon(\mu) := \frac{1}{10} \min_j \left\{1 - |C_j|\right\}$$

then clearly $\varepsilon(\mu) > 0$. We fix an $0 < \varepsilon < \varepsilon(\mu)$. All objects defined below will depend on ε, but we will neglect this fact in the notations.

We split the measure $\mu_d = \mu_{d,1} + \mu_{d,2}$ such that

$$\mu_{d,1} := 2\pi \sum_{j=1}^{N} C_j \delta_{z_j} \ , \qquad \mu_{d,2} := 2\pi \sum_{j=N+1}^{\infty} C_j \delta_{z_j} \ ,$$

where N is chosen such that $2\pi \sum_{j=N+1}^{\infty} |C_j| < \varepsilon/2$. In particular $|\mu_{d,2}|(\mathbf{R}^2) < \varepsilon/2$.

We define

$$h_{d,j}(x) := \frac{1}{2\pi} \int \log \frac{|x-y|}{\langle y \rangle} \mu_{d,j}(\mathrm{d}y) \ , \qquad j = 1, 2 \ , \tag{10}$$

so that $\Delta h_{d,j} = \mu_{d,j}$. Notice that $h_{d,j}(x)$ is well defined for a.e. x, moreover $h_{d,j} \in W^{1,p}_{loc}$ for all $p < 2$ by Jensen's inequality.

Step 2. We split $\mu_c = \mu_{c,1} + \mu_{c,2}$ such that $\mu_{c,1}$ be compactly supported and $|\mu_{c,2}|(\mathbf{R}^2) < \varepsilon/2$. We set $\mu_j := \mu_{d,j} + \mu_{c,j}$, $j = 1, 2$. Then we define

$$h_{c,j}(x) := \frac{1}{2\pi} \int_{\mathbf{R}^2} \log \frac{|x-y|}{\langle y \rangle} \mu_{c,j}(\mathrm{d}y) \ , \qquad j = 1, 2 \ , \tag{11}$$

clearly $h_{c,j} \in W^{1,p}_{loc}$ for all $p < 2$, and $\Delta h_{c,j} = \mu_{c,j}$ (in distributional sense).

Finally, we define

$$h_1 := h_{d,1} + h_{c,1} \ , \qquad h_2 := h_{d,2} + h_{c,2} \ , \qquad h := h_1 + h_2 \tag{12}$$

and clearly $\Delta h_j = \mu_j$. Since $\mu_{d,1}$ and $\mu_{c,1}$ are compactly supported, the estimate (8) is straightforward. We will also need the notation $\nu := \mu_{d,2} + \mu_c = \mu_{c,1} + \mu_2$.

Step 3. For any integer L we define $\Lambda_L := (2^{-L}\mathbf{Z})^2 + (2^{-L-1}, 2^{-L-1})$ to be the shifted and rescaled integer lattice. We define the *dyadic squares of scale* L to be the squares

$$D_k^{(L)} := \left[k_1 - 2^{-L-1}, k_1 + 2^{-L-1}\right) \times \left[k_2 - 2^{-L-1}, k_2 + 2^{-L-1}\right)$$

of side-length 2^{-L} about the lattice points $k = (k_1, k_2) \in \Lambda_L$. The squares

$$\widetilde{D}_k^{(L)} := \left[k_1 - 2^{-L}, k_1 + 2^{-L}\right) \times \left[k_2 - 2^{-L}, k_2 + 2^{-L}\right)$$

of double side-length with the same center k are called *doubled dyadic squares of scale* L. Similarly, the squares

$$\widehat{D}_k^{(L)} := \left[k_1 - 3 \cdot 2^{-L-1}, k_1 + 3 \cdot 2^{-L-1}\right) \times \left[k_2 - 3 \cdot 2^{-L-1}, k_2 + 3 \cdot 2^{-L-1}\right)$$

are called *tripled dyadic squares of scale* L. For a fixed scale L the collection of dyadic squares is denoted by $\mathcal{D}_L$. $\widetilde{\mathcal{D}}_L$ and $\widehat{\mathcal{D}}_L$ denote the set of doubled and tripled dyadic squares, respectively. The elements of $\mathcal{D}_L$ partition $\mathbf{R}^2$ for each L. Notice also that every square $Q \subset \mathbf{R}^2$ can be covered by a doubled dyadic square of area not bigger than a universal constant times $|Q|$.

Lemma 2.4 *There exists $1 \leq M = M(\mu, \varepsilon) < \infty$ such that $|\mu|(Q) < 2\pi(1-\varepsilon)$ for any $Q \in \widehat{\mathcal{D}}_M$.*

Proof. We first notice that the support of $\mu_{d,1}$ consists of finitely many points, hence for large enough L each element of $\widehat{\mathcal{D}}_L$ contains at most one point from this support.

Second, since the measure $|\nu| = |\mu_c| + |\mu_{d,2}|$ does not charge more than $\varepsilon/2$ to any point, we claim that there exists a positive integer $1 \leq M = M(\mu, \varepsilon) < \infty$ such that $|\nu|(D) < \varepsilon$ for any dyadic square of scale M. We can choose $M(\mu, \varepsilon) \geq L$.

This statement is clear by a dyadic decomposition; we start with the partition of $\mathbf{R}^2$ into dyadic squares of scale L. There are just finitely many squares $D \in \mathcal{D}_L$ such that $|\nu|(D) \geq \varepsilon$. We split these squares further into four identical dyadic squares. If this process stops after finitely many steps, then we have reached our M as the scale of the finest decomposition. Now suppose on the contrary that this process never stops. Then we could find a strictly decreasing sequence of nested dyadic squares $D_1 \supset D_2 \supset \ldots$ such that $|\nu|(D_j) \geq \varepsilon$, but $|\nu|$ would charge at least ε weight to their intersection which is a point.

Finally, since $|\mu| = |\mu_{d,1}| + |\nu|$ and every tripled square can be covered by 9 dyadic squares of the same scale, we have $|\mu|(Q) \leq 9\varepsilon + 2\pi \max_j |C_j| < 2\pi(1-\varepsilon)$ for each $Q \in \widehat{\mathcal{D}}_M$ for large enough M. $\square$.

Step 4. Now we turn to the proof of (7) and first we prove it for any doubled dyadic square of big scale. Let $Q = \widetilde{D}_k^{(K)} \in \widetilde{\mathcal{D}}_K$ be a doubled dyadic square with $K \geq M$ and let $\widehat{Q} = \widehat{D}_k^{(K)}$ be the corresponding tripled square with the same center $k \in \Lambda_K$. We split the measure μ as

$$\mu = \mu^{int} + \mu^{ext} := \mathbf{1}_{\widehat{Q}}\mu + \mathbf{1}_{\widehat{Q}^c}\mu$$

with $|\mu| = |\mu^{int}| + |\mu^{ext}|$ and h is decomposed accordingly as $h = h^{int} + h^{ext}$ with

$$h^{\#}(x) := \frac{1}{2\pi}\int_{\mathbf{R}^2} \log\frac{|x-y|}{\langle y\rangle}\mu^{\#}(\mathrm{d}y) \ ,$$

where $\# = $ int, ext. We also define

$$\widetilde{h}^{int}(x) := \frac{1}{2\pi}\int_{\mathbf{R}^2} \log|x-y|\mu^{int}(\mathrm{d}y) = h^{int}(x) + \frac{1}{2\pi}\int_{\mathbf{R}^2}\log\langle y\rangle\mu^{int}(\mathrm{d}y) \ .$$

Let $\mathrm{Av}_Q h^{ext} := |Q|^{-1}\int_Q h^{ext}$ be the average of h^{ext} on Q. A simple calculation shows that

$$\left|h^{ext}(x) - \mathrm{Av}_Q h^{ext}\right| \leq C|\mu|(\mathbf{R}^2) \ , \qquad \forall x \in Q \tag{13}$$

with a universal constant C using that μ^{ext} is supported outside of the tripled square. Therefore

$$\Big(\frac{1}{|Q|}\int_Q e^{2h}\Big)\Big(\frac{1}{|Q|}\int_Q e^{-2h}\Big) \le e^{4C|\mu|(\mathbf{R}^2)}\Big(\frac{1}{|Q|}\int_Q e^{2\widetilde{h}^{int}}\Big)\Big(\frac{1}{|Q|}\int_Q e^{-2\widetilde{h}^{int}}\Big)$$

We split μ^{int} into its positive and negative parts: $\mu^{int} = \mu^{int}_+ - \mu^{int}_-$, we let $\phi_\pm := \frac{1}{2\pi}\int_{\widehat{Q}} \mu^{int}_\pm \ge 0$. By Lemma 2.4 and $K \ge M$ we have $\phi := \phi_+ + \phi_- < (1-\varepsilon)$.

Now we apply Jensen's inequality for the probability measures $(2\pi\phi_\pm)^{-1}\mu^{int}_\pm$ (if $\phi_\pm \neq 0$):

$$\int_Q e^{2\widetilde{h}^{int}} = \int_Q \exp\Big(\frac{1}{2\pi\phi_+}\int_{\widehat{Q}} \log|x-y|^{2\phi_+}\mu^{int}_+(\mathrm{d}y)\Big)\exp\Big(\frac{1}{2\pi\phi_-}\int_{\widehat{Q}} \log|x-y'|^{-2\phi_-}\mu^{int}_-(\mathrm{d}y')\Big)\mathrm{d}x$$

$$\le \int_Q \mathrm{d}x \frac{1}{2\pi\phi_+}\int_{\widehat{Q}} \mu^{int}_+(\mathrm{d}y)\frac{1}{2\pi\phi_-}\int_{\widehat{Q}} \mu^{int}_-(\mathrm{d}y')|x-y|^{2\phi_+}|x-y'|^{-2\phi_-} \le C(\varepsilon)|Q|^{1+\phi_+-\phi_-} \qquad (14)$$

with an ε-dependent constant. When performing the $\mathrm{d}x$ integration, we used the fact that $\phi_- < 1-\varepsilon$, hence the singularity is integrable. Similarly, we have

$$\int_Q e^{-2\widetilde{h}^{int}} \le C(\varepsilon)|Q|^{1-\phi_+ + \phi_-}$$

which completes the proof of (7) for doubled dyadic squares of scale at least M with a K-independent constant.

Step 5. Next, we prove $e^{\pm 2h} \in L^{1+\varepsilon}_{loc}$. We can follow the argument in Step 4. On any square $Q \in \widetilde{\mathcal{D}}_M$ we can use that h^{ext} is bounded by (13) and we can focus on $\exp(\pm 2\widetilde{h}_{int})$. Then we use Jensen's inequality (14) and use the fact that $x \mapsto |x-y|^{-2(1+\varepsilon)\phi_\pm}$ is locally integrable since $\phi_\pm < (1-\varepsilon)$.

Step 6. Now we complete the proof of (7) for all squares $Q \subset K$. Since every square can be covered by a doubled dyadic square of comparable size, we can assume that Q is such a square. If the scale of Q is smaller than M, then $|Q|^{-1} \le 4^{M(\mu,\varepsilon)}$ and we can simply use $e^{\pm 2h} \in L^1_{loc}$ to estimate the integrals.

Step 7. Finally, we prove (9). Let $\widehat{Q}(u) := [u-1, u+1]^2$ and we split the measure μ_2 as

$$\mu_2 = \mu_2^{int} + \mu_2^{ext} := \mathbf{1}_{\widehat{Q}(u)}\mu_2 + \mathbf{1}_{\widehat{Q}(u)^c}\mu_2$$

and the function $h_2 = h_2^{int} + h_2^{ext}$, where

$$h_2^{\#}(x) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \log\frac{|x-y|}{\langle y\rangle}\mu_2^{\#}(\mathrm{d}y) , \qquad \# = \mathrm{int}, \mathrm{ext} .$$

Similarly to the estimates (13) and (14) in Step 4, we obtain

$$\int_{Q(u)} e^{\pm 2h_2} \le C(\varepsilon)\exp\Big(\pm 2\mathrm{Av}_{Q(u)}h_2^{ext}\Big)\exp\Big(2|\mu_2|(\widehat{Q}(u))\log\langle u\rangle\Big)\,,$$

and a simple calculation shows

$$\Big|\mathrm{Av}_{Q(u)}h_2^{ext}\Big| \le \int_{Q(u)}\int_{\widehat{Q}(u)^c}\Big|\log\frac{|x-y|}{\langle y\rangle}\Big|\,|\mu_2^{ext}|(\mathrm{d}y)\mathrm{d}x \le |\mu_2|(\widehat{Q}^c(u))\log\langle u\rangle + C(\varepsilon)\,.$$

From these estimates (9) follows using that $|\mu_2|(\mathbf{R}^2)\le\varepsilon$. $\square$

2.4 Definition of the Pauli operator for measure valued fields

For any real valued function $h\in L^1_{loc}(\mathbf{R}^2)$ we define the following symmetric quadratic form:

$$\pi^h(\psi,\xi) := \pi^h_+(\psi_+,\xi_+) + \pi^h_-(\psi_-,\xi_-)$$

with

$$\pi^h_+(\psi_+,\xi_+) := 4\int \overline{\partial_{\bar z}(e^{-h}\psi_+)}\partial_{\bar z}(e^{-h}\xi_+)e^{2h}\,,\qquad \pi^h_-(\psi_-,\xi_-) := 4\int \overline{\partial_z(e^{h}\psi_-)}\partial_z(e^{h}\xi_-)e^{-2h}$$

on the natural maximal domains

$$\mathcal{D}(\pi^h_\pm) = \Big\{\psi_\pm \in L^2(\mathbf{R}^2)\ :\ \pi^h_\pm(\psi_\pm,\psi_\pm)<\infty\Big\}\,,$$

$$\mathcal{D}(\pi^h) = \mathcal{D}(\pi^h_+)\otimes\mathcal{D}(\pi^h_-) = \Big\{\psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}\in L^2(\mathbf{R}^2,\mathbf{C}^2)\ :\ \pi^h(\psi,\psi)<\infty\Big\}$$

We use $\|\cdot\|$ to denote the usual $L^2(\mathbf{R}^2,\mathrm{d}x)$ or $L^2(\mathbf{R}^2,\mathbf{C}^2,\mathrm{d}x)$ norms. We define the following norms on functions

$$|\!|\!| f |\!|\!|_{h,+} := \Big[\|f\|^2 + \|\partial_{\bar z}(e^{-h}f)e^h\|^2\Big]^{1/2},\qquad |\!|\!| f |\!|\!|_{h,-} := \Big[\|f\|^2 + \|\partial_z(e^{h}f)e^{-h}\|^2\Big]^{1/2}$$

and for a spinor ψ we let

$$|\!|\!|\psi|\!|\!|_h := |\!|\!|\psi_+|\!|\!|_{h,+} + |\!|\!|\psi_-|\!|\!|_{h,-}\,. \tag{15}$$

For any real function $h\in L^1_{loc}$ with $\Delta h\in\overline{\mathcal{M}}$, we define the set

$$\mathcal{C}_h := \Big\{\psi = \begin{pmatrix} g_+e^h\\ g_-e^{-h}\end{pmatrix}\ :\ g_\pm\in C_0^\infty(\mathbf{R}^2)\Big\}\,. \tag{16}$$

Notice that this set depends only on $\mu = \Delta h$: if h, h' are two functions such that $\Delta h = \Delta h' = \mu$ in distributional sense, then $h - h'$ is harmonic, i.e., smooth. Therefore e^h and $e^{h'}$ differ by a smooth multiplicative factor, i.e. $\mathcal{C}_h = \mathcal{C}_{h'}$, hence we can denote this set by $\mathcal{C}_\mu$. Moreover, by Theorem 2.2, for any $\mu \in \overline{\mathcal{M}^*}$ and any compact set K, there exists an $h \in L^1_{loc}$ with $\Delta h = \mu$ on K, and h is unique up to a smooth additive factor. Since the support of $g_\pm$ is compact, the following set is well-defined for all $\mu \in \overline{\mathcal{M}^*}$

$$\mathcal{C}_\mu := \Big\{ \psi = \begin{pmatrix} g_+ e^h \\ g_- e^{-h} \end{pmatrix} \; : \; g_\pm \in C_0^\infty(\mathbf{R}^2), \; \Delta h = \mu \quad \text{on} \; \mathrm{supp}(g_-) \cup \mathrm{supp}(g_+) \Big\} \; . \tag{17}$$

Theorem 2.5 *Let $h \in L^1_{loc}(\mathbf{R}^2)$ be a real valued function such that $\mu := \Delta h \in \overline{\mathcal{M}^*}$. Then*

(i) The quadratic form π^h is nonnegative, symmetric and closed, hence it defines a unique selfadjoint operator H_h

$$(H_h \psi, \xi) := \pi^h(\psi, \xi) \; , \qquad \psi \in \mathcal{D}(H_h), \; \xi \in \mathcal{D}(\pi^h)$$

with domain

$$\mathcal{D}(H_h) := \{ \psi \in \mathcal{D}(\pi^h) \; : \; \pi^h(\psi, \cdot) \in L^2(\mathbf{R}^2, \mathbf{C}^2)' \}$$

(ii) The set $\mathcal{C}_\mu$ is dense in $\mathcal{D}(\pi^h)$ with respect to $\|\!| \cdot |\!\|_h$, i.e., it is a form core of H_h.

(iii) For any L^1_{loc}-functions h and h' with $\Delta h = \Delta h' \in \overline{\mathcal{M}^*}$, the operators H_h and $H_{h'}$ are unitarily equivalent by a $U(1)$-gauge transformation. In particular, the spectral properties of H_h depend only on $\mu = \Delta h$.

Definition 2.6 *For any real function $h \in L^1_{loc}$ with $\mu = \Delta h \in \overline{\mathcal{M}^*}$ the operator H_h will be called the* **Pauli operator with generating potential** *h. For any $\mu \in \overline{\mathcal{M}^*}$ the unitarily equivalent operators $\{H_h \; : \; \Delta h = \mu\}$ are called the* **Pauli operators with a magnetic field** *μ. The Pauli operators for any $\mu \in \overline{\mathcal{M}}$ are defined as $U_\mu^* H U_\mu$ on the core $U_\mu^* C_\mu$, where H is a Pauli operator with field $\mu^* \in \overline{\mathcal{M}^*}$ (see Definition 2.1) and U_μ is defined in (6).*

To complete the definition of the Pauli operator for any magnetic field $\mu \in \overline{\mathcal{M}}$, we need

Theorem 2.7 *For any $\mu \in \overline{\mathcal{M}^*}$, there exists $h \in L^1_{loc}$ with $\Delta h = \mu$. Hence the above definition of H_h actually defines the Pauli operators for any measure valued magnetic field $\mu \in \overline{\mathcal{M}}$.*

Proof of Theorem 2.5. From Corollary 2.3 we know that $e^{\pm h} \in L^2_{loc}$, and we show below that for any doubled dyadic square Q_0 the estimate

$$\Big(\frac{1}{|Q|}\int_Q e^{2h}\Big)\Big(\frac{1}{|Q|}\int_Q e^{-2h}\Big) \le C_3(h, Q_0) \tag{18}$$

analogous to (7) is valid on any square $Q \subset Q_0$, with a (h, Q_0)-dependent constant. These are the two properties of h which we use below.

For any Q_0 one can find a compact set K such that $Q_0 \subset \mathrm{int}(K)$ and $\mu = \Delta h$ restricted to K, $\mu|_K$, has finite total variation. Let $\varepsilon = \varepsilon(\mu|_K)/2$ and we consider h_ε defined in Theorem 2.2. Since $\Delta h = \Delta h_\varepsilon$ on K, we can write $h = h_\varepsilon + \varphi$ with a smooth real function φ depending on h. In particular, for any doubled dyadic square Q_0 the estimate (7) for h_ε implies that (18) is valid for $h = h_\varepsilon + \varphi$ on any square $Q \subset Q_0$.

Part (i). Let $\psi_n = (\psi_{n+}, \psi_{n-})$ be a Cauchy sequence in the norm $\|\!|\cdot|\!\|_h$, i.e., $\psi_n \to \psi$ in $L^2(\mathrm{d}x)$, $\partial_{\bar z}(e^{-h}\psi_{n+}) \to u_+$ in $L^2(e^{2h}\mathrm{d}x)$ and $\partial_z(e^h\psi_{n-}) \to u_-$ in $L^2(e^{-2h}\mathrm{d}x)$. We have to show that $\partial_{\bar z}(e^{-h}\psi_+) = u_+$, $\partial_z(e^h\psi_-) = u_-$. For any $\phi \in C_0^\infty(\mathbf{R}^2)$

$$\int \overline{\phi}u_+ = \lim_{n\to\infty}\int \overline{\phi}\partial_{\bar z}(e^{-h}\psi_{n+}) = -\lim_{n\to\infty}\int \partial_{\bar z}\overline{\phi}\, e^{-h}\psi_{n+} = -\int \partial_{\bar z}\overline{\phi}\, e^{-h}\psi_+$$

hence $\partial_{\bar z}(e^{-h}\psi_+) = u_+$ in distributional sense. Here we used that

$$\Big|\int \overline{\phi}\big(u_+ - \partial_{\bar z}(e^{-h}\psi_{n+})\big)\Big| \le \|\overline{\phi}e^{-h}\|\Big\|u_+ - \partial_{\bar z}(e^{-h}\psi_{n+})\Big\|_{L^2(e^{2h})} \to 0$$

and

$$\Big|\int \partial_{\bar z}\overline{\phi}\, e^{-h}(\psi_+ - \psi_{n+})\Big| \le \Big\|\partial_{\bar z}\overline{\phi}e^{-h}\Big\|\|\psi_+ - \psi_{n+}\| \to 0\ ,$$

which follows from $e^{-h} \in L^2_{loc}$. The proof of the spin-down component is similar. This shows that the form π^h is closed. The rest of the argument is standard (see, e.g., Lemma 1 in [L-S]).

Part (ii). The spin-up and spin-down parts can be treated separately and analogously, so we focus only on the spin-up part.

Step 1. We first show that the set

$$\mathcal{C}_0 := \{f \in \mathcal{D}(\pi^h_+),\ \mathrm{supp}(f)\ \mathrm{compact}\}$$

is dense in $\mathcal{D}(\pi_+^h)$ with respect to $\|\cdot\|_{h,+}$. This is standard: let $\chi(x)$ be a compactly supported smooth cutoff function, $0 \le \chi \le 1$, $\chi(x) \equiv 1$ for $|x| \le 1$, and let $\chi_n(x) := \chi(x/n)$. For any $f \in \mathcal{D}(\pi_+^h)$ we consider $f_n = \chi_n f$, then clearly $\|f - f_n\|_{h,+} \to 0$.

Step 2. We need the following

Lemma 2.8 *Let $f \in \mathcal{C}_0$ then $\nabla(fe^{-h}) \in L^2(e^{2h})$.*

Proof of Lemma 2.8. Let $g := fe^{-h}$. Let Q_1 be a doubled dyadic square that contains a neighborhood of $K := \operatorname{supp}(g)$, and let Q_0 be a doubled dyadic square that strictly contains Q_1 and $|Q_0| = 4|Q_1|$. We define

$$\omega(x) := \begin{cases} e^{2h(x)} & \text{for } x \in Q_1 \\ 1 & \text{for } x \in Q_1^c\,. \end{cases} \tag{19}$$

Lemma 2.9 *The function $\omega(x)$ satisfies the inequality*

$$\Big(\frac{1}{|Q|}\int_Q \omega\Big)\Big(\frac{1}{|Q|}\int_Q \omega^{-1}\Big) \le C_4(h, Q_0) \tag{20}$$

for any square $Q \subset \mathbf{R}^2$, i.e., ω is an A_2-weight (see [G-R, St]).

Proof of Lemma 2.9. It is sufficient to prove (20) for all doubled dyadic squares Q. It is easy to see that one of the following cases occurs: (i) Q is disjoint from Q_1, (ii) $Q \subset Q_0$, (iii) $|Q_1| \le 9|Q|$. In the first case (20) is trivial, in case (ii) it follows from (18). Finally, in case (iii) we have

$$\Big(\frac{1}{|Q|}\int_Q \omega\Big)\Big(\frac{1}{|Q|}\int_Q \omega^{-1}\Big) \le 36^2\Big(1 + \frac{1}{|Q_0|}\int_{Q_0} e^{2h}\Big)\Big(1 + \frac{1}{|Q_0|}\int_{Q_0} e^{-2h}\Big)$$

hence (20) holds with an appropriate constant. $\Box$.

Since $|\nabla g|^2 = 2(|\partial_z g|^2 + |\partial_{\bar z} g|^2)$ and $\omega = e^{2h}$ on $\operatorname{supp}(g)$, Lemma 2.8 follows immediately from

$$\int_{\mathbf{R}^2} |\partial_z g|^2 \omega \le C_5(h, Q_0) \int_{\mathbf{R}^2} |\partial_{\bar z} g|^2 \omega\,. \tag{21}$$

Notice that

$$\widehat{\partial_z g}(\xi) = m(\xi)\widehat{\partial_{\bar z} g}(\xi) \quad \text{with} \quad m(\xi) := \frac{(\xi_1 - i\xi_2)^2}{|\xi|^2}\,,$$

where hat stands for Fourier transform, $\xi \in \mathbf{R}^2$, and $m(\xi)$ is a homogeneous multiplier of degree 0. Hence (21) is just the weighted L^2-inequality for the regular singular integral operator T_m with Fourier multiplier $m(\xi)$ and with weight $\omega \in A_2$ [G-R, St]. $\square$

Step 3. To conclude that $\mathcal{C}_0 \cap e^h C_0^\infty$ is dense in $\mathcal{C}_0$ with respect to $\|\!| \cdot |\!\|_{h,+}$, we use the fact that C_0^∞ is dense in the weighted Sobolev space $W^{1,2}(\omega)$ with the A_2-weight ω (see e.g. [K]). Here we only recall the key point of the proof. Let $g \in W^{1,2}(\omega)$ compactly supported and $g_\varepsilon := J_\varepsilon * g \in C_0^\infty$ where $J_\varepsilon(x) := \varepsilon^{-2} J(x/\varepsilon)$ is a standard mollifier: $0 \le J \le 1$, $\int J = 1$, J smooth, compactly supported. Then the functions $|\nabla g_\varepsilon| \le J_\varepsilon * |\nabla g|$, have an L^2-integrable majorant by the weighted maximal inequality [St] applied to $|\nabla g| \in L^2(\omega)$, hence $g_\varepsilon \to g$ in $W^{1,2}(\omega)$ as $\varepsilon \to 0$. Notice that every g_ε is supported on a common compact neighborhood of the support of g.

Part (iii). Since $\Delta h = \Delta h'$, we can write $h' = h + \varphi$ with a smooth real function φ. We define λ as the harmonic conjugate of φ, $\nabla\lambda = \nabla^\perp\varphi$, which exists and is smooth by $\Delta\varphi = 0$. By $\partial_{\bar z}(\varphi + i\lambda) = 0$ we have

$$\pi^h(\psi,\psi) = \pi^{h'}\left(e^{-i\lambda}\psi, e^{-i\lambda}\psi\right), \qquad \psi \in \mathcal{C}_\mu ,$$

and then by the density of $\mathcal{C}_\mu$ we obtain the same relation for all $\psi \in \mathcal{D}(\pi^h)$. $\square$

Proof of Theorem 2.7. Since $|\mu|$ is finite on every bounded set, we can find a sequence of disjoint rings, $R_j := \{x \ : \ r_j \le |x| \le r_j + 2\delta_j\}$, $j = 1, 2, \ldots$, with appropriate widths $2\delta_j > 0$ and radii $r_j \to \infty$ (as $j \to \infty$), such that $\sum_j |\mu|(R_j) < \infty$. For $j = 0$ we set $r_j = \delta_j = 0$. Let $0 \le \chi_j \le 1$ $(j = 0, 1, \ldots)$ be smooth functions such that $\chi_j(x) \equiv 1$ for $r_j + 2\delta_j \le |x| \le r_{j+1}$ and $\chi_j(x) \equiv 0$ for $|x| \le r_j + \delta_j$ or $|x| \ge r_{j+1} + \delta_{j+1}$. Notice that the supports of χ_j are disjoint.

We define $\mu_j := \mu \cdot \mathbf{1}\{x \ : r_j + 2\delta_j \le |x| \le r_{j+1}\}$, By Theorem 2.2 there exist $h_j \in \bigcap_{p<2} W^{1,p}_{loc}$, $e^{\pm h_j} \in L^2_{loc}$ with $\Delta h_j = \mu_j$. We notice that

$$\Delta\Big(\sum_j \chi_j h_j\Big) = \nu + \sum_j \chi_j \mu_j$$

where ν is absolutely continuous, $\nu = N(x)\mathrm{d}x$ with

$$N = \sum_j \Big[2\nabla\chi_j \cdot \nabla h_j + h_j \Delta\chi_j\Big] \in L^1_{loc}$$

We can find a decomposition $N = N_1 + N_2$, $N_1 \in L^\infty_{loc}$ and $N_2 \in L^1(\mathbf{R}^2)$.

Let $\kappa := \mu - \sum_j \chi_j \mu_j - N_2(x)\mathrm{d}x$, then $\kappa \in \mathcal{M}$ since $N_2 \in L^1(\mathbf{R}^2)$ and the measure $\mu - \sum_j \chi_j \mu_j$ belongs to $\mathcal{M}$ since it vanishes on the complement of $\bigcup_j R_j$, it has a total variation smaller than $|\mu|$ on each R_j and $\sum_j |\mu|(R_j) < \infty$. It is also clear that κ does not charge more to any point than μ does since $0 \le \chi_j \le 1$ and they have disjoint supports, hence $\kappa \in \mathcal{M}^*$.

By Theorem 2.2 there is $k \in \bigcap_{p<2} W^{1,p}_{loc}$, $e^{\pm k} \in L^2_{loc}$ such that $\Delta k = \kappa$. We define $h^* := k + \sum_j \chi_j h_j$, clearly $h^* \in \bigcap_{p<2} W^{1,p}_{loc}$ and $\mu = \Delta h^* - N_1(x)\mathrm{d}x$.

By the Poincaré formula there exists $\mathbf{A} \in L^\infty_{loc}$ with $\nabla^\perp \cdot \mathbf{A} = -N_1$. For any fixed $p < \infty$, one can find $\widetilde{\mathbf{A}} \in L^p_{loc}$ with $\nabla^\perp \cdot \widetilde{\mathbf{A}} = -N_1$, $\nabla \cdot \widetilde{\mathbf{A}} = 0$ by Lemma 1.1. (ii) [L]. But then $\widetilde{\mathbf{A}}^\perp = (-\widetilde{A}_2, \widetilde{A}_1)$ is curl-free, hence $\widetilde{\mathbf{A}}^\perp = \nabla \widetilde{h}$ for some $\widetilde{h} \in W^{1,p}_{loc}$ by Lemma 1.1. (i) [L]. Then $\Delta \widetilde{h} = N_1$, hence $h := h^* - \widetilde{h} \in L^1_{loc}$ satisfies $\Delta h = \mu$. $\square$

Finally, we have to verify that the Pauli operator H_h defined in this Section coincides with the standard Pauli operator if $\mathbf{A} \in L^2_{loc}$, modulo a gauge transformation.

Proposition 2.10 *Let $\mathbf{A} \in L^2_{loc}$ and let $P_\mathbf{A}$ be the operator defined in Section 2.1. We assume that $\nabla^\perp \cdot \mathbf{A}$ (in distributional sense) is a measure and that $\mu := \nabla^\perp \cdot \mathbf{A} \in \overline{\mathcal{M}}$. Then $\mu \in \overline{\mathcal{M}^*}$, in fact μ has no discrete component. Moreover, if $\Delta h = \mu$ with some $h \in L^1_{loc}$, then the operator H_h defined in Theorem 2.5 is unitarily equivalent to $P_\mathbf{A}$.*

Remark: $\mathbf{A} \in L^2_{loc}$ does not imply that $\nabla^\perp \cdot \mathbf{A}$ is even locally a measure of finite variation. One example is the radial gauge $\mathbf{A}(x) := \Phi(|x|)|x|^{-2}x^\perp$, $x^\perp := (-x_2, x_1)$, that generates the radial field

$$B(x) := \sum_{n=1}^\infty \frac{(-4)^n}{n} \cdot \mathbf{1}(2^{-n} \le |x| < 2^{-n+1}).$$

with flux $\Phi(r) := \int_{|x| \le r} B(x)\mathrm{d}x$. One can easily check that $\int_{|x|\le 1} |B(x)|\mathrm{d}x = \infty$ but $\mathbf{A} \in L^2_{loc}$.

However, if $\nabla^\perp \cdot \mathbf{A} \ge 0$ as a distribution, then it is a (positive) Borel measure $\mu \in \overline{\mathcal{M}^*}$ (see [L-L]).

Proof. First we show that $\mu = \nabla^\perp \cdot \mathbf{A}$ has no discrete component. Suppose, on the contrary, that $\mu(\{x\}) \neq 0$ for some x, and we can assume $x = 0$, $\mu(\{0\}) > 0$. Let χ be a radially symmetric smooth function on $\mathbf{R}^2$, $0 \le \chi \le 1$, $\mathrm{supp}\chi \subset \{|x| \le 2\}$, $\chi(x) \equiv 1$ for

$|x| \le 1$, $|\nabla\chi| \le 2$, and let $\chi_n(x) := \chi(2^n x)$. Clearly $-\int \mathbf{A}\cdot\nabla^\perp\chi_n = \int \chi_n \mathrm{d}\mu \to \mu(\{0\})$ as $n\to\infty$. Using polar coordinates, we have, for large enough n,

$$\frac{1}{2}\mu(\{0\}) \le -\int \mathbf{A}\cdot\nabla^\perp\chi_n \le 4\int_{2^{-n}}^{2^{-n+1}}\int_0^{2\pi} |\mathbf{A}(s,\theta)|\mathrm{d}\theta\,\mathrm{d}s$$

$$\le 4\sqrt{2\pi}\bigg(\int_{2^{-n}}^{2^{-n+1}}\int_0^{2\pi} |\mathbf{A}(s,\theta)|^2\mathrm{d}\theta\, s\,\mathrm{d}s\bigg)^{1/2} \le 4\sqrt{2\pi}\bigg(\int |\mathbf{A}(x)|^2\cdot \mathbf{1}(2^{-n}\le |x|\le 2^{-n+1})\mathrm{d}x\bigg)^{1/2}$$

hence $\int_{|x|\le 1}|\mathbf{A}|^2 = \infty$. The proof also works if we assume only $\mu\in\overline{\mathcal{M}}$ instead of $\mu\in\mathcal{M}$.

Now we prove the unitarity. Without loss of generality we can assume that $\nabla\cdot\mathbf{A} = 0$ by part (ii) Lemma 1.1. of [L]. Let $\mathbf{A}_h := \nabla^\perp h$, then $\nabla^\perp\cdot\mathbf{A}_h = \mu$, $\nabla\cdot\mathbf{A}_h = 0$ and $\mathbf{A}_h\in L^1_{loc}$ by Corollary 2.3. Since $\nabla^\perp\cdot(\mathbf{A}-\mathbf{A}_h) = 0$, there exists $\lambda\in W^{1,1}_{loc}$ such that $\mathbf{A} = \mathbf{A}_h + \nabla\lambda$ by part (i) Lemma 1.1. of [L]. Taking the divergence, we see that $\Delta\lambda = 0$. Let φ be a smooth harmonic conjugate of λ, $\nabla\lambda = \nabla^\perp\varphi$. A simple calculation shows that

$$\int |\boldsymbol{\sigma}\cdot(-i\nabla+\mathbf{A})\psi|^2 = \pi^{h+\varphi}(\psi,\psi) = \pi^h\left(e^{i\lambda}\psi, e^{i\lambda}\psi\right)$$

for all $\psi\in\mathcal{D}(p_{\mathbf{A}}) = \mathcal{D}(\pi^{h+\varphi})$, i.e., the quadratic forms $p_{\mathbf{A}}$ and π^h are unitarily equivalent. □

2.5 Pauli operator generated by both potentials

Theorem 2.7 showed that every measure $\mu\in\overline{\mathcal{M}^*}$ can be generated by an h-potential, $\Delta h = \mu$, and we defined the Pauli operators. However, it may be useful to combine the scalar potential with the usual vector potential $\mathbf{A}\in L^2_{loc}$ to generate the given magnetic field. In this way one has more freedom in choosing the potentials. Typically, the singularities can be easier handled by the h-potential, and the standard $h = \frac{1}{2\pi}\log|\cdot| * \mu$ formula is (locally) available. But this formula exhibits a strong non-locality of h, and the truncation method of the proof of Theorem 2.7 is not particularly convenient in practice. Large distance behavior of the bulk magnetic field is better described by a vector potential. In this section we give such a unified definition of the Pauli operator.

For any $h\in L^1_{loc}$, $\mathbf{A}\in L^2_{loc}$ we define the quadratic form

$$\pi^{h,\mathbf{A}}(\psi,\psi) := \int \Big|(-2i\partial_{\bar z}+a)(e^{-h}\psi_+)\Big|^2 e^{2h} + \int \Big|(-2i\partial_z+\bar a)(e^{h}\psi_+)\Big|^2 e^{-2h}$$

on the maximal domain

$$\mathcal{D}(\pi^{h,\mathbf{A}}) := \Big\{ \psi \in L^2(\mathbf{R}^2, \mathbf{C}^2) \ : \ \|\psi\|_{h,\mathbf{A}} < \infty \Big\} ,$$

where $a = A_1 + iA_2$ and

$$\|\psi\|_{h,\mathbf{A}} := \Big[\|\psi\|^2 + \pi^{h,\mathbf{A}}(\psi,\psi) \Big]^{1/2} .$$

Let

$$\mathcal{P}^* := \Big\{ (h, \mathbf{A}) \ : \ h \in L^1_{loc}, \ e^{\pm h}\mathbf{A} \in L^2_{loc}, \ \Delta h \in \overline{\mathcal{M}^*}, \nabla^\perp \cdot \mathbf{A} \in \overline{\mathcal{M}} \Big\}$$

be the set of admissible potential pairs. The measure $\mu := \Delta h + \nabla^\perp\cdot\mathbf{A} \in \overline{\mathcal{M}}$ is called the magnetic field generated by $(h, \mathbf{A})$. We recall from Corollary 2.3 that $(h, \mathbf{A}) \in \mathcal{P}^*$ implies $h \in \bigcap_{p<2} W^{1,p}_{loc}$ and $e^{\pm h} \in L^2_{loc}$, moreover, $e^{\pm h}\mathbf{A} \in L^2_{loc}$ implies $\mathbf{A} \in L^2_{loc}$.

Since $\nabla^\perp\cdot\mathbf{A}$ has no discrete component (Proposition 2.10), the measure μ generated by $(h, \mathbf{A}) \in \mathcal{P}^*$ is in $\overline{\mathcal{M}^*}$. In particular, the set of measures generated by a potential pair from $\mathcal{P}^*$ is the same as the set of measures generated by only L^1_{loc} h-potentials (Theorem 2.7).

Theorem 2.11 *(i) (Self-adjointness). Assume that $(h, \mathbf{A}) \in \mathcal{P}^*$ and let $\mu := \Delta h + \nabla^\perp \cdot \mathbf{A}$. Then $\pi^{\mathbf{A},h}$ is a nonnegative symmetric closed form, hence it defines a unique self-adjoint operator $H_{h,\mathbf{A}}$.*

(ii) (Core). The set $\mathcal{C}_\mu$ (see (17)) is dense in $\mathcal{D}(\pi^{h,\mathbf{A}})$ with respect to $\| \cdot \|_{h,\mathbf{A}}$, i.e., it is a form core for $H_{h,\mathbf{A}}$.

(iii) (Consistency). If $(h, \mathbf{A}) \in \mathcal{P}^*$ and $\widetilde{h} \in L^1_{loc}$ such that $\Delta h + \nabla^\perp\cdot\mathbf{A} = \Delta\widetilde{h}$, then $H_{h,\mathbf{A}}$ is unitary equivalent to $H_{\widetilde{h}}$ defined in Theorem 2.5.

Definition 2.12 *For any $(h, \mathbf{A}) \in \mathcal{P}^*$ the operator $H_{h,\mathbf{A}}$ is called the* **Pauli operator with a potential pair** *$(h, \mathbf{A})$.*

Notice that Proposition 2.10 and (iii) of Theorem 2.11 guarantees that the Pauli operators with the same magnetic field are unitarily equivalent, irrespectively which definition we use.

Proof of Theorem 2.11. Part (i). The proof that $\pi^{h,\mathbf{A}}$ is closed is very similar to the proof of part (i) of Theorem 2.5. The operators $\partial_{\bar z}$ and ∂_z should be replaced by $\partial_{\bar z} + ia$ and $\partial_z - i\overline{a}$, but the extra terms with a can always be estimated by the local L^2 norm of $e^{\pm h}\mathbf{A}$.

Part (ii). Step 1. We need the following preliminary observation. Since $\mathbf{A} \in L^2_{loc}$, we can consider the decomposition $\mathbf{A} = \widetilde{\mathbf{A}} + \nabla\widetilde{\lambda}$, $\nabla \cdot \widetilde{\mathbf{A}} = 0$, $\widetilde{\mathbf{A}} \in L^2_{loc}$, $\widetilde{\lambda} \in W^{1,2}_{loc}$ (see Lemma 1.1 [L]). $\widetilde{\mathbf{A}}$ and $\widetilde{\lambda}$ are called the divergence-free and the gradient component of $\mathbf{A} \in L^2_{loc}$, and notice that $\widetilde{\mathbf{A}}$ is unique up to a smooth gradient, since if $\widetilde{\mathbf{A}} + \nabla\widetilde{\lambda} = \widetilde{\mathbf{A}}' + \nabla\widetilde{\lambda}'$ then $0 = \nabla \cdot (\mathbf{A} - \mathbf{A}') = \Delta(\widetilde{\lambda}' - \widetilde{\lambda})$, i.e., $\widetilde{\lambda}' - \widetilde{\lambda}$ is smooth.

Moreover, if $\mathbf{A}e^{\pm h} \in L^2_{loc}$, then $\widetilde{\mathbf{A}}e^{\pm h} \in L^2_{loc}$ as well. To see this, we fix a compact set K and a compact set K^* whose interior contains K, then we choose a cutoff function $0 \leq \varphi_K \leq 1$ with $\varphi_K \equiv 1$ on K and $\operatorname{supp} \varphi_K \subset K^*$. We let $\mathbf{A}_K := \varphi_K \mathbf{A} \in L^2$ and let λ be defined via its Fourier transform

$$\widehat{\lambda}(\xi) := \frac{\xi \cdot \widehat{\mathbf{A}}_K(\xi)}{|\xi|^2}, \qquad \xi \in \mathbf{R}^2$$

i.e., $-\Delta\lambda = \nabla \cdot \mathbf{A}$. Then $\nabla\lambda$ is obtained from $\mathbf{A}_K$ by the action of a singular integral operator whose multiplier is $\xi \otimes \xi/|\xi|^2$. Choose ω as in (19), where Q_0 is a dyadic square containing K^*, then $\omega \in A_2$. Hence, by the weighted L^2-inequality we have

$$\int |\nabla\lambda|^2\omega \leq C(\omega) \int |\mathbf{A}_K|^2\omega = C(\omega) \int |\mathbf{A}_K|^2 e^{2h}$$

with some ω-dependent constant. In particular $\nabla\lambda \in L^2_{loc}(e^{2h})$. The proof of $\nabla\lambda \in L^2_{loc}(e^{-2h})$ is identical. Now $\widetilde{\lambda}$ satisfies $\Delta\widetilde{\lambda} = -\Delta\lambda$ on K, i.e. $\widetilde{\lambda}$ and λ differ by an additive smooth function, hence $\nabla\widetilde{\lambda} \in L^2_{loc}(e^{\pm 2h})$, which means that $\widetilde{\mathbf{A}}e^{\pm h} \in L^2_{loc}$.

Step 2. We show that $\mathcal{C}_\mu \subset \mathcal{D}(\pi^{h,\mathbf{A}})$ if $(h, \mathbf{A}) \in \mathcal{P}^*$, $\mu = \Delta h + \nabla^\perp \cdot \mathbf{A}$. Let $\psi \in \mathcal{C}_\mu$ be compactly supported on K and let K^* be a compact set whose interior contains K. Since μ restricted to K^* has finite total variation, we can apply Theorem 2.2 for the restricted measure to construct a function $h^* \in L^1_{loc}$ such that $\Delta h^* = \mu = \Delta h + \nabla^\perp \cdot \widetilde{\mathbf{A}}$ on K^*. Then there is a real function χ such that $\widetilde{\mathbf{A}} = \nabla^\perp(h^* - h) + \nabla\chi$ (Lemma 1.1. of [L]). After taking the divergence, we see that χ is harmonic on K^*. Let $\varphi \in C^\infty$ be its harmonic conjugate, $\nabla\chi = \nabla^\perp\varphi$. We have the identity

$$\pi^{h,\mathbf{A}}(e^{-i\widetilde{\lambda}}\psi, e^{-i\widetilde{\lambda}}\psi) = \pi^{h,\widetilde{\mathbf{A}}}(\psi, \psi) = \pi^{h^*+\varphi}\Big(\psi, \psi\Big) \tag{22}$$

for any ψ supported on K^*. Since $\psi \in \mathcal{C}_\mu$, we can write $\psi_\pm = g_\pm e^{\pm h^*}$ and we see that the right hand side of (22) is finite, hence $e^{-i\widetilde{\lambda}}\psi \in \mathcal{D}(\pi^{h,\mathbf{A}})$. But by Schwarz inequality

$$\pi^{h,\mathbf{A}}(e^{-i\widetilde{\lambda}}\psi, e^{-i\widetilde{\lambda}}\psi) \geq \frac{1}{2}\pi^{h,\mathbf{A}}(\psi, \psi) - 4\sum_{\pm} \|g_\pm\|^2_\infty \int_{K^*} |\nabla\widetilde{\lambda}|^2 e^{\pm 2h}$$

hence $\psi \in \mathcal{D}(\pi^{h,\mathbf{A}})$ by Step 1.

Step 3. We now show that $\mathcal{C}_\mu$ is dense in $\mathcal{D}(\pi^{h,\mathbf{A}})$ with respect to $\|\!| \cdot |\!\|_{h,\mathbf{A}}$ if $(h,\mathbf{A}) \in \mathcal{P}^*$, $\mu = \Delta h + \nabla^\perp \cdot \mathbf{A}$. We first notice that it is sufficient to show that $\mathcal{C}_\mu$ is dense in the set

$$\overline{\mathcal{C}}_0 := \{\psi \in \mathcal{D}(\pi^{h,\mathbf{A}}), \,:\, \operatorname{supp}(\psi) \text{ compact}\}$$

similarly to Step 1 of the proof of Theorem 2.5 (ii). So let $\psi \in \mathcal{D}(\pi^{h,\mathbf{A}})$ be supported on a compact set K. As in Step 2, we let $h^* \in L^1_{loc}$ be a function such that $\Delta h^* = \mu$ on a compact neighborhood K^* of K, $K \subset \operatorname{int}(K^*)$. As before, we have $\widetilde{\mathbf{A}} = \nabla^\perp(h^* - h) + \nabla\chi$ with a harmonic χ and let $\varphi \in C^\infty(K^*)$ be its harmonic conjugate, $\nabla\chi = \nabla^\perp\varphi$. The identity (22) is now written as

$$\pi^{h,\mathbf{A}}(\psi,\psi) = \pi^{h^*+\varphi}\Big(e^{i\widetilde{\lambda}}\psi, e^{i\widetilde{\lambda}}\psi\Big) \tag{23}$$

for any ψ supported on K^*. In particular, $\psi \in \mathcal{D}(\pi^{h,\mathbf{A}})$ implies $e^{i\widetilde{\lambda}}\psi \in \mathcal{D}(\pi^{h^*+\varphi})$.

We define the set

$$\widetilde{\mathcal{C}}_\mu := \left\{ \psi = \begin{pmatrix} g_+ e^h \\ g_- e^{-h} \end{pmatrix} \;:\; g_\pm \in L^\infty_0,\ \Delta h = \mu \quad \text{on } \operatorname{supp}(g_-) \cup \operatorname{supp}(g_+) \right\}$$

where L^∞_0 denotes the set of bounded, compactly supported functions. The set $\widetilde{\mathcal{C}}_\mu$ is well defined, see the remark before the definition (17).

Since $\mathcal{C}_\mu$ is dense in $\mathcal{D}(\pi^{h^*+\varphi})$ with respect to $\|\!| \cdot |\!\|_{h^*+\varphi}$ by part (ii) of Theorem 2.5, we can find a sequence of spinors $\xi_n \in \mathcal{C}_\mu$ such that $\|\!|\xi_n - e^{i\widetilde{\lambda}}\psi|\!\|_{h^*+\varphi} \to 0$. We can assume that all ξ_n are supported in K^* (see remark at the end of Step 3 of the proof Theorem 2.2 (ii)). But then $\|\!|e^{-i\widetilde{\lambda}}\xi_n - \psi|\!\|_{h,\mathbf{A}} \to 0$ again by (23), in particular the set $\overline{\mathcal{C}}_1 := \mathcal{D}(\pi^{h,\mathbf{A}}) \bigcap \widetilde{\mathcal{C}}_\mu$ is dense in $\Big(\mathcal{D}(\pi^{h,\mathbf{A}}), \|\!| \cdot |\!\|_{h,\mathbf{A}}\Big)$.

Finally, we show that $\mathcal{C}_\mu$ is dense in $\Big(\overline{\mathcal{C}}_1, \|\!| \cdot |\!\|_{h,\mathbf{A}}\Big)$. Let $\chi \in \overline{\mathcal{C}}_1$, i.e., $\chi_\pm = g_\pm e^{\pm h}$ with some compactly supported bounded functions $g_\pm$.

Notice that if g is a bounded function, then $ge^{\pm h}\mathbf{A} \in L^2_{loc}$ since $e^{\pm h}\mathbf{A} \in L^2_{loc}$. In particular $(\partial_{\bar z} + ia)g_+ \in L^2(e^{2h})$ implies $\partial_{\bar z} g_+ \in L^2(e^{2h})$. But then $\nabla g_+ \in L^2(e^{2h})$ by Lemma 2.8 and similarly for g_-. We focus only on the spin-up part, the spin-down part is similar.

Let $g^{(\varepsilon)} := J_\varepsilon * g_+$, where J_ε is a standard mollifier (see Step 3. of the proof of Theorem 2.5 (ii)). Recall that $\|g^{(\varepsilon)}\|_\infty \le \|g_+\|_\infty$ and the functions $|\nabla g^{(\varepsilon)}| \le J_\varepsilon * |\nabla g_+|$ have an $L^2(e^{2h})$-integrable majorant using the weighted maximal inequality. By passing to a subsequence

$g^{(\varepsilon)} \to g_+$, $\nabla g^{(\varepsilon)} \to \nabla g_+$ in $L^2(e^{2h})$ and $g^{(\varepsilon)} \to g_+$ a.e. as $\varepsilon \to 0$. Therefore

$$\int \left|(\partial_{\bar z} + ia)(g^{(\varepsilon)} - g_+)\right|^2 e^{2h} + \left\|(g^{(\varepsilon)} - g_+)e^h\right\|^2$$

$$\le 2\int \left|\nabla(g^{(\varepsilon)} - g_+)\right|^2 e^{2h} + 2\int |\mathbf{A}|^2 |g^{(\varepsilon)} - g_+|^2 e^{2h} + \left\|(g^{(\varepsilon)} - g_+)e^h\right\|^2 \to 0$$

as $\varepsilon \to 0$ since $e^h \in L^2_{loc}$ and $e^h\mathbf{A} \in L^2_{loc}$.

Part (iii). Since $\Delta\widetilde{h} \in \overline{\mathcal{M}^*}$, we know that $\widetilde{h} \in L^2_{loc}$ (Corollary 2.3). Since $\nabla^\perp \cdot (\nabla^\perp h - \nabla^\perp \widetilde{h} + \mathbf{A}) = 0$, there exists $\lambda \in W^{1,2}_{loc}$ with $\nabla^\perp(h - \widetilde{h}) + \mathbf{A} = \nabla\lambda$. A simple calculation shows that

$$\pi^{h,\mathbf{A}}(\psi,\psi) = \pi^{\widetilde{h}}(e^{i\lambda}\psi, e^{i\lambda}\psi)\ . \qquad \square$$

3 Aharonov-Casher theorem

We prove the following extension of the Aharonov-Casher theorem:

Theorem 3.1 *Let $\mu \in \overline{\mathcal{M}}$ and we assume that $\mu^* \in \mathcal{M}^*$. Let $\Phi := \frac{1}{2\pi}\int \mu^*$. The dimension of the kernel of any Pauli operator H with magnetic field μ is given*

$$dimKer\, H = \begin{cases} [|\Phi|] & if \quad \Phi \notin \mathbf{Z} \quad or \quad \Phi = 0 \\ [|\Phi|] \ or \ [|\Phi|] - 1 & if \quad \Phi \in \mathbf{Z} \setminus \{0\} \end{cases}$$

(here $[a]$ denotes the integer part of a). In case of nonzero integer Φ both cases can occur, but if, additionally, μ^* has a compact support or has a definite sign, then always $dimKer\, H = [|\Phi|] - 1$. In all cases the kernel is in the eigenspace of σ_3: $Ker(H) \subset \{\psi \ : \ \sigma_3\psi = -s\psi\}$ with $s = sign(\Phi)$.

Proof. We can assume that $\mu \in \mathcal{M}^*$ and we choose $h := h_\varepsilon$ with some $\varepsilon < \varepsilon(\mu)$. By (iii) of Theorem (2.5) it is sufficient to consider the operator H_h. We can also assume that $\Phi \ge 0$.

Suppose first that Φ is not integer, let $\{\Phi\} = \Phi - [\Phi]$ be its fractional part. Choose $\varepsilon < \min\left\{\varepsilon(\mu), \{\Phi\}/3, (1 - \{\Phi\})/3\right\}$. Any normalized eigenspinor ψ with $\pi^h(\psi,\psi) = 0$ must be in the form $\psi = (e^h g_+, e^{-h} g_-)$ where g_+ is holomorphic and g_- is antiholomorphic.

First we show that $g_+ = 0$. Let $u \in \mathbf{R}^2$ with $|u| \geq R(\varepsilon, \mu) + 1$,

$$1 = \|\psi\|^2 \geq \int_{Q(u)} e^{2h}|g_+|^2 \geq \Big(\int_{Q(u)} e^{h_1}|g_+|\Big)^2 \Big(\int_{Q(u)} e^{-2h_2}\Big)^{-1}$$

then by Theorem 2.2 and subharmonicity of $|g_+|$ we see that $|g_+(u)| \leq C \int_{Q(u)} |g_+| \leq C\langle u\rangle^{-\Phi+2\varepsilon} \to 0$ as $u \to \infty$, hence $g_+ = 0$.

A similar calculation shows that $|g_-(u)| \leq C\langle u\rangle^{\Phi+2\varepsilon}$, i.e., g_- must be a polynomial of degree at most $[\Phi]$ since $\Phi + 2\varepsilon < [\Phi] + 1$. However, a polynomial of degree $[\Phi]$ would give

$$\int e^{-2h}|g_-|^2 \geq C \sum_{k\in\Lambda_0, |k|\geq R} |k|^{2[\Phi]} \int_{Q(k)} e^{-2h} \geq C \sum_{k\in\Lambda_0, |k|\geq R} |k|^{2[\Phi]} \Big(\int_{Q(k)} e^{2h}\Big)^{-1}$$

$$\geq C \sum_{k\in\Lambda_0, |k|\geq R} |k|^{2([\Phi]-\Phi-2\varepsilon)} = \infty$$

for some large enough R and various constants C.

On the other hand, the functions $g_-(z) = 1, z, \ldots, z^{[\Phi]-1}$ all give normalizable spinors since for these choices

$$\int e^{-2h}|g_-|^2 \leq C \sum_{k\in\Lambda_0} |k|^{2([\Phi]-\Phi-1)} \int_{Q(k)} e^{-2h_2} \leq C \sum_{k\in\Lambda_0} |k|^{2([\Phi]-\Phi-1+\varepsilon)} < \infty\,. \tag{24}$$

If Φ is integer, then the same arguments work except (24); in fact $g_-(z) = z^{\Phi-1}$ may or may not give normalizable solutions.

If μ is compactly supported and $\Phi \geq 0$ is integer then from the definition of h (10)-(12) we see that $\Big|h(x) - \Phi \log|x|\Big|$ is bounded for all large enough $|x|$. Similarly one can easily verify that if $\mu \geq 0$, then $h(x) \leq \Phi \log\langle x\rangle + C$ since $\log|x-y| \leq \log\langle x\rangle + \log\langle y\rangle + C$. In both cases $z^{\Phi-1}e^{-h}$ is not L^2-normalizable at infinity.

However, if μ can change sign and not compactly supported then there could be $\Phi \in \mathbf{Z}$ zero energy states. For example the radial field (with $\beta > 0$, $N \in \mathbf{N}$)

$$B(x) = \begin{cases} 2(N+\beta)e^{-2} & \text{for} \quad |x| \leq e \\ -\beta(|x|\log|x|)^{-2} & \text{for} \quad e < |x| \end{cases},$$

with $\Phi = \frac{1}{2\pi}\int B = N$, is generated by a radial potential $h(x)$ such that $h(x) = N\log|x| + \beta\log\log|x|$ for large x and is regular for small x. The threshold state $z^{\Phi-1}e^{-h}$ is normalizable only for $\beta > \frac{1}{2}$, so in this case the dimension of the kernel is Φ, otherwise $\Phi - 1$. $\square$

This theorem requires $\mu \in \mathcal{M}$. We will see in Section 4 that the Aharonov-Casher theorem need not be true for magnetic fields with infinite total variation. However, the proof above still works for magnetic fields that can be decomposed into the sum of a component in $\mathcal{M}$ and a component with a regularly behaving generating potential. We just remark one possible extension:

Corollary 3.2 *Suppose that $\mu \in \overline{\mathcal{M}}$ can be written as $\mu = \mu_{rad} + \widetilde{\mu}$ such that $\widetilde{\mu} \in \mathcal{M}$ and μ_{rad} is a rotationally symmetric Borel measure (i.e., $\mu_{rad} = \mu_{rad} \circ R$ for any rotation R in $\mathbf{R}^2$ around the origin). We can assume that $\mu_{rad}(\{0\}) = 0$ by including the possible delta function at the origin into $\widetilde{\mu}$. We assume that that $\Phi_{rad} := \lim_{R\to\infty} \Phi(R) := \lim_{R\to\infty} \frac{1}{2\pi} \int_{|x|\le R} \mu_{rad}(\mathrm{d}x)$ exists (possibly infinite) and $h_{rad}(x) := \Phi(|x|) \log |x|$ satisfies $\nabla h_{rad} \in L^\infty_{loc}$. Then all statements of Theorem 3.1 for the Pauli operator with magnetic field μ are valid with $\Phi := \Phi_{rad} + \frac{1}{2\pi} \int \widetilde{\mu}^*(\mathrm{d}x)$.*

Proof. Let $\mathbf{A}_{rad} := \nabla^\perp h_{rad} \in L^\infty_{loc}$ and let $\widetilde{h}$ be the generating function of $\widetilde{\mu}$ given in Theorem 2.2 for some $\varepsilon < \varepsilon(\widetilde{\mu})$. Then $(\widetilde{h}, \mathbf{A}_{rad}) \in \mathcal{P}^*$ with a magnetic field μ, hence the Pauli operators are well defined and unitarily equivalent. Clearly $\pi^{\widetilde{h}, \mathbf{A}_{rad}}(\psi, \psi) = \pi^h(\psi, \psi)$ with $h = h_{rad} + \widetilde{h}$, hence any zero energy state ψ must be in the form $\psi = (e^h g_+, e^{-h} g_-)$ where $g_\pm$ are (anti)holomorphic. Now we can follow the proof of Theorem 3.1. We use the estimates (7), (9) for the $\widetilde{h}$ part of the generating potential and we estimate $h_{rad}(x) := \Phi(|x|) \log |x|$ by $\lim_{R\to\infty} \Phi(R)$ for large R, $\lim_{R\to 0} |\Phi(R)| = 0$ for small R and by $\Phi \in L^\infty_{loc}(\mathbf{R})$ for intermediate R. $\square$

4 Counterexample

In this section we present the construction of the Counterexample 1.1. For simplicity, the magnetic field will be only bounded and not continuous, but it will be easy to see that a small mollification does not modify the estimates.

Let $\delta < 1/10$ be a fixed small number and $N_k = 10k$ for $k = 1, 2, \ldots$. We denote the N_k-th roots of unity by $\zeta_{k,j} := \exp(2\pi i j / N_k)$, $j = 1, 2, \ldots N_k$. Let $D_{k,n,j} := \{x \ : \ |x - n\zeta_{k,j}| \le \delta\}$ be the disk of radius δ about $n\zeta_{k,j}$, let $\overline{D}_{k,n,j} := \{x \ : \ |x - n\zeta_{k,j}| \le 2\delta\}$ be the twice bigger disk.

Let $0 < \varepsilon < 1/4$ be fixed. The magnetic field B is given as $B := B_0 + \widetilde{B} - \widehat{B}$ with

$$B_0(x) := 2(1+\varepsilon)\delta^{-2}\mathbf{1}(|x| \le \delta)$$

$$\widetilde{B} := \sum_{k=1}^{\infty} \widetilde{B}_k, \qquad \widetilde{B}_k := \sum_{n=4^k+1}^{4^k+2^k} \widetilde{B}_{k,n}, \qquad \widetilde{B}_{k,n}(x) := \sum_{j=1}^{N_k} 2\delta^{-2}\mathbf{1}(x \in D_{k,n,j})$$

and

$$\widehat{B} := \sum_{k=1}^{\infty} \widehat{B}_k, \qquad \widehat{B}_k := \sum_{n=4^k+1}^{4^k+2^k} \widehat{B}_{k,n}, \qquad \widehat{B}_{k,n}(x) := \frac{1}{2\pi}\int_0^{2\pi} \widetilde{B}_{k,n}(|x|e^{i\theta})\mathrm{d}\theta$$

The field $\widetilde{B}$ consists of uniform field "bumps" with flux 2π localized on the disks $D_{k,n,j}$ around points $n\zeta_{k,j}$ that are located on concentric circles of radius n. The field $\widehat{B}$ is the radial average of $\widetilde{B}$. The field $B_k = \widetilde{B}_k - \widehat{B}_k$ is called the k-th *band*. The relation (5) is straightforward by construction.

We define the potential function $h := h_0 + \widetilde{h} - \widehat{h}$, with

$$h_0(x) := \frac{1}{2\pi}\int_{\mathbf{R}^2} \log|x-y| B_0(y)\mathrm{d}y$$

$$\widetilde{h} := \sum_{k=1}^{\infty} \widetilde{h}_k, \qquad \widetilde{h}_k := \sum_{n=4^k+1}^{4^k+2^k} \widetilde{h}_{k,n}, \qquad \widetilde{h}_{k,n}(x) := \frac{1}{2\pi}\int_{\mathbf{R}^2} \log|x-y|\widetilde{B}_{k,n}(y)\mathrm{d}y - N_k \log n$$

and

$$\widehat{h} := \sum_{k=1}^{\infty} \widehat{h}_k, \qquad \widehat{h}_k := \sum_{n=4^k+1}^{4^k+2^k} \widehat{h}_{k,n}, \qquad \widehat{h}_{k,n}(x) := \frac{1}{2\pi}\int_0^{|x|} \frac{\mathrm{d}r}{r}\int_{|y|\le r} \widehat{B}_{k,n}(y)\mathrm{d}y\ .$$

Clearly $\Delta\widetilde{h}_k = \widetilde{B}_k$, $\Delta\widehat{h}_k = \widehat{B}_k$. Easy computations yield the following relations:

$$h_0(x) = (1+\varepsilon)\log|x| \qquad \text{for} \quad |x| \ge \delta\ ,$$

$$\widetilde{h}_{k,n}(x) = \sum_{j=1}^{N_k} \log\Big|\zeta_{k,j} - \frac{x_1 + ix_2}{n}\Big| = \log\Big|1 - \Big(\frac{x_1+ix_2}{n}\Big)^{N_k}\Big| \qquad \text{for} \quad x \notin \bigcup_{j=1}^{N_k} D_{k,n,j}\ ,$$

$$\widehat{h}_{k,n}(x) = \begin{cases} 0 & \text{for} \quad |x| \le n-\delta \\ N_k\Big[\log\frac{|x|}{n} + O(n^{-1})\Big] & \text{for} \quad |x| \ge n-\delta\ . \end{cases}$$

The infinite sums in the definition of $\widetilde{h}$ and $\widehat{h}$ are absolutely convergent, hence $h \in L^\infty_{loc}$. The sum of the $\widetilde{h}_k(x)$'s converges since

$$\sum_{k=1}^{\infty}\sum_{n=4^k+1}^{4^k+2^k} \Big|\frac{x}{n}\Big|^{N_k} < \infty$$

for each fixed x and $\widehat{h}_k(x)$ is actually zero for all but finite k. Therefore we know that $\Delta h = B$ in distributional sense. Moreover, we can rearrnge the sums and write

$$h = h_0 + \sum_{k=1}^{\infty} h_k, \qquad h_k := \sum_{n=4^k+1}^{4^k+2^k} h_{k,n}\ , \qquad h_{k,n} := \widetilde{h}_{k,n} - \widehat{h}_{k,n}\ .$$

A short calculation shows that for each k_0

$$\sum_{\substack{k=1\\ k\neq k_0}}^{\infty} |h_k(x)| = O(1) \qquad \text{for} \quad 3\cdot 4^{k_0-1} - 1 \le |x| \le 3\cdot 4^{k_0} + 1\ . \tag{25}$$

Hence the size of $h(x)$ is determined by $h_0(x)$ and the band nearest to x.

Now we show that $\int e^{-2h} = \infty$, in fact $\int_D e^{-2h} = \infty$ where $D := \bigcup_{k,n,j}\big(\overline{D}_{k,n,j} \setminus D_{k,n,j}\big)$. The proof of $\int e^{2h} = \infty$ is similar but easier. We fix $k \ge 1$, $4^k + 1 \le m \le 4^k + 2^k$, $1 \le \ell \le N_k$ and let $x \in \overline{D}_{k,m,\ell} \setminus D_{k,m,\ell}$. Then

$$h_{k,n}(x) = \begin{cases} \log\Big|1 - \Big(\frac{n}{x_1+ix_2}\Big)^{N_k}\Big| + N_k O(n^{-1}) & \text{for} \quad n < m \\ \log\Big|1 - \Big(\frac{x_1+ix_2}{n}\Big)^{N_k}\Big| & \text{for} \quad n > m\ . \end{cases}$$

Writing $x_1 + ix_2 = m\zeta_{k,\ell} + (\varrho_1 + i\varrho_2)$, $\varrho = (\varrho_1,\varrho_2) \in \mathbf{R}^2$, $\delta \le |\varrho| \le 2\delta$ and expanding $h_{k,n}(x)$ around $m\zeta_{k,\ell}$ up to second order in ϱ we easily obtain that $h_{k,n}(x) \le N_k O(n^{-1})$ for each $n \ne m$ if δ is small enough, hence $h_k(x) \le h_{k,m}(x) + O(1)$.

Moreover, $h_{k,m}(x) = \log\Big|1 - [(x_1 + ix_2)/m]^{N_k}\Big| + O(1)$ since $|\widehat{h}_{k,m}(x)| = O(1)$ for any $m - 2\delta \le |x| \le m + 2\delta$. Hence

$$\int e^{-2h} \ge C \sum_{k=1}^{\infty} \sum_{m=4^k+1}^{4^k+2^k} \frac{1}{m^{2(1+\varepsilon)}} \sum_{\ell=1}^{N_k} \int_{\overline{D}_{k,m,\ell}\setminus D_{k,m,\ell}} \Big|1 - \Big(\frac{x}{m}\Big)^{N_k}\Big|^{-2} \mathrm{d}x$$

$$= C \sum_{k=1}^{\infty} \sum_{m=4^k+1}^{4^k+2^k} \frac{N_k}{m^{2(1+\varepsilon)}} \int_{\delta\le|\varrho|\le 2\delta} \Big|1 - \Big(1 + \frac{\varrho_1 + i\varrho_2}{m}\Big)^{N_k}\Big|^{-2} \mathrm{d}\varrho \ge C \sum_{k=1}^{\infty} \frac{1}{N_k} 2^{k(1-4\varepsilon)} = \infty\ .$$

Finally, we have to show that $e^{-h}\bar{f} \notin L^2(\mathbf{R}^2)$ for any entire function f.

First we show that f cannot have zeros. Suppose that a is (one of) its zero closest to the origin, i.e., $f(z) = (z-a)^m g(z)$, g is entire, $g(0) \ne 0$, $m \ge 1$. Let $A_k := \{x\ :\ 3\cdot 4^k - 1 \le |x| \le 3\cdot 4^k + 1\}$, then $h(x) = h_0(x) + O(1)$ for all $x \in A_k$ by (25). Hence for a large enough K

$$\int e^{-2h}|f|^2 \ge C \sum_{k=K}^{\infty} \int_{A_k} e^{-2h_0(x)} |x-a|^{2m} |g(x)|^2 \mathrm{d}x$$

$$\geq C\sum_{k=K}^{\infty}4^{2k(m-1-\varepsilon)}\int_{A_k}|g(x)|^2\mathrm{d}x \geq C|g(0)|^2\sum_{k=K}^{\infty}4^{2k(m-1-\varepsilon)}\cdot 4^k = \infty$$

using that $|g|^2$ is subharmonic and the area of A_k is of order 4^k.

Now, since f has no zeros, we can write $f = e^{\varphi}$ and we would like to show that φ is constant. It is enough to show that $R := \mathrm{Re}\,\varphi$ is constant and we can assume $R(a) = 0$. Suppose that $\nabla R(a) \neq 0$ for some $a \in \mathbf{C}$. Let z_k be the point where the maximum of R over the closed disk $D_k := \{|x| \leq 3\cdot 4^k\}$ is attained. Since R is harmonic, $|z_k| = 3\cdot 4^k$. Using (25) and the subharmonicity of $|e^{2\varphi}|$, we have

$$\int e^{-2h}|f|^2 \geq C\sum_{k=1}^{\infty}4^{-2k(1+\varepsilon)}\int_{|x-z_k|\leq 1}|e^{2\varphi(x)}|\mathrm{d}x \geq C\sum_{k=1}^{\infty}4^{-2k(1+\varepsilon)}e^{2R(z_k)}\ . \tag{26}$$

From the Poisson formula we easily obtain $|\nabla R(a)| \leq 4^{-k}\max_{D_k} R = 4^{-k}R(z_k)$ for large enough k. Hence $R(z_k) \geq 4^k|\nabla R(a)|$ and the integral in (26) is infinite. $\square$.

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